Title: Relativity - Lecture 221108

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Collection: Relativity (2022/2023)

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KERR SOLUTION

$$ds^2 = -dt^2 + \frac{2Mr}{\rho^2}(dt - a\sin^2\theta\, d\phi)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + (r^2+a^2)\sin^2\theta\, d\phi^2$$

where: $\begin{cases} \rho^2(r,\theta) := r^2 + a^2\cos^2\theta \\ \Delta(r) := r^2 + a^2 - 2Mr \end{cases}$

- $ds^2 \underset{r\to\infty}{\sim} ds^2_{\text{Mink}}$ + corrections
- $\begin{cases} M = \text{mass as seen from } \infty \\ J \equiv Ma = \text{ang. mom. — " —} \end{cases}$

If $a \to 0$ we get back Schwarzschild

If $M \to 0$ & a = fixed we get Mink. in ellipsoidal coords where $(r,t) = 0$ is a disk, not a pt. with ring-like boundary at $(r,t,\theta) = (0,0,\frac{\pi}{2})$

Physical sing at $\rho^2 = 0 \Rightarrow r = 0, \theta = \pi/2$

↳ ring-like

Horizon(s) at $\Delta(r) = 0$

- $a^2 > M^2$, no horizon → naked sing.
  Nonphysical
- $a^2 = M^2$ extremal case
- $a^2 < M^2$ physical → $r_\pm = M \pm \sqrt{M^2 - a^2}$

focus on exterior horizon r_+

$J \equiv Ma$ = ang. mom. — u —

2 killing v.f.

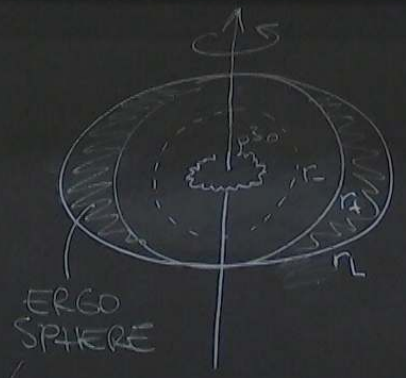
- $\xi^a = \partial_t$ stationarity
- $\psi^a = \partial_\phi$ axisymmetry

Where does ξ^a become spacelike?

The surface at which this happens is called STANIONARITY LIMIT SPHERE

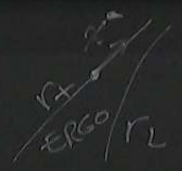
$\xi^\mu \xi_\mu = 0$ @ $r_L = M + \sqrt{M^2 - a^2 \cos^2\theta} \geq r_+$

SLS is Timelike everywhere (except @ poles)



↳ is where we cannot stay stationary b.c. $|\dot{\phi}| > 0$ and of same sign of a

- SLS can be crossed two ways



Back to geometry

- Area of outer horizon

$$A = \int_{r=r_+} \sqrt{g_{\theta\theta} g_{\phi\phi}}\, d\theta d\phi$$

$$= 4\pi(r_+^2 + a^2)$$

$$= 8\pi\left(M^2 + \sqrt{M^4 - J^2}\right)$$

- $\chi^a = \xi^a + \Omega_H \psi^a$

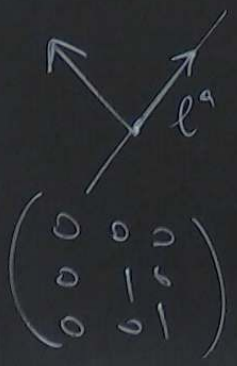
$\Omega_H = \dfrac{a}{r_+^2 + a^2}$ const $\Rightarrow \chi^a$ is also a killing

ANGULAR VELOCITY

This is special b.c.

