

Title: Statistical Physics - Lecture 221130

Speakers:

Collection: Statistical Physics (2022/2023)

Date: November 30, 2022 - 9:00 AM

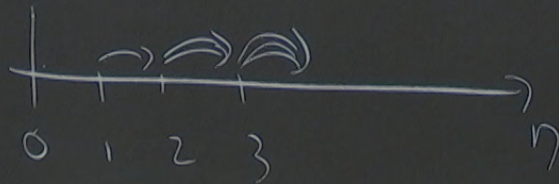
URL: <https://pirsa.org/22110019>

Birth Death Processes

$$\dot{P}_n = (n-1)P_{n-1} - 2nP_n + (n+1)P_{n+1}$$

$$\rightarrow \text{Pure Birth} : \dot{P}_n = (n-1)P_{n-1} - nP_n$$

$$P_n(t) = e^{-t}(1-e^{-t})^{n-1}$$



$$P_n(t) = e^{-t} \overbrace{(1-e^{-t})}^x^{n-1}$$

0. $\langle n \rangle = et$

1. ultrawide distribution

$$n \rightarrow \infty \quad P_n(t) \approx e^{-t} e^{-ne^{-t}}$$

$$\approx e^{-t} e^{-n/n^*}$$

$$n^* = e^t$$

2. Moments $M_k = \langle n^k \rangle$

$$\langle n \rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n e^{-t} x^{n-1}$$

$$= e^{-t} \frac{d}{dx} \sum_{n=1}^{\infty} x^n$$

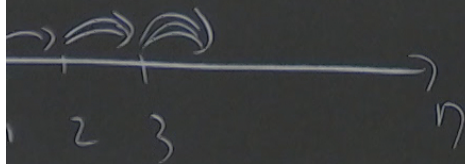
$$= e^{-t} \frac{d}{dx} \frac{x}{1-x}$$

$$= e^{-t} \left(\frac{1}{1-x} + \frac{x}{(1-x)^2} \right) = e^{-t}$$

Death Processes

$$P_{n-1} - 2nP_n + (n+1)P_{n+1}$$

$$\dot{P}_n = (n-1)P_{n-1} - nP_n$$



$$P_n(t) = e^{-t} \overbrace{(1-e^{-t})}^x{}^{n-1} = e^{-t} x^{n-1}$$

0. $\langle n \rangle = e^t$

1. ultrawide distribution

$$n \rightarrow \infty \quad P_n(t) \approx e^{-t} e^{-ne^{-t}}$$

$$\approx e^{-t} e^{-n/n^*}$$

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2. Moments

$$\langle n \rangle =$$

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$$= e^{-t} \frac{d}{dx} \sum_{n=1}^{\infty} x^n$$

$$= e^{-t} \frac{d}{dx} \frac{x}{1-x}$$

$$= e^{-t} \left(\frac{1}{1-x} + \frac{x}{(1-x)^2} \right) = e^{-t}$$

$$\langle n^2 \rangle = \sum n^2 P_n$$

$$= \sum n n P_n$$

$$= \sum n n e^{-t} x^{n-1}$$

$$= e^{-t} \frac{d}{dx} x \frac{d}{dx} \sum x^n$$

$$= 2e^{2t} - e^t$$

Standard
deviation

$$\sqrt{\langle n^2 \rangle - \langle n \rangle^2}$$

$$= \sqrt{e^{2t} - e^t}$$

$$\approx e^t$$

ps

$P_{n+1}P_{n+1}$

$P_{n-1} - nP_n$

P_n

$$P_n(t) = e^{-t} \underbrace{\left(\underbrace{1 - e^{-t}}_{(1-\varepsilon) = e^{-\varepsilon}} \right)^{n-1}}_X = e^{-t} X^{n-1}$$

0. $\langle n \rangle = et$
 1. ultrawide distribution

2. Moments

$\langle n \rangle =$

$$n \rightarrow \infty \quad P_n(t) \approx e^{-t} e^{-ne^{-t}}$$

$$\approx e^{-t} e^{-n/n^*}$$

$$n^* = e^t$$

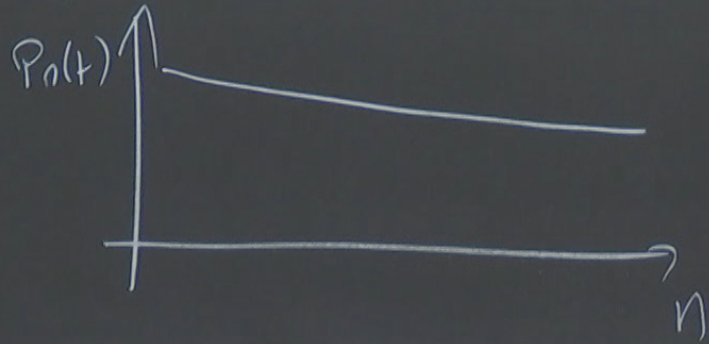
$(1-\varepsilon)^n = e^{-\varepsilon n}$
 $(1-\varepsilon)^{n-1} = e^{-\varepsilon(n-1)} = e^{-\varepsilon n + \varepsilon} = e^{-\varepsilon n} e^{\varepsilon}$

$0. \langle n \rangle = \varepsilon t$
 1. ultrawide distribution

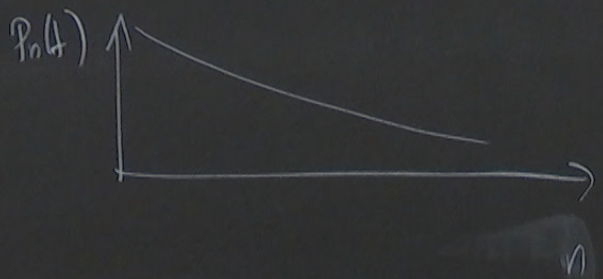
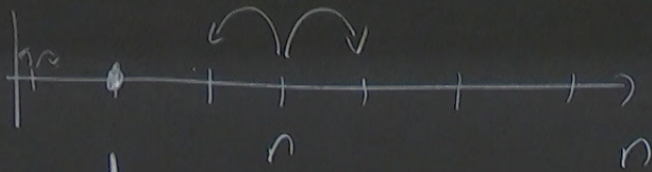
$\langle n \rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n$

$n \rightarrow \infty \quad P_n(t) \approx e^{-t} e^{-n e^{-t}}$
 $\approx e^{-t} e^{-n/n^*}$

$n^* = e^t$



$= e^{-t} \frac{d}{dx} \sum_{n=1}^{\infty} x^n$
 $= e^{-t} \frac{d}{dx} \frac{x}{1-x}$
 $= e^{-t} \left(\frac{1}{1-x} + \frac{x}{(1-x)^2} \right)$



