Title: Moduli space of cactus flowers

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Series: Mathematical Physics

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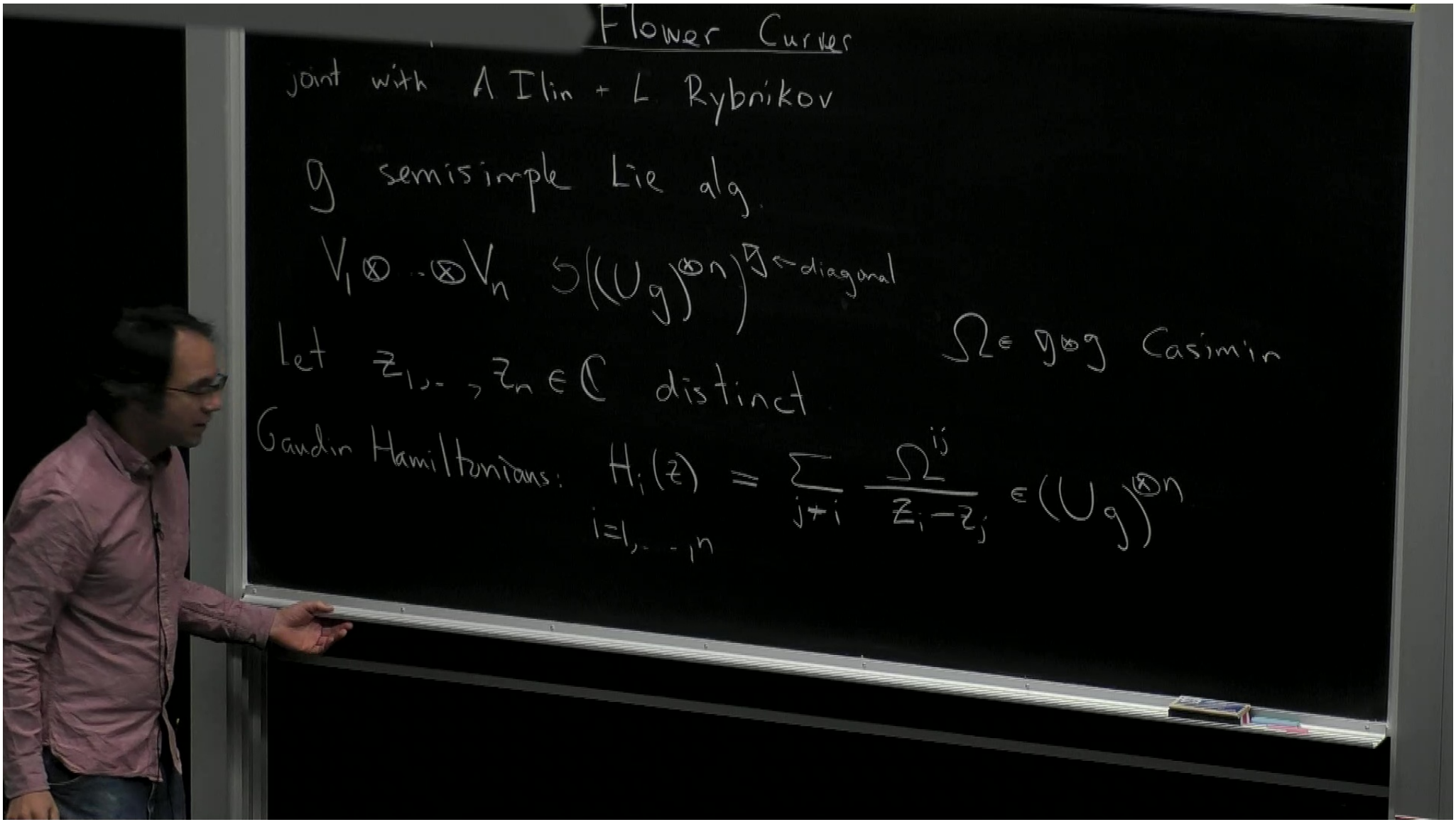
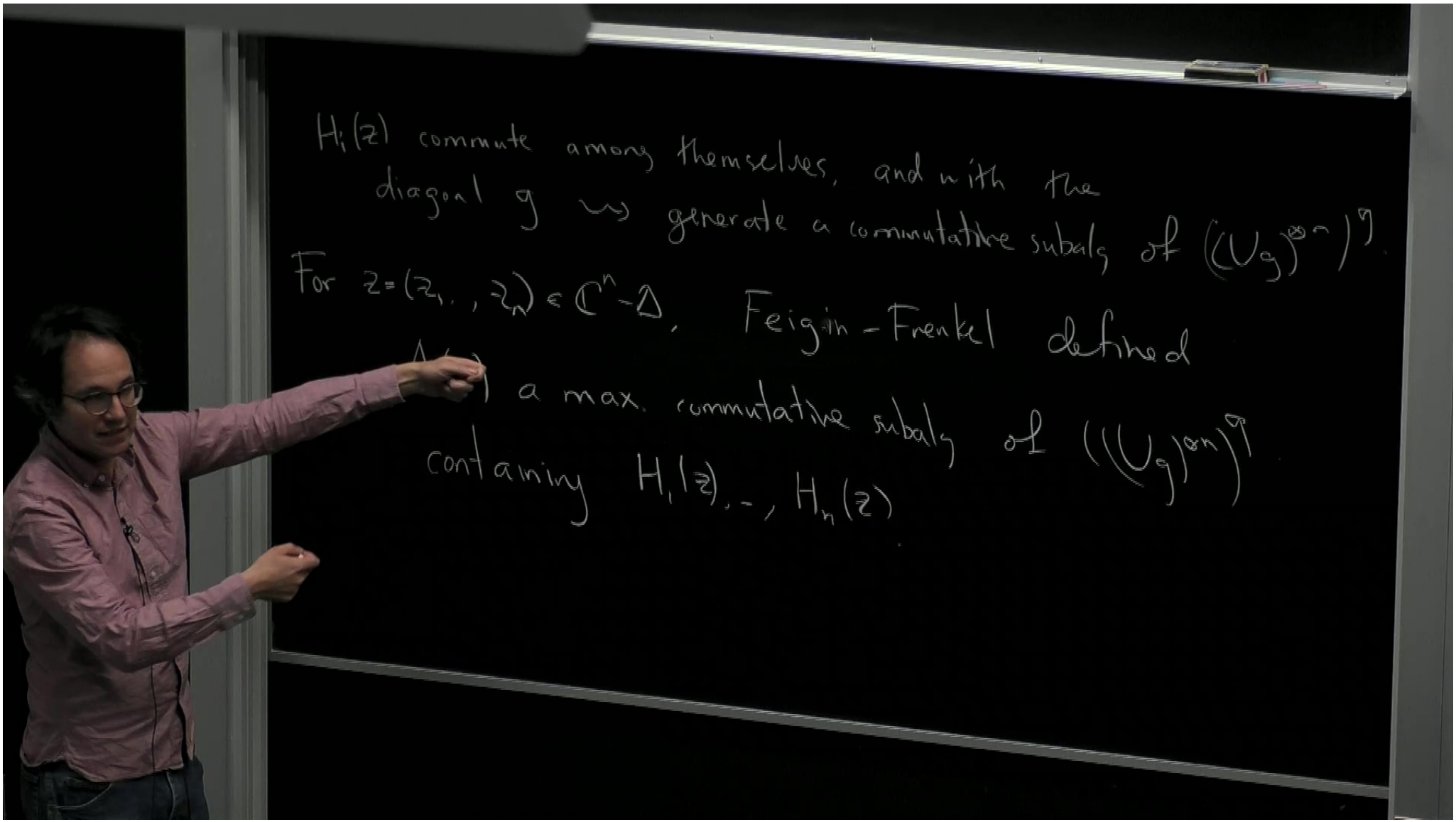
Abstract: The Deligne-Mumford moduli space of genus 0 curves plays many roles in representation theory. For example, the fundamental group of its real locus is the cactus group which acts on tensor products of crystals.