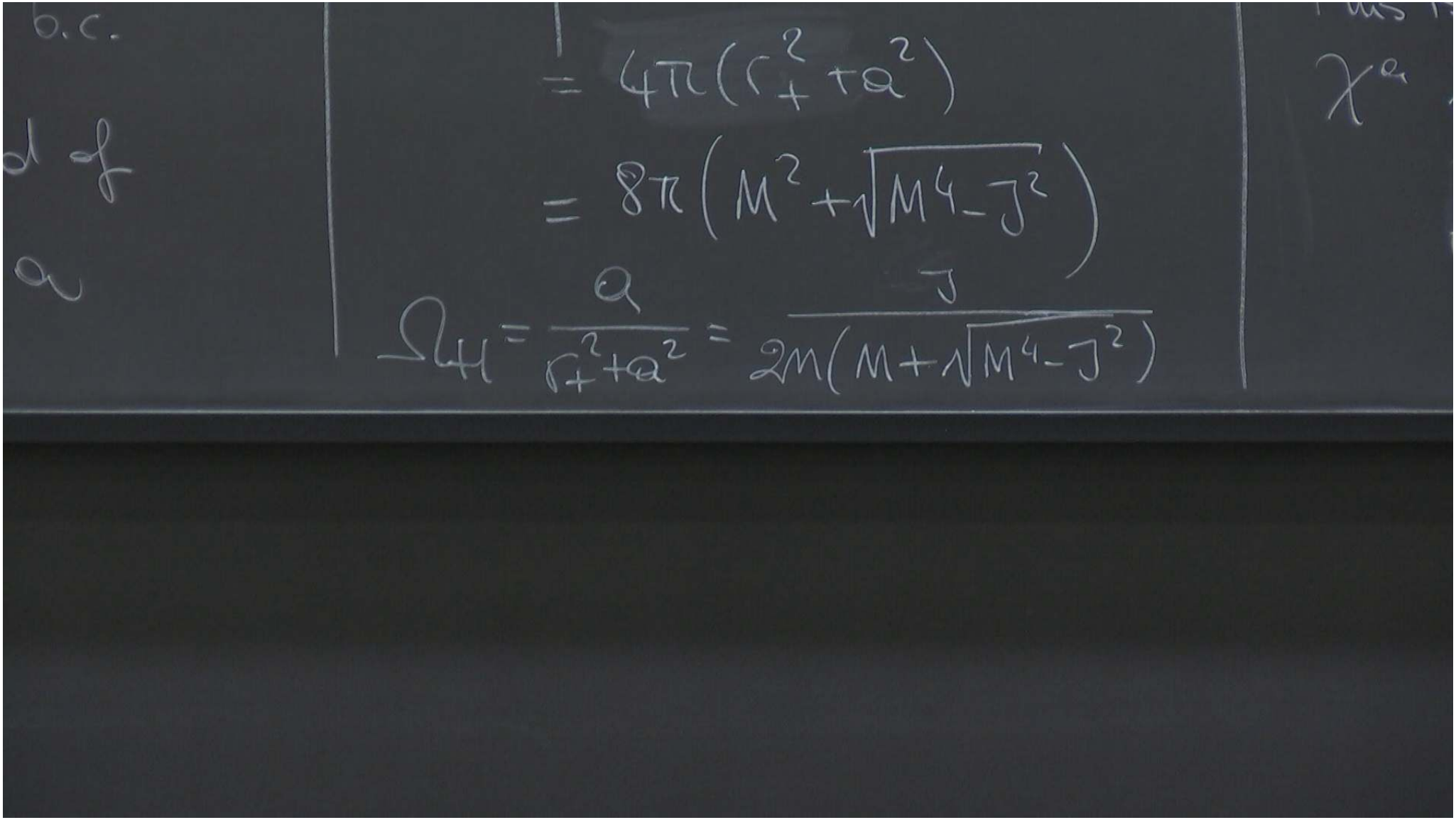
χ^a is null & future directed & tangent (normal) to the outer horizon $r = r_+$

$$\chi^a \chi_a \Big|_{r_+} = 0$$



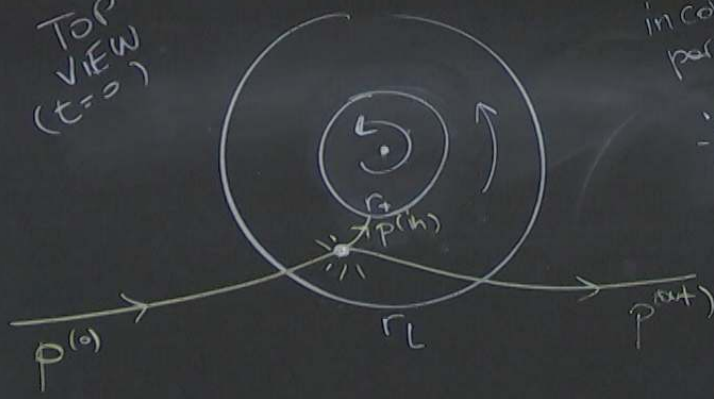
$\ell^a \propto \ell^a$ parallel to itself