≥ 1

$$P_n(t) = \frac{1}{(1+t)^2} y^n$$

$$y^n = \left(\frac{t}{1+t}\right)^n = \left(\frac{1}{1+t}\right)^n \approx 1 - n/t$$

$$e^{-n/t} \approx e^{-n/n^*} \quad n^* = t$$

$$\begin{aligned} 0. \quad \langle n \rangle &= \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n \frac{1}{(1+t)^2} y^n \\ &= 1 \\ &= \frac{1}{(1+t)^2} \frac{d}{dy} \sum_{n=1}^{\infty} y^n \\ &= \frac{1}{(1+t)^2} \frac{d}{dy} \frac{y}{1-y} \\ &= \frac{1}{(1+t)^2} \frac{1}{(1-y)^2} \\ &= 1 \end{aligned}$$

$$\begin{aligned}
 0. \quad \langle n \rangle &= \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n \frac{1}{(1+t)^2} y^{n-1} \\
 &= 1 \\
 &= \frac{1}{(1+t)^2} \frac{d}{dy} \sum_{n=1}^{\infty} y^n \\
 &= \frac{1}{(1+t)^2} \frac{d}{dy} \frac{y}{1-y} \\
 &= \frac{1}{(1+t)^2} \frac{1}{(1-y)^2} \\
 &= 1
 \end{aligned}$$

$$t \equiv e^{-n/n^*} \quad n^* = t$$

$$\begin{aligned}
 2. \quad \langle n^2 \rangle &= 1 + 2t \\
 \sqrt{\langle n^2 \rangle - \langle n \rangle^2} &\propto \sqrt{t}
 \end{aligned}$$

3. Extinction for Sure

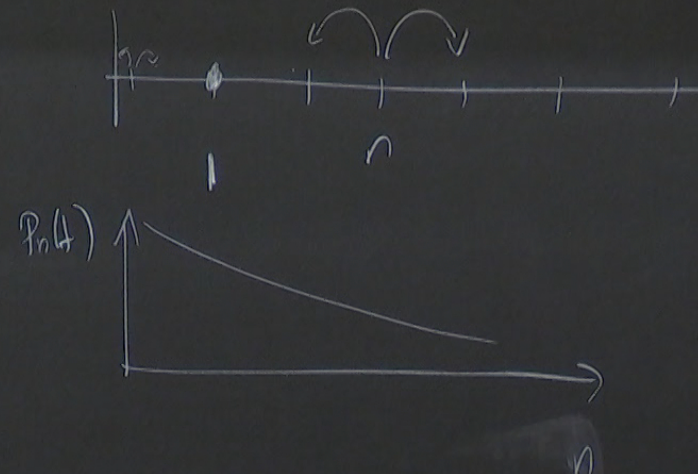
$$\begin{aligned}
 S(t) &\equiv \text{Survival prob.} \\
 &= 1 - P_0(t) \\
 &= \frac{1}{1+t}
 \end{aligned}$$

Birth Death Process

\Rightarrow generating fn $\frac{\partial P}{\partial t} = (z-1)^2 \frac{\partial P}{\partial z}$

for $P_n(t) = \delta_{n,1}$

$$\begin{cases} P_n(t) = \left(\frac{t}{1+t}\right)^{n-1} \frac{1}{(1+t)^2} & n \geq 1 \\ P_0(t) = \frac{t}{1+t} \end{cases}$$



$$P_n(t) = \frac{1}{(1+t)^2} y^n \quad y^n = \left(\frac{t}{1+t}\right)^n$$

$$0. \langle n \rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n \frac{1}{(1+t)^2} y^{n-1}$$

$$= 1$$

$$= \frac{1}{(1+t)^2} \frac{d}{dy} \sum_{n=1}^{\infty} y^n$$

$$= \frac{1}{(1+t)^2} \frac{d}{dy} \frac{y}{1-y}$$

$$= \frac{1}{(1+t)^2} \frac{1}{(1-y)^2}$$

$$= 1$$

$$\approx e^{-n/n^*}$$

$$n^* = t$$

$$2. \langle n^2 \rangle = 1 + 2t$$

$$\sqrt{\langle n^2 \rangle - \langle n \rangle^2} \propto \sqrt{t}$$

3. Extinction for Sure

$S(t) \equiv$ survival prob.

$$= 1 - P_0(t)$$

$$= \frac{1}{1+t}$$

Population Dynamics

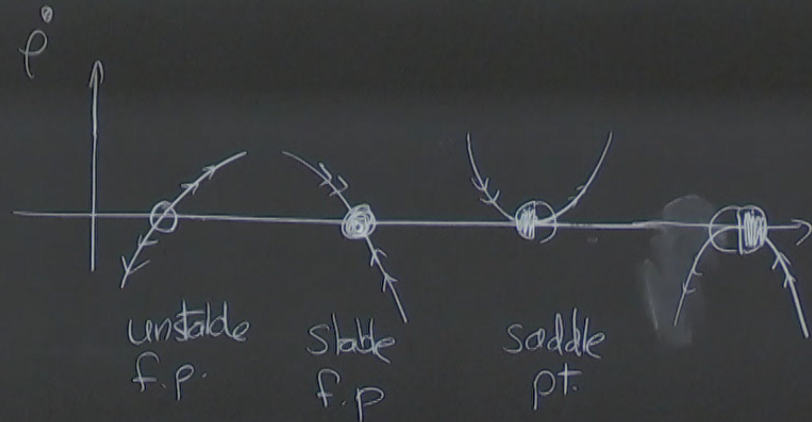
$$\dot{p} = f(p)$$

one species

$$\left. \begin{aligned} \dot{p}_1 &= f(p_1, p_2) \\ \dot{p}_2 &= g(p_1, p_2) \end{aligned} \right\}$$

two species

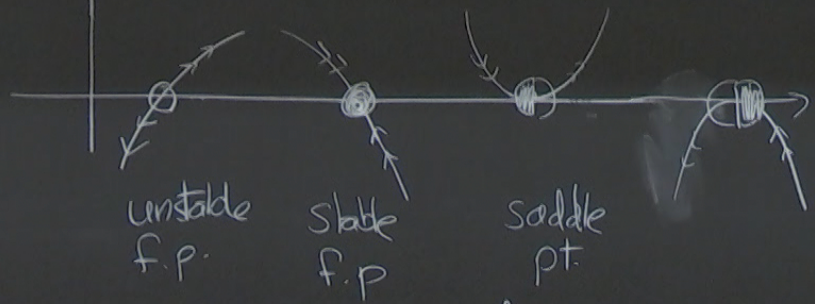
n species



$$N e^{-\lambda t} \approx e^{-\lambda t} \quad \lambda > 0$$

1d \leftrightarrow one species

2d \leftrightarrow two species

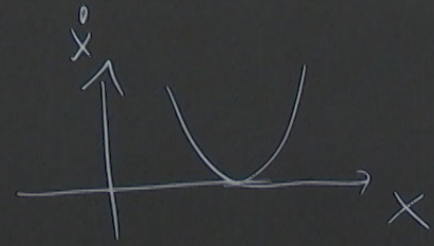


unstable f.p.

stable f.p.

