Title: Vertex algebras of divisors in toric Calabi-Yau threefolds from perverse coherent extensions

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Series: Mathematical Physics

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Abstract: I'll explain work in progress, joint with Miroslav Rapcak, on geometric constructions of vertex algebras associated to divisors in toric Calabi-Yau threefolds, in terms of moduli stacks of objects in certain exotic abelian subcategories of complexes of coherent sheaves on the underlying threefold. These vertex algebras were originally proposed by Gaiotto-Rapcak, and constructed mathematically in the example of affine space by Rapcak-Soibelman-Yang-Zhao, building on Schiffmann-Vasserot's proof of the AGT conjecture. We give a geometric explanation and generalization of the quivers with potential that feature in the latter results, and outline the analogous construction of vertex algebras in this setting.

Zoom link: https://pitp.zoom.us/j/93749710253?pwd=Y3NUTHBXZ3FPQUdPU0E0d0ttVzFFdz09

Grojnowski, Nakajima – Heisenberg algs + Hilbert schemes
Schiffmann, Vasserot – Cherednik algs, W algs and
Rapčák, Seiberg, Yang, Zhao – CoHAs, Vortex algs and

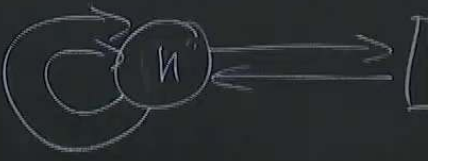
Nekrasov – Seiberg-Witten prepotential + instanton counting
Alday, Gaiotto, Tachikawa – Liouville correlation fns from 4d
Belavin, Bershtein, Feigin, Litvinov, Tarnopolsky – Instanton moduli
Bershtein, Feigin, Litvinov – Coupling CFTs + Nakajima–Yoshioka
Gaiotto, Rapčák – VAs at the corner
Creutzig, Gaiotto – VAs for S duality / Arakawa, Creutzig, Fe
(Galakov), Li, Yamazaki, – ..., Quiver Yangians, ...

rojnowski, Nakajima – Heisenberg algs + Hilbert schemes of points on surfac
chiffmann, Vasserot – Cherednik algs, W algs and instantons
pčák, Seibelman, Yang, Zhao – CoHAs, Vortex algs and instantons

Nekrasov – Seiberg-Witten prepotential + instanton counting
lday, Gaiotto, Tachikawa – Liouville correlation f^n's from 4d gage theory
lavin, Bershtein, Feigin, Litvinov, Tarnopolsky – Instanton moduli + Coset CFT
ershtein, Feigin, Litvinov – Coupling CFTs + Nakajima–Yoshioka blowup eq-ns
aiotto, Rapčák – VAs at the corner
reutzig, Gaiotto – VAs for S duality / Arakawa, Creutzig, Feigin – Urod algebras
(Gaiotto), Li, Yamazaki – ..., Quiver Yangians, ...

Intro: Let S smooth proj. surface / $\mathbb{C}$

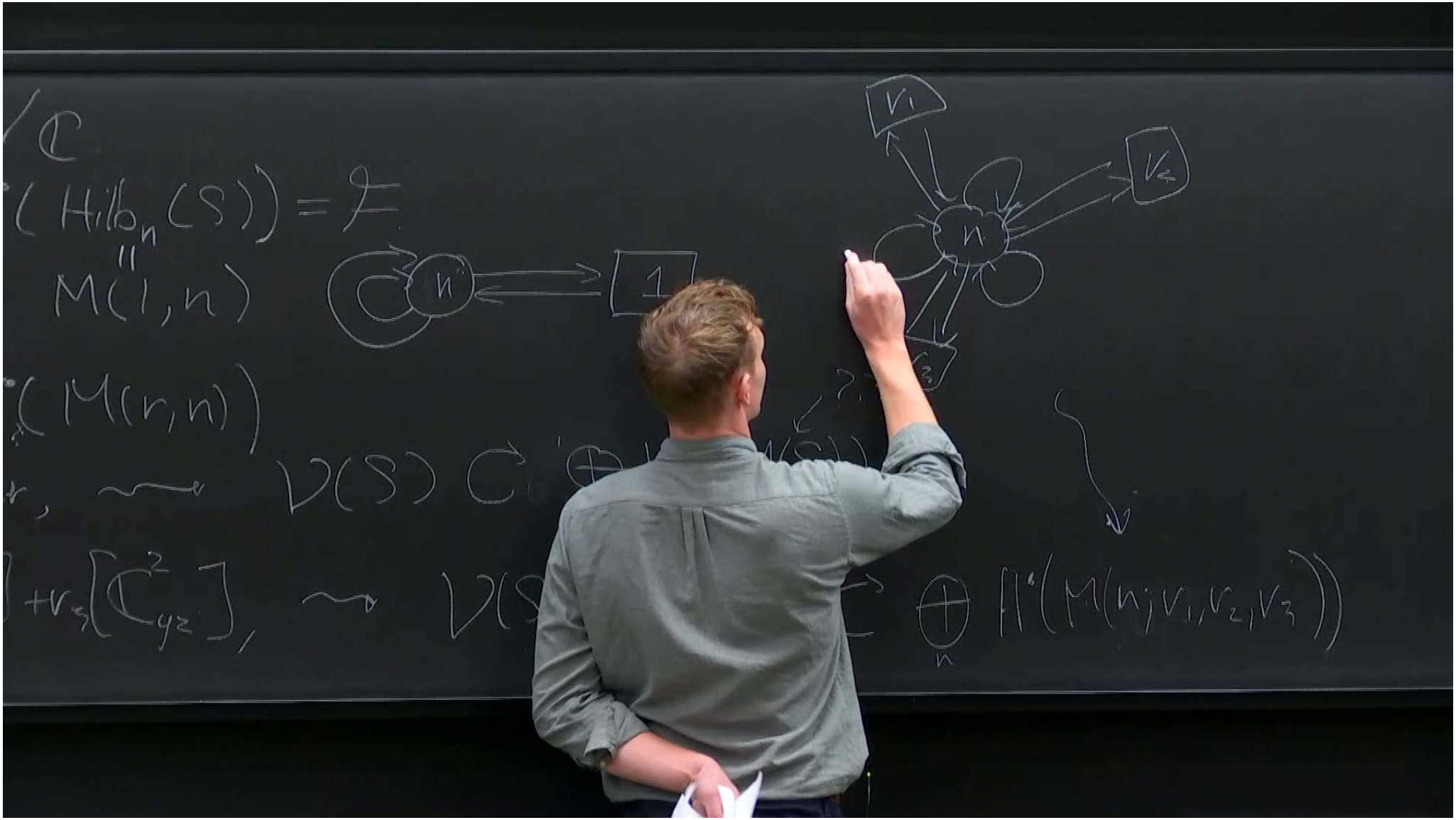
Then $\mathrm{Heis}_{H^*(S)} \subset \bigoplus_{n \in \mathbb{N}} H^*(\mathrm{Hilb}_n(S)) = \mathcal{F}$

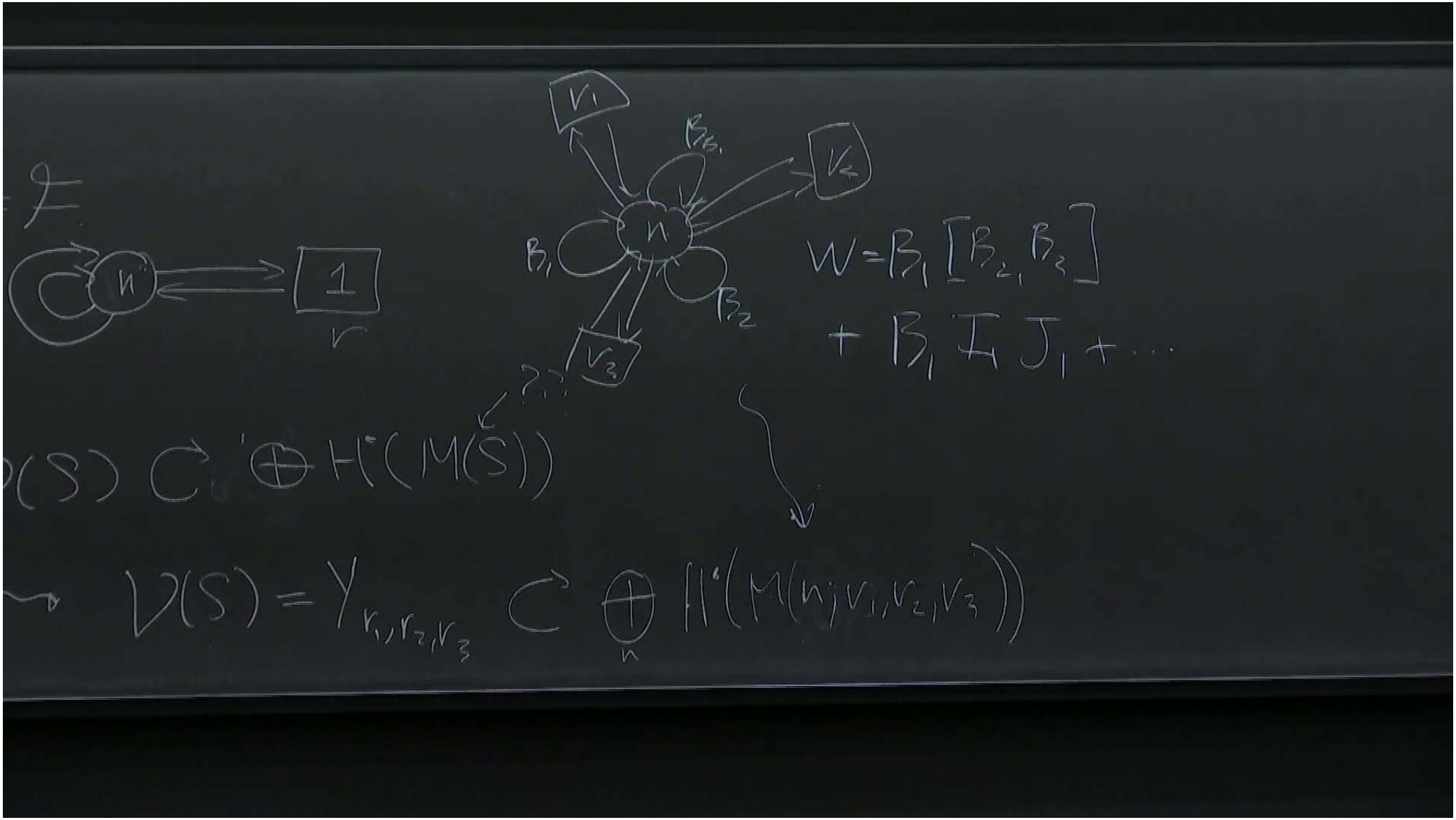
$S = \mathbb{C}^2$ $\quad \mathrm{Heis} \quad M(1,n)$ 

[AGT, S.V] $\quad W(\mathfrak{gl}_r) \subset \bigoplus H^*(M(r,n))$

[GR] X toric CY3, S a divisor, $\rightsquigarrow$ $V(S)$ C

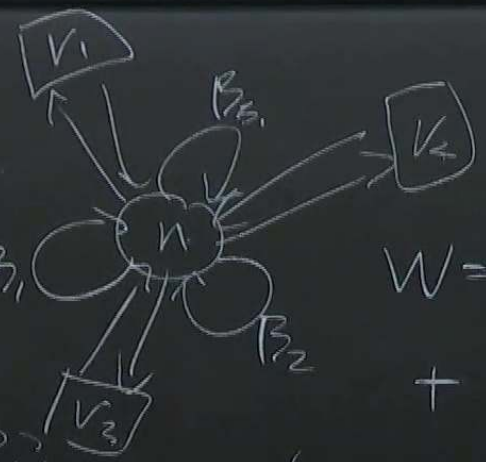
[RS.YZ] $X = \mathbb{C}^3$, $S = r_1[\mathbb{C}^2_{xy}] + r_2[\mathbb{C}^2_{xz}] + r_3[\mathbb{C}^2_{yz}]$, $\rightsquigarrow$


M(1,n)
n
1
n
V1
V2
V3


n
1
r
V_1
V_2
V_3
$B_{1,1}$
B_1
B_2
$W = B_1[B_2, B_2] + B_1 I J_1 + \dots$
$H^*(M(S))$
$V(S) = Y_{r_1,r_2,r_3} \subset \bigoplus_n H^*(M(n; r_1, r_2, r_3))$

$r_1 = r_2 = 0$

the dR ref.



