Title: Statistical Physics - Lecture 221130

Speakers:

Collection: Statistical Physics (2022/2023)

Date: November 30, 2022 - 9:00 AM

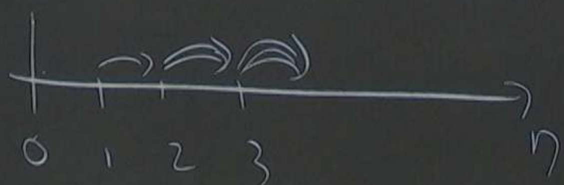
URL: https://pirsa.org/22110019

Birth Death Processes

$$\dot{P}_n = (n-1)P_{n-1} - 2nP_n + (n+1)P_{n+1}$$

→ Pure Birth : $\dot{P}_n = (n-1)P_{n-1} - nP_n$

$$P_n(t) = e^{-t}(1-e^{-t})^{n-1}$$



$$P_n(t) = e^{-t}\big(\overbrace{1-e^{-t}}^{x}\big)^{n-1}$$

0. $\langle n\rangle = e^t$
1. ultrawide distribution

$n\to\infty \quad P_n(t) \simeq e^{-t}\, e^{-ne^{-t}}$

$$\simeq e^{-t}\, e^{-n/n^*}$$

$$n^* = e^t$$

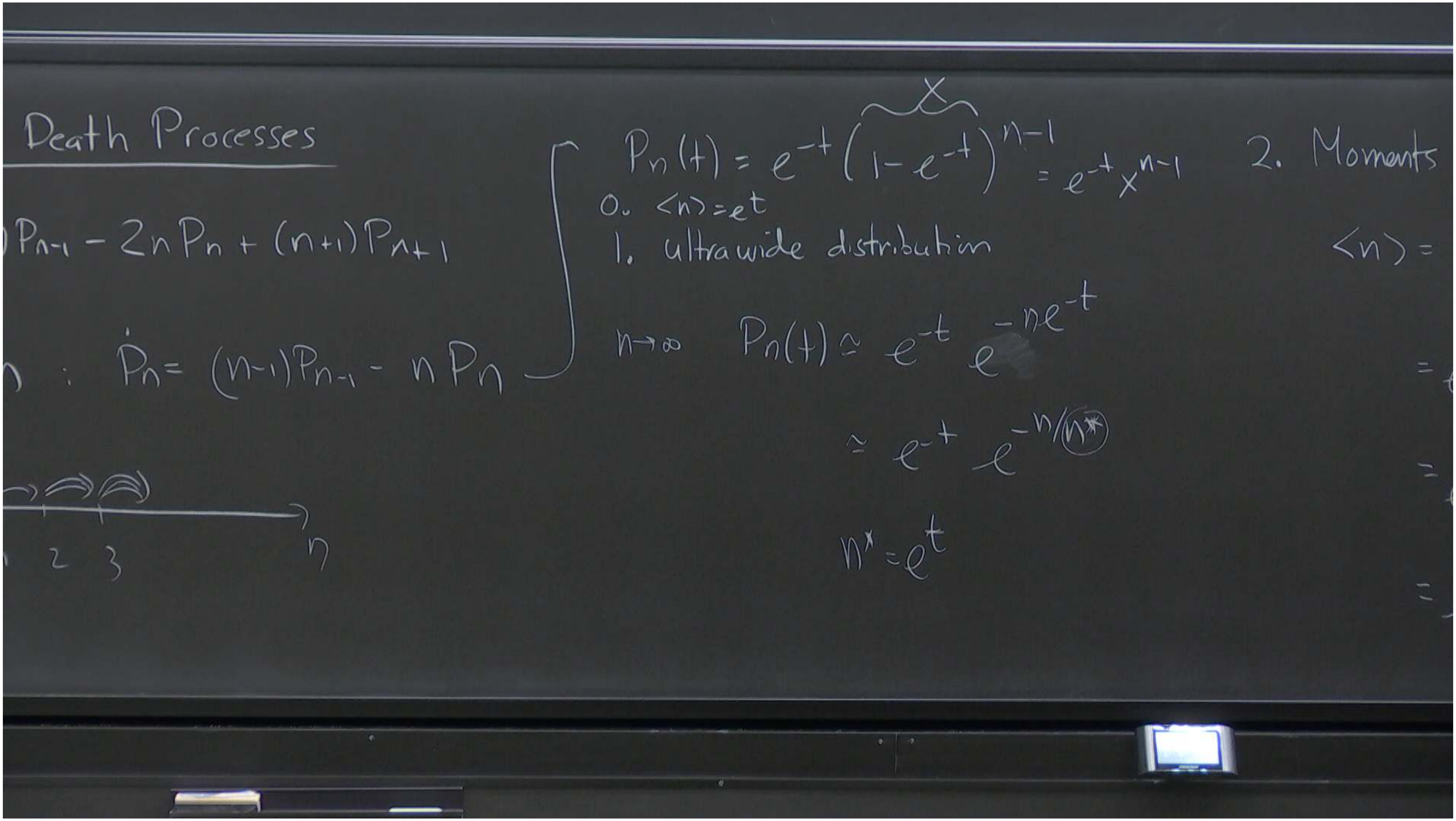
2. Moments $\quad M_k = \langle n^k\rangle$

$$\langle n\rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n\, e^{-t} x^{n-1}$$

$$= e^{-t} \frac{\partial}{\partial x} \sum_{1}^{\infty} x^n$$

$$= e^{-t} \frac{\partial}{\partial x} \frac{x}{1-x}$$

$$= e^{-t}\left(\frac{1}{1-x} + \frac{x}{(1-x)^2}\right) = e^t$$


Death Processes
$P_{n-1} - 2n P_n + (n+1) P_{n+1}$
$\dot{P}_n = (n-1) P_{n-1} - n P_n$
1 2 3 n
$P_n(t) = e^{-t} (1 - e^{-t})^{n-1} = e^{-t} x^{n-1}$
x
0. $\langle n \rangle = e^t$
1. ultrawide distribution
$n \to \infty \quad P_n(t) \simeq e^{-t} e^{-n e^{-t}}$
$\simeq e^{-t} e^{-n/n^*}$
$n^* = e^t$
2. Moments
$\langle n \rangle =$

