Title: Vertex algebras of divisors in toric Calabi-Yau threefolds from perverse coherent extensions

Speakers: Dylan Butson

Series: Mathematical Physics

Date: November 04, 2022 - 1:30 PM

URL: https://pirsa.org/22110046

Abstract: I'll explain work in progress, joint with Miroslav Rapcak, on geometric constructions of vertex algebras associated to divisors in toric Calabi-Yau threefolds, in terms of moduli stacks of objects in certain exotic abelian subcategories of complexes of coherent sheaves on the underlying threefold. These vertex algebras were originally proposed by Gaiotto-Rapcak, and constructed mathematically in the example of affine space by Rapcak-Soibelman-Yang-Zhao, building on Schiffmann-Vasserot's proof of the AGT conjecture. We give a geometric explanation and generalization of the quivers with potential that feature in the latter results, and outline the analogous construction of vertex algebras in this setting.

Zoom link: https://pitp.zoom.us/j/93749710253?pwd=Y3NUTHBXZ3FPQUdPU0E0d0ttVzFFdz09

Grojnowski, Nakajima – Heisenberg alg's + Hilbert schemes
Schiffmann, Vasserot – Cherednik algs, W algs and
Rapčák, Soibelman, Yang, Zhao – CoHAs, Vortex algs and i

Nekrasov – Seiberg-Witten prepotential + instanton counting.
Alday, Gaiotto, Tachikawa – Liouville correlation f^{n}s from 4d
Belavin, Bershtein, Feigin, Litvinov, Tarnopolsky – Instanton moduli
Bershtein, Feigin, Litvinov – Coupling CFTs + Nakajima-Yoshioka
Gaiotto, Rapčák – VAs at the corner.
Creutzig, Gaiotto – VAs for S duality / Arakawa, Creutzig, Fe
(Galakov), Li, Yamazaki – ..., Quiver Yangians, ...

rojnowski, Nakajima – Heisenberg alg's + Hilbert schemes of points on surfac

chiffmann, Vasserot – Cherednik algs, W algs and instantons

pčák, Soibelman, Yang, Zhao – CoHAs, Vortex algs and instantons

Nekrasov – Seiberg-Witten prepotential + instanton counting

lday, Gaiotto, Tachikawa – Liouville correlation f^{n}s from 4d gauge theory

lavin, Bershtein, Feigin, Litvinov, Tarnopolsky – Instanton moduli + Coset CFT

Bershtein, Feigin, Litvinov – Coupling CFTs + Nakajima–Yoshioka blowup eq^{n}s

aiotto, Rapčák – VAs at the corner

reutzig, Gaiotto – VAs for S duality | Arakawa, Creutzig, Feigin – Urod algebra

(Galakov), Li, Yamazaki – ..., Quiver Yangians, ...

Intro: Let S smooth proj. surface $/\mathbb{C}$.

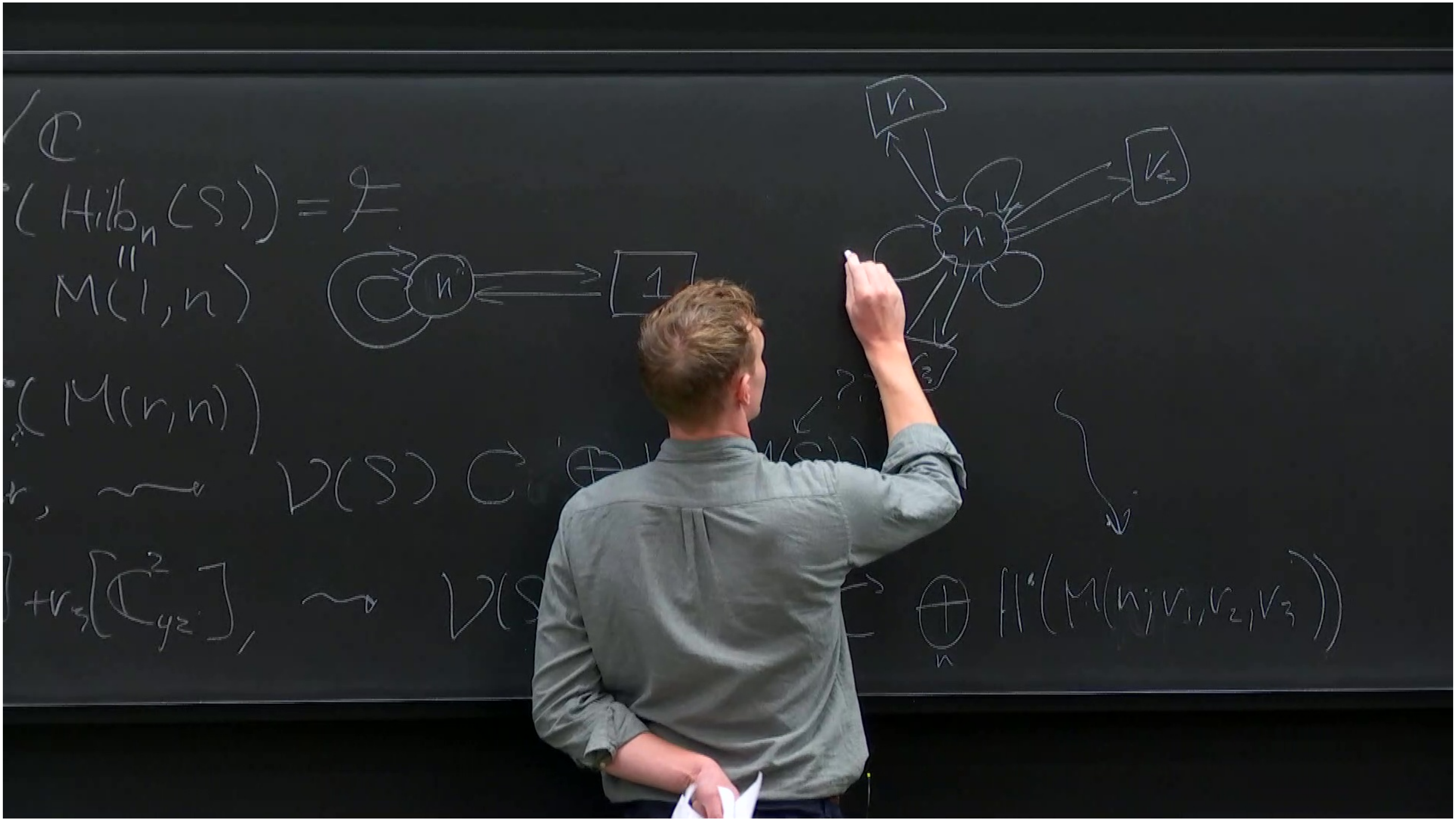
Then $\mathrm{Heis}_{H^\bullet(S)} \circlearrowright \bigoplus_{n\in\mathbb{N}} H^\bullet(\mathrm{Hilb}_n(S)) = \mathcal{F}$

$S=\mathbb{C}^2$ $\quad \mathrm{Heis} \qquad \mathrm{Hilb}_n(S) = M(1,n)$

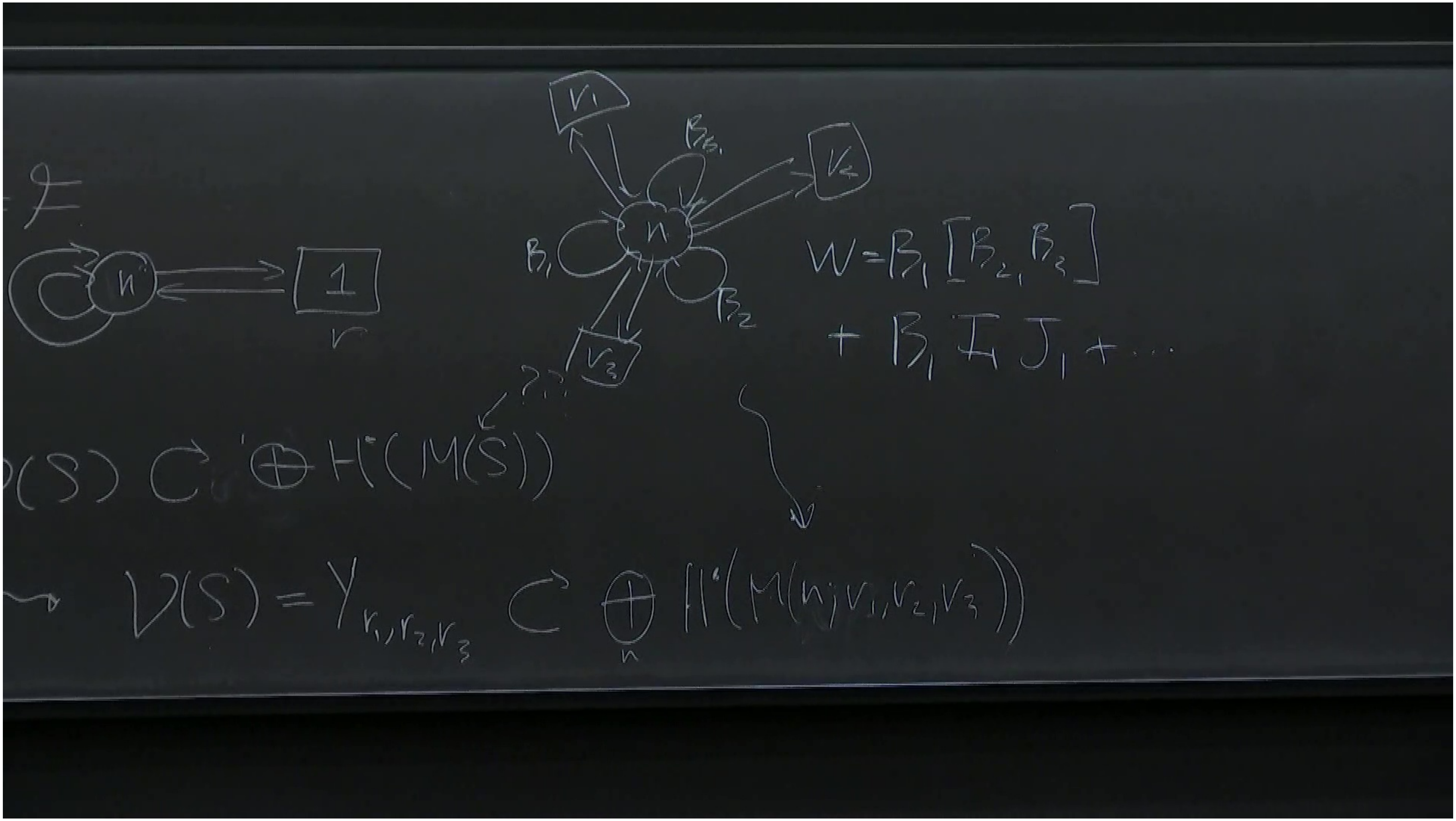
[AGT, S.V] $\quad W^k(\mathfrak{gl}_r) \circlearrowright \bigoplus H^\bullet_{T}(M(r,n))$