I will discuss a variant on this space which parametrizes "cactus flower curves". The fundamental group of the real locus of this space is the virtual cactus group. This moduli space of cactus flower curves is also the parameter space for inhomogeneous Gaudin algebras.

Zoom link: https://pitp.zoom.us/j/96658223425?pwd=NUxRN2FsdWJ1SWtHMlRDcTdHMGNPQT09


Flower Curves
joint with A Ilin + L Rybnikov
$\mathfrak{g}$ semisimple Lie alg
$V_1 \otimes \cdots \otimes V_n \circlearrowleft ((U\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$ ← diagonal
$\Omega \in \mathfrak{g} \otimes \mathfrak{g}$ Casimir
let $z_1, \ldots, z_n \in \mathbb{C}$ distinct
Gaudin Hamiltonians: $H_i(z) = \sum_{j \neq i} \frac{\Omega^{ij}}{z_i - z_j} \in (U\mathfrak{g})^{\otimes n}$
$i = 1, \ldots, n$


$H_i(z)$ commute among themselves, and with the
diagonal $\mathfrak{g}$ $\rightsquigarrow$ generate a commutative subalg of $((U\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$.
For $z=(z_1,\dots,z_n)\in\mathbb{C}^n-\Delta$, Feigin-Frenkel defined
$A(z)$ a max. commutative subalg of $((U\mathfrak{g})^{\otimes n})^{\mathfrak{g}}$
containing $H_1(z),\dots,H_n(z)$

<u>Theorem</u> [Aguirre-Felder-Veselov, Shuvalov, Rybnikov].

The map

descends/ extends to

$$\begin{tikzcd}
\mathbb{C}^n \setminus \Delta \arrow[r] \arrow[d] & \text{subalg of } ((\mathbb{U}\mathfrak{g})^{*n})^{\mathfrak{g}} \\
\mathbb{C}^n \setminus \Delta / \mathbb{C}^\times \ltimes \mathbb{C} \arrow[ur] \arrow[d, equal] & \\
(\mathbb{P}^1)^{n+1} \setminus \Delta / PGL_2 \subset \overline{M}_{0,n+1} \arrow[uu] & \leftarrow \text{Deligne-Mumford space.}
\end{tikzcd}$$

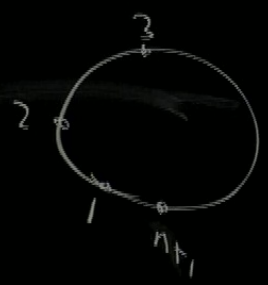
Theorem [Aguirre-Felder-Veselov, Shuvalov, Rybnikov].