1

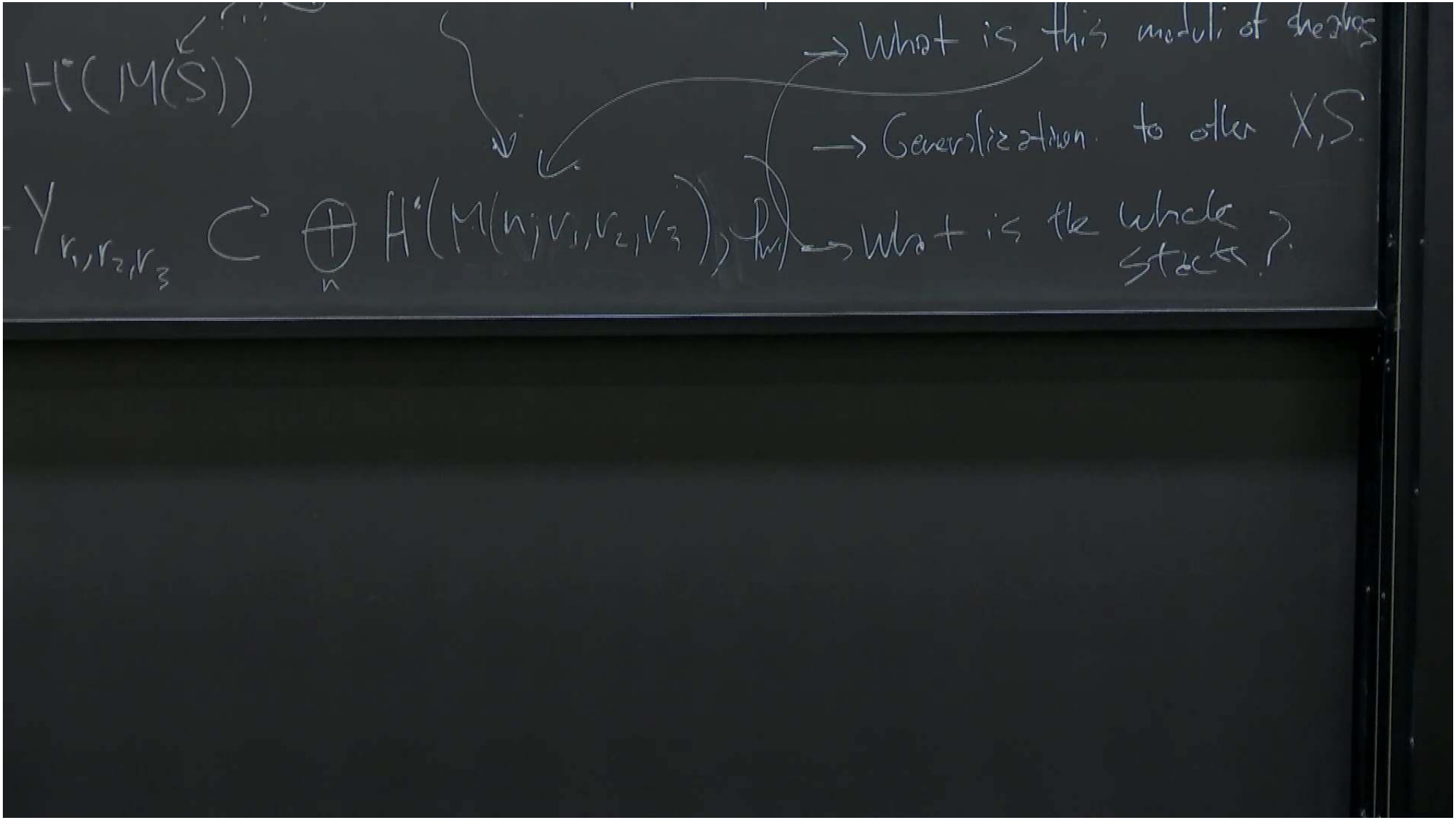
r

$W = B_1 [B_2, B_3] + B_1 I_1 J_1 + \dots$

$\oplus H^*(M(S))$

→ What is this moduli of the rhs

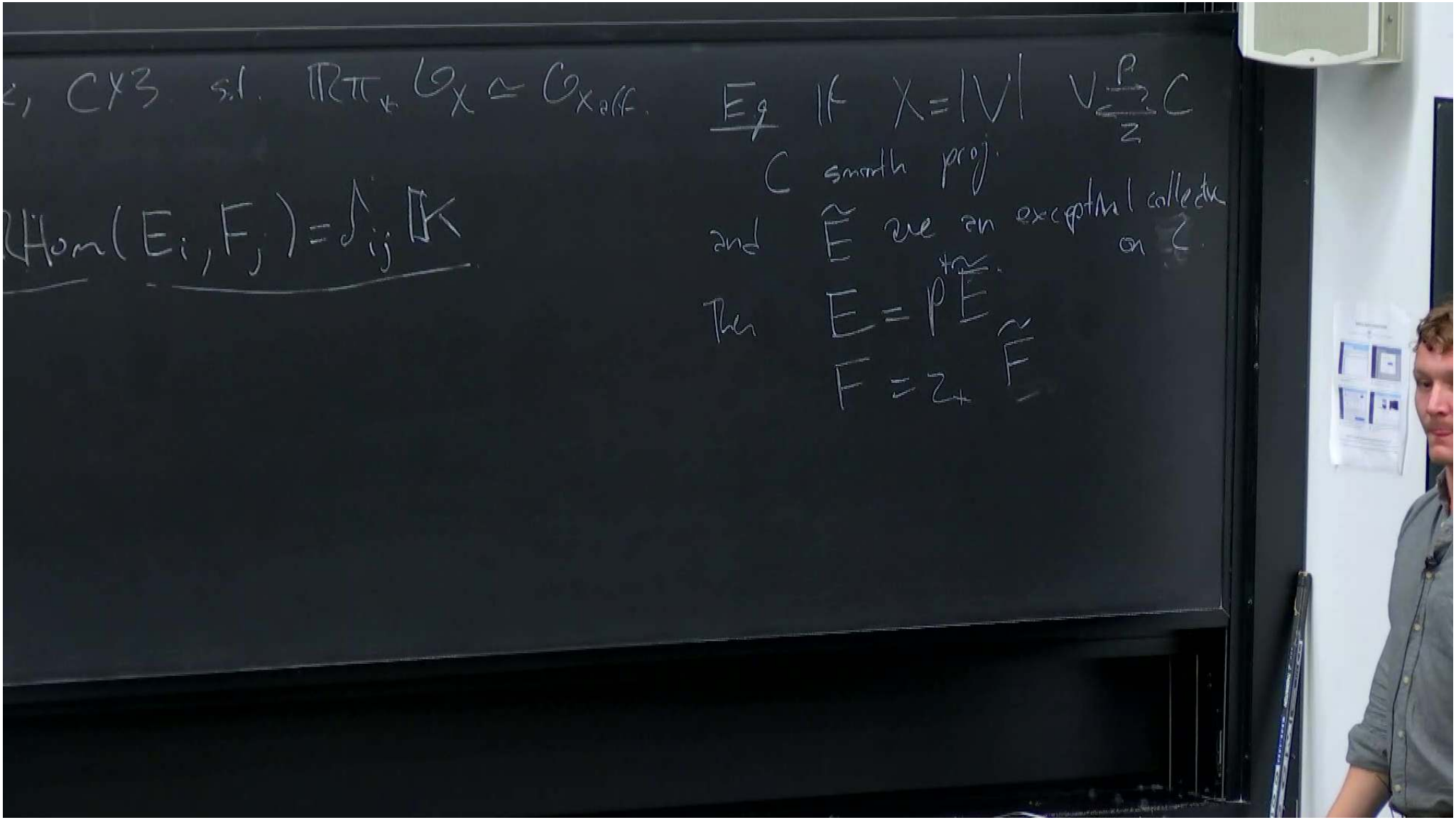
$= Y_{r_1, r_2, r_3} \subset \bigoplus_n H^*(M(n; r_1, r_2, r_3))$


$H^*(M(S))$
$Y_{r_1,r_2,r_3} \subset \bigoplus_n H^*(M(n;r_1,r_2,r_3), \mathbb{Q})$
What is this moduli of sheaves
Generalization to other X, S
What is the whole story?

Let $X \xrightarrow{\pi} X_{\text{aff}}$ a small resolution of X_{aff} an affine, toric, C

Fix $E = \bigoplus_{i \in I} E_i \in \text{Coh}(X)$ proj. gen., $\Lambda = \text{End}(E)$.

$F = \bigoplus_{i \in I} F_i \in D^b \text{Coh}_{cs}(X)$ simple, cctlg supp,


$\exists$, CY3 s.t. $R\pi_* \mathcal{O}_X \simeq \mathcal{O}_{X\,\text{aff}}$.
$R\mathrm{Hom}(E_i, F_j) = \Lambda_{ij}$ ℝ
Eg If $X = |V|$ $V \xrightarrow{\rho} C$
C smooth proj.
and $\tilde{E}$ are an exceptional collection on Σ.
Then $E = \rho^* \tilde{E}$
$F = \Sigma_* \tilde{F}$

resolution of X_{aff} an affine, toric, CY3 s.t. $\mathbb{R}\pi_* \mathcal{O}_X \simeq \mathcal{O}_{X_{aff}}$.

proj. gen., $\Lambda = End(E)$.

$D^b_{coh}(X)$ ~~simple~~, cctlg supp, s.t. $\mathbb{R}Hom(E_i, F_j) = \Lambda_{ij}[k]$, $\Sigma = Ext^*(F,F)$

Eq

and

Then

$X \xrightarrow{\pi} X_{\mathrm{aff}}$ a small resolution of X_{aff} an affine, toric, CY3

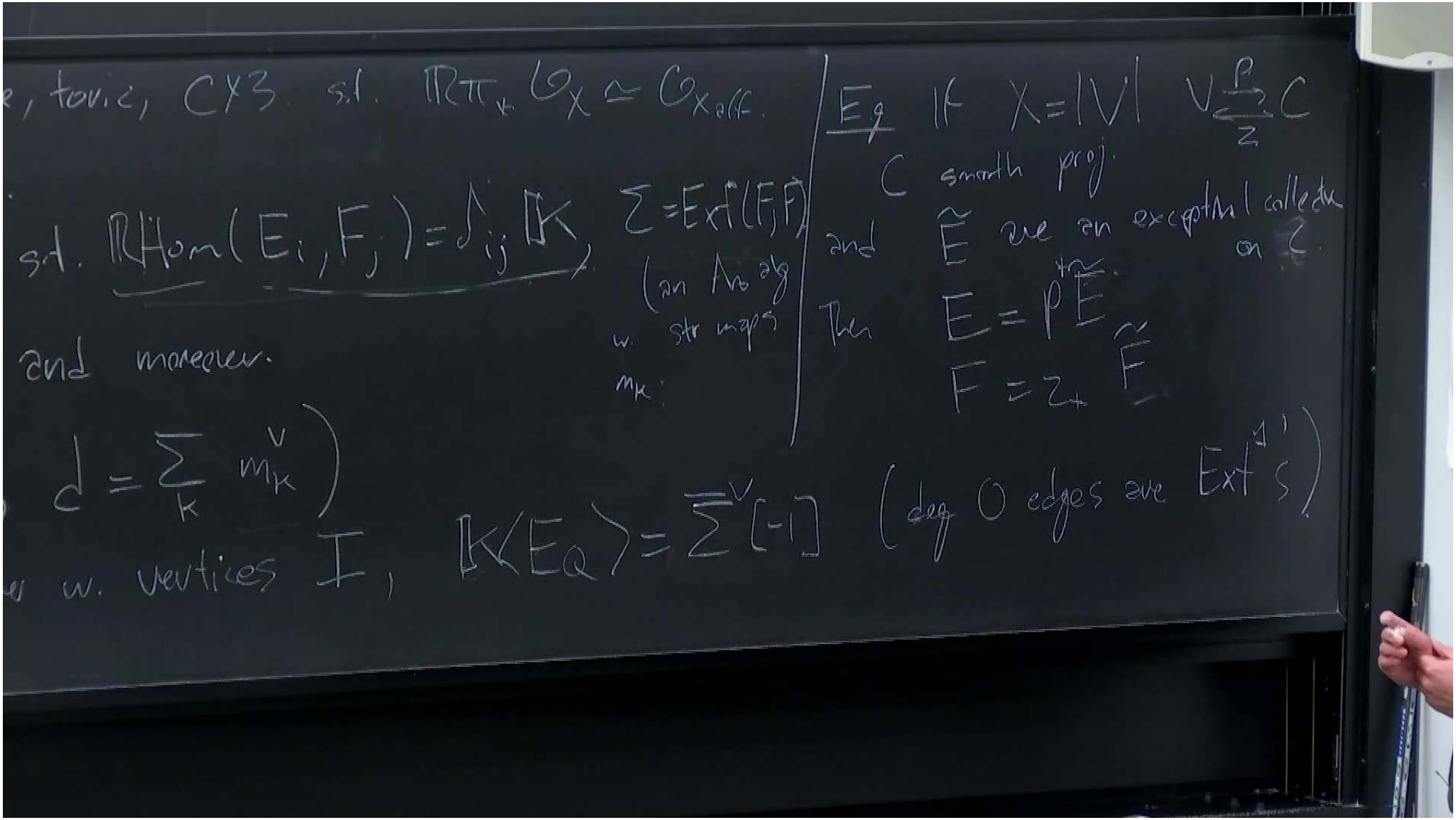
$E = \bigoplus_{i \in I} E_i \in \mathrm{Coh}(X)$ (T-inv?) proj. gen., $\Lambda = \mathrm{End}(E)$.