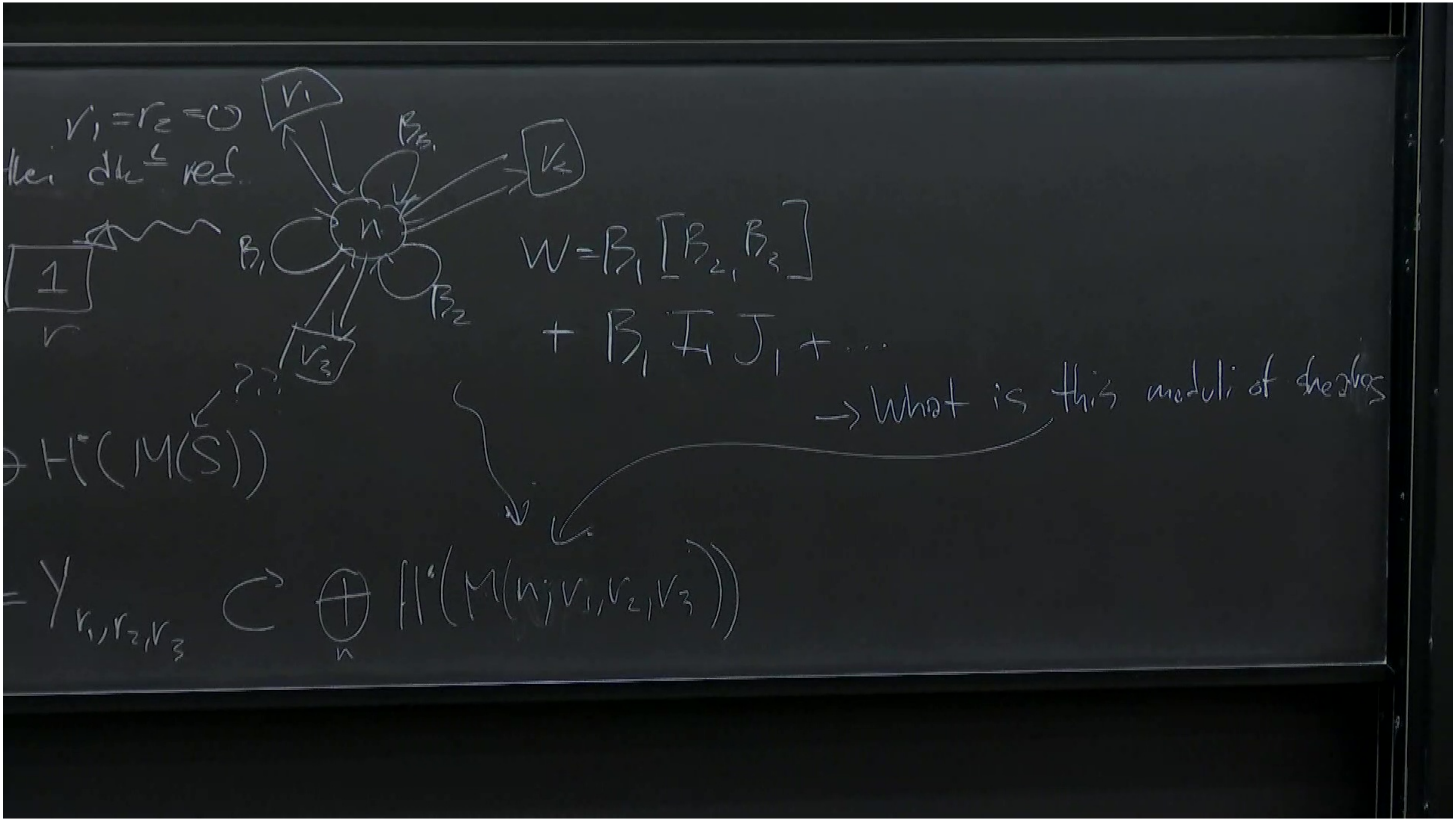
[GR] X toric CY3, S a divisor, $\rightsquigarrow$ $V(S) \circlearrowright$

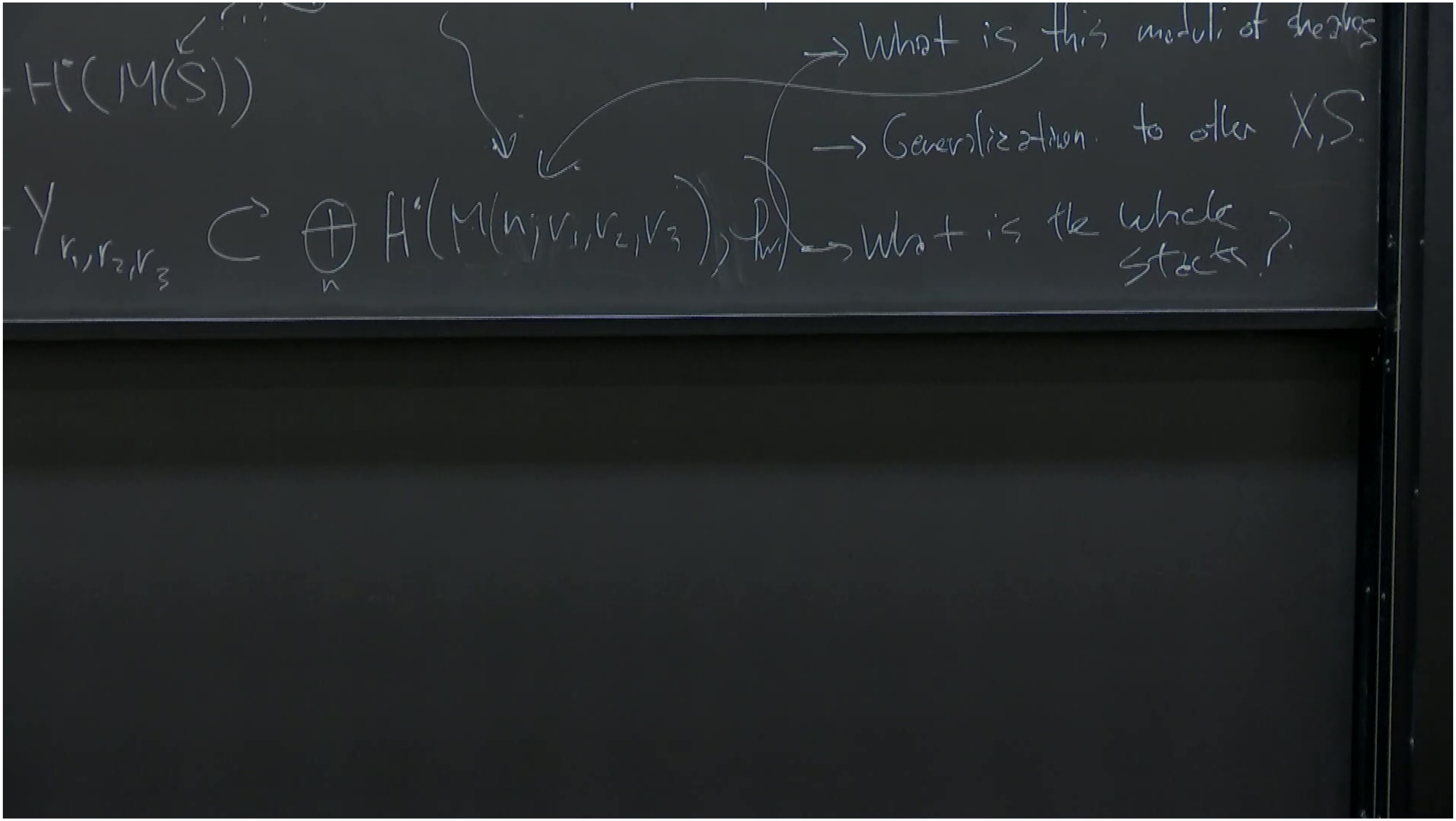
[RS.YZ] $X=\mathbb{C}^3$, $S = r_1[\mathbb{C}^2_{xy}] + r_2[\mathbb{C}^2_{xz}] + r_3[\mathbb{C}^2_{yz}]$, $\rightsquigarrow$



M(1,n)
M(r,n)
V(S)
M(n;v1,v2,v3)


v1
v2
v3
n
1
r
B1
B2
B3
W = B1 [B2, B3]
+ B1 I1 J1 + ...
??
(S) ⊂ ⊕ H*(M(S))
V(S) = Y r1,r2,r3 ⊂ ⊕n H*(M(n; v1, v2, v3))

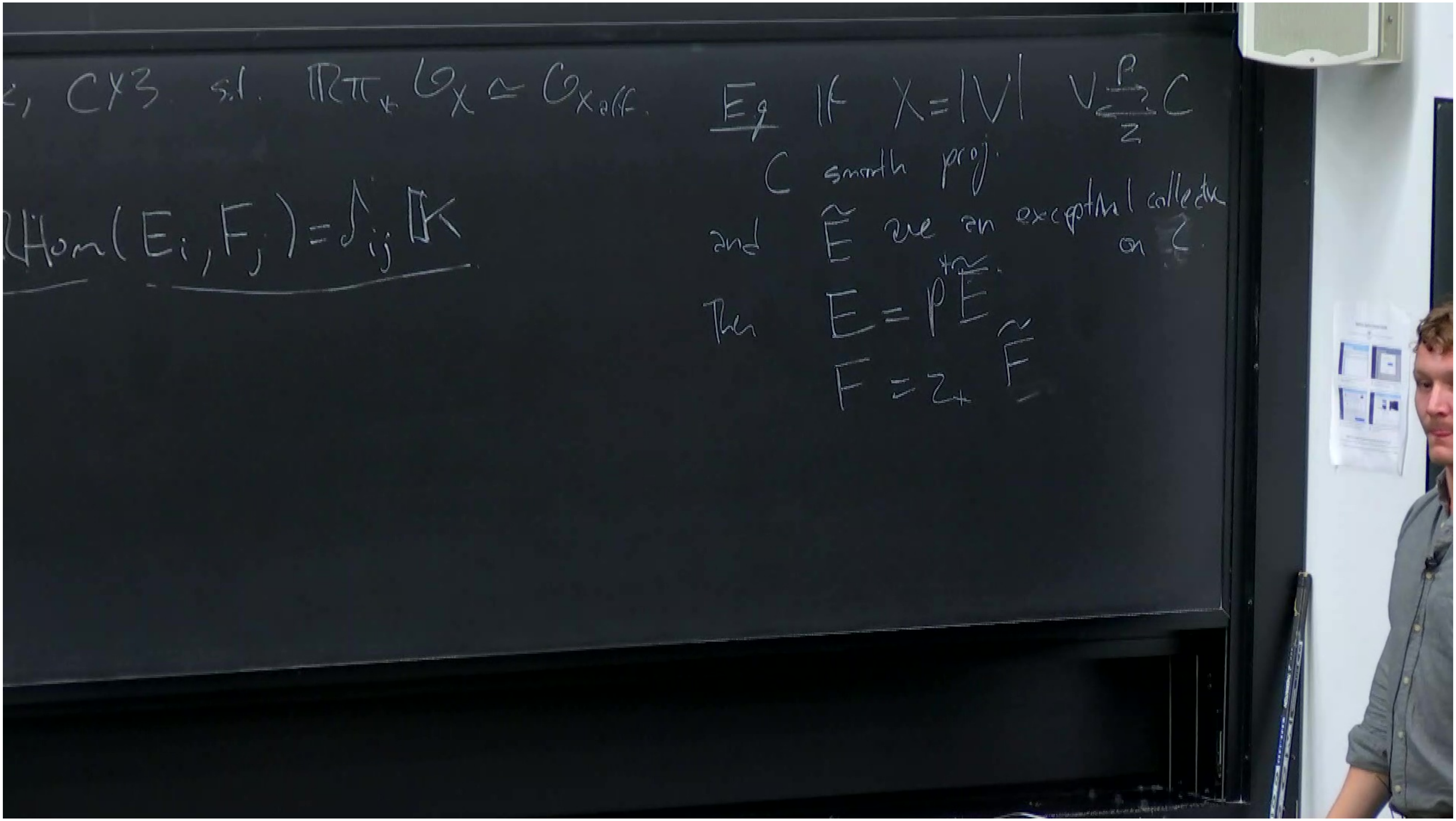

$r_1 = r_2 = 0$
1
r
n
r_1
r_2
r_3
B_1
B_2
B_3
$W = B_1 [B_2, B_3]$
$+ B_1 I_1 J_1 + \dots$
→ What is this moduli of sheaves
$H^*(M(S))$
$Y_{r_1, r_2, r_3} \subset \bigoplus_n H^*(M(n; r_1, r_2, r_3))$


H•(M(S))
Y_{r1,r2,r3} ⊂ ⊕_n H•(M(n; r1,r2,r3))
→ What is this moduli of sheaves
→ Generalization to other X, S.
→ What is the whole stack?

let $X \xrightarrow{\pi} X_{aff}$ a small resolution of X_{aff} an affine, toric, C

Fix $E = \bigoplus_{i \in I} E_i \in Coh(X)$ proj. gen., $\Lambda = End(E)$.