$\ell^a \ell_a = 0$ normal to itself


b.c.
$= 4\pi(r_+^2 + a^2)$
$= 8\pi\left(M^2 + \sqrt{M^4 - J^2}\right)$
$\Omega_H = \frac{a}{r_+^2 + a^2} = \frac{J}{2M(M + \sqrt{M^4 - J^2})}$

SLS is Timelike everywhere (except @ poles)

Penrose Process



incoming particle: $E^{(0)} = -p_\mu^{(0)} \xi^\mu > 0$ (conserved)

$p_\mu^{(0)} = p_\mu^{(in)} + p_\mu^{(out)}$

$\hookrightarrow \xi^\mu p_\mu^{(0)} = \xi^\mu p_\mu^{(in)} + \xi^\mu p_\mu^{(out)}$

$E^{(0)} = E^{(in)} + E^{(out)}$

$E^{(in)} < 0$

only possible because $\xi = \partial_t$ is SPACELIKE in ergosph (while p^μ is always timelike)

$> E^{(o)}$

$= E^{(ih)} < 0$

process extracts
a rotating BH

Let's look at χ^a and
the associated conserved
quantity:

$$p^{(o)}_\mu \chi^\mu = -E^{(o)} + \Omega_H L^{(o)}$$

↑ angular momentum

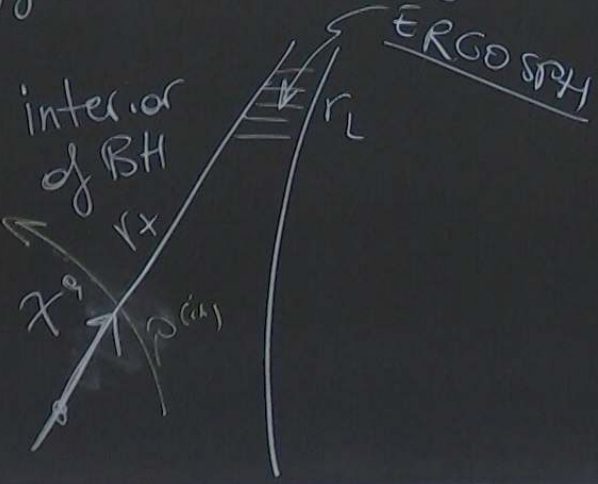
ved)

$\Rightarrow E^{(out)} > E^{(0)}$

tot energy conserv

$$\boxed{\delta M = E^{(in)} < 0}$$

the Penrose process extracts energy from a rotating BH



Let's look at χ^a and the associated conserved quantity:

$$p^{(o)}_\mu \chi^\mu = -E^{(o)} + \Omega_H L^{(o)}$$

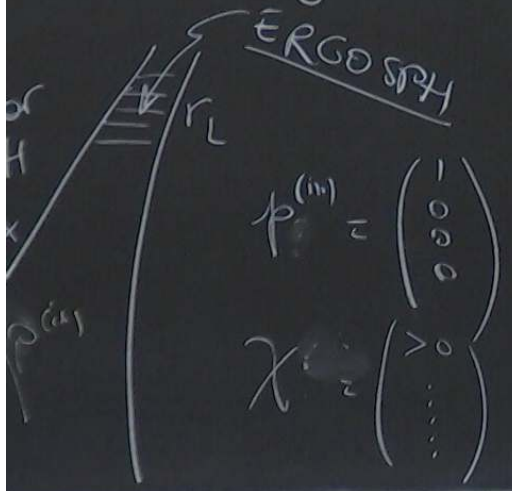
$L^{(o)}$: angular momentum

$$p^{(in)}_\mu \chi^\mu < 0$$

$> E^{(o)}$

$= E^{(in)} < 0$

e process extracts
a rotating BH



Let's look at χ^a and the associated conserved quantity:

$$p_\mu^{(o)} \chi^\mu = -E^{(o)} + \Omega_H L^{(o)}$$

$L^{(o)}$ ↕ angular momentum

$p_\mu^{(o)}$, χ^μ both future directed

$$p_\mu^{(in)} \chi^\mu < 0$$

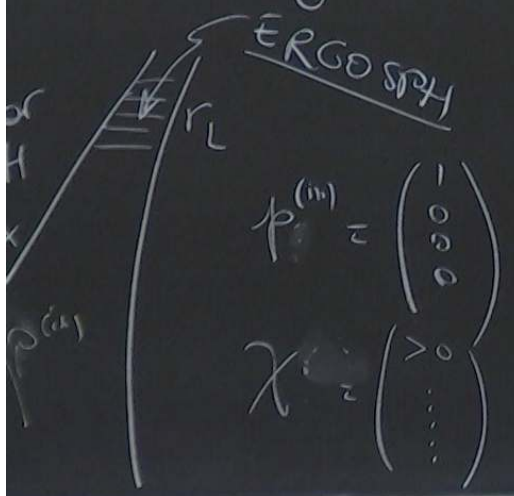
$$-E^{(in)} + \Omega_H L^{(in)} \Rightarrow \Omega_H L^{(in)} < E^{(in)} < 0$$

penrose

$> E^{(o)}$

$= E^{(in)} < 0$

e process extracts
a rotating BH



Let's look at χ^a and the associated conserved quantity:

$$p^{(o)}_\mu \chi^\mu = -E^{(o)} + \Omega_H L^{(o)}$$

$L^{(o)}$ ↕ angular momentum

both future directed

$$p^{(in)}_\mu \chi^\mu < 0$$

$$\| $$

$$-E^{(in)} + \Omega_H L^{(in)} \Rightarrow \Omega_H L^{(in)} < E^{(in)} < 0$$

penrose

$$\boxed{\Omega_H \delta J = \Omega_H \delta L < E^{(in)} = \delta M < 0}$$

$\boxed{\Omega_H \delta J < \delta M}$ iff $\boxed{\delta A > 0}$

$A = 16\pi M_{irr}^2$

$$\frac{\delta A}{8\pi} = 2M\delta M + \frac{2M^3\delta M - J\delta J}{\sqrt{M^4 - J^2}}$$

$$= \underbrace{\frac{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}{\sqrt{M^4 - J^2}}}_{= 1/\kappa} \left(\delta M - \underbrace{\frac{J}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}}_{\Omega_H} \delta J \right)$$

SURFACE GRAVITY $\kappa = \sqrt{\nabla^\mu V \nabla_\mu V}\Big|_{\mathcal{H}}$, $V = \sqrt{-\chi^\mu \chi_\mu}$

δM iff $\boxed{\delta A > 0}$

$$A = 16\pi M_{irr}^2$$

$$\boxed{\delta M = \frac{\kappa \delta A}{8\pi} + \Omega_H \delta J}$$

$$1 + \frac{2M^3 \delta M - J \delta J}{\sqrt{M^4 - J^2}}$$

$$\left(M^2 + \sqrt{M^4 - J^2}\right)\left(\delta M - \underbrace{\frac{J}{2M\left(M^2 + \sqrt{M^4 - J^2}\right)}}_{\Omega_H} \delta J\right)$$