$\ldots^{-t} x^{n-1}$

2. Moments $M_k = \langle n^k \rangle$

$$\langle n \rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n\, e^{-t} x^{n-1}$$

$$= e^{-t} \frac{\partial}{\partial x} \sum_{1}^{\infty} x^n$$

$$= e^{-t} \frac{\partial}{\partial x} \frac{x}{1-x}$$

$$= e^{-t} \left(\frac{1}{1-x} + \frac{x}{(1-x)^2} \right) = e^{t}$$

$$\langle n^2 \rangle = \sum n^2 P_n$$

$$= \sum n\, n P_n$$

$$= \sum n\, n\, e^{-t} x^{n-1}$$

$$= e^{-t} \frac{\partial}{\partial x} x \frac{\partial}{\partial x} \sum x^n$$

$$= 2 e^{2t} - e^{t}$$

Standard deviation $\sqrt{\langle n^2\rangle - \langle n\rangle^2}$

$$= \sqrt{e^{2t} - e^{t}}$$

$$\simeq e^{t}$$

$$P_n(t) = e^{-t}\big(\underbrace{1-e^{-t}}_{X}\big)^{n-1} = e^{-t} X^{n-1}$$

$(1-\varepsilon) = e^{\varepsilon}$

0. $\langle n \rangle = e^t$

1. ultrawide distribution

$$n \to \infty \quad P_n(t) \simeq e^{-t}\, e^{-n e^{-t}}$$

$$\simeq e^{-t}\, e^{-n/n^*}$$

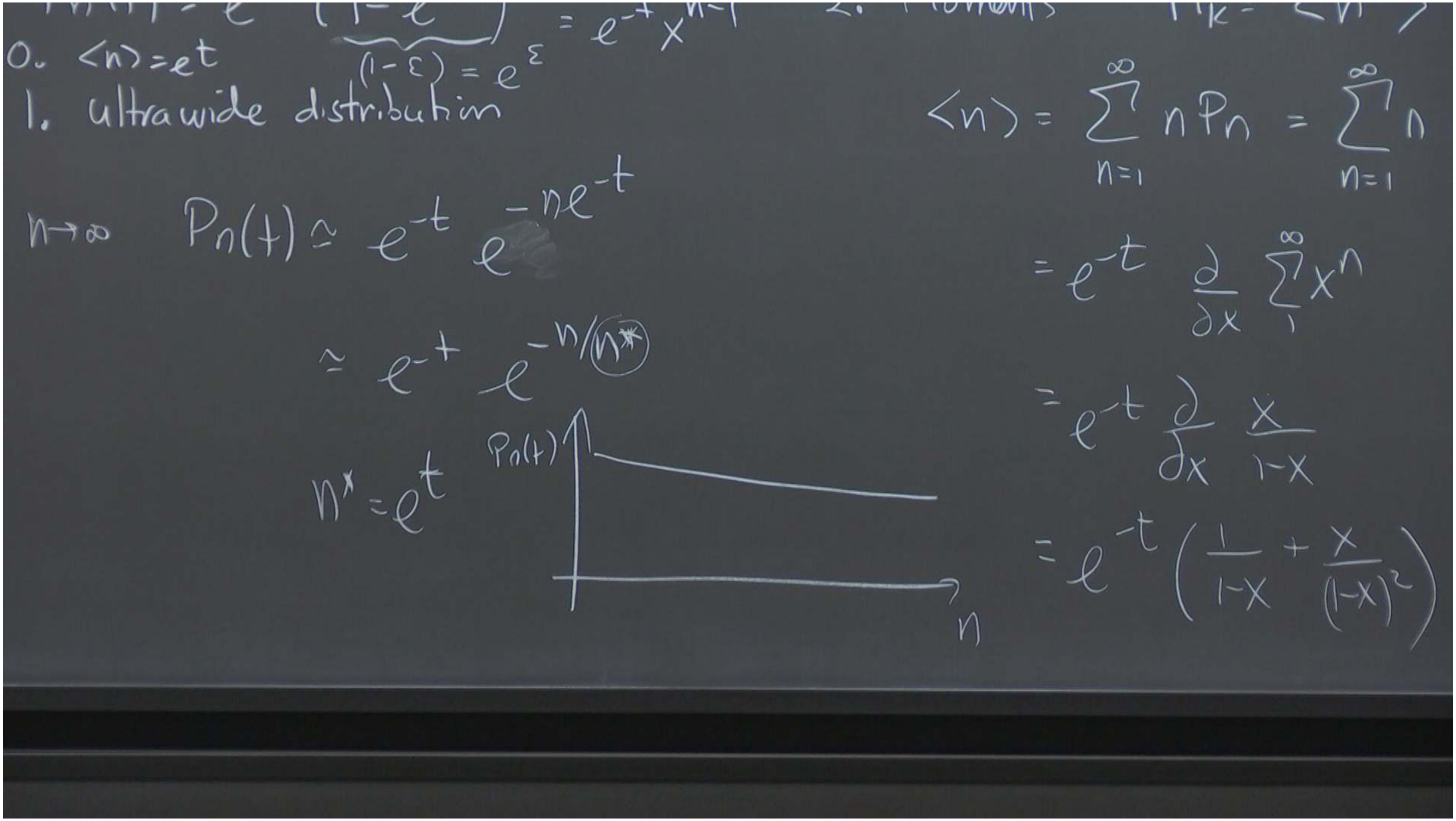
$$n^* = e^t$$

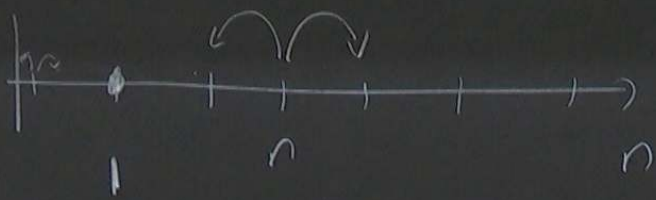
2. Moments

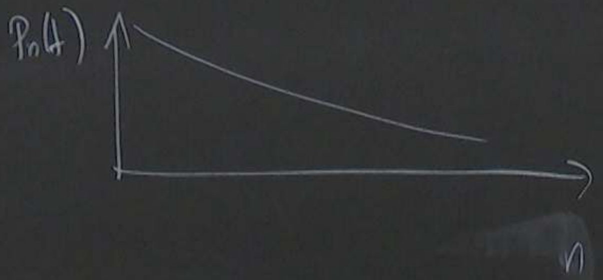
$\langle n \rangle =$

$(n+1)P_{n+1}$

$P_{n-1} - n P_n$


0. <n> = e^t
(1-ε) = e^ε
= e^-t x^(n-1)
1. ultrawide distribution
n→∞ Pn(t) ≃ e^-t e^(-ne^-t)
≃ e^-t e^(-n/n*)
n* = e^t
Pn(t)
n
<n> = Σ n=1 ∞ n Pn = Σ n=1 ∞ n
= e^-t ∂/∂x Σ 1 ∞ x^n
= e^-t ∂/∂x x/(1-x)
= e^-t (1/(1-x) + x/(1-x)^2)