$F = \bigoplus_{i \in I} F_i \in D^b Coh_{cs}(X)$ simple, cptly supp,



, CY3 s.t. $\mathbb{R}\pi_* \mathcal{O}_X \simeq \mathcal{O}_{X_{aff}}$.
$\mathbb{R}\mathrm{Hom}(E_i, F_j) = \delta_{ij}\,\mathbb{K}$
Eg If $X = |V|$ $V \underset{z}{\overset{p}{\rightleftarrows}} C$
C smooth proj.
and $\tilde{E}$ are an exceptional collection on C.
Then $E = p^* \tilde{E}$
$F = z_* \tilde{E}$

resolution of X_{aff} an affine, toric, CY3 s.t. $\mathbb{R}\pi_* \mathcal{O}_X \simeq \mathcal{O}_{X_{aff}}$.

proj. gen., $\Lambda = \mathrm{End}(E)$.

$\mathrm{Coh}_{cs}(X)$ ~~simple~~, cctly supp, s.t. $\mathbb{R}\mathrm{Hom}(E_i, F_j) = \delta_{ij}\mathbb{K}$, $\Sigma = \mathrm{Ext}^\bullet(F, F)$

E

and

Then

...t $X \xrightarrow{\pi} X_{aff}$ a small resolution of X_{aff} an affine, toric, CY3

...x $E = \bigoplus_{i \in I} E_i \in Coh(X)$ (Tilting) ~~proj.~~ gen., $\Lambda = End(E)$.

$F = \bigoplus_{i \in I} F_i \in D^b Coh_{cs}(X)$ ~~simple~~, cptly supp, s.t. $\mathbb{R}Hom(E_i,$ ...

$S = \bigoplus_{i \in I} \mathbb{K}_i$ and moreover.

...n, Λ, Σ are augmented over S

$$\Lambda = \Sigma^! = Hom_\Sigma(S,S) = \left(\otimes_S^{\bullet} \Sigma^\vee[-1],\ d = \sum_k m_k^\vee \right)$$

...oshot: $\Lambda = \mathbb{K}\langle Q_X, W_X \rangle$ where Q_X is DG quiver w. vertices I

e, toric, CY3 s.t. $\mathbb{R}\pi_* \mathcal{O}_X \simeq \mathcal{O}_{X_{\mathrm{aff}}}$.

s.t. $\mathbb{R}\mathrm{Hom}(E_i, F_j) = \delta_{ij}\mathbb{K}$, $\Sigma = \mathrm{Ext}^\bullet(F,F)$ (an A_∞ alg w. str maps m_k:

and moreover.

$d = \sum_k m_k^\vee$)

w. vertices I, $\mathbb{K}\langle E_Q\rangle = \Sigma^\vee[-1]$ (deg 0 edges are Ext^1's)

<u>Eg</u> If $X = |V|$ $V \underset{z}{\overset{p}{\rightleftarrows}} C$

C smooth proj.

and $\tilde{E}$ are an exceptional collection on C.

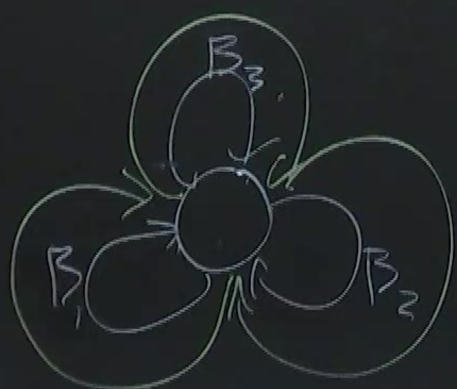
Then $E = p^*\tilde{E}$

$F = z_*\tilde{F}$

E.g.: $X = \mathbb{C}^3$, $E = \mathcal{O}$, $F = \mathcal{O}_{p}$,

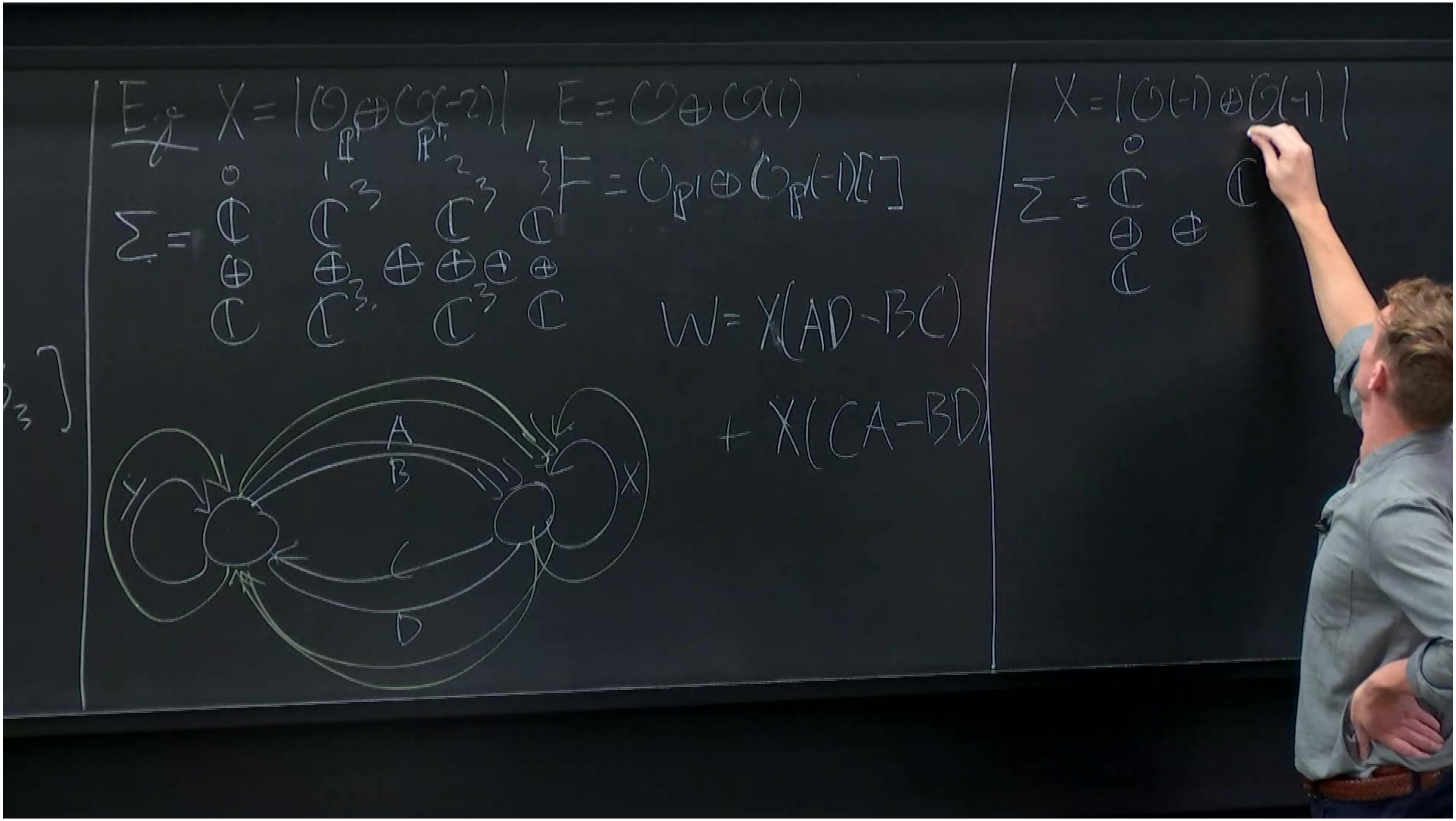
$\Lambda = \mathbb{C}[x,y,z]$,

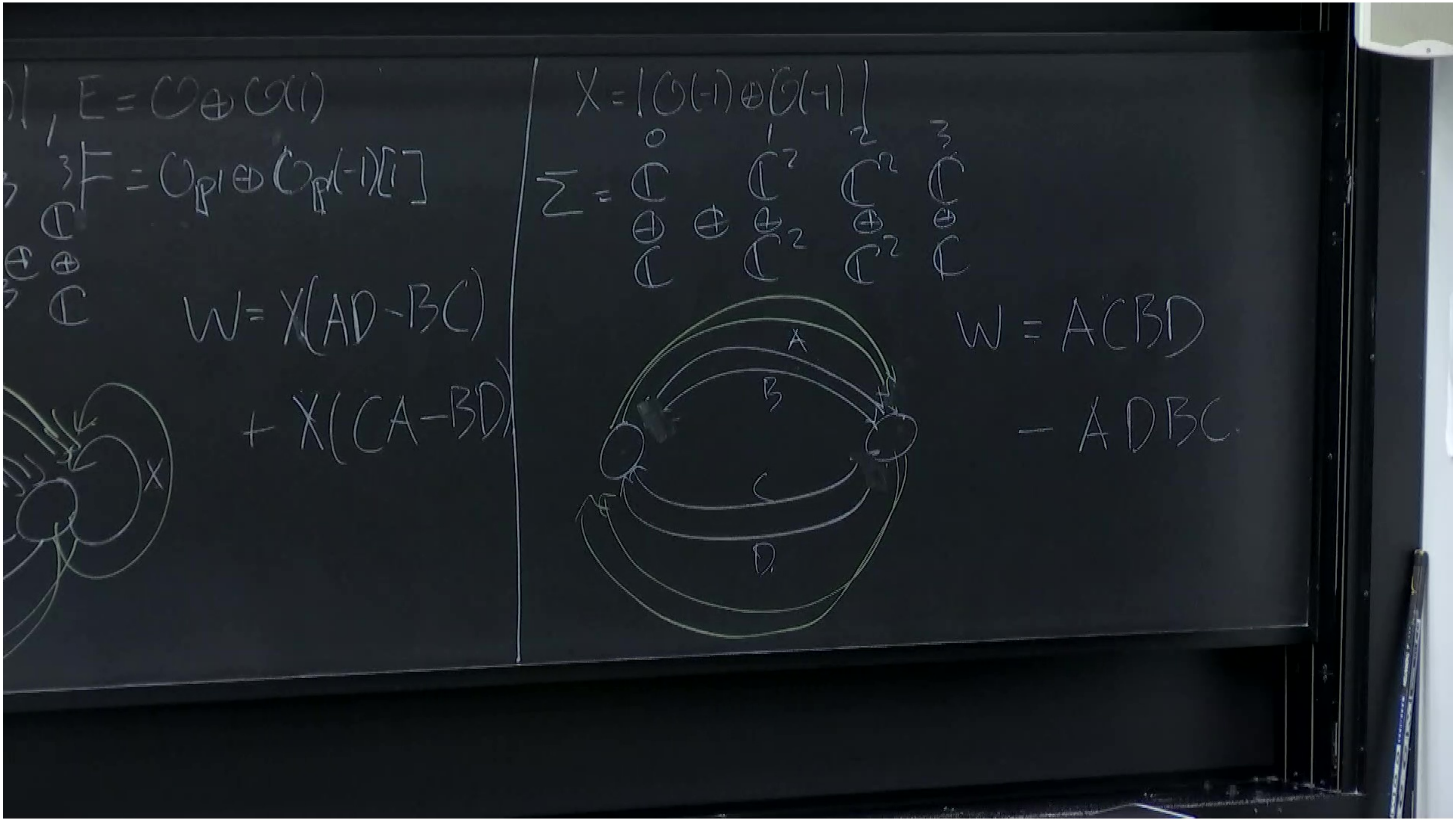
$\Sigma = \overset{0}{\mathbb{C}} \oplus \overset{1}{\mathbb{C}^3} \oplus \overset{2}{\mathbb{C}^3} \oplus \overset{3}{\mathbb{C}}$