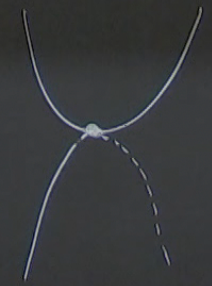
saddle pt.

$$\dot{X} = (X-1)^2$$



- (un)stable f.p

- saddle pts



- (un)stable spiral points



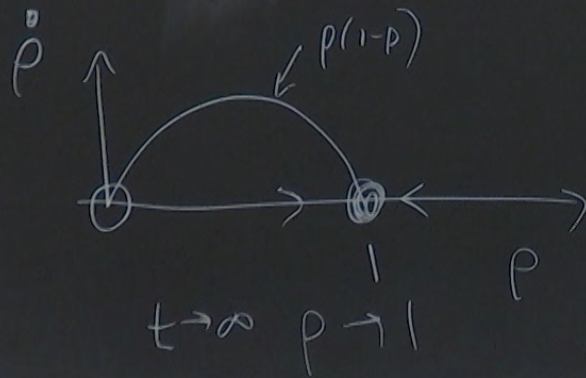
- Centers



Logistic Growth

$$\dot{p} = p - p^2$$

$$\dot{p} = p(1-p)$$



$$\frac{dp}{p(1-p)} = dt$$

$$dp \left(\frac{1}{p} + \frac{1}{1-p} \right) = dt$$

$$\ln p/p_0 + \ln \frac{1-p}{1-p_0} = t$$

$$\ln \frac{p(1-p)}{p_0(1-p_0)} = t$$

$$\rightarrow p(t)$$

Two Species Competition

$$\dot{A} = A(1 - A - \epsilon B)$$