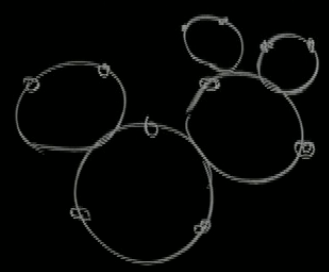
The map $\mathbb{C}^n \setminus \Delta \longrightarrow$ subalg of $((U_q)^{\otimes n})^q$

descends/extends to

$$\begin{tikzcd} \mathbb{C}^n \setminus \Delta \arrow[d] \arrow[r] & \text{subalg of } ((U_q)^{\otimes n})^q \\ \mathbb{C}^n \setminus \Delta / \mathbb{C}^\times \ltimes \mathbb{C} \arrow[d, equal] \arrow[ur] & \\ (\mathbb{P}^1)^{n+1} \setminus \Delta / PGL_2 \subset \overline{M}_{0,n+1} \arrow[uu] & \end{tikzcd}$$

← Deligne-Mumford space.



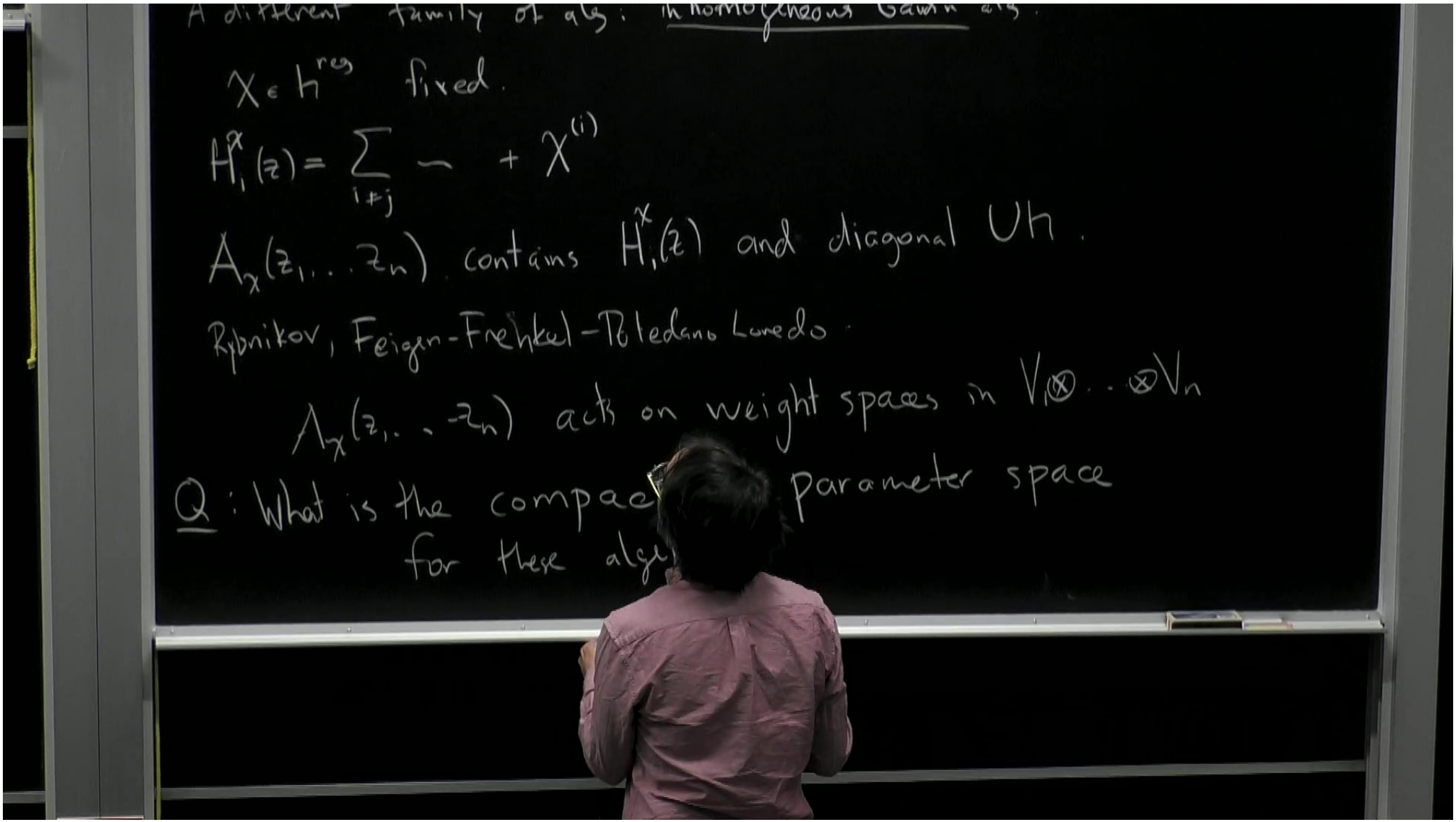


If $z \in \overline{M}_{0,n+1}(\mathbb{R})$, then $A(z)$ is generated by Hermitian operators and acts with simple spectrum on $\mathrm{Hom}_{\mathfrak{g}}(V_{\mathrm{AH}}, V_1 \otimes \cdots \otimes V_n)$

$\rightsquigarrow$ can study the monodromy of the eigenlines

$\rightsquigarrow$ get an action of $\pi_1^{S_n}(\overline{M}_{0,n+1}(\mathbb{R})) = C_n$ cactus group.