$W = B_1[B_2, B_3]$

E.g. $X = |\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)|$, $E = \mathcal{O}$


Eg X = |O ⊕ O(-2)|, E = O ⊕ O(1)
P¹ P¹
F = O_P¹ ⊕ O_P¹(-1)[1]
0 1 2 3
Σ = C C³ C³ C
⊕ ⊕ ⊕ ⊕ ⊕ ⊕
C C³ C³ C
W = X(AD - BC)
+ X(CA - BD)
A
B
C
D
X
Y
X = |O(-1) ⊕ O(-1)|
0
Σ = C C
⊕ ⊕
C



E = O ⊕ O(1)
F = O_P1 ⊕ O_P1(-1)[1]
W = X(AD - BC)
+ X(CA - BD)
X
X = [O(-1) ⊕ O(-1)]
0 1 2 3
Σ = C C^2 C^2 C
⊕ ⊕ ⊕ ⊕ ⊕
C C^2 C^2 C
A
B
C
D
W = ACBD
- ADBC

[Bridgeland, Van den Bergh, Bondal, Orlov, . . .]

$$D^b\mathrm{Coh}(X)^{\mathbb{C}^\times} \underset{E\otimes_\Lambda}{\overset{\mathrm{Hom}(E,-)}{\rightleftarrows}} D_{perf}(\Lambda) \underset{\mathrm{Hom}_\Sigma(S,-)}{\overset{\otimes_\Lambda S}{\rightleftarrows}} D_{f.d.}(\Sigma)$$

$$E_i \longmapsto P_i \longmapsto S_i$$

$$F_i \longmapsto S_i \longmapsto I_i = \Sigma_i^\vee$$

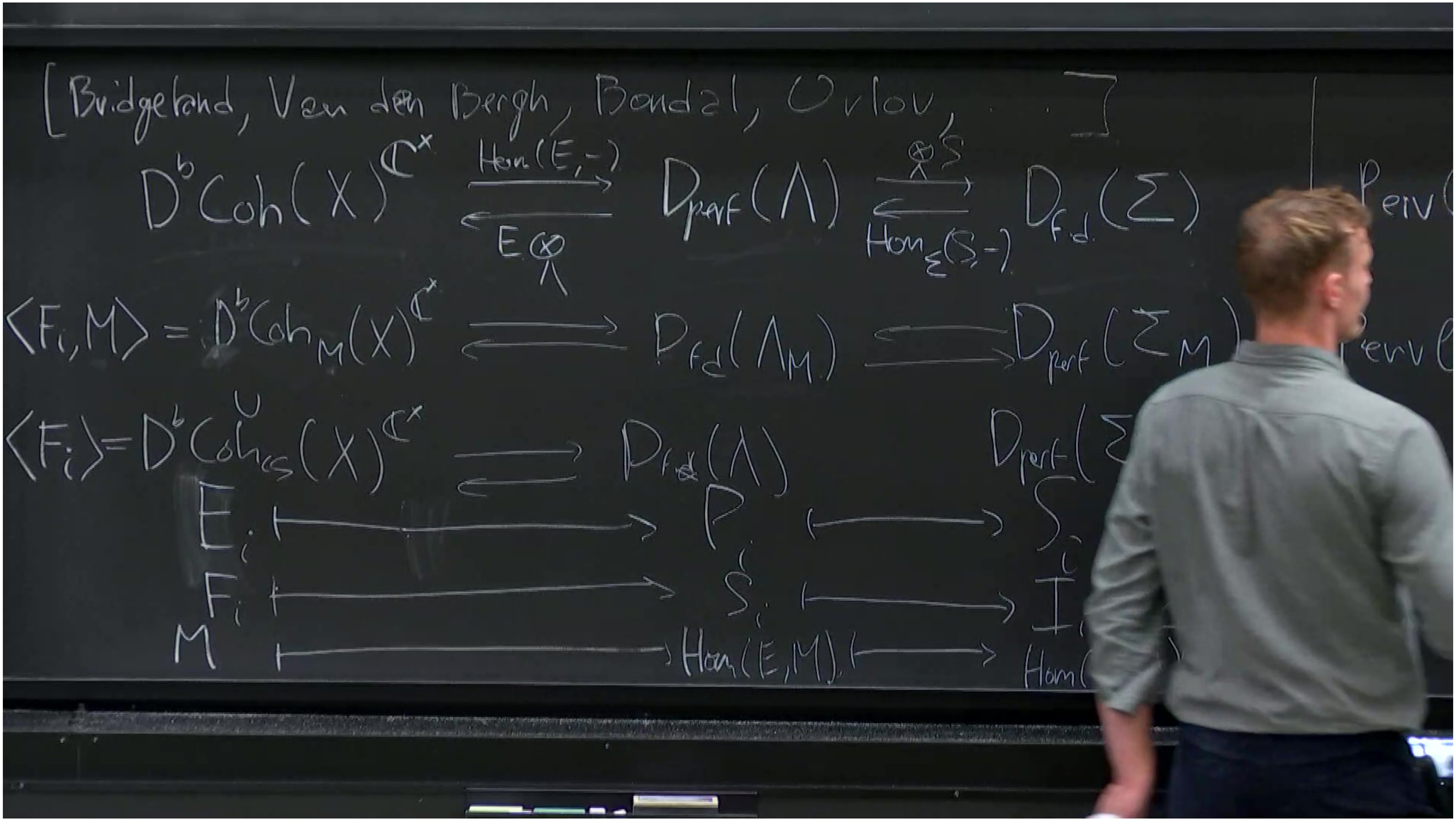
[Bridgeland, Van den Bergh, Bondal, Orlov, ...]

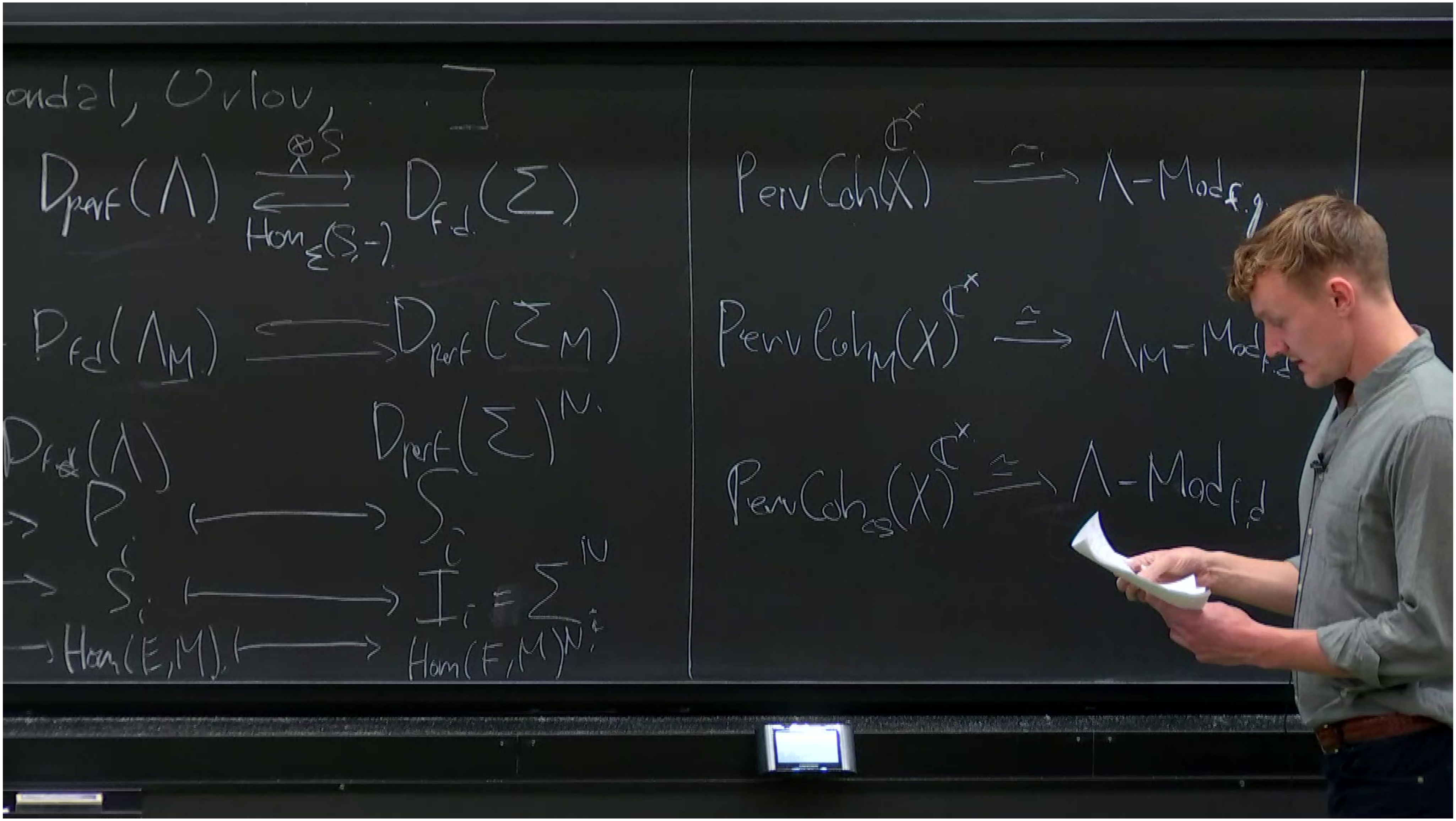
$$D^b\mathrm{Coh}(X)^{\mathbb{C}^\times} \underset{E \otimes_\Lambda}{\overset{\mathrm{Hom}(E,-)}{\rightleftarrows}} D_{perf}(\Lambda) \underset{\mathrm{Hom}_\Sigma(S,-)}{\overset{\otimes_\Lambda S}{\rightleftarrows}} D_{f.d.}(\Sigma)$$