$F = \bigoplus_{i \in I} F_i \in D^b \mathrm{Coh}_{\mathrm{cs}}(X)$ simple, cotlg supp, s.t. $\mathbb{R}\mathrm{Hom}(E_i,$

Λ, Σ are agraded over $S = \bigoplus_{i \in I} \mathbb{K}_i$ and moreover.

$\Lambda = \Sigma^! = \mathrm{Hom}_{\Sigma}(S, S) = \left(\bigotimes_S^{\bullet} \Sigma^{\vee}[-1], d = \sum_k m_k^{\vee} \right)$

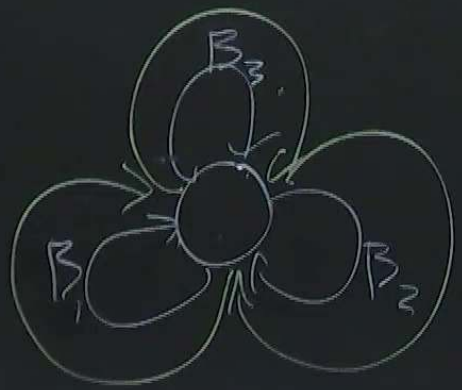
pshot: $\Lambda = \mathbb{K}\langle Q_X, W_X \rangle$ where Q_X is DG quiver w. vertices


, toric, CY3 s.t. $R\pi_* \mathcal{O}_X \simeq \mathcal{O}_{X\text{aff}}$.
s.t. $R\mathrm{Hom}(E_i, F_j) = \Lambda_{ij}[k]$, $\Sigma = \mathrm{Ext}^*(F,F)$ (an A_∞ alg w. str maps m_k)
and moreover.
$d = \sum_k m_k^V$)
w. vertices I, $\mathbb{K}(E_Q) = \Sigma^V[-1]$ (deg 0 edges are Ext^1's)
Eg If $X = |V|$ $V \xrightarrow{\rho} C$
C smooth proj.
and $\tilde{E}$ are an exceptional collection on $\tilde{C}$.
Then $E = \rho^* \tilde{E}$
$F = \tilde{\Sigma}_* \tilde{F}$

Ex: $X = \mathbb{C}^3$, $E = \mathcal{O}$, $F = \mathcal{O}_{p^1}$

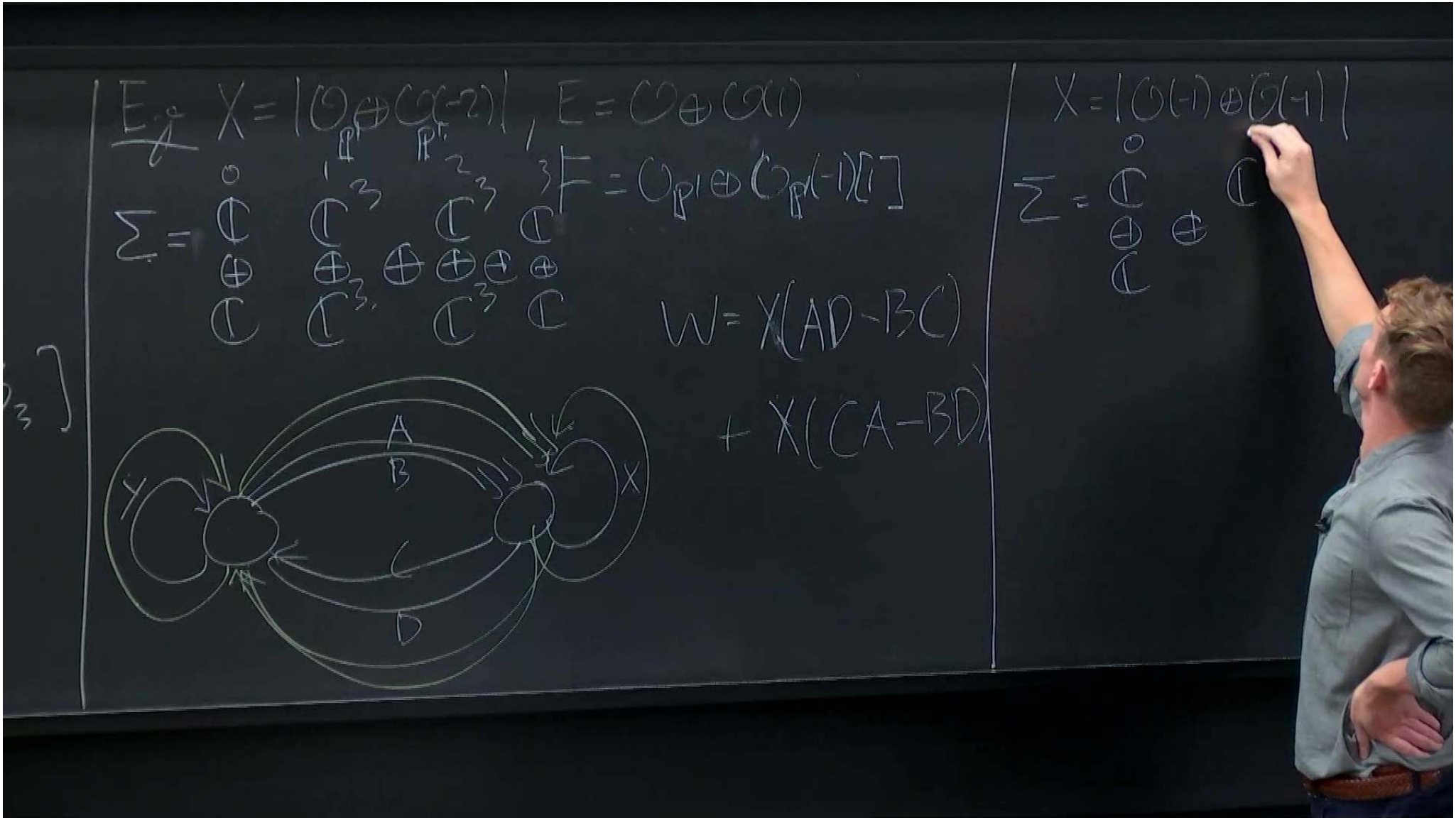
$\Lambda = \mathbb{C}[x, y, z]$

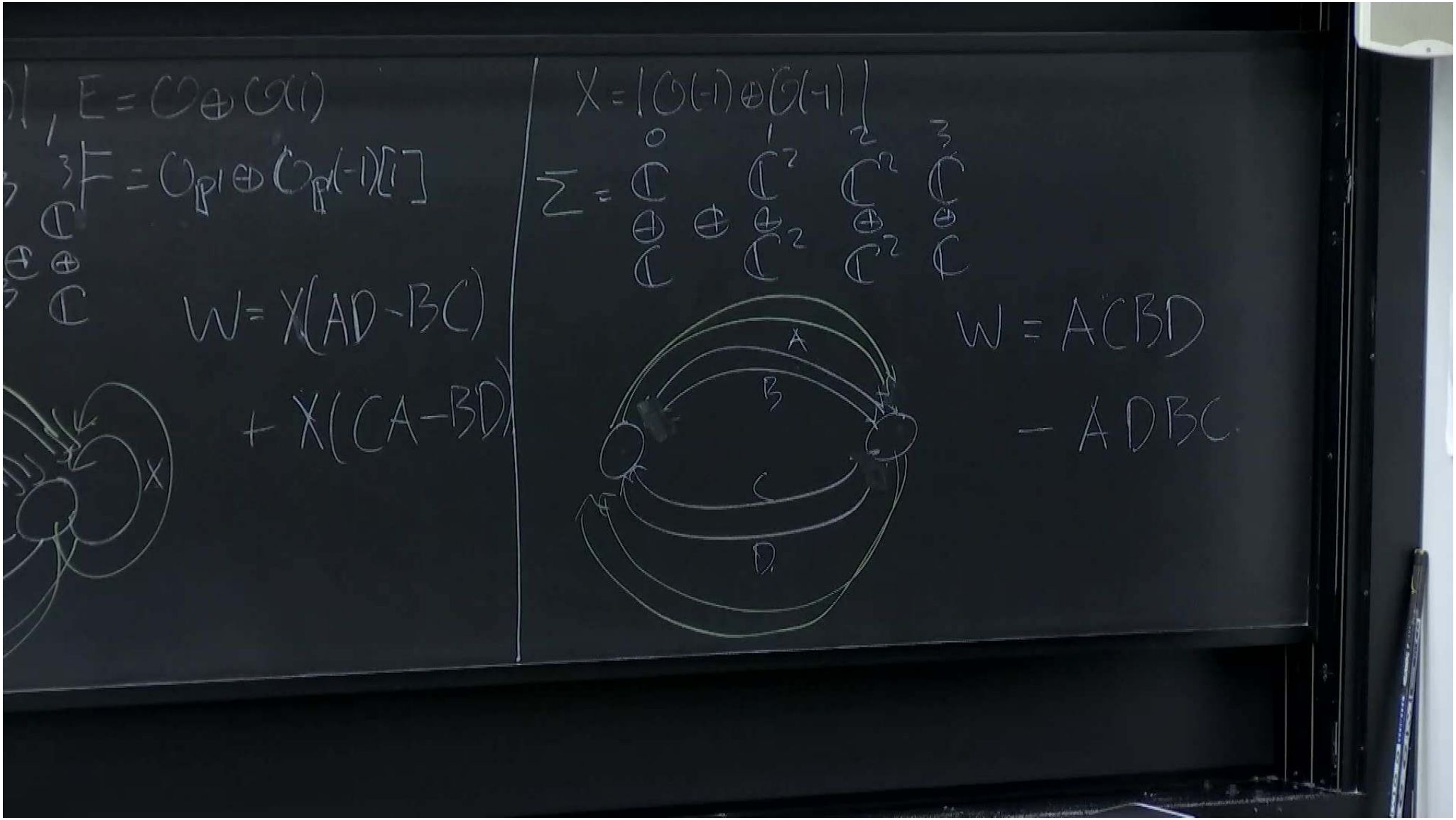
$\Sigma = \mathbb{C} \oplus \mathbb{C}^3 \oplus \mathbb{C}^3 \oplus \mathbb{C}$