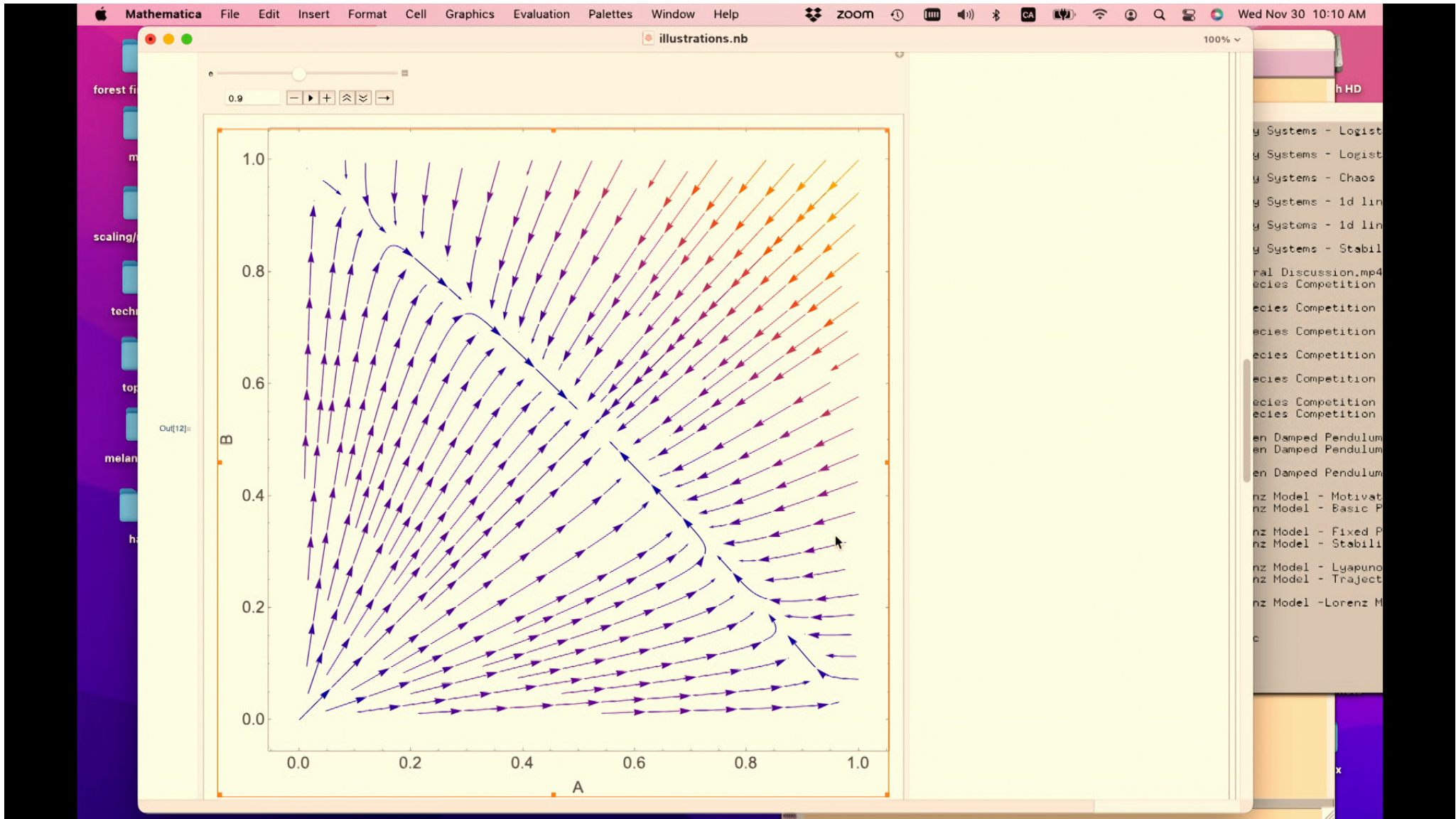
$$\dot{B} = B(1 - B - \epsilon A)$$

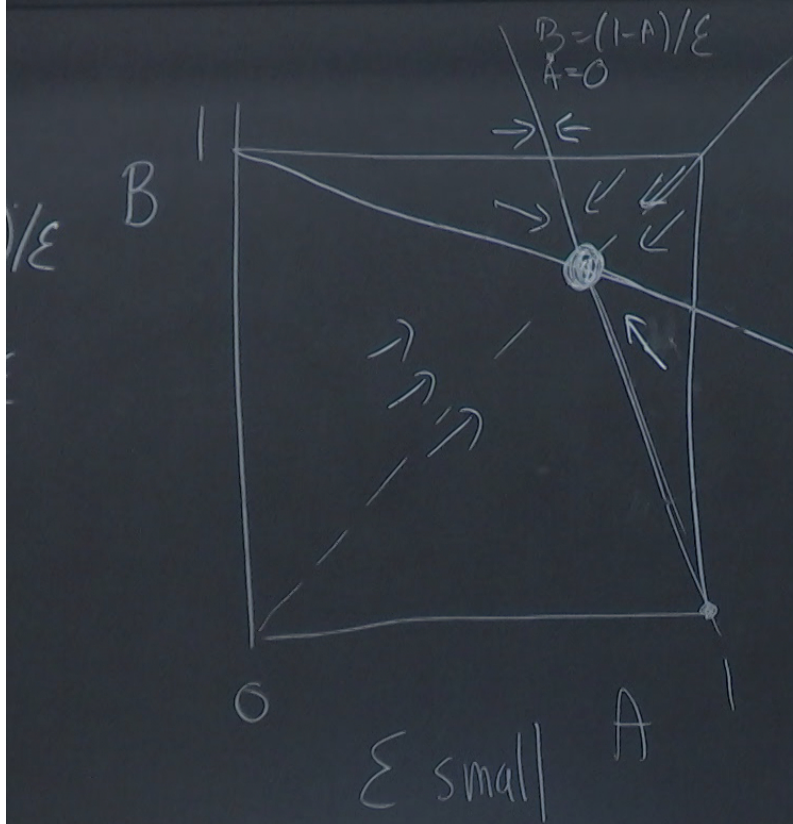
1. find nullclines
2. find fixed points
3. infer global behavior

nullclines $\dot{A} = 0, \dot{B} = 0$

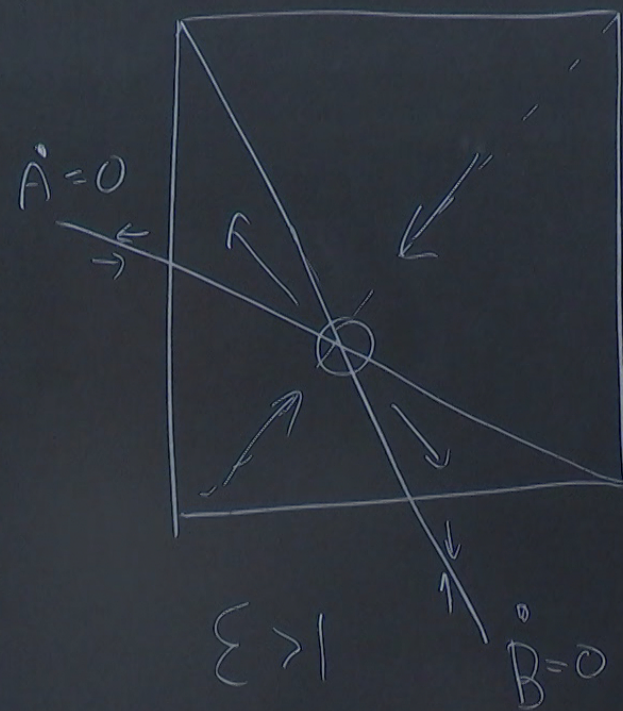
$$\dot{A} = 0 \Rightarrow A = 0 ; B = (1 - A) / \epsilon \quad B$$

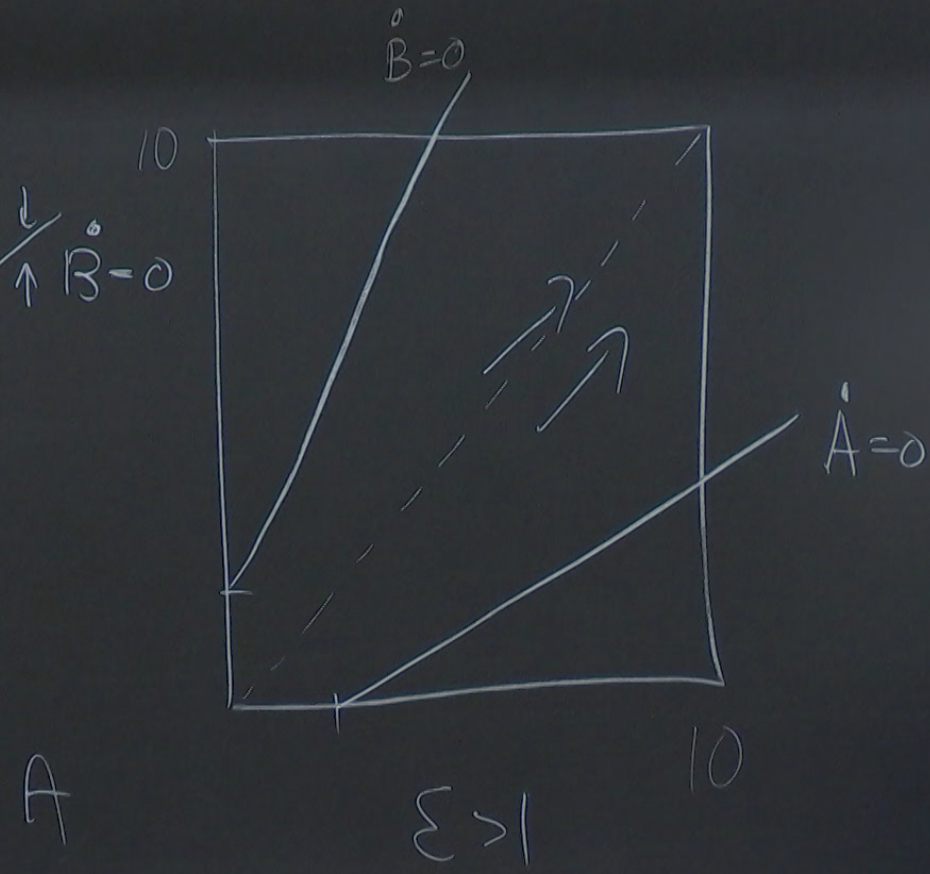
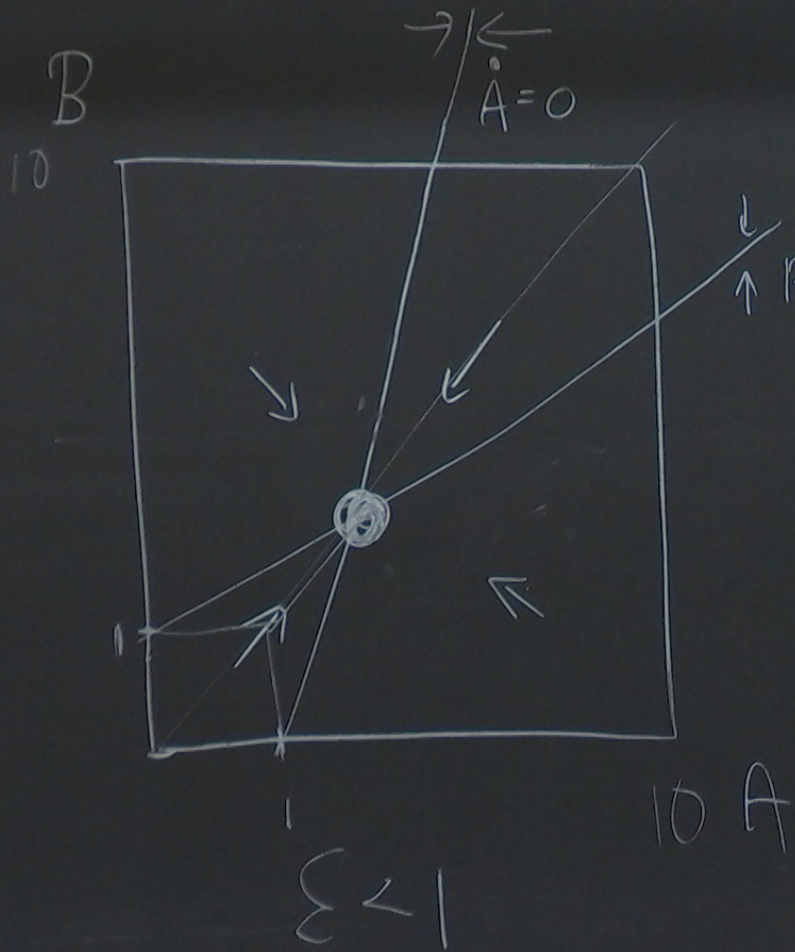
$$\dot{B} = 0 \Rightarrow B = 0 ; A = (1 - B) / \epsilon$$