$$E_i \longmapsto P_i \longmapsto S_i$$

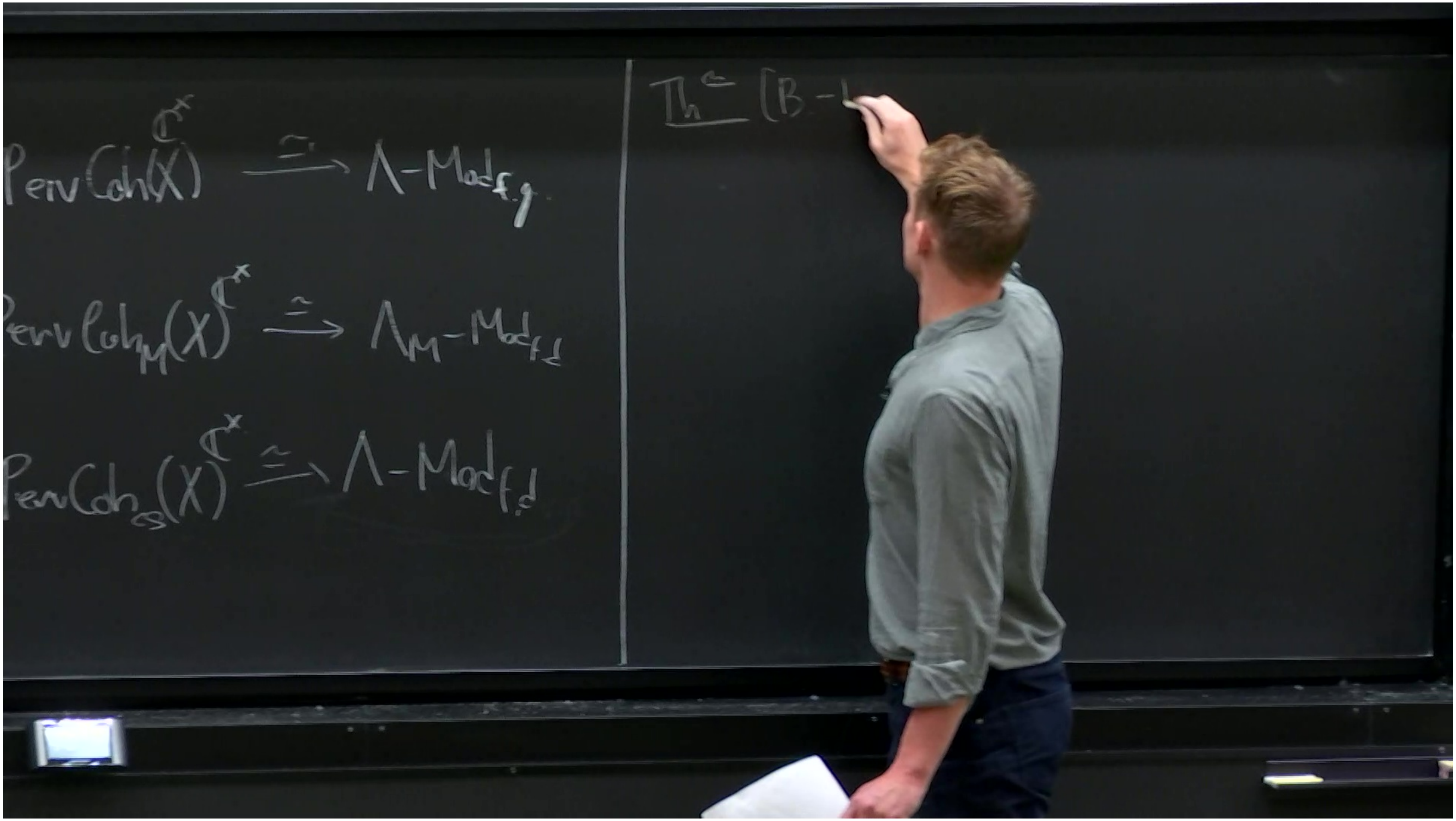
$$F_i \longmapsto S_i \longmapsto I_i = \Sigma_i^{\vee}$$

$$M \longmapsto \mathrm{Hom}(E,M) \longmapsto \mathrm{Hom}(F,M)^{\vee}$$


[Bridgeland, Van den Bergh, Bondal, Orlov, ...]
$D^b Coh(X)^{\mathbb{C}^\times} \underset{E \otimes_\Lambda}{\overset{Hom(E,-)}{\rightleftarrows}} D_{perf}(\Lambda) \underset{Hom_\Sigma(S,-)}{\overset{\otimes_\Lambda S}{\rightleftarrows}} D_{f.d.}(\Sigma)$
$\langle F_i, M \rangle = D^b Coh_M(X)^{\mathbb{C}^\times} \rightleftarrows D_{fd}(\Lambda_M) \rightleftarrows D_{perf}(\Sigma_M)$
$\cup$
$\langle F_i \rangle = D^b Coh_{cs}(X)^{\mathbb{C}^\times} \rightleftarrows D_{fd}(\Lambda)$
$D_{perf}(\Sigma$
$E_i \longmapsto P_i \longmapsto S_i$
$F_i \longmapsto S_i \longmapsto I_i$
$M \longmapsto Hom(E,M) \longmapsto Hom($
Perv(
Perv(



ondal, Orlov, ...]
Dperf(Λ)
Hom_Σ(S,-)
Dfd(Σ)
Dfd(ΛM)
Dperf(ΣM)
Dfd(Λ)
Dperf(Σ)^N
Hom(E,M)
Hom(F,M)^N
PervCoh(X)
Λ-Mod f.g.
PervCoh_M(X)
ΛM-Mod f.d.
PervCoh_cs(X)
Λ-Mod f.d


PervCoh(X) ≃ Λ-Mod f.g.
PervCoh_M(X) ≃ Λ_M-Mod f.d
PervCoh_cs(X) ≃ Λ-Mod f.d
Thm ≃ [B.-

$$\mathrm{Coh}(X)^{\mathbb{C}^\times} \xrightarrow{\simeq} \Lambda\text{-}\mathrm{Mod}_{f.g.}$$

$$\mathrm{Coh}_M(X)^{\mathbb{C}^\times} \xrightarrow{\simeq} \Lambda_M\text{-}\mathrm{Mod}_{f.d.}$$

$$\mathrm{Coh}_{cs}(X)^{\mathbb{C}^\times} \xrightarrow{\simeq} \Lambda\text{-}\mathrm{Mod}_{f.d.}$$

Thm [B.-Rapčák] There is an equivalence of stacks:

$$\mathfrak{M}(Q_M, W_M) \longrightarrow \mathfrak{M}_{\mathrm{PervCoh}_M(X)^{\mathbb{C}^\times}}$$

$$(V_i, B_{ij}) \longmapsto \bigoplus_i V_i \otimes K(I_i), \quad d = \sum_{ij} B_{ij} \otimes b_{ij}.$$

$\mathrm{Coh}(X)^{\mathbb{C}^\times} \xrightarrow{\simeq} \Lambda\text{-}\mathrm{Mod}_{f.g.}$

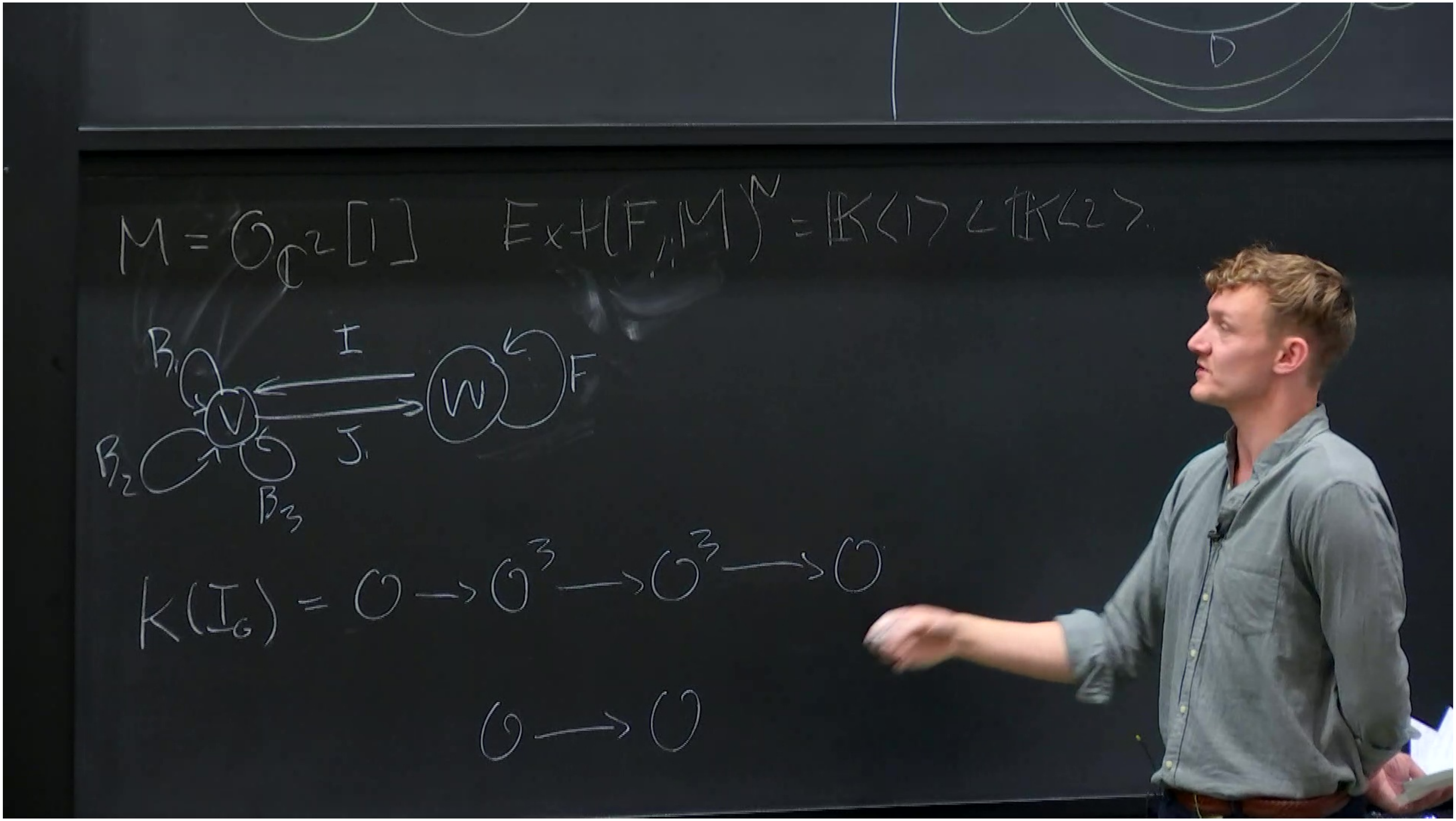
$\mathrm{Coh}_M(X)^{\mathbb{C}^\times} \xrightarrow{\simeq} \Lambda_M\text{-}\mathrm{Mod}_{f.d}$

$\mathrm{Coh}_{cs}(X)^{\mathbb{C}^\times} \xrightarrow{\simeq} \Lambda\text{-}\mathrm{Mod}_{f.d}$

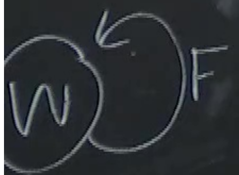
Thm [B-Rapčák] There is an equivalence of stacks

$$\mathfrak{M}(Q_M, W_M) \longrightarrow \mathfrak{M}_{\mathrm{PervCoh}_M(X)}^{\mathbb{C}^\times}$$

$$(V_i, B_{ij}) \longmapsto \bigoplus V_i \otimes K(I_i), \quad d = \sum_{ij} B_{ij} \otimes b_{ij}.$$


M = 𝒪_ℂ²[1]
B₁
B₂
B₃
I
J
V
W
F
K(I₀) = 𝒪 → 𝒪³ → 𝒪³ → 𝒪
𝒪 → 𝒪

$$\mathrm{Ext}(F, M)^\vee = \mathbb{K}\langle 1\rangle \oplus \mathbb{K}\langle 2\rangle.$$