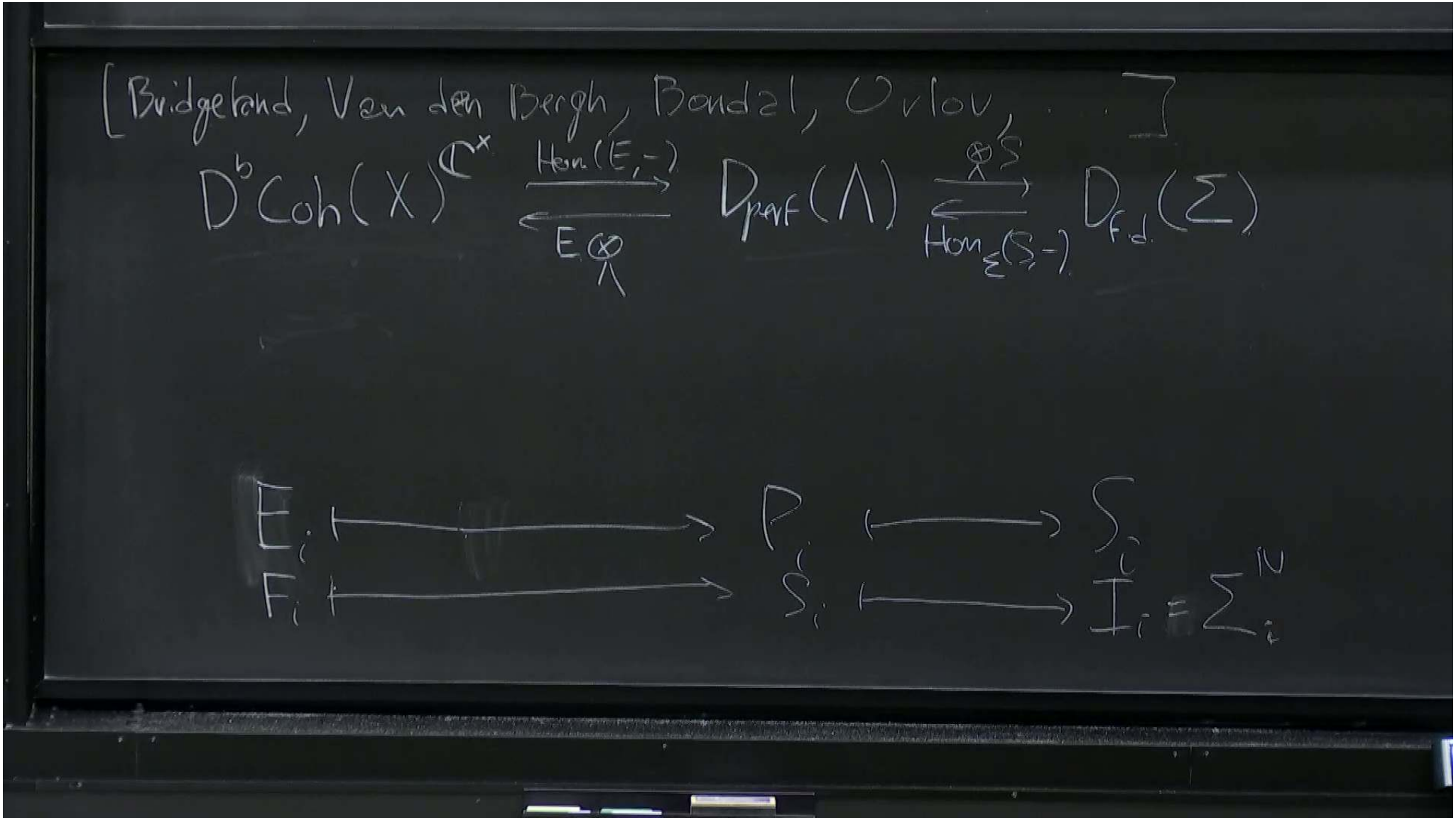


$W = B_1[B_2, B_3]$

Ex: $X = |\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-2)|$, $E = C$


Ex X = |O ⊕ O(-2)|, E = O ⊕ O(1)
F = O_P1 ⊕ O_P1(-1)[1]
W = X(AD - BC) + X(CA - BD)
X = |O(-1) ⊕ O(-1)|
A
B
C
D
X


E = O ⊕ O(1)
F = O_P1 ⊕ O_P1(-1)[1]
W = X(AD - BC)
+ X(CA - BD)
X
X = |O(-1) ⊕ O(-1)|
0 1 2 3
Σ = C C² C² C
⊕ ⊕ ⊕ ⊕ ⊕
C C² C² C
W = ACBD
- ADBC
A
B
C
D


[Bridgeland, Van den Bergh, Bondal, Orlov, ...]
$D^b\mathrm{Coh}(X)^{\mathbb{C}^\times}$
$\mathrm{Hom}(E,-)$
$E\otimes -$
$D_{\mathrm{perf}}(\Lambda)$
$-\otimes S$
$\mathrm{Hom}_\Sigma(S,-)$
$D_{f.d.}(\Sigma)$
$E_i \longmapsto P_i \longmapsto S_i$
$F_i \longmapsto S_i \longmapsto I_i = \Sigma^{iN}$

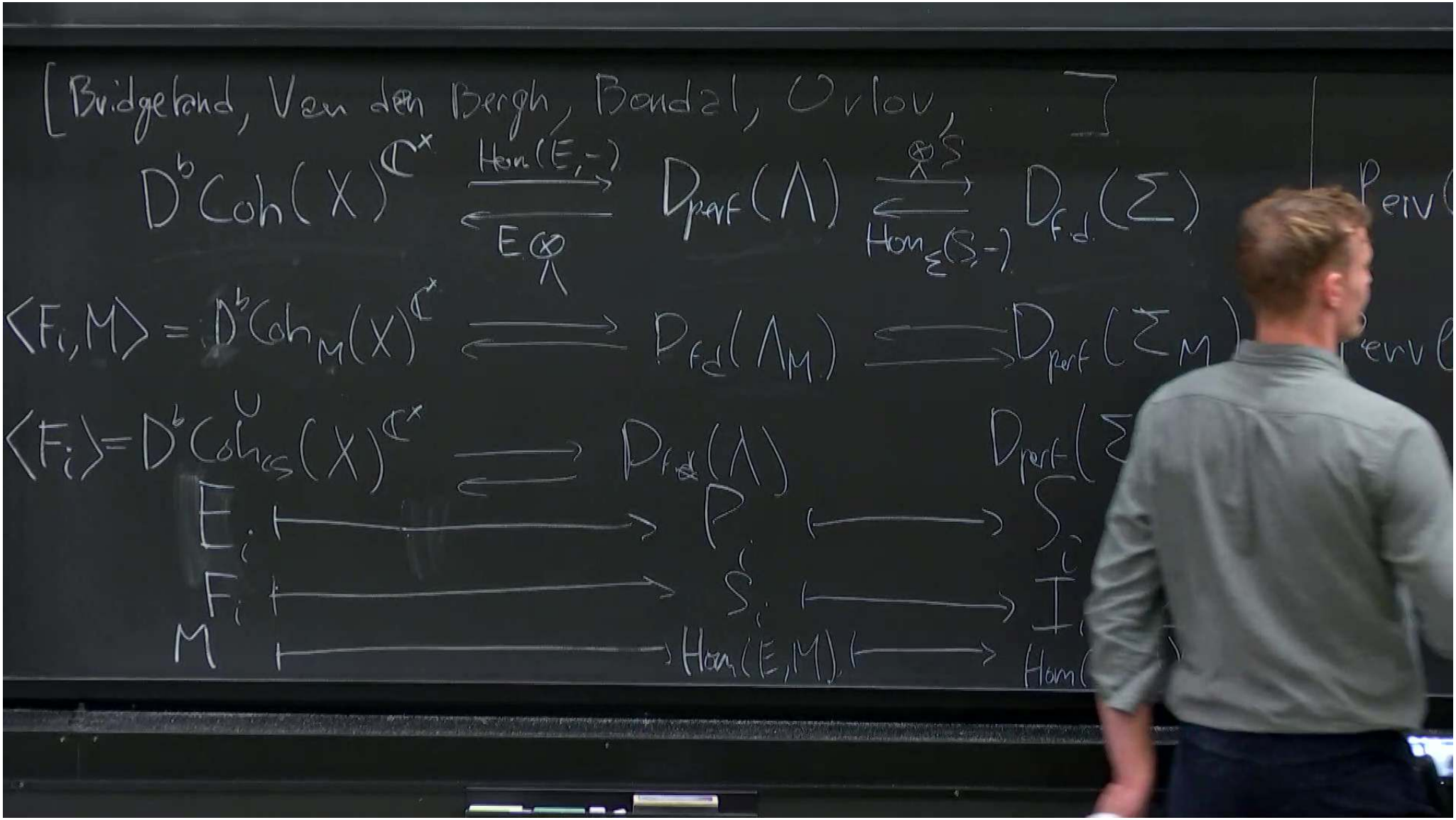
[Bridgeland, Van den Bergh, Bondal, Orlov, ...]

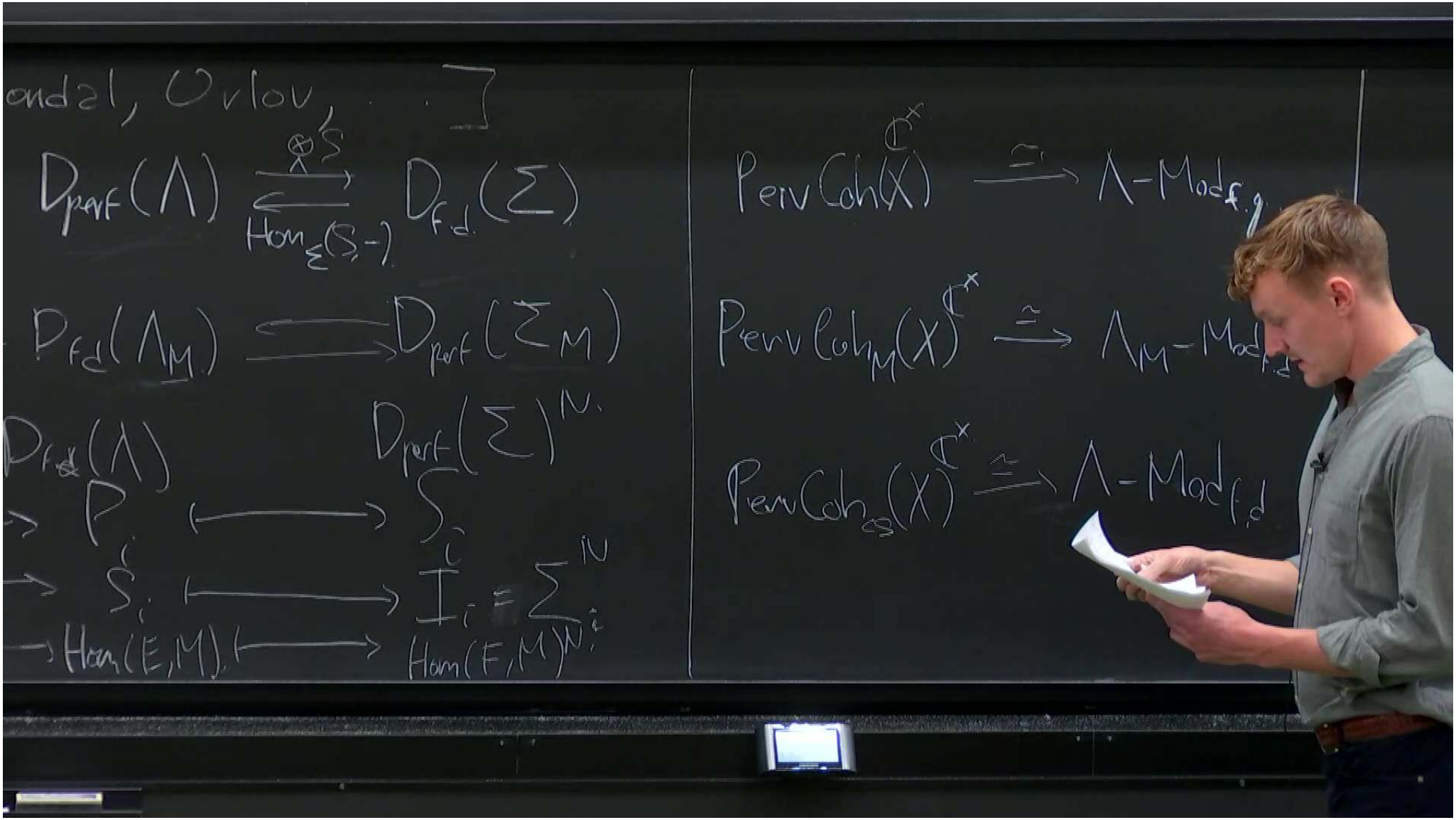
$$D^b\mathrm{Coh}(X)^{\mathbb{C}^*} \underset{E\otimes(-)}{\overset{\mathrm{Hom}(E,-)}{\rightleftarrows}} D_{\mathrm{perf}}(\Lambda) \underset{\mathrm{Hom}_\Sigma(S,-)}{\overset{-\otimes_\Lambda S}{\rightleftarrows}} D_{f.d.}(\Sigma)$$