$\dot{B} = 0$





Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help zoom 100% Wed Nov 30 10:23 AM

illustrations.nb

0.1

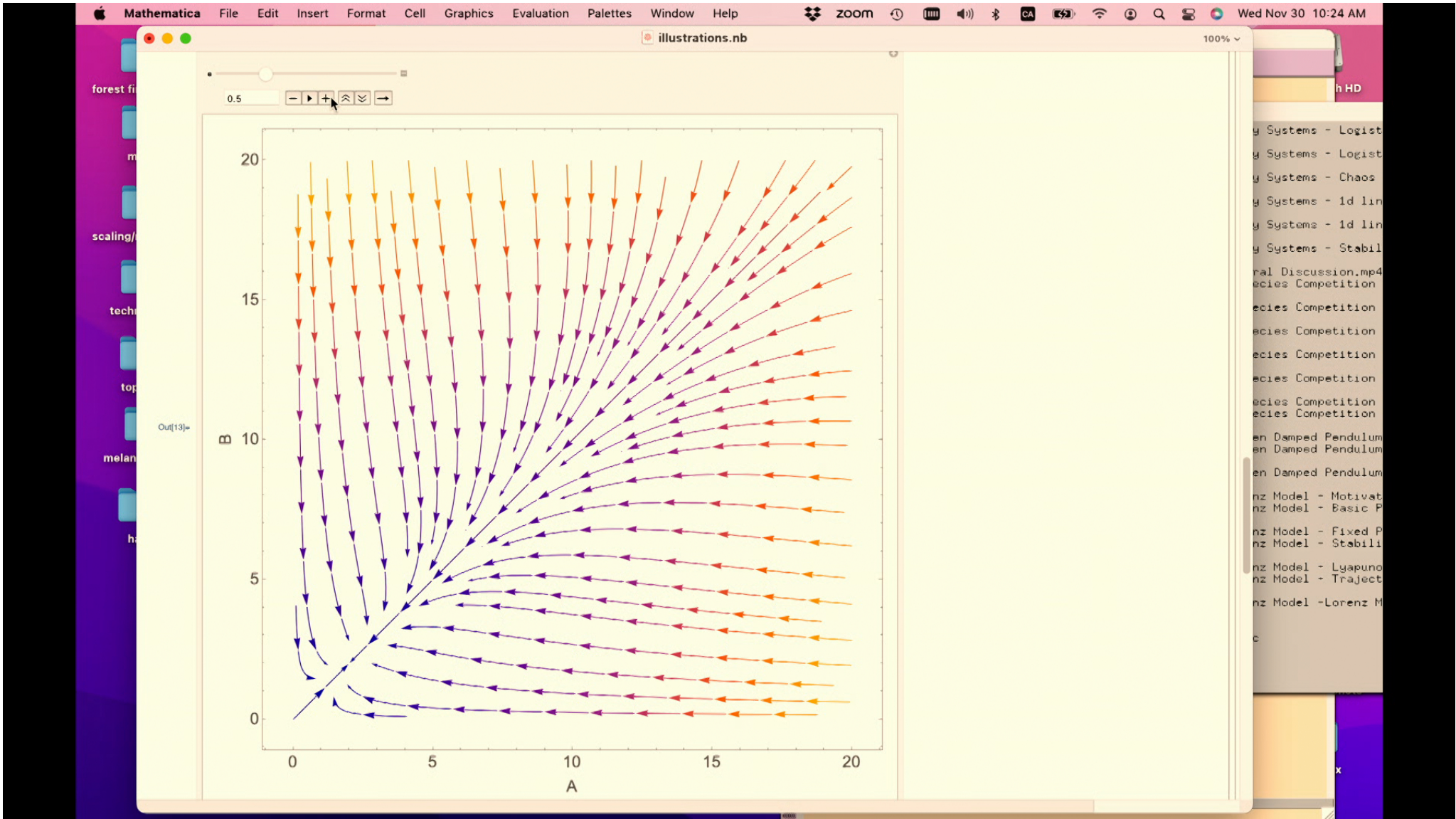
Out[13]=

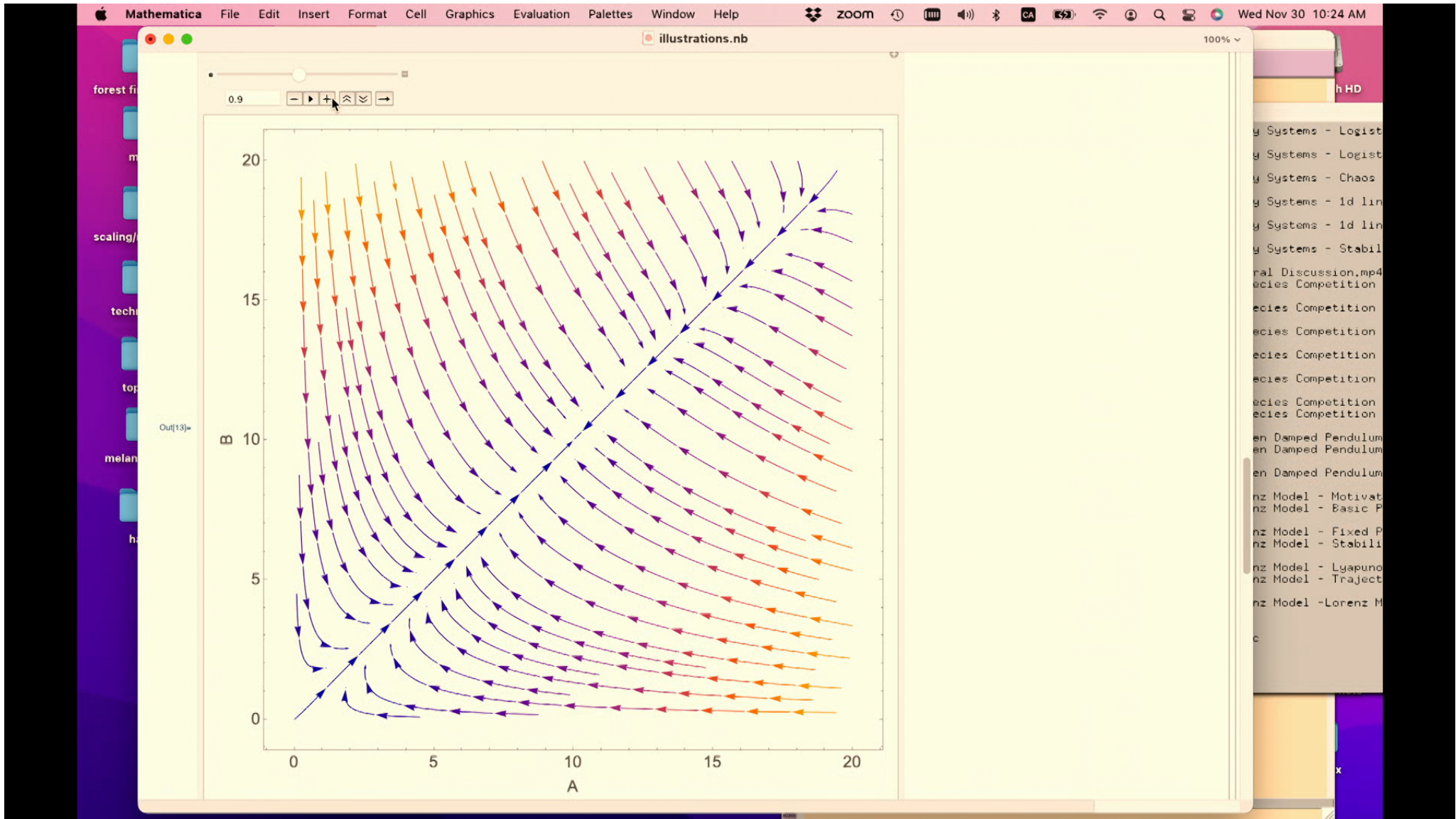
A

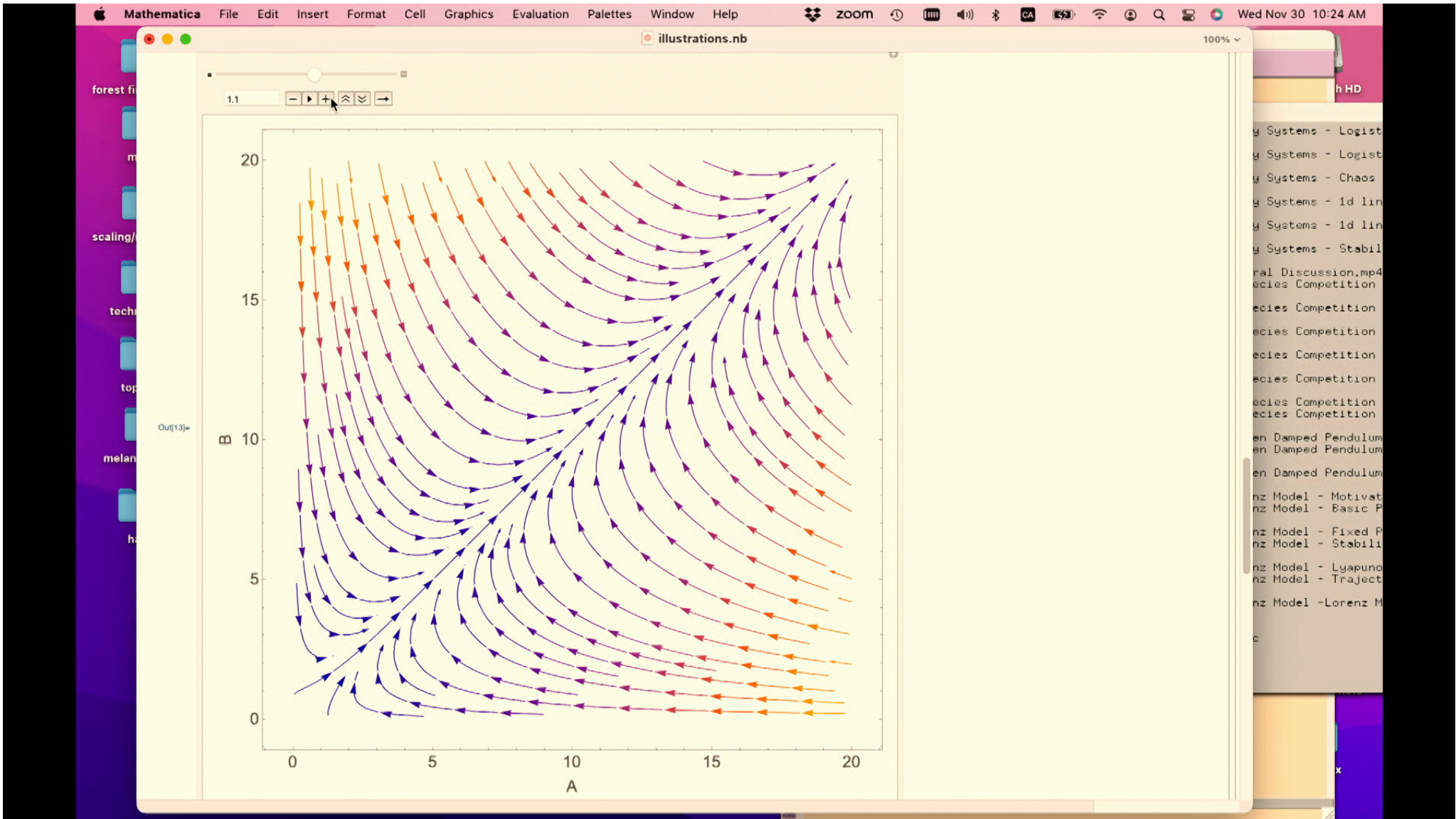
0 5 10 15 20

0 5 10 15 20

The image shows a Mathematica notebook window titled 'illustrations.nb' displaying a phase plane plot. The plot has a horizontal axis labeled 'A' and a vertical axis labeled 'm', both ranging from 0 to 20. A vector field is shown with arrows indicating the direction of flow. Several trajectories are plotted, starting from various points and moving towards a stable point near the origin. The trajectories are colored in a gradient from blue to orange. The plot is displayed in a window with a zoom level of 100%. The Mathematica interface includes a menu bar (File, Edit, Insert, Format, Cell, Graphics, Evaluation, Palettes, Window, Help) and a toolbar with various icons. The notebook's sidebar on the left shows a list of