[Halacheva - K. - Rybnikov - Weekes]


A different family of alg: inhomogeneous Gaudin alg.
$\chi \in \mathfrak{h}^{reg}$ fixed.
$H_i^{\chi}(z) = \sum_{i \neq j} \frac{\ }{\ } + \chi^{(i)}$
$A_\chi(z_1, \ldots z_n)$ contains $H_i^{\chi}(z)$ and diagonal $U\mathfrak{h}$.
Rybnikov, Feigin-Frenkel-Toledano Laredo.
$A_\chi(z_1, \ldots z_n)$ acts on weight spaces in $V_1 \otimes \ldots \otimes V_n$
Q: What is the compact parameter space
for these alg

$$\mathbb{C}^n \setminus \Delta / \mathbb{C} \longrightarrow \text{subalg of } (U_q \mathfrak{g})^{\otimes n}$$

$$\circlearrowleft ? \longleftarrow$$

V vector space $\quad V \hookrightarrow \mathbb{C}^n$

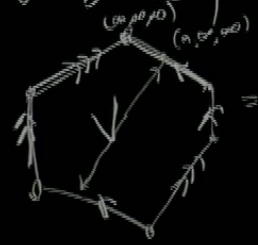
Matroid Schubert variety: $\quad \overline{V} :=$ closure of V inside $(\mathbb{P}^1)^n$.

<u>eg</u> $\quad V = \{ a+b=c \} \subset \mathbb{C}^3 \qquad \overline{V} = \{ a+b=c \ ; \ a,b,c \in \mathbb{P}^1 \}$

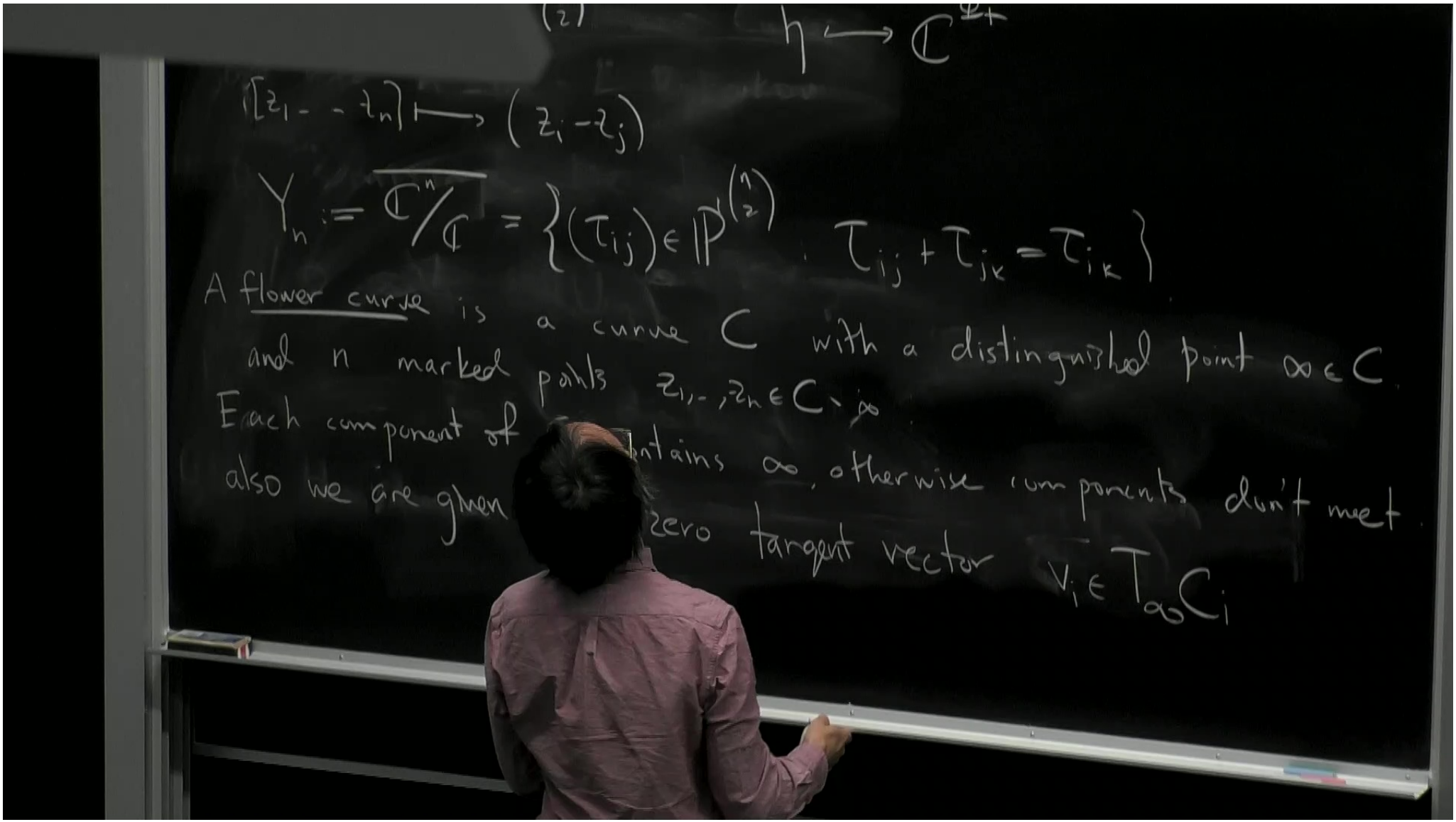
$$= V \cup \underbrace{\{ \infty + b = \infty \}}_{=c} \cup \underbrace{\{ a + \infty = \infty \}}_{=c} \cup \underbrace{\{ \infty + \infty = c \}}_{=c}$$