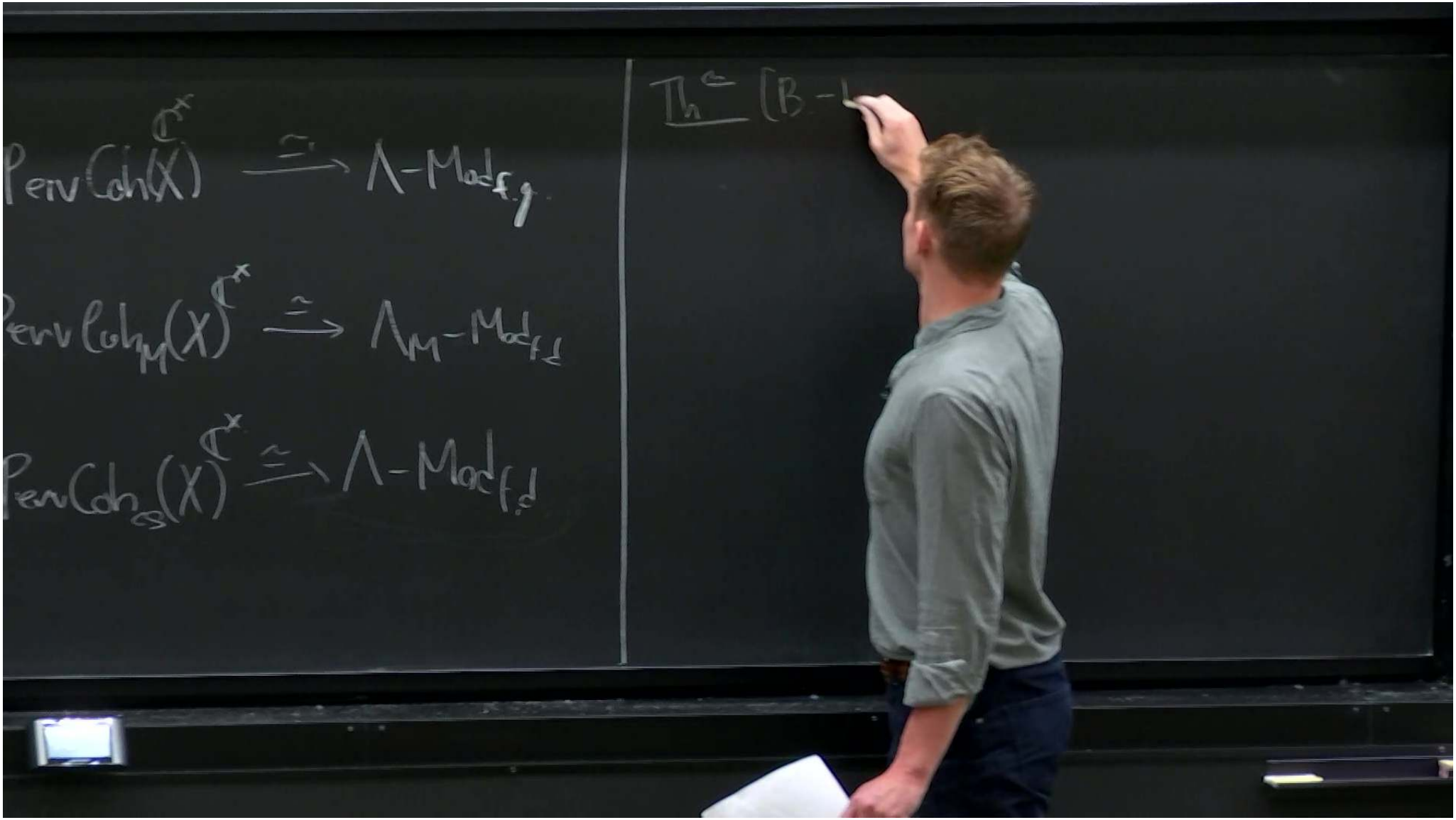
$$E_i \longmapsto P_i \longmapsto S_i$$

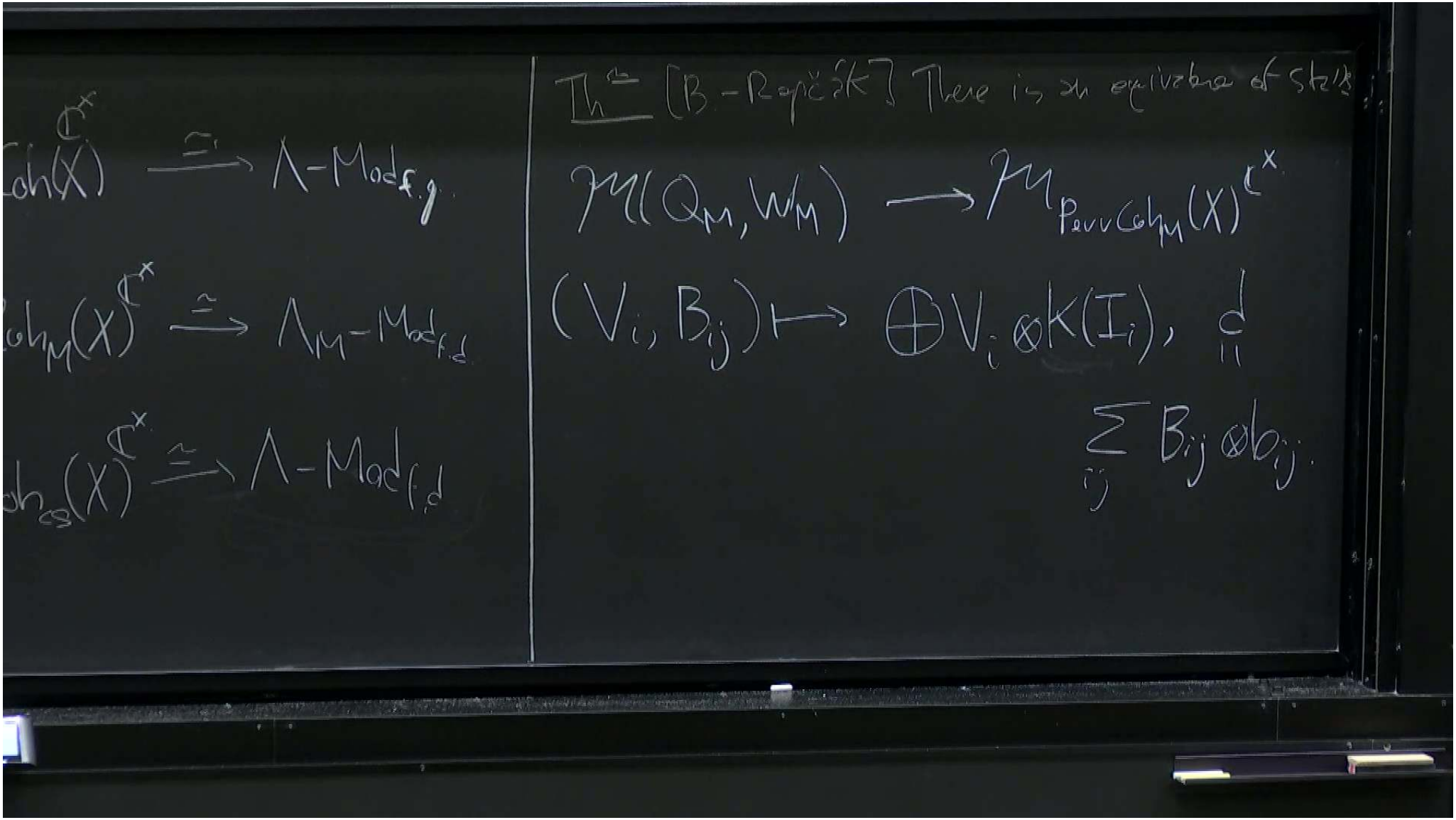
$$F_i \longmapsto S_i \longmapsto I_i = \Sigma^{iv}$$

$$M \longmapsto \mathrm{Hom}(E,M) \longmapsto \mathrm{Hom}(F,M)^{v_i}$$


[Bridgeland, Van den Bergh, Bondal, Orlov, ...]
$D^b\mathrm{Coh}(X)^{\mathbb{C}^*}$
$\mathrm{Hom}(E,-)$
$E\otimes$
$D_{\mathrm{perf}}(\Lambda)$
$\otimes S$
$\mathrm{Hom}_\Sigma(S,-)$
$D_{f.d.}(\Sigma)$
$\langle F_i, M\rangle = D^b\mathrm{Coh}_M(X)^{\mathbb{C}^*}$
$D_{fd}(\Lambda_M)$
$D_{\mathrm{perf}}(\Sigma_M)$
$\langle F_i\rangle = D^b_{cs}\mathrm{Coh}(X)^{\mathbb{C}^*}$
$D_{fd}(\Lambda)$
$E_i \mapsto P_i \mapsto S_i$
$F_i \mapsto S_i \mapsto I_i$
$M \mapsto \mathrm{Hom}(E,M) \mapsto \mathrm{Hom}($


ondal, Orlov, ...]
$D_{perf}(\Lambda)$
$\otimes S$
$\mathrm{Hom}_\Sigma(S,-)$
$D_{f.d.}(\Sigma)$
$D_{fd}(\Lambda_M) \rightleftarrows D_{perf}(\Sigma_M)$
$D_{perf}(\Sigma)^{N.}$
$P_{fd}(\Lambda)$
$P \longmapsto S$
$S \longmapsto I = \sum^N$
$\mathrm{Hom}(E,M) \longmapsto \mathrm{Hom}(F,M)^N$
$\mathrm{Perv}\,\mathrm{Coh}(X) \xrightarrow{\Gamma^+} \Lambda\text{-Mod}_{f.g.}$
$\mathrm{Perv}\,\mathrm{Coh}_M(X) \xrightarrow{\Gamma^+} \Lambda_M\text{-Mod}$
$\mathrm{Perv}\,\mathrm{Coh}_{cs}(X) \xrightarrow{\Gamma^+} \Lambda\text{-Mod}_{f.d.}$


Perv(Coh(X))
Λ-Mod
Perv(Coh_M(X))
Λ_M-Mod
Perv(Coh_S(X))
Λ-Mod


Coh(X)^{C^*} ≃ Λ-Mod_{f.g.}
Coh_M(X)^{C^*} ≃ Λ_M-Mod_{f.d}
Coh_{cs}(X)^{C^*} ≃ Λ-Mod_{f.d}
Th ≃ [B.-Rapčák] There is an equivalence of stacks
M(Q_M, W_M) → M_{PervCoh_M(X)^{C^*}}
(V_i, B_ij) ↦ ⊕ V_i ⊗ k(I_i), d
Σ_ij B_ij ⊗ b_ij.