≥ 1

$$P_n(t) = \frac{1}{(1+t)^2}\, y^n$$

$$y^n = \left(\frac{t}{1+t}\right)^n = \frac{1}{(1+1/t)^n} \approx 1 - n/t$$

$$\sim e^{-n/t} = e^{-n/n^*} \qquad n^* = t$$

$$0. \quad \langle n \rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n \frac{1}{(1+t)^2}\, y^{n-1}$$

$$= 1$$

$$= \frac{1}{(1+t)^2} \frac{\partial}{\partial y} \sum_{n=1}^{\infty} y^n$$

$$= \frac{1}{(1+t)^2} \frac{\partial}{\partial y} \frac{y}{1-y}$$

$$= \frac{1}{(1+t)^2} \frac{1}{(1-y)^2}$$

$$= 1$$

0. $\langle n \rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n \frac{1}{(1+t)^2} y^{n-1}$

$= 1$

$= \frac{1}{(1+t)^2} \frac{\partial}{\partial y} \sum_{n=1}^{\infty} y^n$

$= \frac{1}{(1+t)^2} \frac{\partial}{\partial y} \frac{y}{1-y}$

$= \frac{1}{(1+t)^2} \frac{1}{(1-y)^2}$

$= 1$

$/t \simeq e^{-n/n^*} \quad n^* = t$

2. $\langle n^2 \rangle = 1 + 2t$

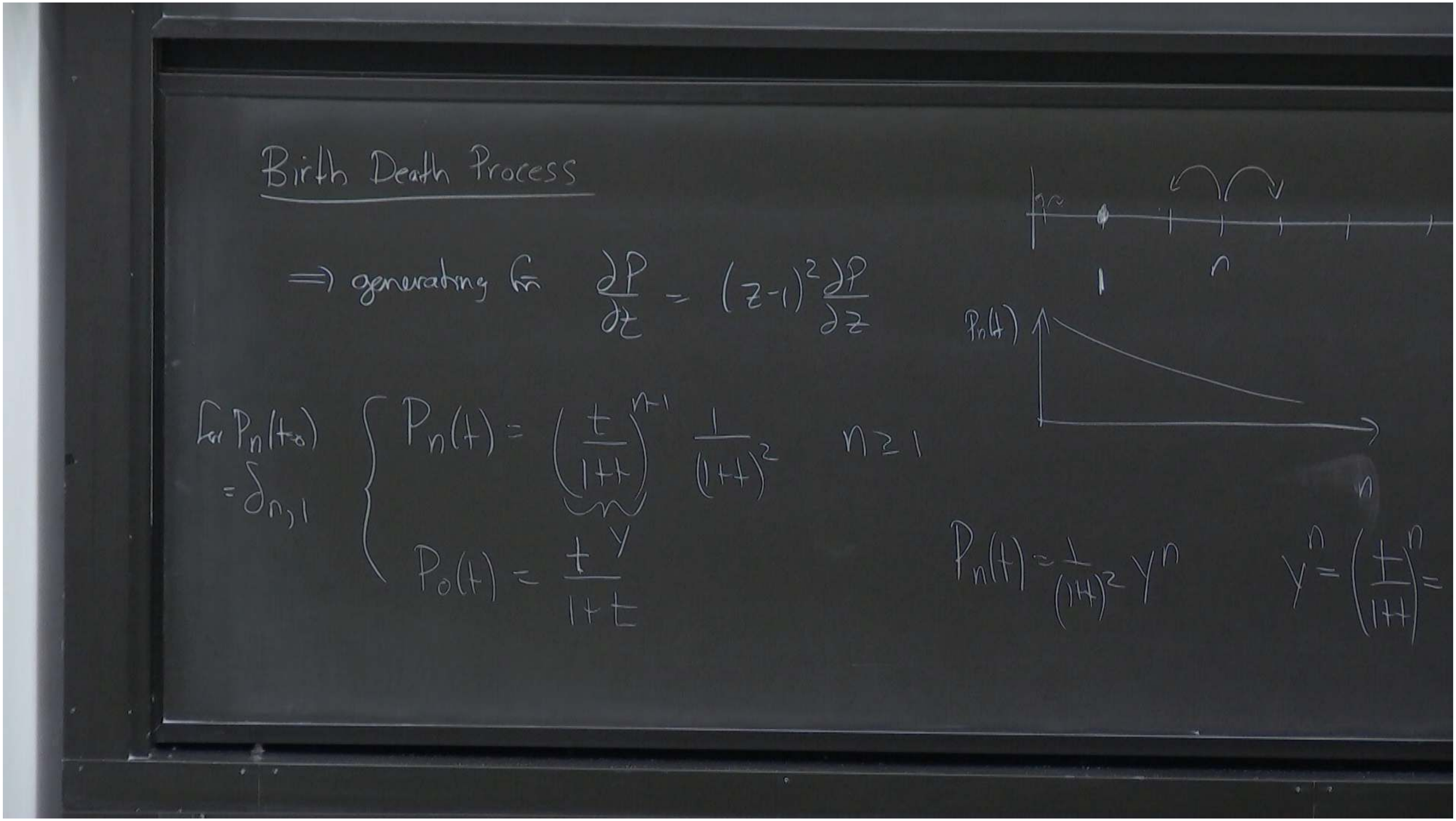
$\sqrt{\langle n^2 \rangle - \langle n \rangle^2} \propto \sqrt{t}$

3. Extinction for Sure

$S(t) \equiv$ survival prob.