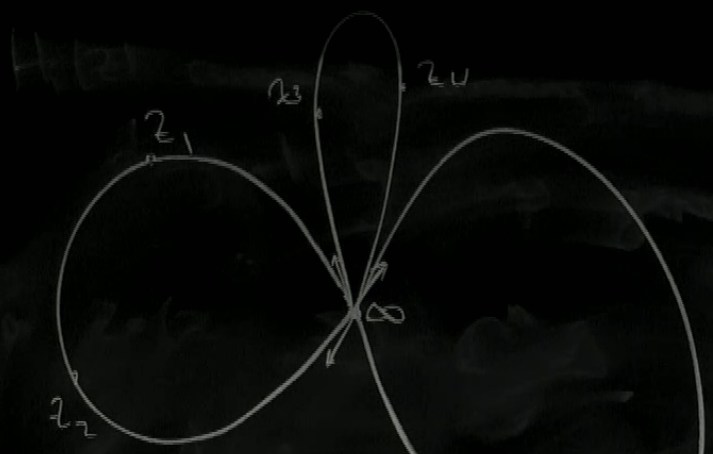
$$\cup \{ \infty + \infty = \infty \}, \ pt$$

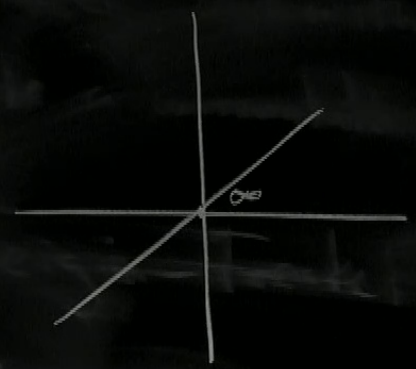
$\overline{V}(\mathbb{R}) =$

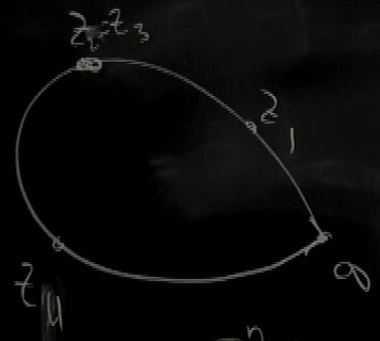


$=$ torus with two pts identified.


$\eta \longmapsto \mathbb{C}^{\mathcal{L}_+}$
$[z_1 \dots z_n] \longmapsto (z_i - z_j)$
$Y_n := \overline{\mathbb{C}^n / \mathbb{C}} = \{ (\tau_{ij}) \in \mathbb{P}^{\binom{n}{2}} : \tau_{ij} + \tau_{jk} = \tau_{ik} \}$
A flower curve is a curve C with a distinguished point $\infty \in C$
and n marked points $z_1, \dots, z_n \in C \setminus \infty$
Each component of
ntains ∞, otherwise components don't meet
also we are given
zero tangent vector $v_i \in T_\infty C_i$