$\mathrm{Coh}(X) \xrightarrow{\sim} \Lambda\text{-Mod}_{f.g.}$

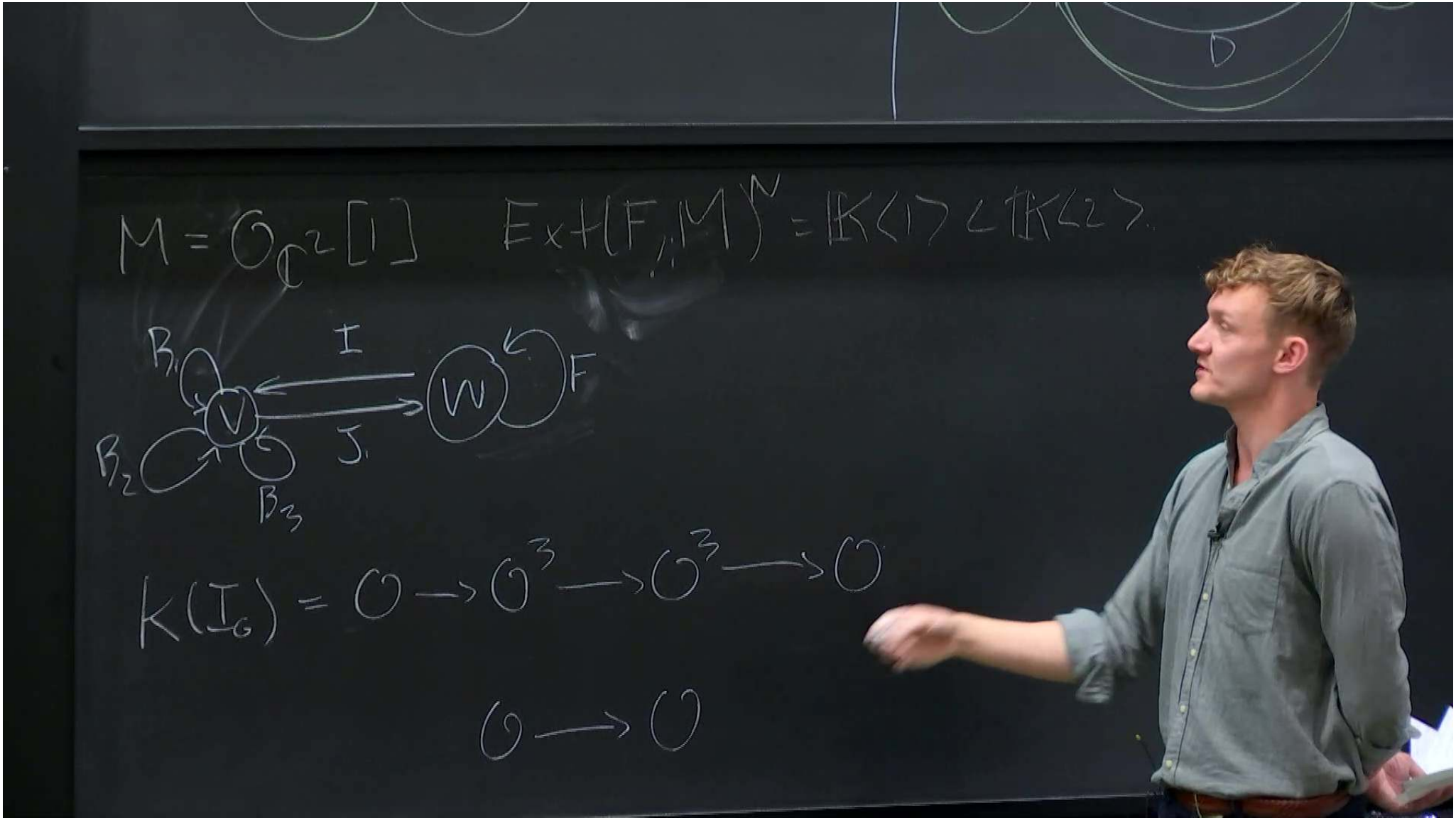
$\mathrm{Coh}_M(X)^{\mathbb{C}^\times} \xrightarrow{\sim} \Lambda_M\text{-Mod}_{f.d.}$

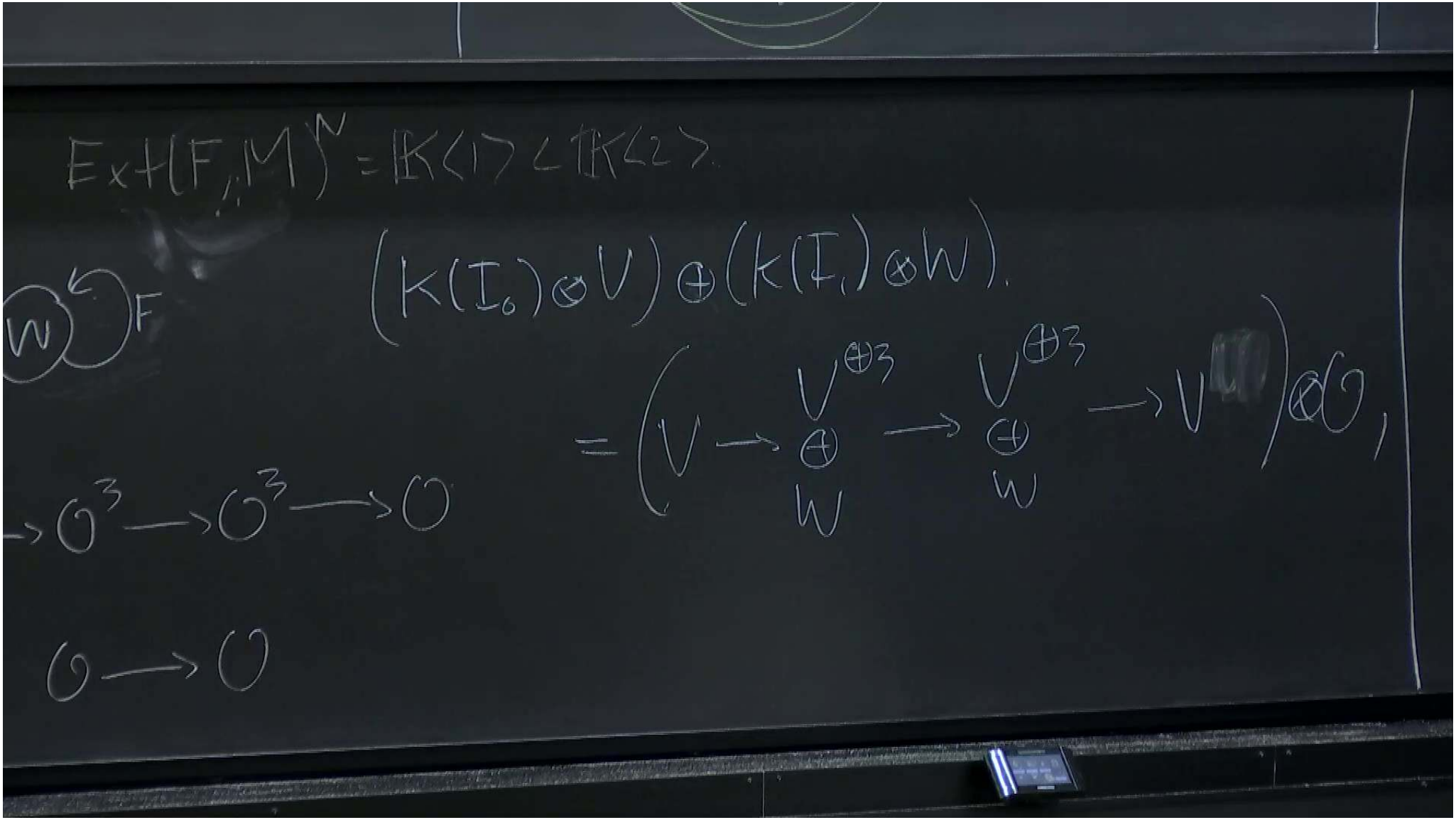
$\mathrm{Coh}_{cs}(X)^{\mathbb{C}^\times} \xrightarrow{\sim} \Lambda\text{-Mod}_{f.d.}$

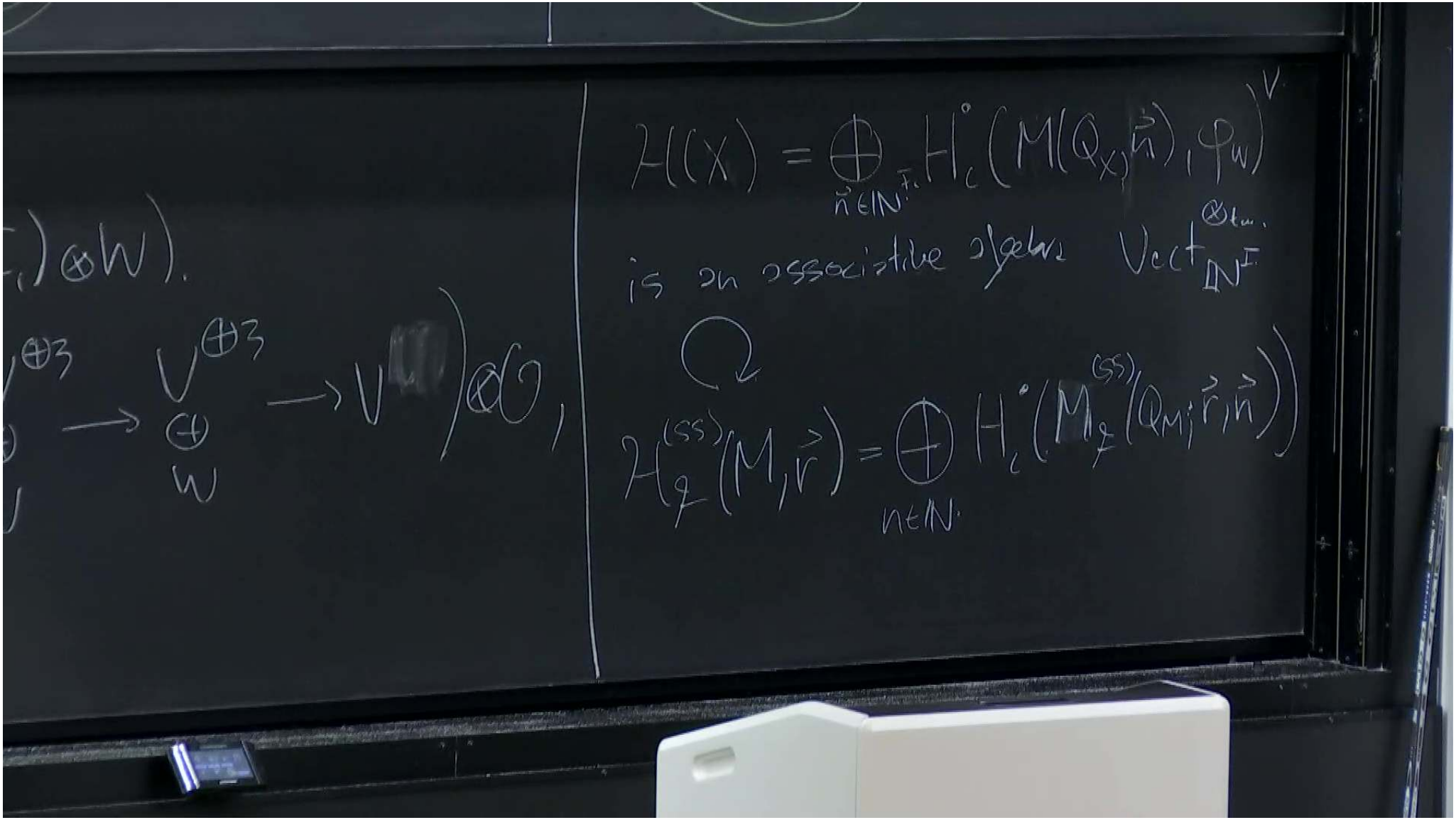
Th [B.-Rapčák] There is an equivalence of stacks