$= 1 - P_0(t)$

$= \frac{1}{1+t}$


Birth Death Process
⟹ generating fn $\frac{\partial P}{\partial t} = (z-1)^2 \frac{\partial P}{\partial z}$
for $P_n(t_0) = \delta_{n,1}$
$P_n(t) = \left(\frac{t}{1+t}\right)^{n-1} \frac{1}{(1+t)^2}$ $n \geq 1$
$P_0(t) = \frac{t}{1+t}$
$P_n(t)$
$P_n(t) = \frac{1}{(1+t)^2} y^n$
$y^n = \left(\frac{t}{1+t}\right)^n =$

0. $\langle n \rangle = \sum_{n=1}^{\infty} n P_n = \sum_{n=1}^{\infty} n \frac{1}{(1+t)^2} y^{n-1}$

$= 1$

$= \frac{1}{(1+t)^2} \frac{\partial}{\partial y} \sum_{n=1}^{\infty} y^n$

$= \frac{1}{(1+t)^2} \frac{\partial}{\partial y} \frac{y}{1-y}$

$\frac{\partial}{\partial y} \frac{y}{1-y} = \frac{1}{1-y} + \frac{y}{(1-y)^2}$

$= \frac{1}{(1+t)^2} \frac{1}{(1-y)^2}$

$= 1$

$\equiv e^{-n/n^*} \quad n^* = t$

2. $\langle n^2 \rangle = 1 + 2t$

$\sqrt{\langle n^2 \rangle - \langle n \rangle^2} \propto \sqrt{t}$

3. Extinction for Sure

$S(t) \equiv$ survival prob.

$= 1 - P_0(t)$

$= \frac{1}{1+t}$

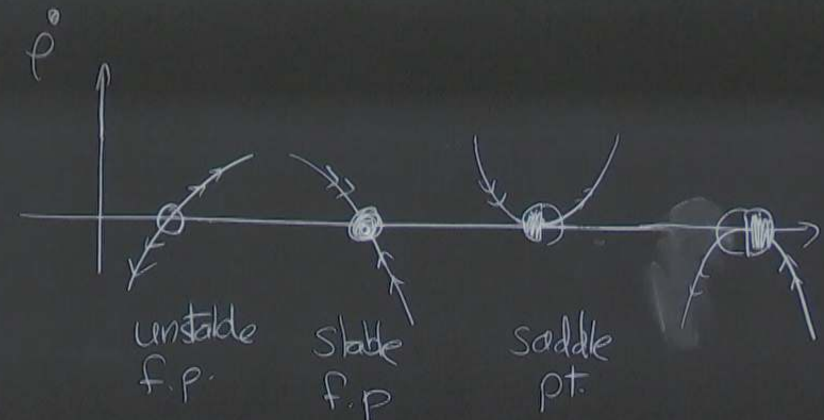
Population Dynamics

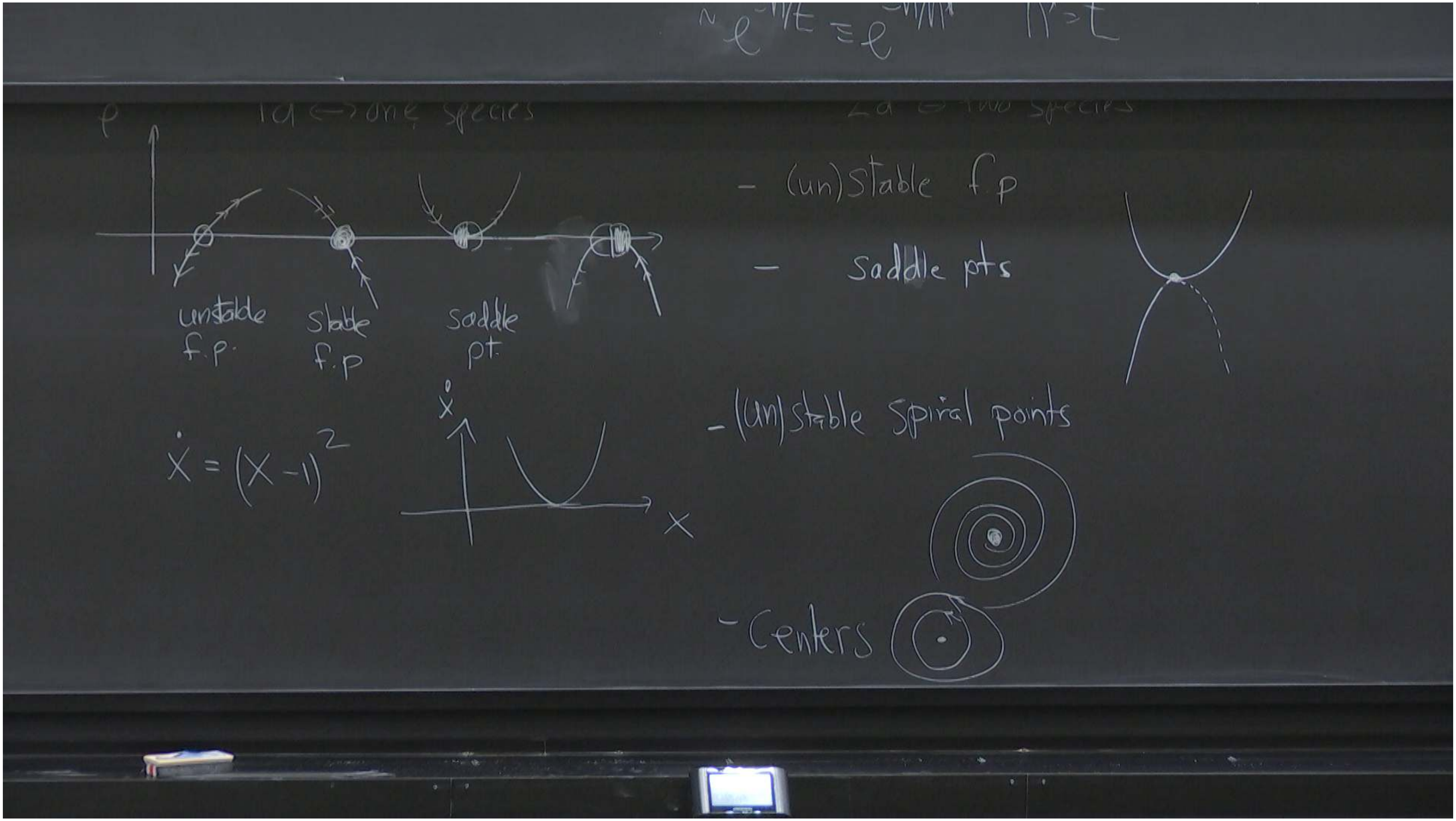
$$\dot{\rho} = f(\rho)$$ one species

$$\left.\begin{aligned}\dot{\rho}_1 &= f(\rho_1, \rho_2)\\ \dot{\rho}_2 &= g(\rho_1, \rho_2)\end{aligned}\right\}$$ two species