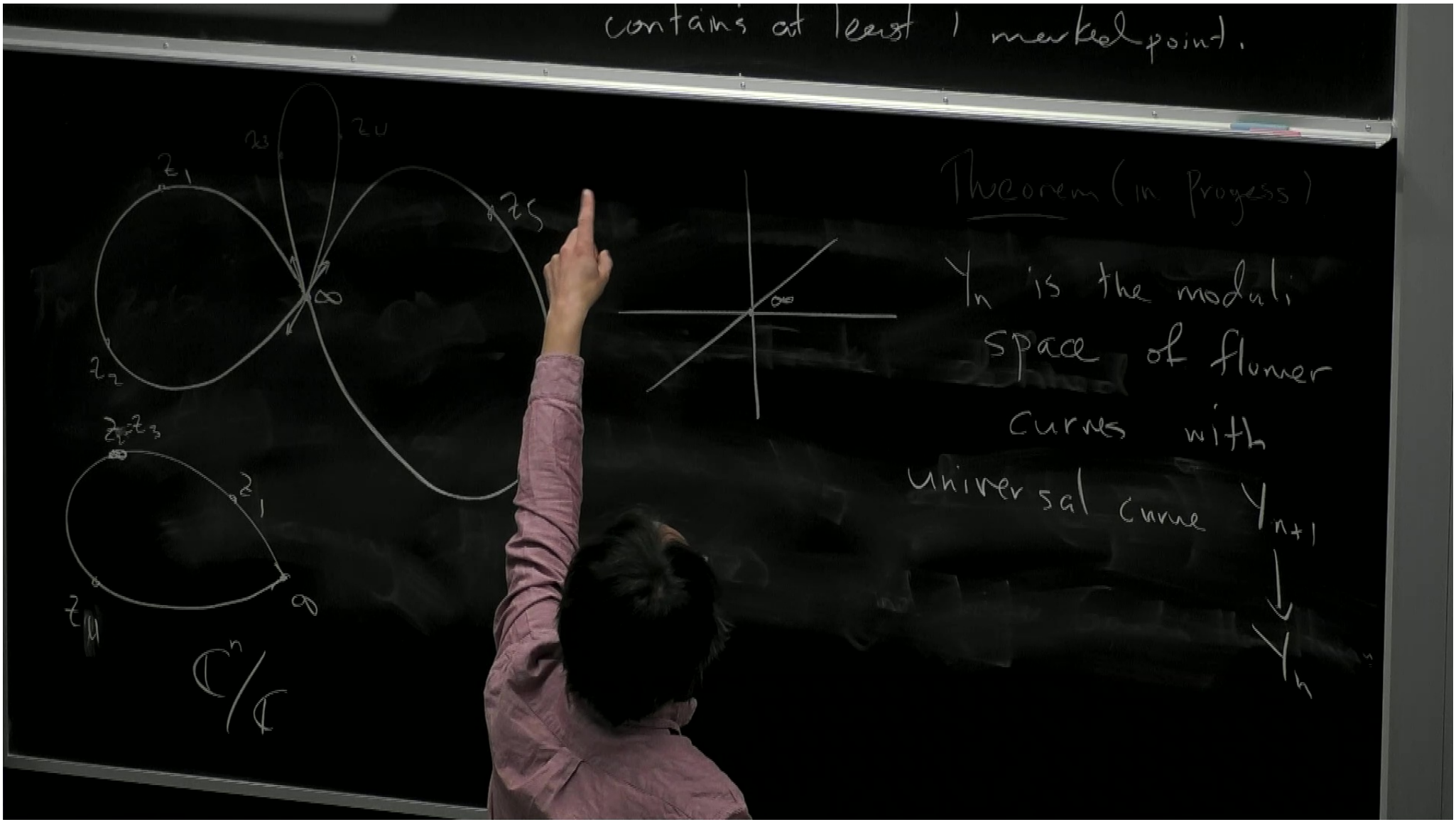


$\mathbb{C}^n / \mathbb{C}$

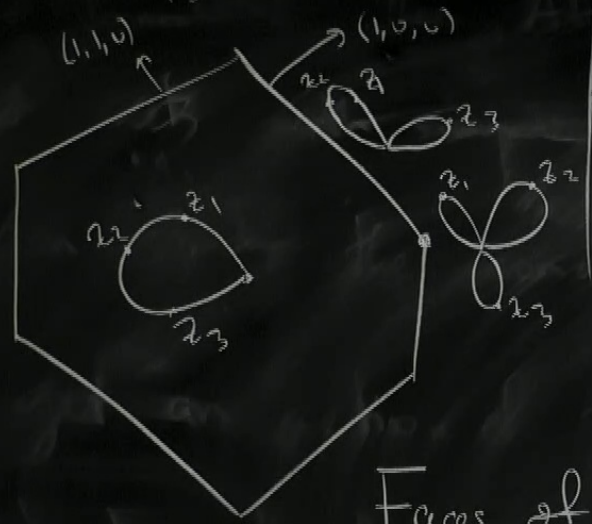
<u>Theorem</u> (in progress)

Y_n is the moduli space of flower curves with universal curve

$$\begin{array}{c} Y_{n+1} \\ \downarrow \\ Y_n \end{array}$$


contains at least 1 marked point.
Theorem (in Progress)
Y_n is the moduli space of flower curves with universal curve $Y_{n+1} \to Y_n$
$\mathbb{C}^n/\mathbb{C}$

$n=3 \quad Y_3 = \overline{\{a+b=c\}}$



P_n = permutahedron

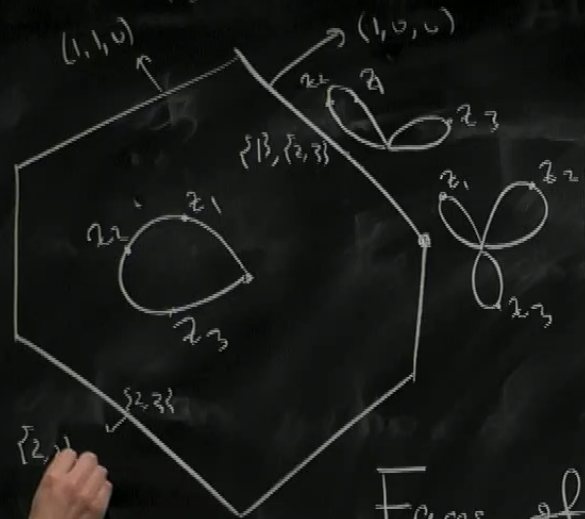
$= \mathrm{Conv}(S_n \rho) \subset \{(a_1, \dots, a_n) : a_1 + \dots + a_n = 0\}$ $\quad \rho = (n-1, n-3, \dots, -n+1)$

Has facets labelled by subsets $S \subset \{1, \dots, n\}$

$S \longrightarrow \delta_S = (0 \cdots 1 \cdots 1 \; 0)$

Faces of permutahedron $\cong$ ordered set decompositions $\{1, \dots, n\} = S_1 \cup \dots \cup S_m$

$n=3 \quad Y_3 = \overline{\{a+b=c\}}$



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Faces of permutahedron $\cong$ ordered set decompositions $\{1, \dots, n\} = S_1 \cup \dots \cup S_m$

Two faces are parallel if

$\{S_1, \dots, S_m\} = \{S_1', \dots, S_m'\}$

$P_n/\sim$ = identify parallel faces of all dimensions

stratification of $P_n/\sim$ by unordered set decompositions of $\{1,\ldots$

[Bartholdi - Enriques - Etingof - Rains]

<u>Theorem</u>

$$Y_n(\mathbb{R}) \cong P_n/\sim$$

The <u>virtual symmetric group</u> is the quotient of $S_n * S_n$

$w \quad s_i$

modulo $\quad w s_i w^{-1} = s_{w(i)}$

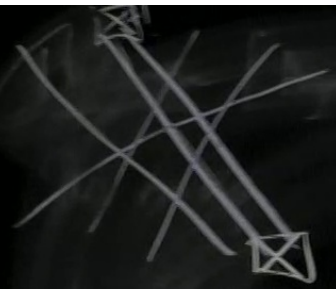
whenever $w(i+1) = w(i)+1$

point $\infty \in C$.

ts don't meet.