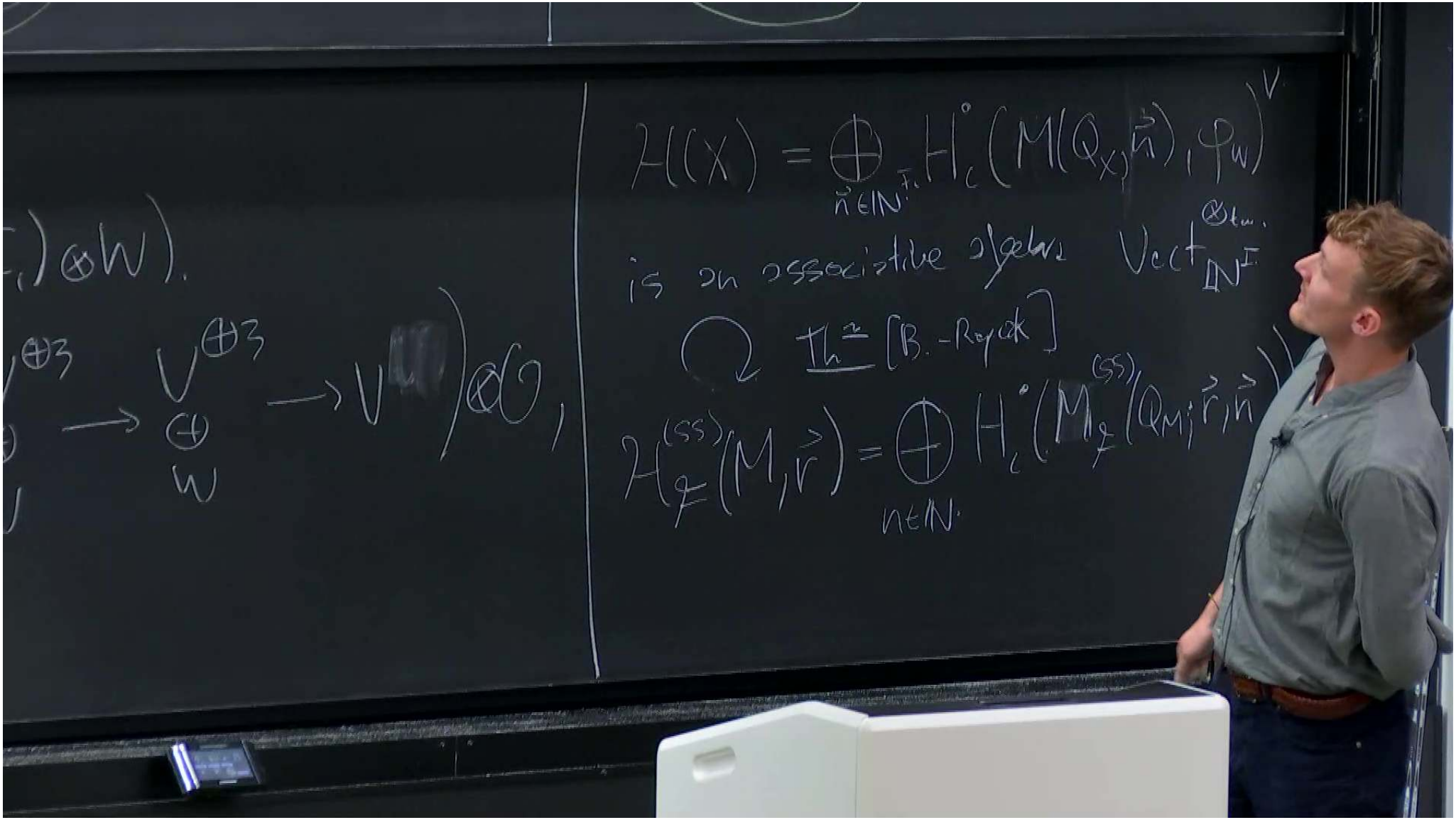
$$\mathcal{M}(Q_M, W_M) \longrightarrow \mathcal{M}_{\mathrm{Perv}\mathrm{Coh}_M(X)}{}^{\mathbb{C}^\times}$$

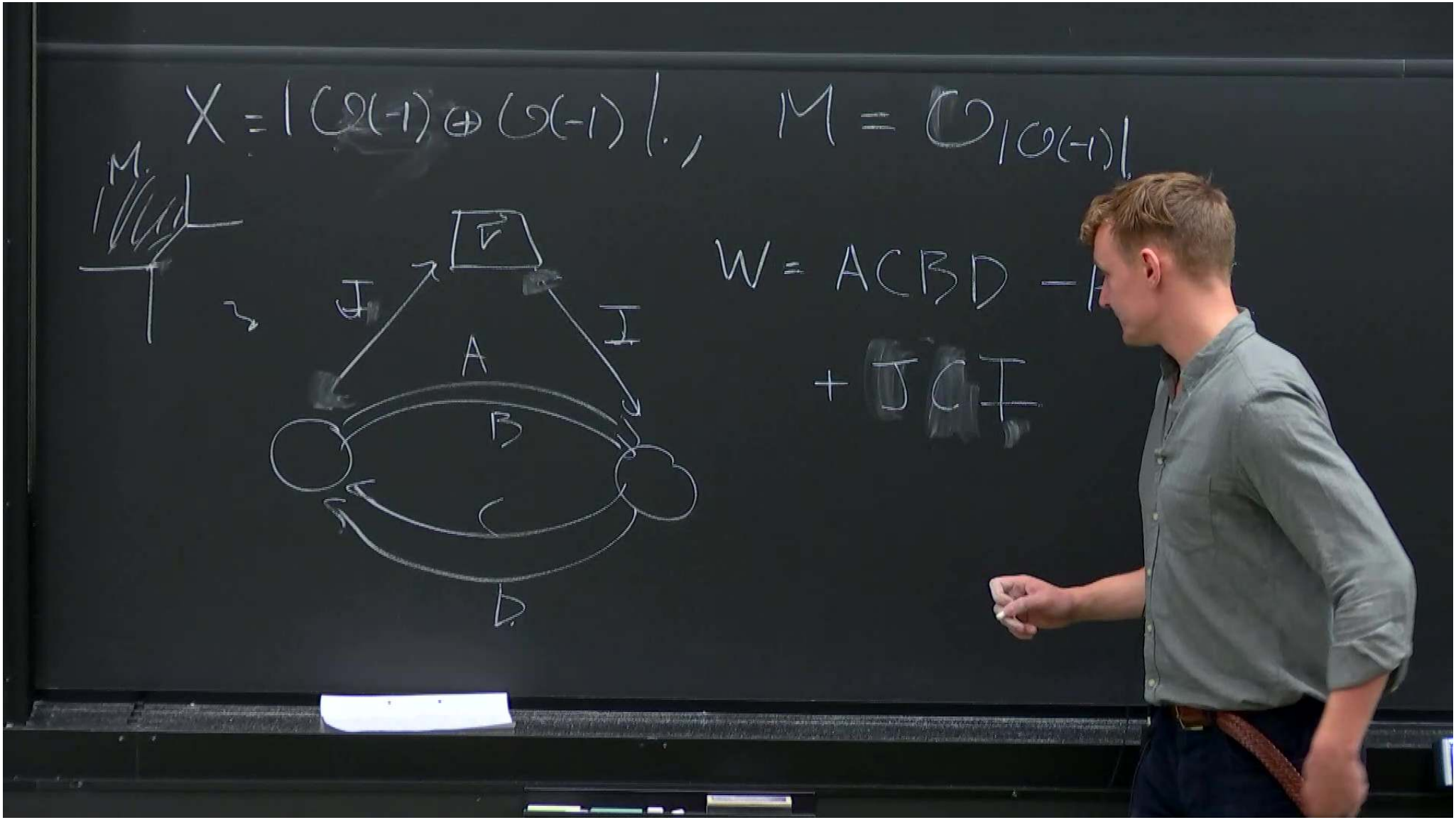
$$(V_i, B_{ij}) \longmapsto \bigoplus_i V_i \otimes K(I_i), \quad \sum_{ij} B_{ij} \otimes b_{ij}.$$

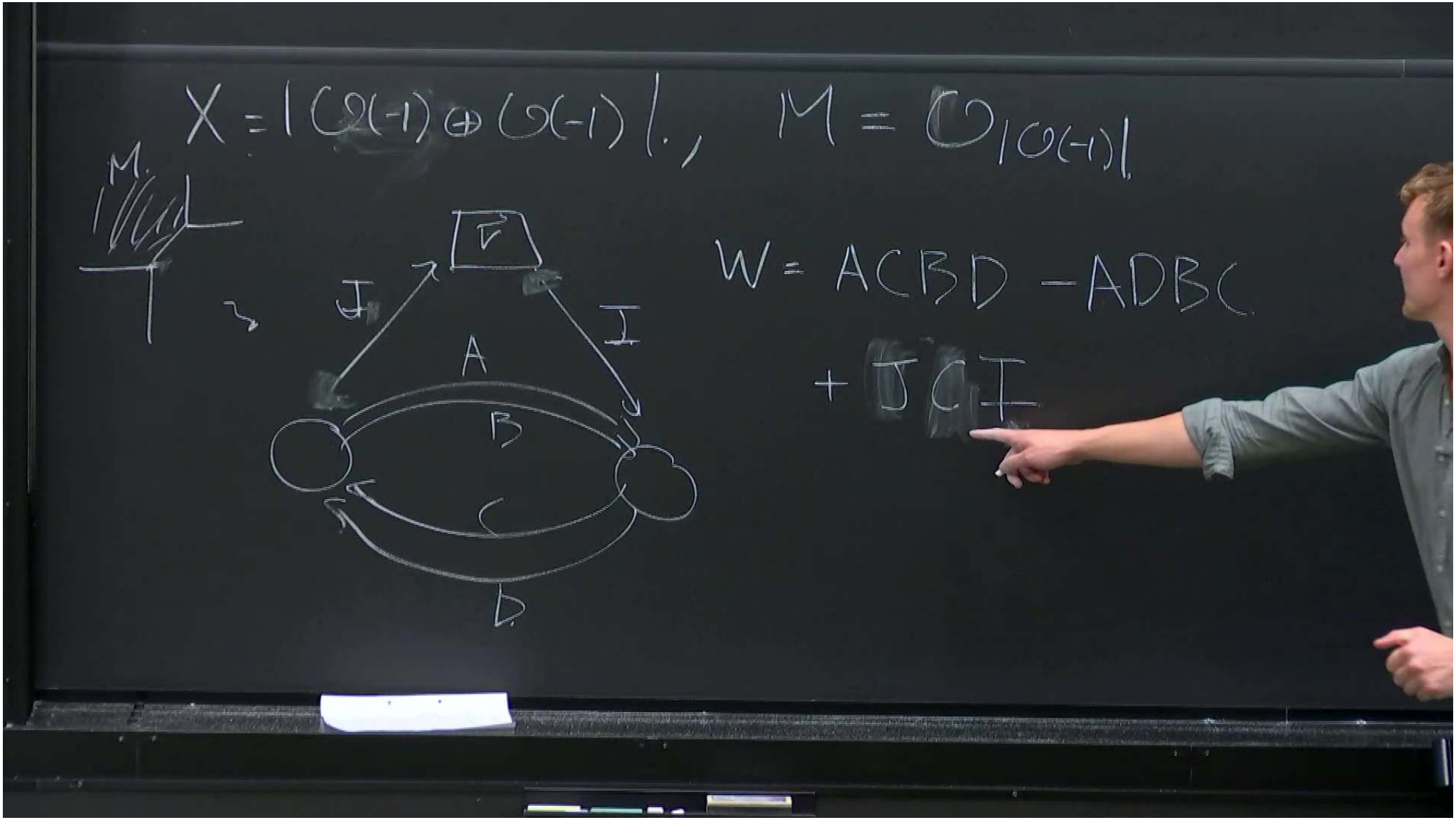

M = O_{C^2}[1]
Ext(F, M)^N = Hom(K⟨1⟩, K⟨2⟩)
B1
I
V
W
F
J
B2
B3
K(I_G) = O → O^3 → O^3 → O
O → O

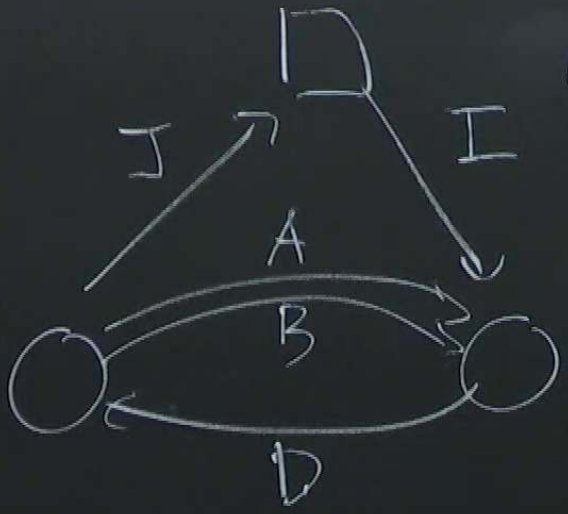

$Ext(F,M)^N = \mathbb{K}\langle 1\rangle \subset \mathbb{K}\langle 2\rangle$
$(k(I_0)\otimes V)\oplus(k(I_1)\otimes W)$
W
F
$\to O^3 \to O^3 \to O$
$O \to O$
$V \to V^{\oplus 3} \to V^{\oplus 3} \to V$
$\oplus$
W
$\otimes O,$


$\mathcal{H}(X) = \bigoplus_{\vec{n} \in \mathbb{N}^{F_1}} H_c^\bullet(M(Q_X, \vec{n}), \varphi_W)^{\otimes_{\mathrm{Ver}} \mathrm{Vect}_{\mathbb{N}^F}}$
is an associative algebra
$\mathcal{H}_q^{(ss)}(M, \vec{r}) = \bigoplus_{n \in \mathbb{N}} H_c^\bullet(M_q^{(ss)}(Q_M, \vec{r}, n))$