$\vdots$

n species

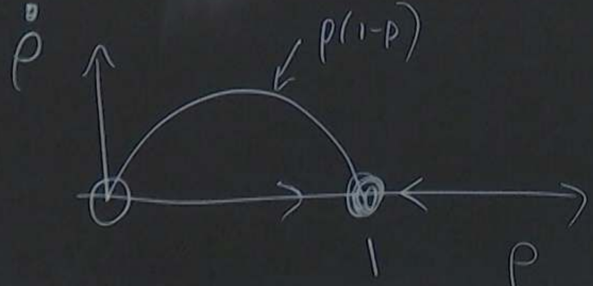



1d ⟷ one species
2d ⟷ two species
unstable f.p.
stable f.p
saddle pt.
$\dot{x} = (x-1)^2$
- (un)stable f.p
- saddle pts
- (un)stable spiral points
- Centers

Logistic Growth

$$\dot{\rho} = \rho - \rho^2$$

$$\dot{\rho} = \rho(1-\rho)$$



$$t \to \infty \quad \rho \to 1$$

$$\frac{d\rho}{\rho(1-\rho)} = dt$$

$$d\rho\left(\frac{1}{\rho} + \frac{1}{1-\rho}\right) = dt$$

$$\ln \rho/\rho_0 + \ln \frac{1-\rho}{1-\rho_0} = t$$

$$\ln \frac{\rho(1-\rho)}{\rho_0(1-\rho)} = t$$

$$\to \rho(t)$$

Two Species Competition

$$\dot{A} = A(1 - A - \varepsilon B)$$

$$\dot{B} = B(1 - B - \varepsilon A)$$

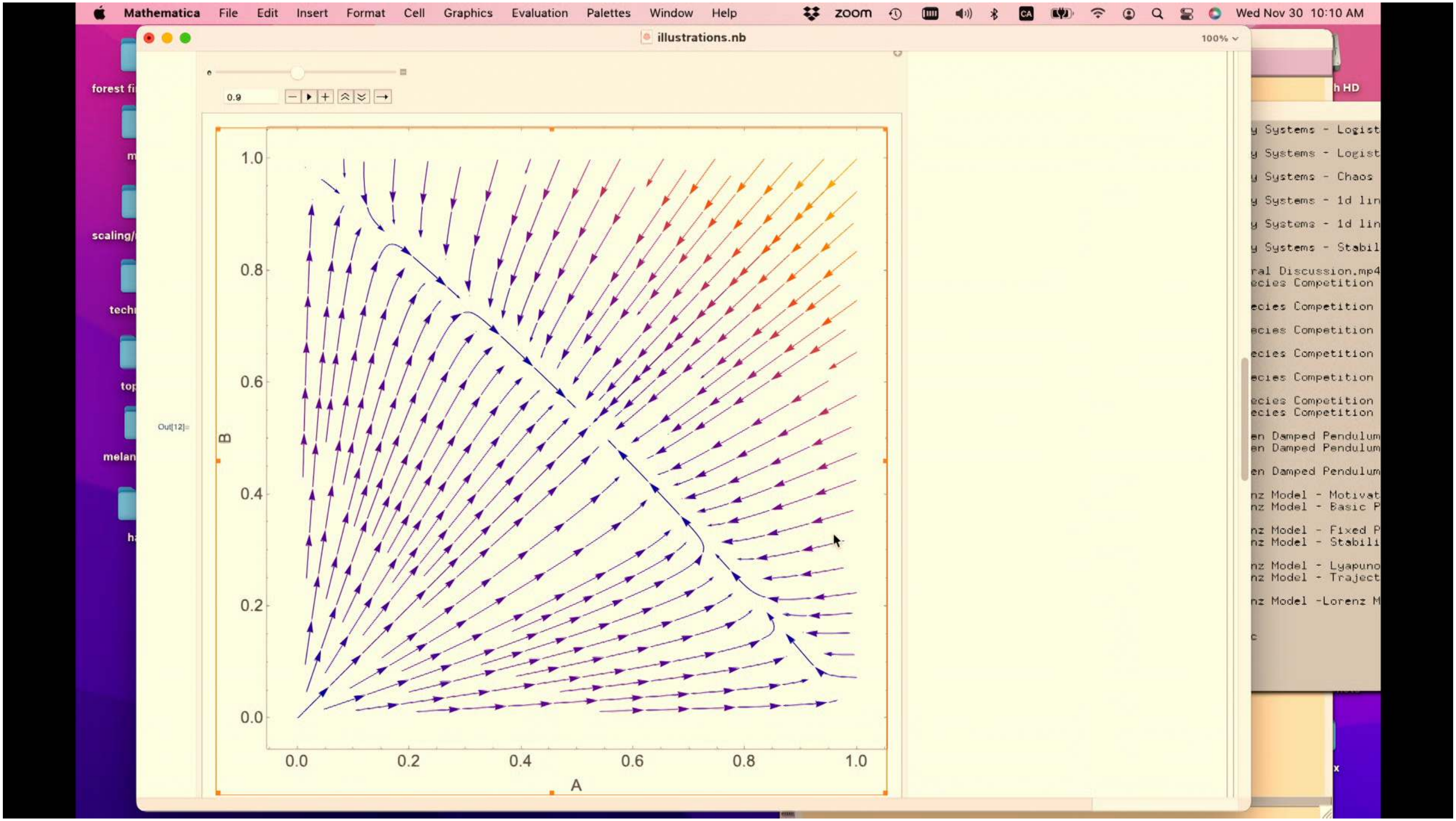
1. find nullclines
2. find fixed points
3. infer global behavior