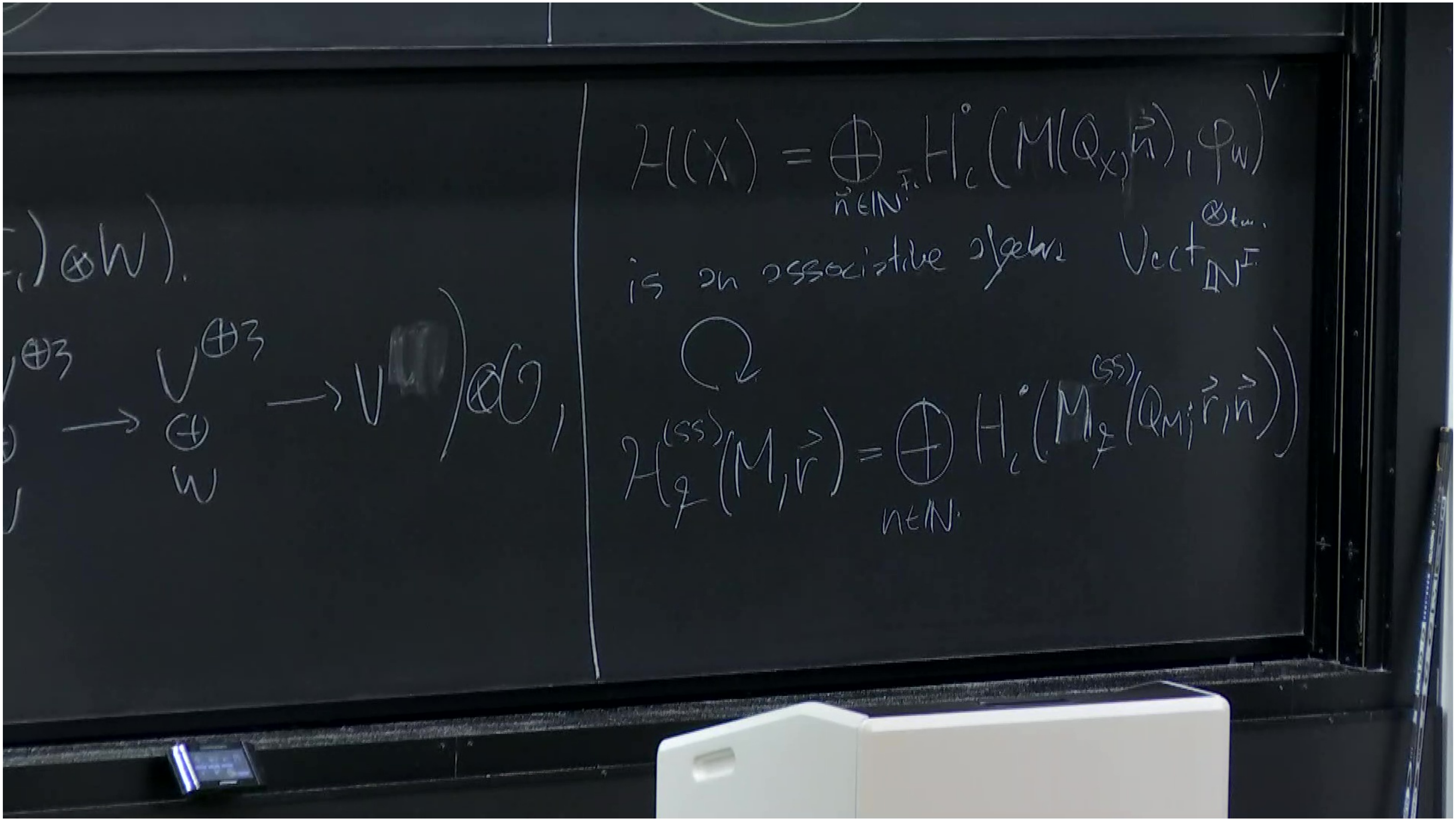


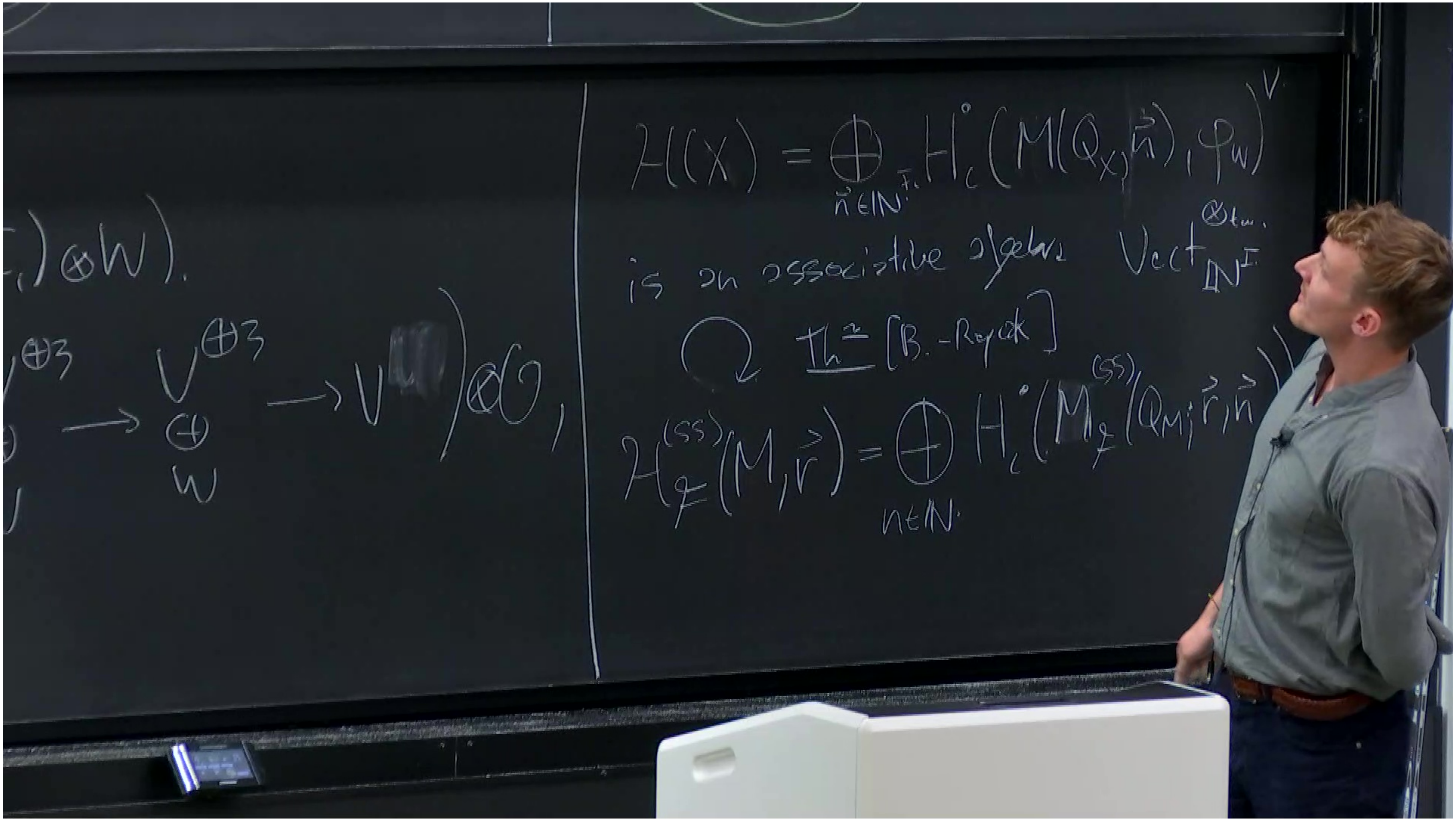
$$(K(I_0)\otimes V)\oplus(K(I_1)\otimes W).$$

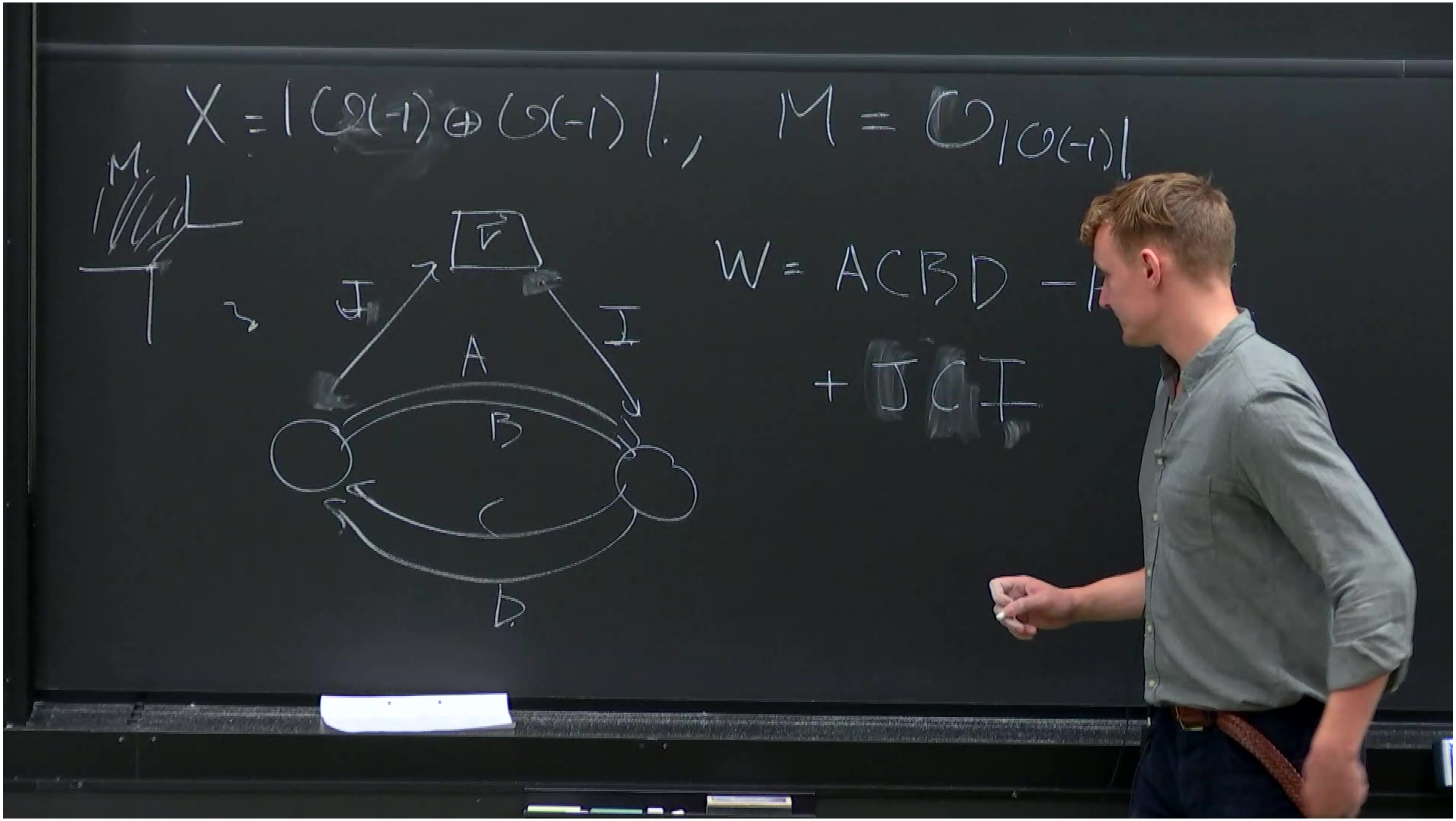
$$= \left(V \longrightarrow \begin{matrix} V^{\oplus 3} \\ \oplus \\ W \end{matrix} \longrightarrow \begin{matrix} V^{\oplus 3} \\ \oplus \\ W \end{matrix} \longrightarrow V\right)\otimes \mathcal{O},$$