∞C_i

kolpoint.



$$\vee S_n \longrightarrow S_n$$

$$w \longmapsto w$$

$$s_i \longmapsto s_i$$

$$P_\vee S_n = \ker(\vee S_n \to S_n)$$

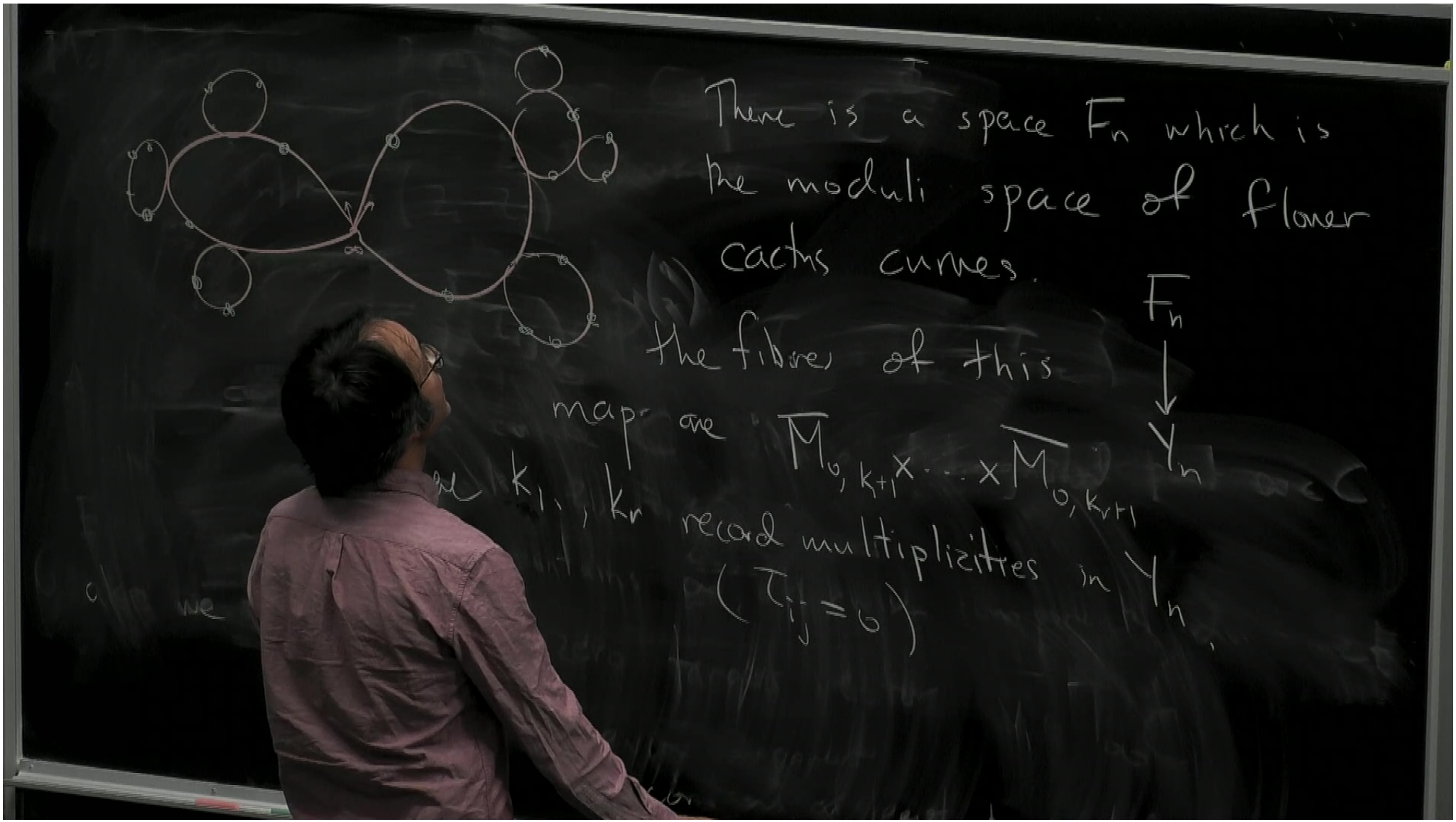
<u>Theorem</u> [BEER]

$P_\vee S_n = \pi_1(Y_n(\mathbb{R}))$ + all higher homotopy groups

$\vee S_n = \pi_1^{S_n}(Y_n(\mathbb{R}))$

A <u>flower cactus curve</u> is a cactus curve as

each component is a nodal curve with distinct


There is a space F_n which is
the moduli space of flower
cactus curves.
F_n
$\downarrow$
Y_n
the fibres of this
map are $\overline{M}_{0,k_1+1} \times \cdots \times \overline{M}_{0,k_r+1}$
$k_1, \ldots, k_r$ record multiplicities in Y_n.
$(\tau_{ij} = 0)$

There is another space "Mau-Woodward space" Q_n

which bigger $Q_n \longrightarrow F_n \longrightarrow Y_n$.