files, and the right side shows a list of topics related to dynamical systems, such as 'Logistic Systems', 'Chaos', '1d linear', 'Stability', 'Damped Pendulum', 'Lorenz Model', and 'Trajectory'.







Prey Predator Dynamics

$$\dot{A} = A - AB$$

$$\dot{B} = -B + AB$$

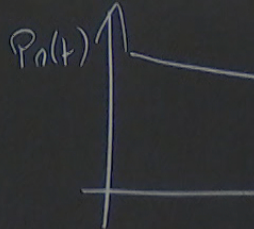
Lotka-Volterra model

$$P_n(t) = e^{-t} \overbrace{(1 - e^{-t})}^x{}^{n-1}$$

$\rho. \langle n \rangle = e^t$
1. ultrawide distribution

$$n \rightarrow \infty \quad P_n(t) \approx e^{-t} e^{-ne^{-t}}$$

$$\approx e^{-t} e^{-n/n^*}$$

$$n^* = e^t$$


finite

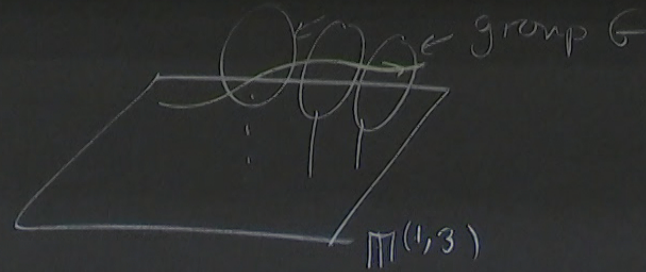
$$g(x) \in SU(2)$$

$$A_\mu(x) \rightarrow g(x) \cdot [A_\mu(x)]$$

$$A(x) = A_\mu(x) \cdot dx^\mu$$

\uparrow
2x2 matrices

connexion on
a fiber bundle



$$A(x) \rightarrow g(x) \left(A(x) + i dx^\mu \frac{\partial}{\partial x^\mu} \right) g^{-1}(x) = A_g(x)$$

compact notation

$$F_{\mu\nu}[A] \rightarrow F_{\mu\nu}[A_g] = g \cdot F_{\mu\nu} \cdot g^{-1}$$