nullclines $\dot{A} = 0, \dot{B} = 0$

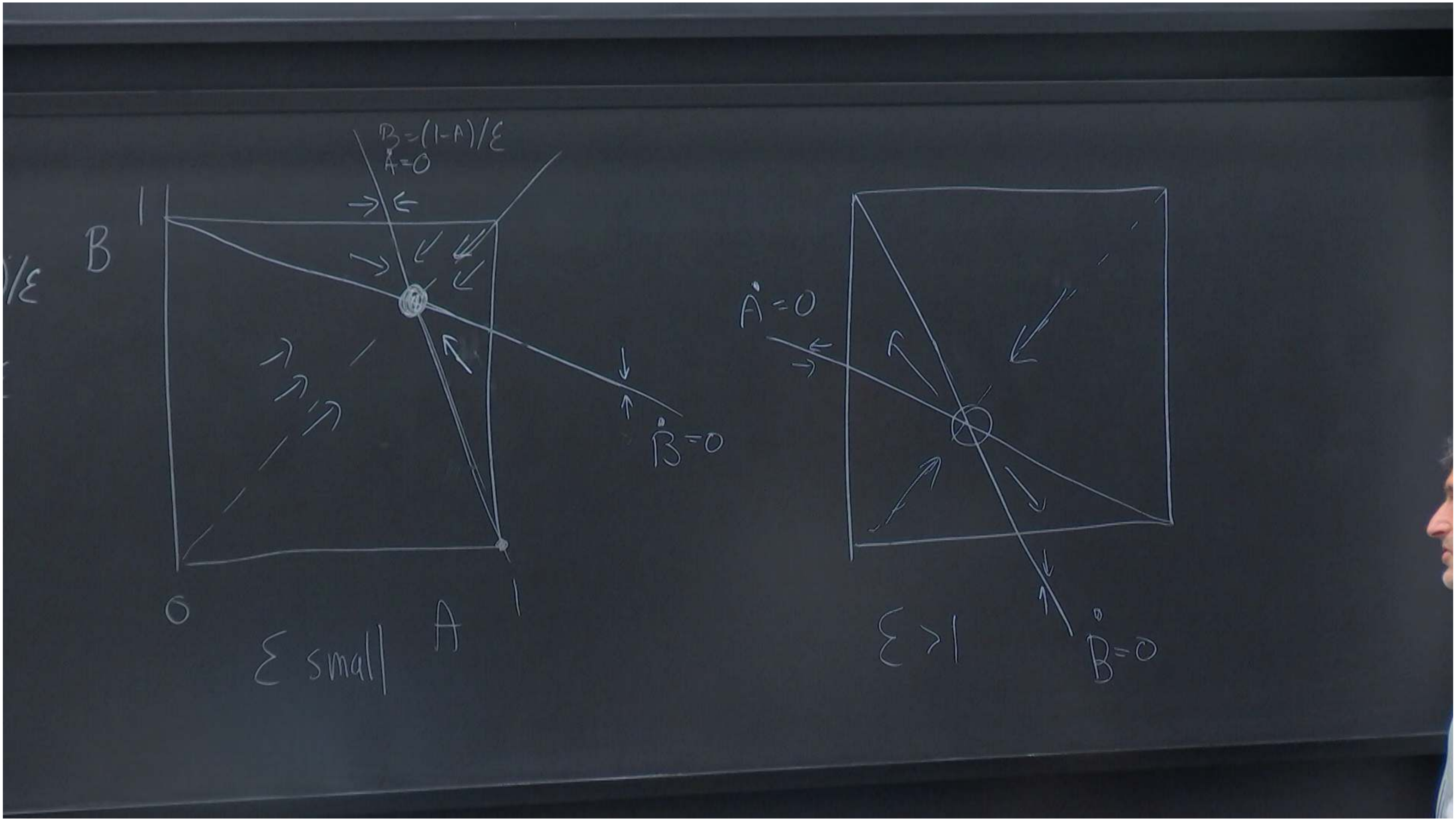
$\dot{A} = 0 \Rightarrow A = 0 \; ; \; B = (1-A)/\varepsilon$