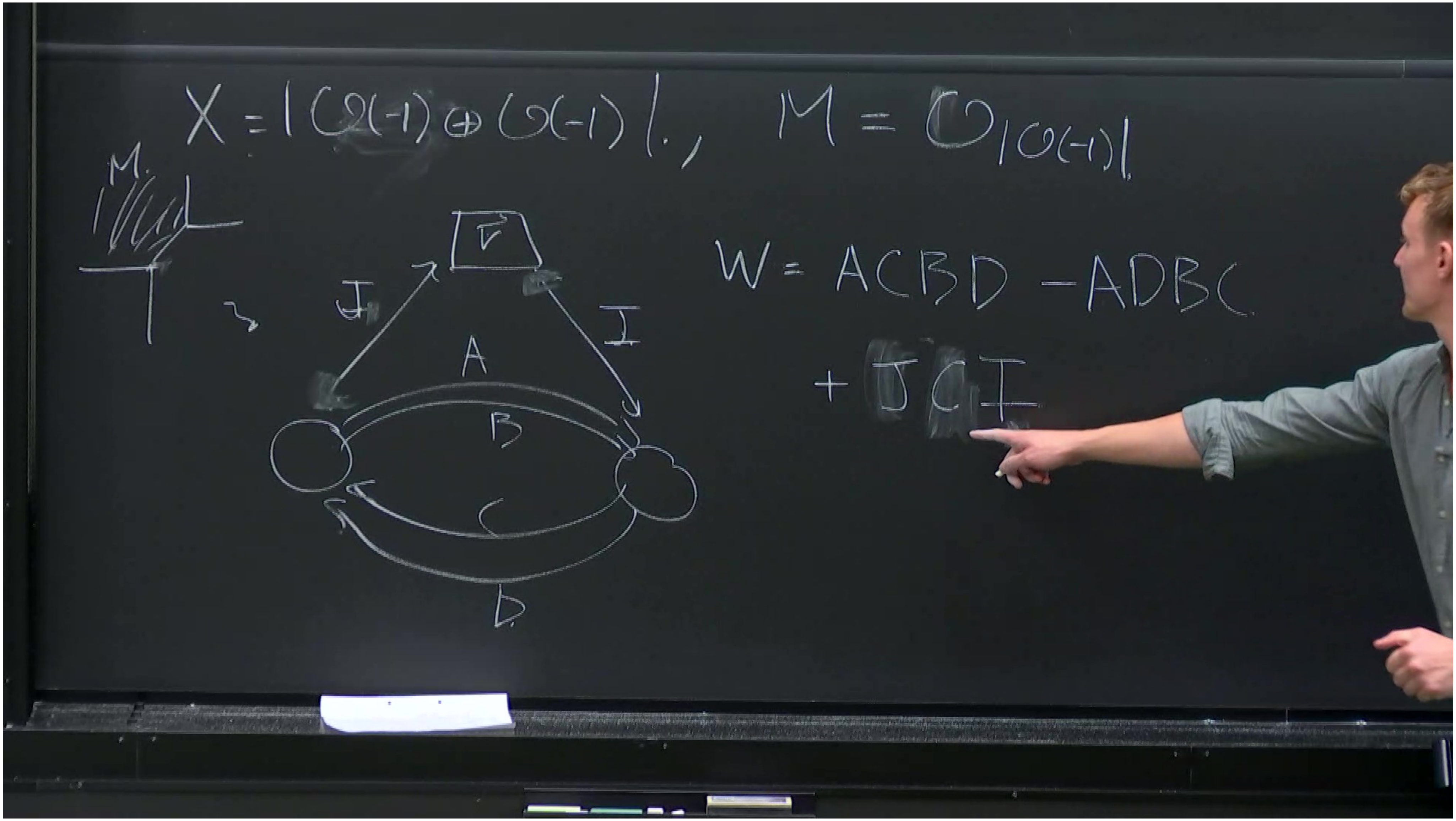
$$\longrightarrow \mathcal{O}^3 \longrightarrow \mathcal{O}^3 \longrightarrow \mathcal{O}$$

$$\mathcal{O} \longrightarrow \mathcal{O}$$

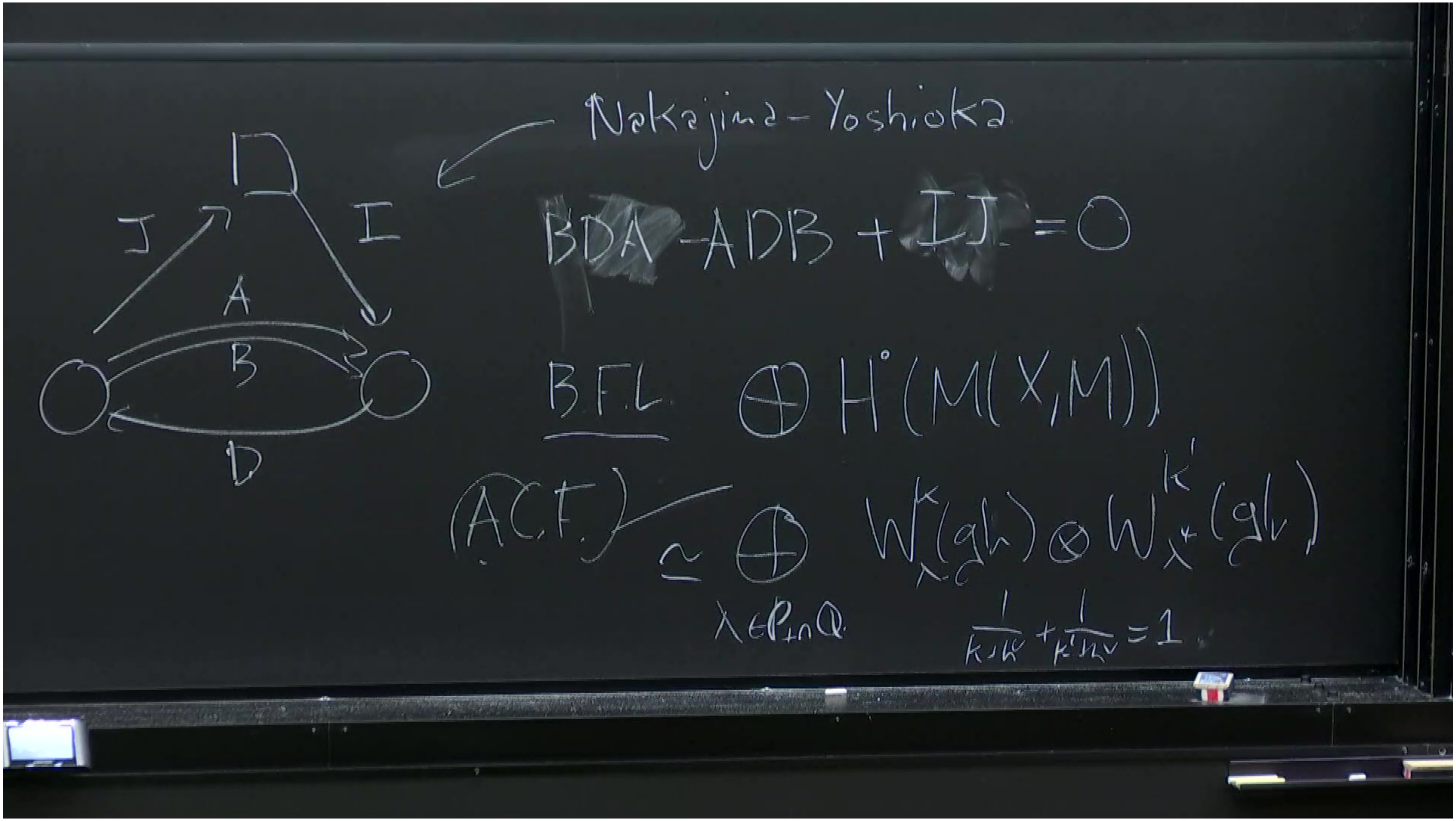

$\mathcal{H}(X) = \bigoplus_{\vec{n} \in \mathbb{N}^{I}} H_c^{\bullet}(M(Q_X, \vec{n}), \varphi_W)^{\vee}$
is an associative algebra $\mathrm{Vect}^{\otimes_{tw}}_{\mathbb{N}^{I}}$
$\mathcal{H}_q^{(ss)}(M, \vec{r}) = \bigoplus_{n \in \mathbb{N}} H_c^{\bullet}(\mathfrak{M}_q^{(ss)}(Q_M; \vec{r}, \vec{n}))$

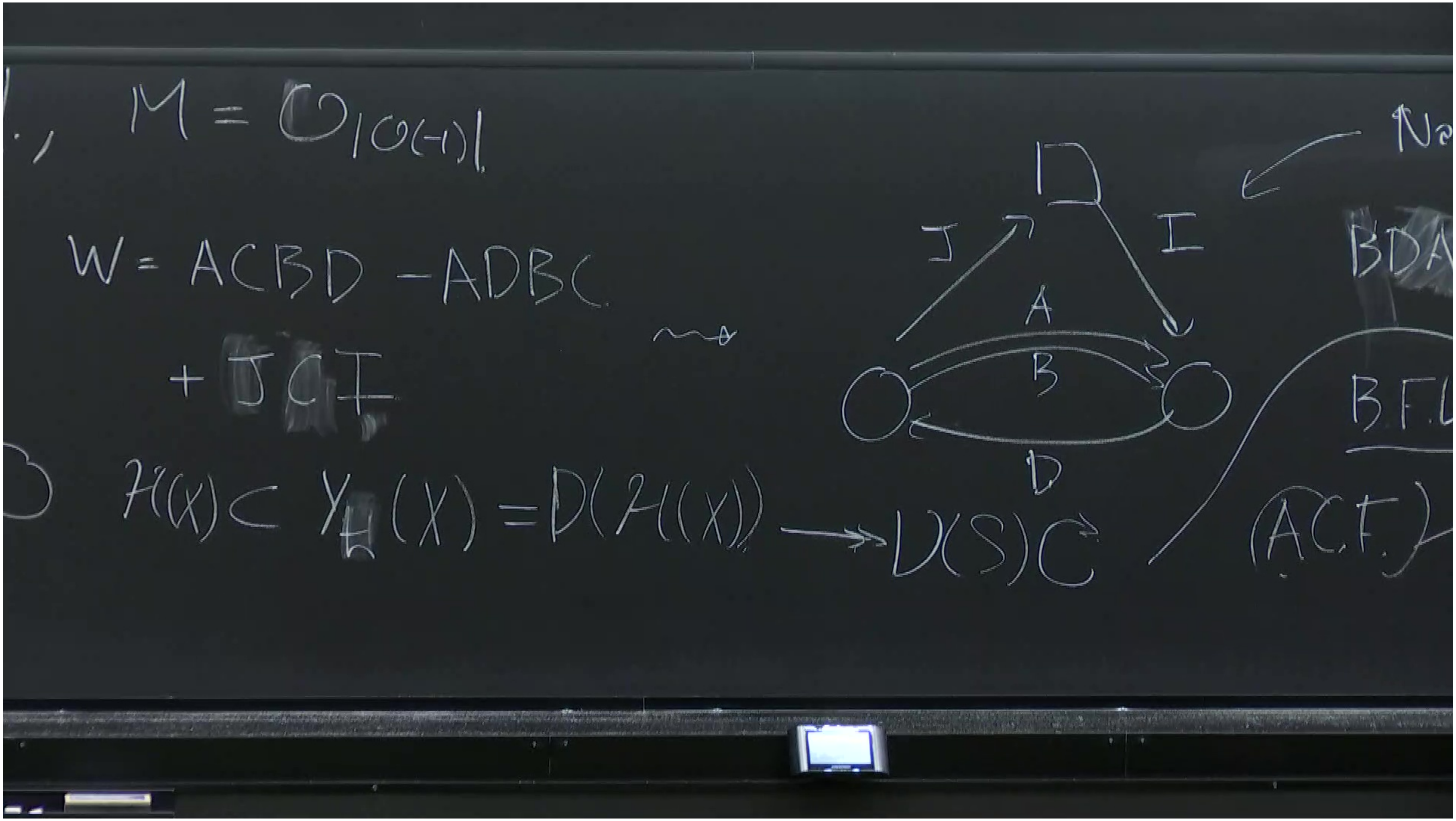

$\mathcal{H}(X) = \bigoplus_{\vec{n} \in \mathbb{N}^{F_i}} H_c^{\circ}(\mathcal{M}(Q_X, \vec{n}), \varphi_W)^{\vee}$
is an associative algebra $\mathrm{Vect}^{\otimes tw.}_{\mathbb{N}^{I}}$
$\mathcal{H}^{(SS)}_{\mathcal{F}}(M, \vec{r}) = \bigoplus_{n \in \mathbb{N}} H_c^{\circ}(\mathcal{M}^{(SS)}_{\mathcal{F}}(Q_M; \vec{r}, \vec{n}))$