$\overline{M}_{0,n+1} :=$ closure of $\mathbb{C}^n \setminus \Delta$ in $\prod_V \mathbb{P}(\mathbb{C}^n/V)$

V all irreducible subspaces spanned by roots

de Concini-Procesi construction

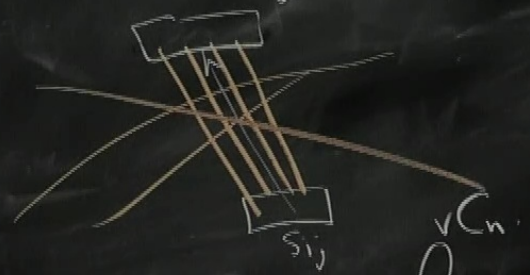
$Q_n =$ closure of $\mathbb{C}^n \setminus \Delta$ in $\prod \mathbb{P}(\mathbb{C}^n/V) \times (\mathbb{P}^1)^{\Phi_+}$

$\rotatebox{90}{$\leftarrow$}$

F_n

The virtual cactus group is the quotient of $S_n * C_n$ by

$$w s_{ij} = s_{w(i)\,w(j)}\, w \quad \text{if} \quad w(i+k) = w(i) + k \quad \text{for } k = 1, \dots, j-i$$



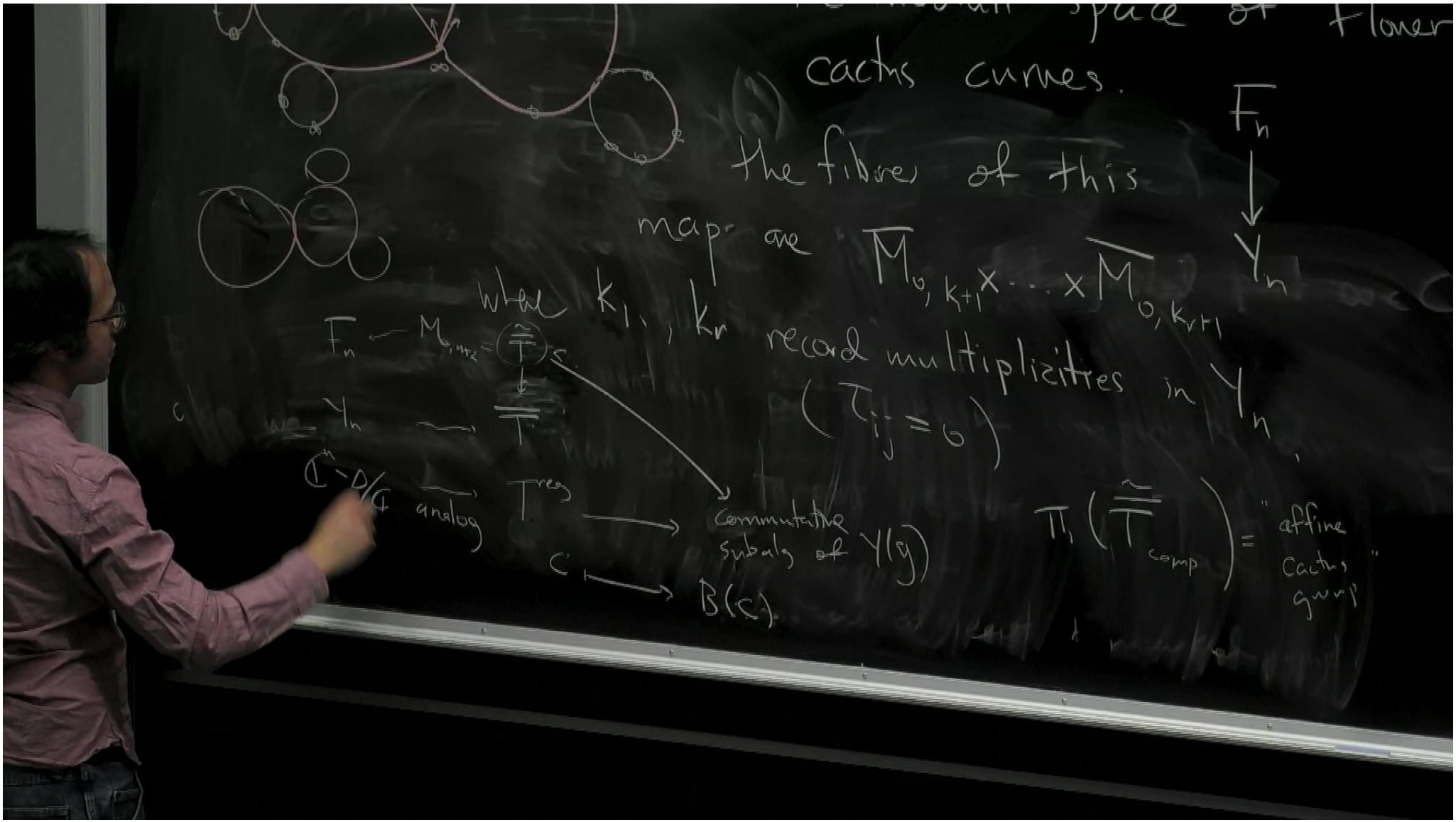
Theorem

$$\pi_1^{S_n}(F_n(\mathbb{R})) \cong vC_n$$

$$C_n \circlearrowright B_1 \otimes \cdots \otimes B_n \circlearrowleft S_n$$

vC_n $\circlearrowright$ stupid

$$V_1 \otimes \cdots \otimes V_n$$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where $k_1, \ldots, k_r$ record multiplicities in Y_n
$(\tau_{ij} = 0)$
F_n
Y_n
$F_n \leftarrow M_{0,m_2} = \hat{\overline{T}}$
$Y_n \longrightarrow \overline{T}$
$\mathbb{C}^n/G$ analog
T^{reg}
$\longrightarrow$ Commutative subalg of $Y(\mathfrak{g})$
$c \longmapsto B(c)$
$\pi_1(\hat{\overline{T}}_{comp}) =$ "affine cactus group"