$\dot{B} = 0 \Rightarrow B = 0 \; ; \; A = (1-B)/\varepsilon$

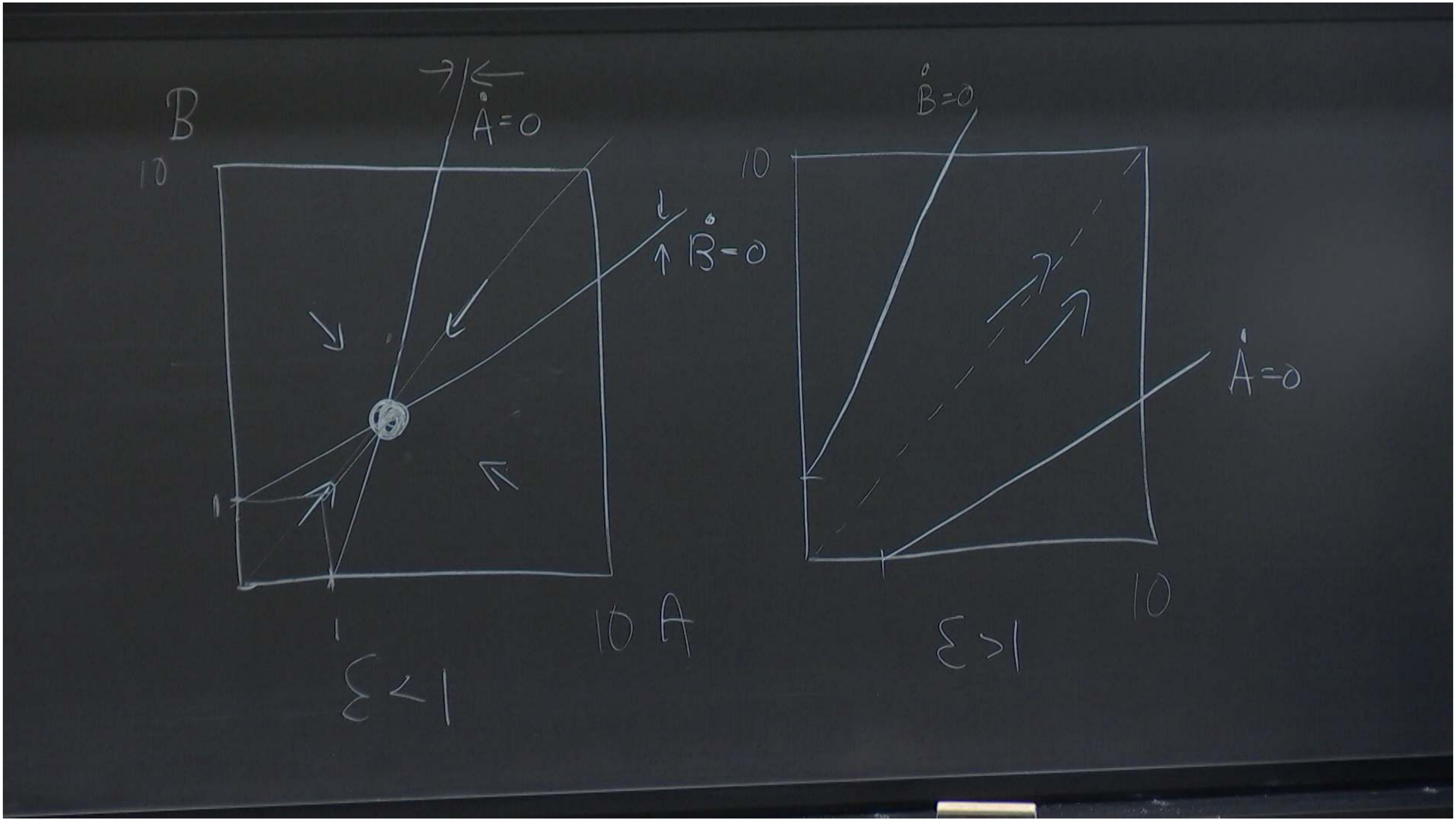
B



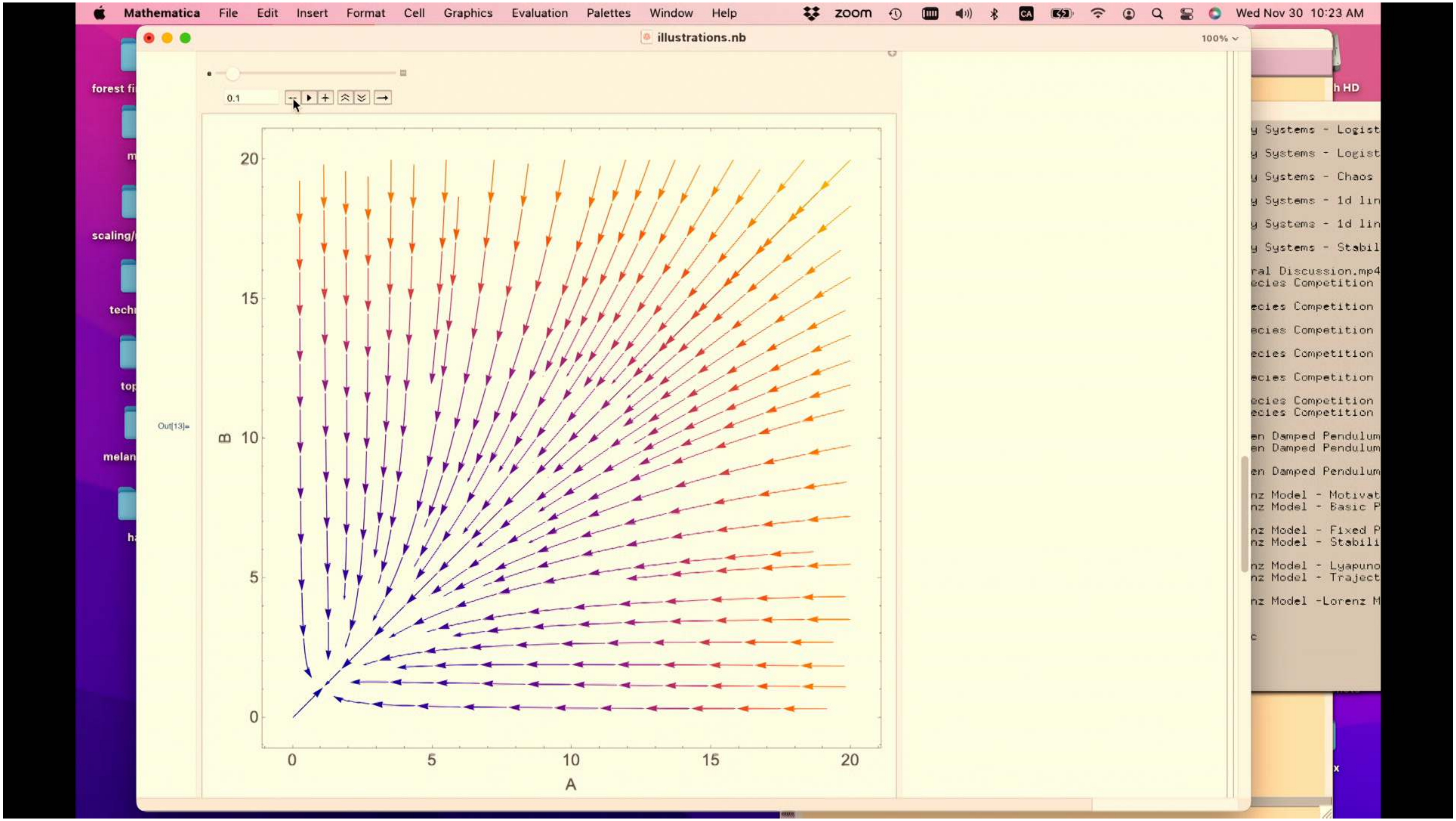
Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help
Wed Nov 30 10:10 AM
illustrations.nb
0.9
Out[12]=
1.0
0.8
0.6
0.4
0.2
0.0
B
0.0 0.2 0.4 0.6 0.8 1.0
A