$$\sqrt{M^4 - J^2}$$

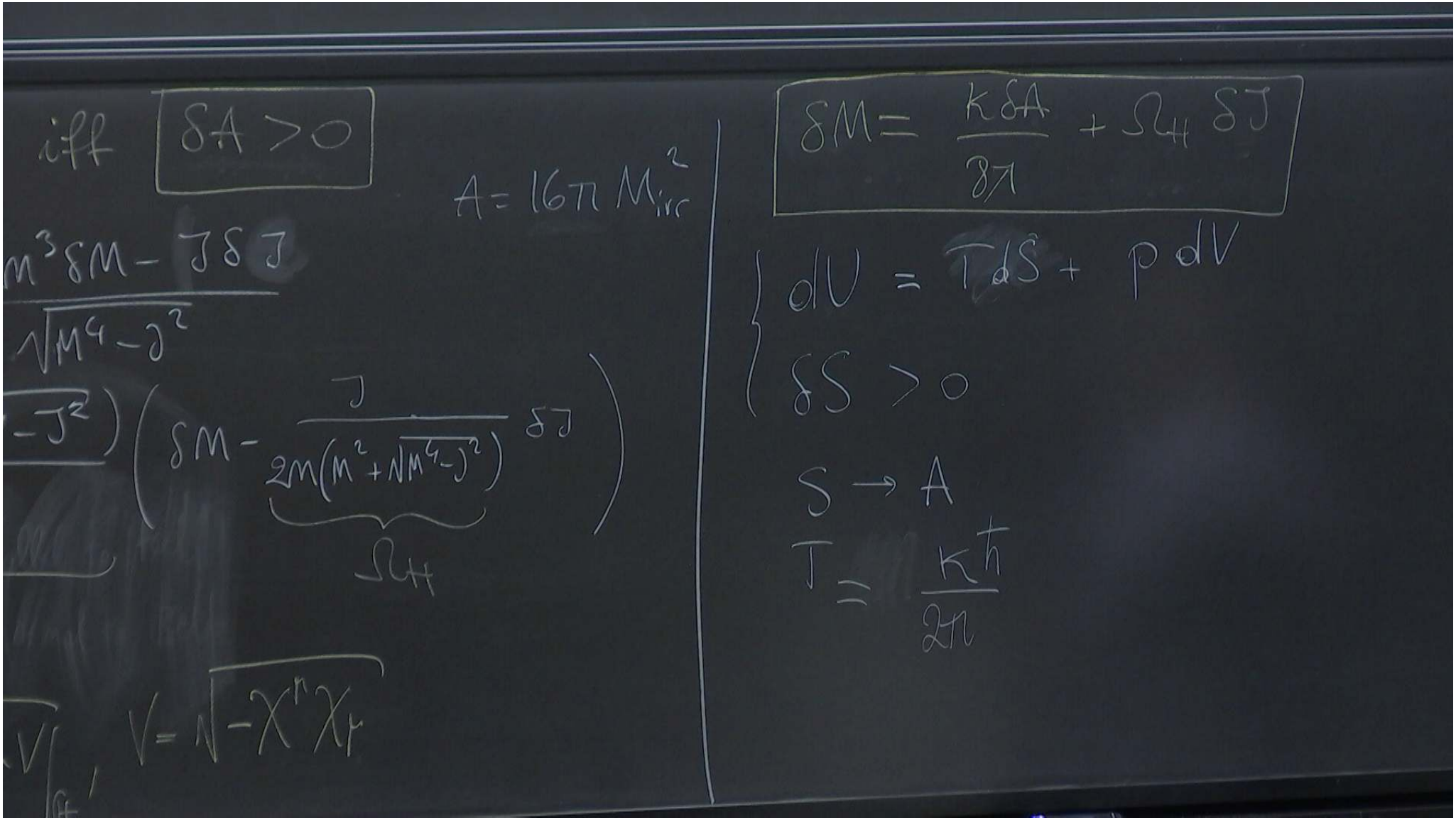
$$= \frac{1}{\kappa}$$

$$\left.\chi^\mu \nabla_\mu V\right|_{\mathcal{H}}, \quad V = \sqrt{-\chi^\mu \chi_\mu}$$

$$\begin{cases} dU = T dS + p\, dV \\ \delta S > 0 \end{cases}$$

$S \to A$

$T \to \kappa$


iff $\delta A > 0$
$A = 16\pi M_{irr}^2$
$\delta M = \frac{\kappa \delta A}{8\pi} + \Omega_H \delta J$
$M^3 \delta M - J \delta J$
$\sqrt{M^4 - J^2}$
$\left(\delta M - \frac{J}{2M(M^2 + \sqrt{M^4 - J^2})} \delta J\right)$
Ω_H
$\begin{cases} dU = T dS + p dV \\ \delta S > 0 \end{cases}$
$S \to A$
$T = \frac{\kappa \hbar}{2\pi}$
$V = \sqrt{-\chi^\mu \chi_\mu}$