$\mathcal{H}(X) = \bigoplus_{\vec{n} \in \mathbb{N}^I} H_c^\bullet(M(Q_X, \vec{n}), \varphi_W)^{\otimes_{\mathrm{tw}}}$
is an associative algebra
$\mathrm{Vect}_{\mathbb{N}^I}$
Th$^{\simeq}$ [B.-Rozek]
$\mathcal{H}^{(ss)}_{\zeta}(M, \vec{r}) = \bigoplus_{n \in \mathbb{N}} H_c^\bullet(M^{(ss)}_{\zeta}(Q_M, \vec{r}, n))$


X = |O(-1) ⊕ O(-1)|,
M = O|O(-1)|
M
V
J
I
A
B
C
W = ACBD
+ JCI


X = |O(-1) ⊕ O(-1)|, M = O|O(-1)|
M
J
I
A
B
C
D
W = ACBD − ADBC
+ JCI



Nakajima – Yoshioka

$$BDA - ADB + IJ = 0$$

BFL $\bigoplus H^\bullet(M(X,M))$

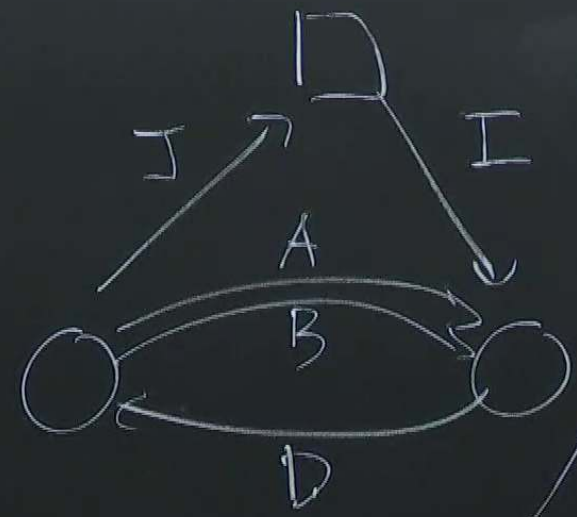
(A.C.F.) $\simeq \bigoplus_{\lambda \in P_+ \cap Q} W_\lambda^k(g_N) \otimes W_{\lambda^t}^{k'}(g_N)$

$\frac{1}{k+N} + \frac{1}{k'+N} = 1$

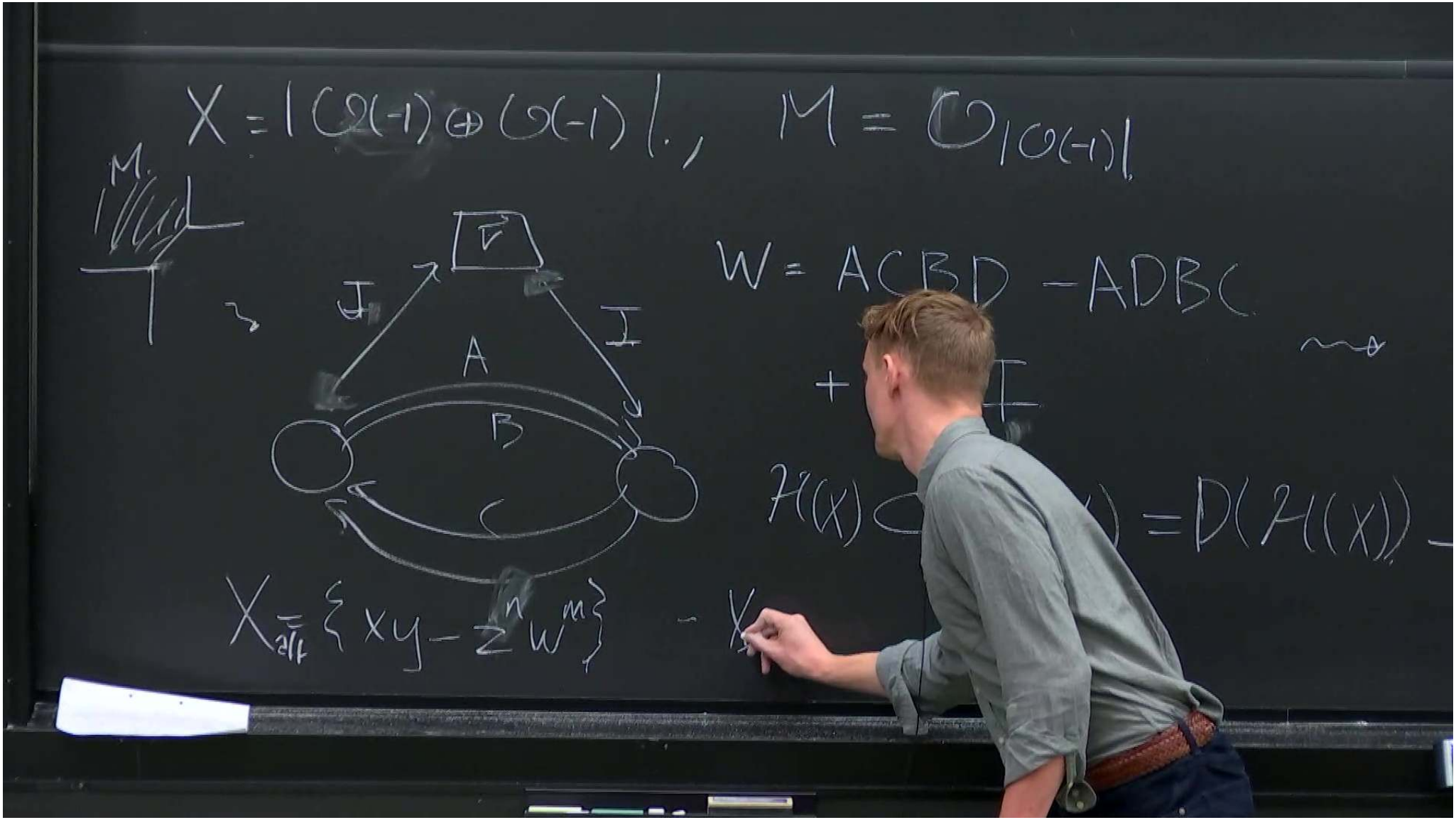
$M = \mathbb{C}_{|\mathcal{O}(-1)|}$

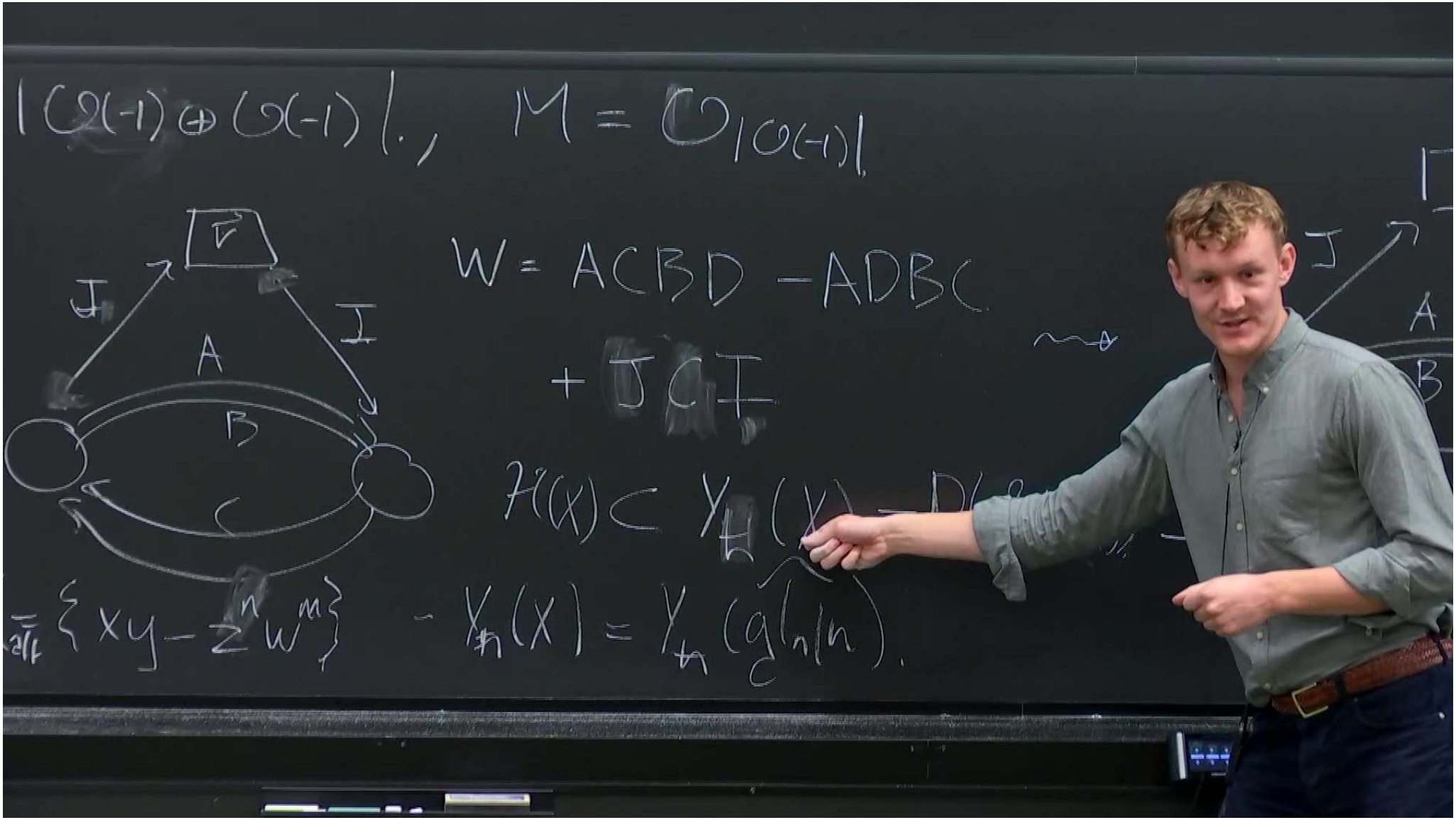
$W = ACBD - ADBC$

$+ JCI$

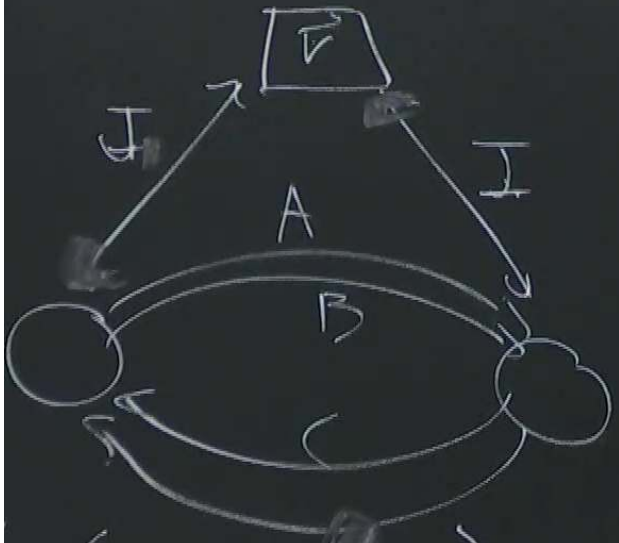


$\mathcal{F}(X) \subset Y_{\Sigma}(X) = D(\mathcal{F}((X)) \twoheadrightarrow D(S)C$


X = |O(-1) ⊕ O(-1)|, M = O|O(-1)|
M.
Γ
J
I
A
B
C
W = ACBD − ADBC
+
=D(P((X))
X = {xy − z^n w^m}


W = ACBD − ADBC
+ JCI
J
I
A
B
C
Γ

$|\mathcal{O}(-1)\oplus\mathcal{O}(-1)|$, $M = \mathcal{O}_{|\mathcal{O}(-1)|}$

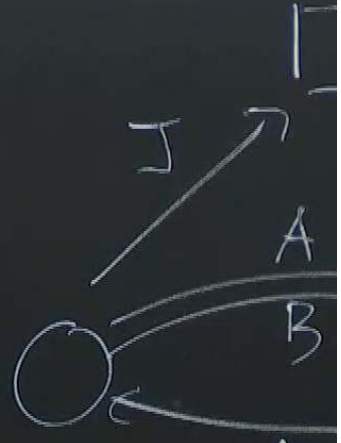


$W = ACBD - ADBC$

$+ JCI$

$\mathcal{H}(X) \subset Y_n(X) = D(\mathcal{H}(X))$

$\overline{\mathcal{H}}\{xy - z^n w^m\}$ $Y_n(X) = Y_n(\widehat{g_{n/n}})$.



$\to V(S)$