$$\text{Tr}[F_{\mu\nu} F^{\mu\nu}] \rightarrow \text{Tr}[g F_{\mu\nu} g^{-1} g F^{\mu\nu} g^{-1}] = \text{Tr}[F_{\mu\nu} F^{\mu\nu}]$$

$$\int D[A]$$

Formal

$\mathcal{P}(A_g) = \mathcal{P}(A)$ for any g Landau $\partial^\mu A_\mu^a(x) = 0$, Feynman $\partial^\mu A_\mu^a(x) = \xi^a(x)$

$\mathcal{P} = \{ \text{Physical Configurations} \} = \mathcal{A} / \mathcal{G}$
 $= \{ \text{orbits} \}$

$G \otimes M^{1,3}$ | $\int \mathcal{A} \rightarrow \int \mathcal{P}$ integration over orbits

$$F_{\mu\nu}[A] \rightarrow F_{\mu\nu}[A_g] = g \cdot F_{\mu\nu} \cdot g^{-1} \quad \text{Tr}[F_{\mu\nu} F^{\mu\nu}] \rightarrow \text{Tr}[g F_{\mu\nu} g^{-1} \cdot g F^{\mu\nu} g^{-1}] = \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \quad \mathcal{D}[A_g]$$

Big Space of A configurations

$$\mathcal{A} = \{ \{ A_\mu^a(x) \mid x \in M^{1,3}, a=1,2,3, \mu=0,1,2,3 \} \}$$

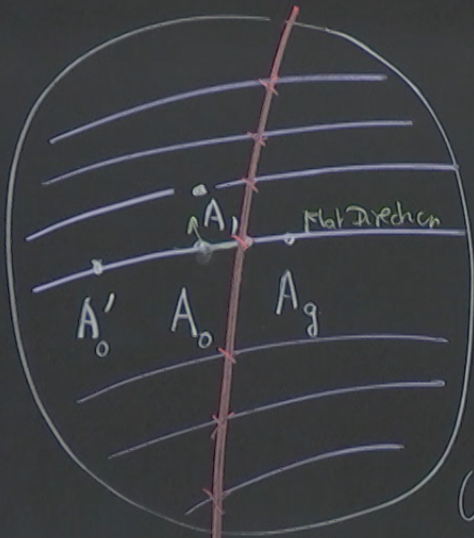
$$\mathcal{S} = \{ \text{Physical} \} = \{ \text{orbits} \}$$

$$\mathcal{G} = \{ g(x), g \in \text{SU}(2) \} \quad \text{space of gauge transformations}$$