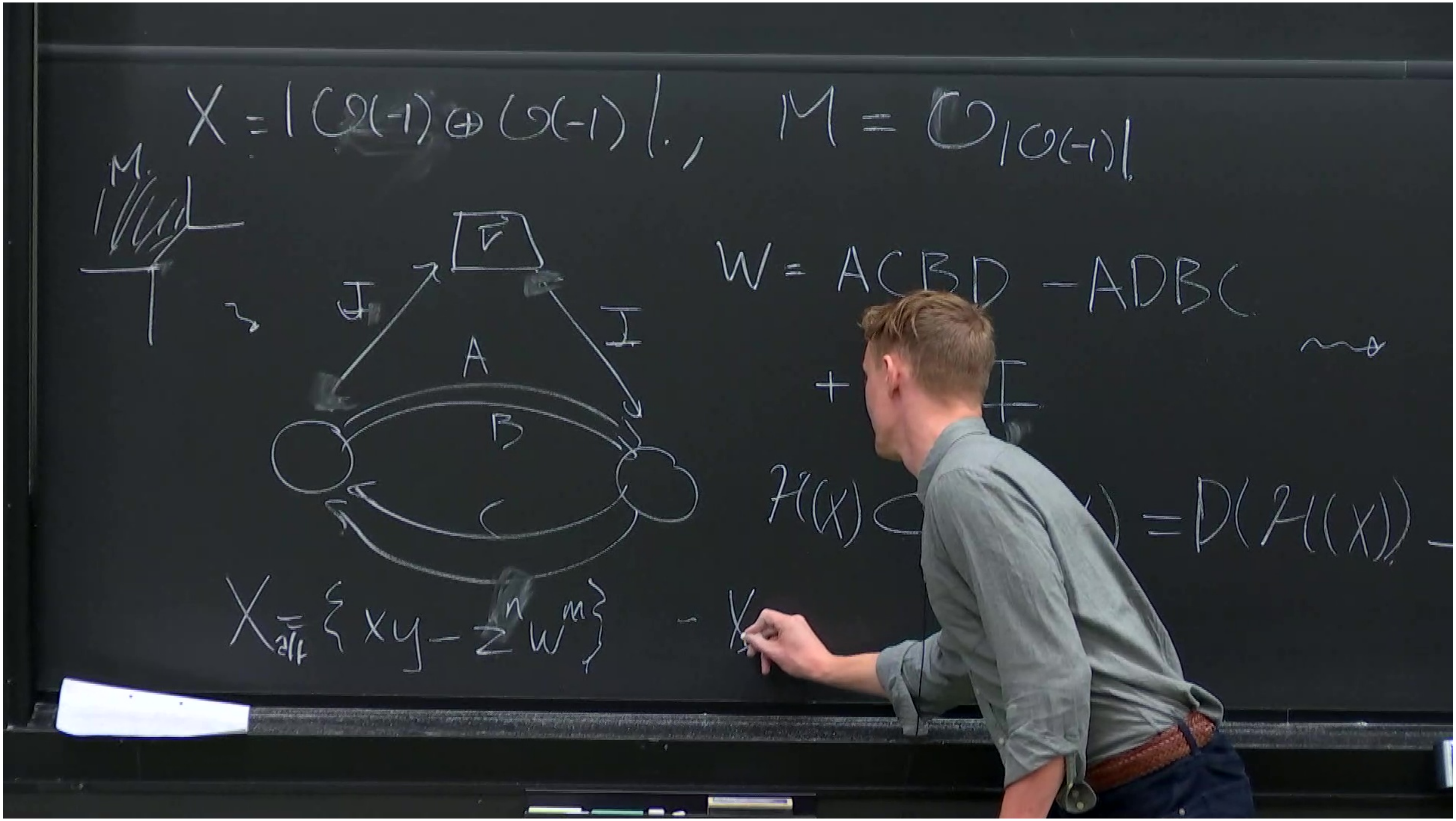

X = |O(-1) ⊕ O(-1)|.,
M = O_{|O(-1)|}
W = ACBD −
+ JCI
J
I
A
B
C
D
M.

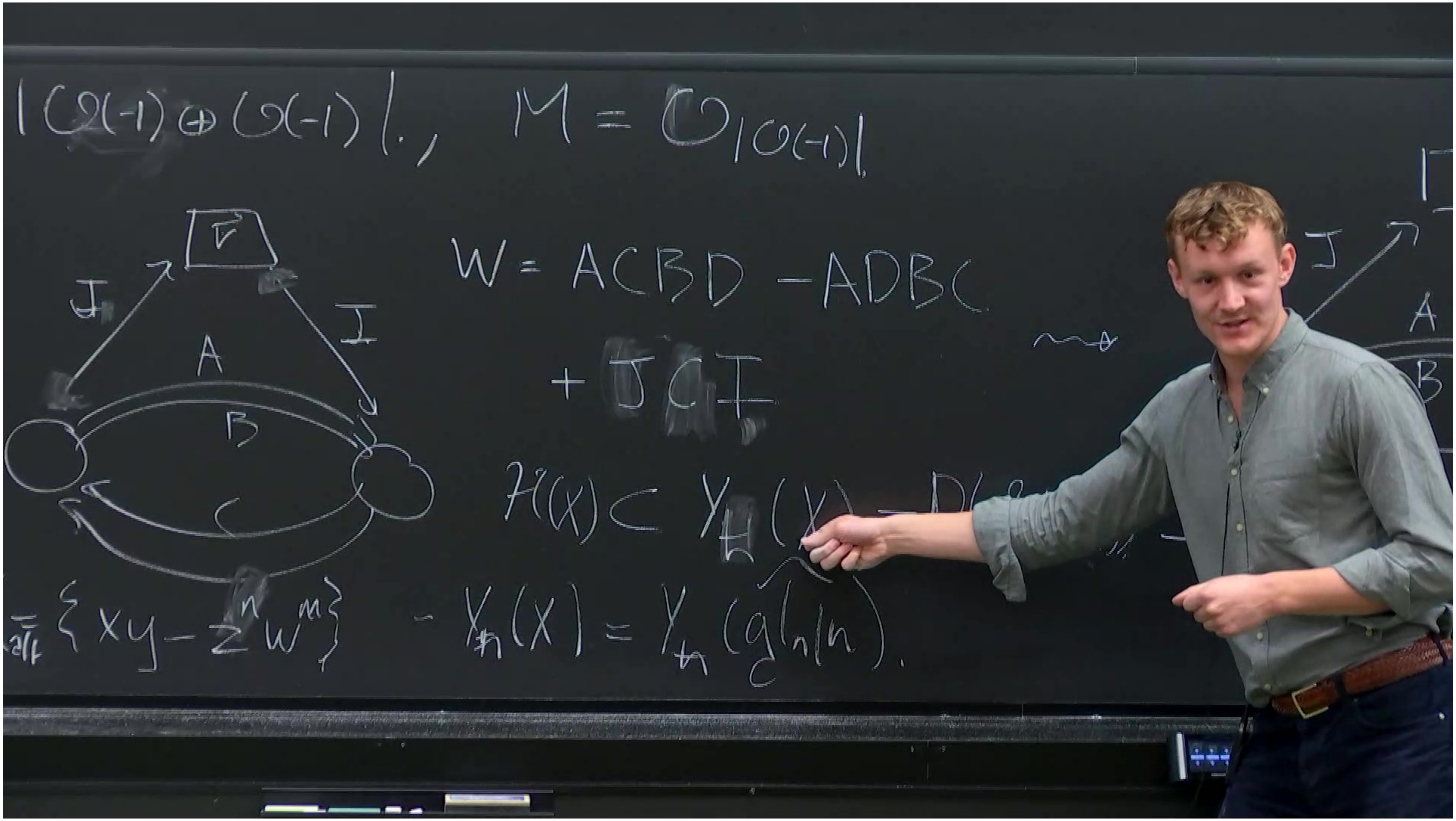


X = |O(-1) ⊕ O(-1)|., M = O_{|O(-1)|}
W = ACBD − ADBC
+ JCI
M.
J
I
A
B
C
D


Nakajima-Yoshioka
J
I
A
B
D
BDA − ADB + IJ = 0
B.F.L.
⊕ H°(M(X,M))
(A.C.F.)
≃ ⊕ W^k_λ(gl_r) ⊗ W^k'_λ*(gl_r)
λ∈P_+∩Q
1/(k+h^∨) + 1/(k'+h^∨) = 1

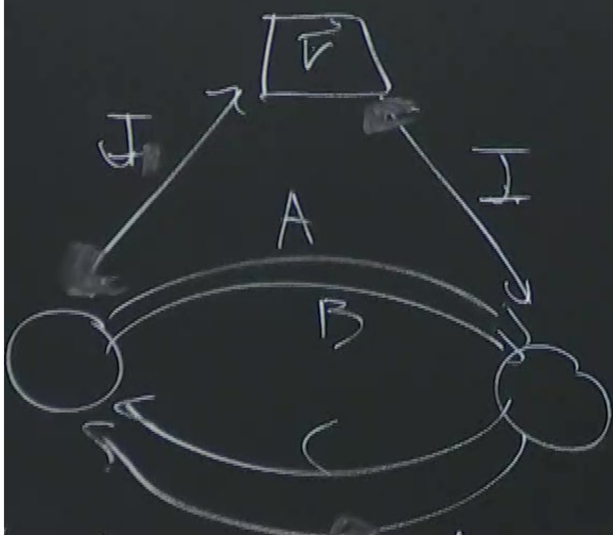

M = O|O(-1)|
W = ACBD - ADBC
+ JCI
H(X) ⊂ Y_H(X) = D(H(X))
D(S)C
J
I
A
B
D
BDA
B.F.L
(A.C.F.)


X = |O(-1) ⊕ O(-1)|,
M = O|O(-1)|
W = ACBD - ADBC
A
B
C
J
I
X = {xy - z^n w^m}

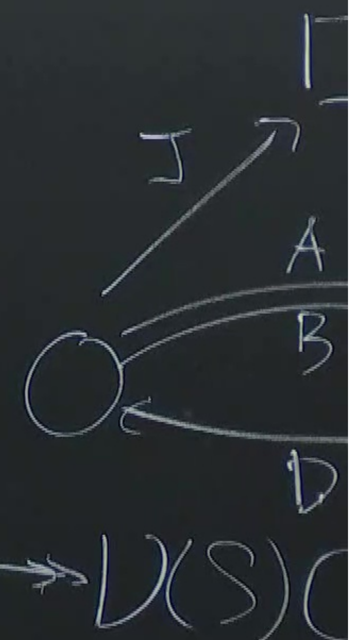


M =
W = ACBD − ADBC
+ JCI
J
I
A
B
C

$|\mathcal{O}(-1)\oplus\mathcal{O}(-1)|.,\quad M = \mathcal{O}_{|\mathcal{O}(-1)|}$



$$W = ACBD - ADBC + JCI \rightsquigarrow$$



$$\mathcal{H}(X) \subset Y_\hbar(X) = D(\mathcal{H}(X)) \longrightarrow D(S)($$

$\{xy - z^n w^m\}$

$$Y_\hbar(X) = Y_\hbar(\widehat{\mathfrak{gl}_{n|n}}).$$