$$A \text{ in } \mathcal{A} \rightarrow A_g = A' \Leftrightarrow A \equiv A'$$

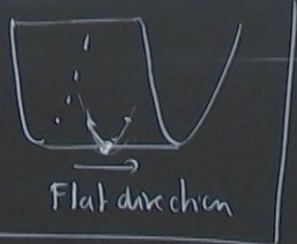
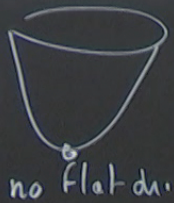
$$g \text{ in } \mathcal{G} \quad \text{orbit of } g = \{ \text{all } A_g \text{ for a given } g \}$$

$$\mathcal{G} = G^{\otimes M^{1,3}}$$



\mathcal{A}

gauge fixing slice \perp orbits



$$\int \mathcal{D}[A]$$

choosing a transverse slices
||

choose a gauge Fixing condition

$$F^a(x) = \partial^\mu A_\mu^a(x) = 0$$

$$F[A] = \left\{ \partial^\mu A_\mu^a(x) : a, x \right\}$$

Functional of the $A_\mu^a(x) \rightarrow F^a(x) := \partial^\mu A_\mu^a(x)$

$$[A] \delta[F[A]] \cdot \left| \text{Det}[F'[A]] \right|$$

↑
Faddeev-Popov Determinant
Feynman-DeWitt

$$Z^a(x) = \prod_x \prod_x \delta[\partial^\mu A_\mu^a(x)]$$

↑
Dirac δ -function

$$F[A]=0 \quad \text{For each orbit, } \exists A_0: F[A_0]=0$$

$$\int \mathcal{D}[A] \int_{\text{orbits}} \delta[A-A_0]$$

$$A_\mu(x) = A_\mu^a(x) \cdot t_a, \quad \alpha(x) = \alpha^a(x) t_a$$

$$[A_\mu, \alpha] = [A_\mu, \alpha]^a t_a = A_\mu^a \cdot \alpha^b \cdot i \epsilon_{abc} \cdot t_c$$

$$\begin{aligned} \partial^\mu [A_\mu, \alpha(x)] &= \partial^\mu (A_\mu^a(x) \cdot \alpha^b(x)) \cdot i \epsilon_{abc} t_c \\ &= [\partial^\mu A_\mu^a(x)] \alpha^b(x) + A_\mu^a(x) \cdot (\partial^\mu \alpha^b(x)) \end{aligned}$$

Diff operator which depends

$$\int \mathcal{D}[A] = \int \mathcal{D}[g] \int \mathcal{D}[A] \delta[F[A]] \cdot |\text{Det}(F'[A])|$$

$$\underbrace{\int \mathcal{D}[g]}_{\text{Vol}(G) \otimes M^{1,3}}$$

Volume of the whole G
independent of A

↑
F.D. Functional Determinant
of a differential operator
which depends on A
and of F , gauge fixing
choice

operator which depends on A

$$\left(F[A] \right) |_{+}$$

Functional Determinant
differential operator
which depends on A
of F , gauge fixing
choice

$$\exp\left(\frac{i}{2g^2\hbar} S[A]\right) \quad \boxed{\hbar = 1}$$

↑
Dynamics of quantum theory

$$\text{Observable } [A] \quad \langle \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \rangle$$

$$A = (\overset{\downarrow}{A}_0, \vec{A})$$

 Φ_F

Maxwell $A_\mu(x)$

$\partial^\mu A_\mu$ Landau (Lorentz)

$\vec{\nabla} \cdot \vec{A}$ Coulomb

$A_0 = 0$ Axial

 Φ_A

$$A = (\overset{\downarrow}{A_0}, \vec{A})$$

Maxwell $A_\mu(x)$

$\partial^\mu A_\nu$ Landau (Lorentz)

$\vec{\nabla} \cdot \vec{A}$ Coulomb

$A_0 = 0$ Axial

⋮

A_μ^a $e^{i\alpha(x)}$

3 x 4 components

3 gauge fixing cond.

Φ_F

Fundame

$\Phi_F \xrightarrow{g} g \Phi$

Φ_A

$\Phi_A \rightarrow g \cdot \Phi$

$$A = (A_0, \vec{A})$$

Maxwell $A_\mu(x)$

$\partial^\mu A_\nu$ Landau (Lorentz)

$\vec{\nabla} \cdot \vec{A}$ Coulomb

$A_0 = 0$ Axial

⋮

$$e^{i\alpha(x)}$$

A_μ^a 3x4 components $13 - 3 = 10$
3 gauge fixing cond.

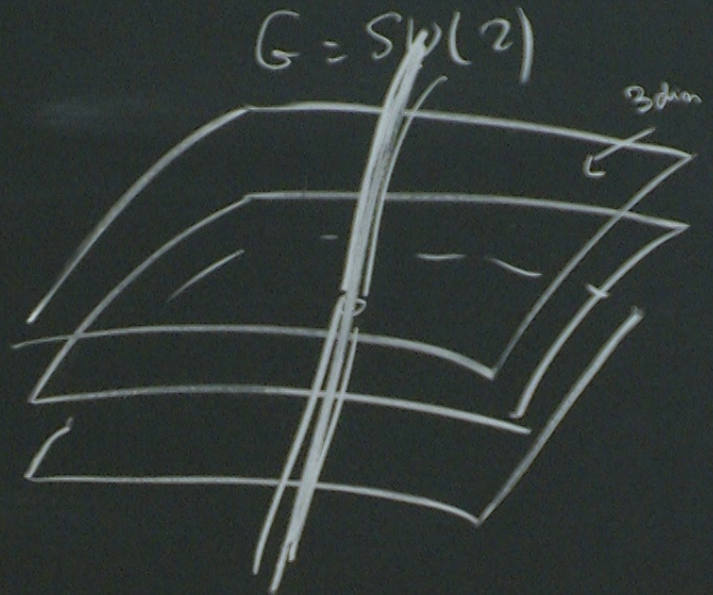
Φ_F

Φ_A

Fundamental repr

$$\Phi_F \xrightarrow{g} g \Phi_F$$

$$\Phi_A \rightarrow g \cdot \Phi_A \cdot g^{-1} \text{ (Trivial U(1))}$$



3 + 3 = 6
E, B

$$F_{\mu\nu} = \begin{pmatrix} & & & \\ & & & \\ & & & \\ & & & \end{pmatrix}$$

$$A = (\overset{\downarrow}{A}_0, \vec{A})$$

Maxwell A_μ

$\partial^\mu A_\nu$ Landau

$\vec{\nabla} \cdot \vec{A}$ Coulomb

$A_0 = 0$ Axial

\vdots

$A_\mu^a e^{i\alpha^a}$
3 x 4
3 gauge