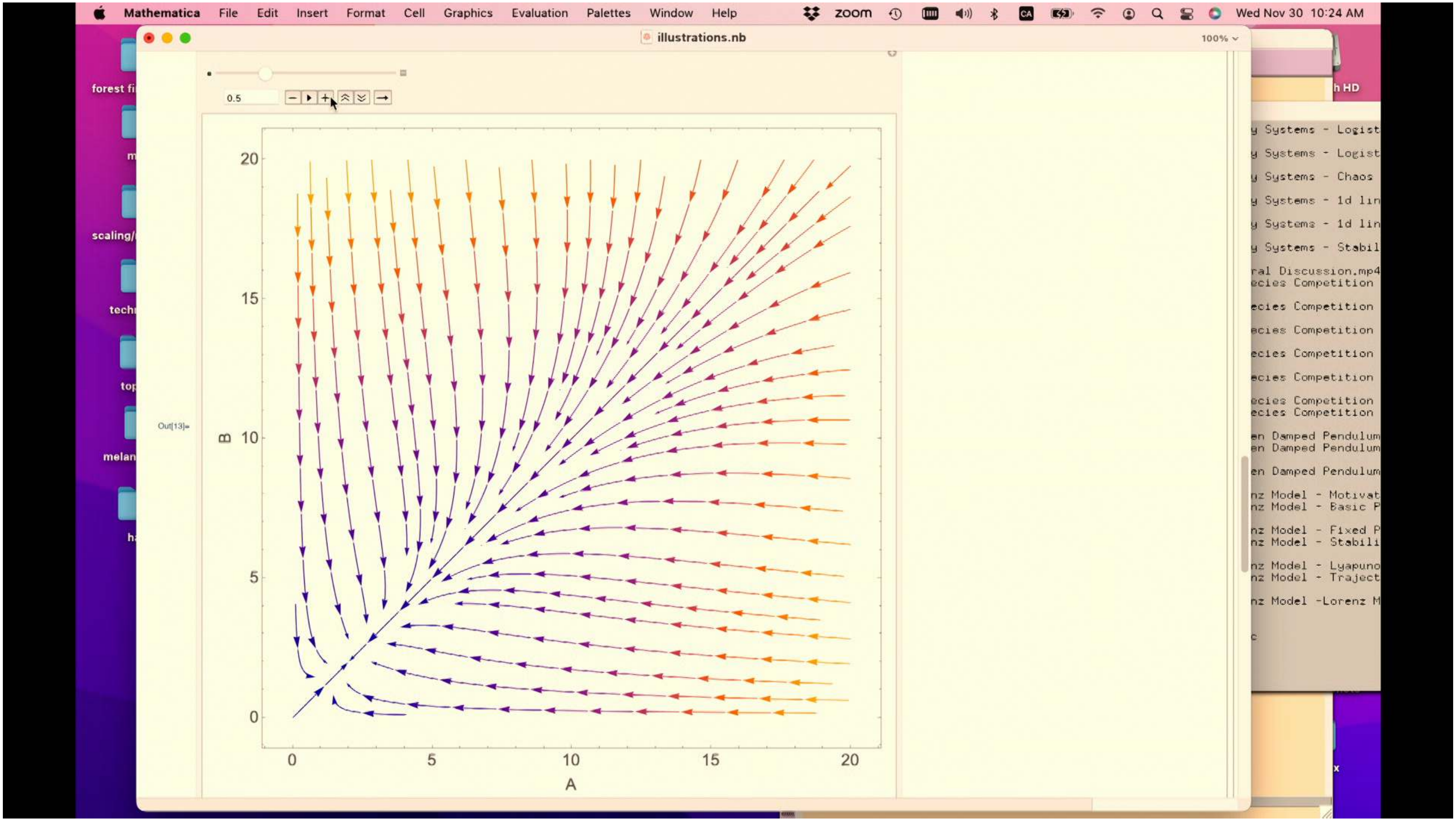


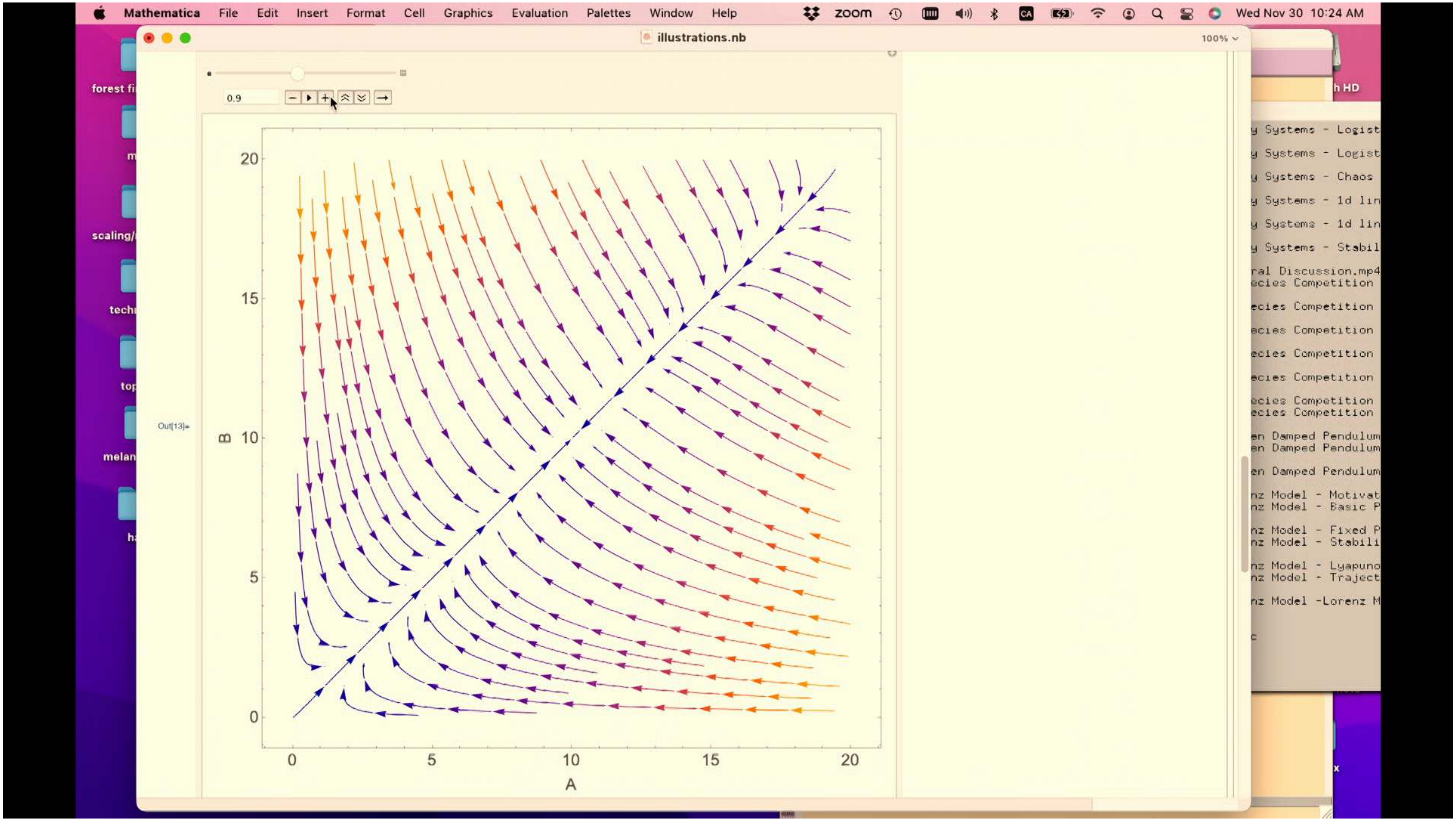
B = (1-A)/ε
Ȧ=0
1
B
0
A
1
Ḃ=0
ε small
Ȧ=0
ε>1
Ḃ=0

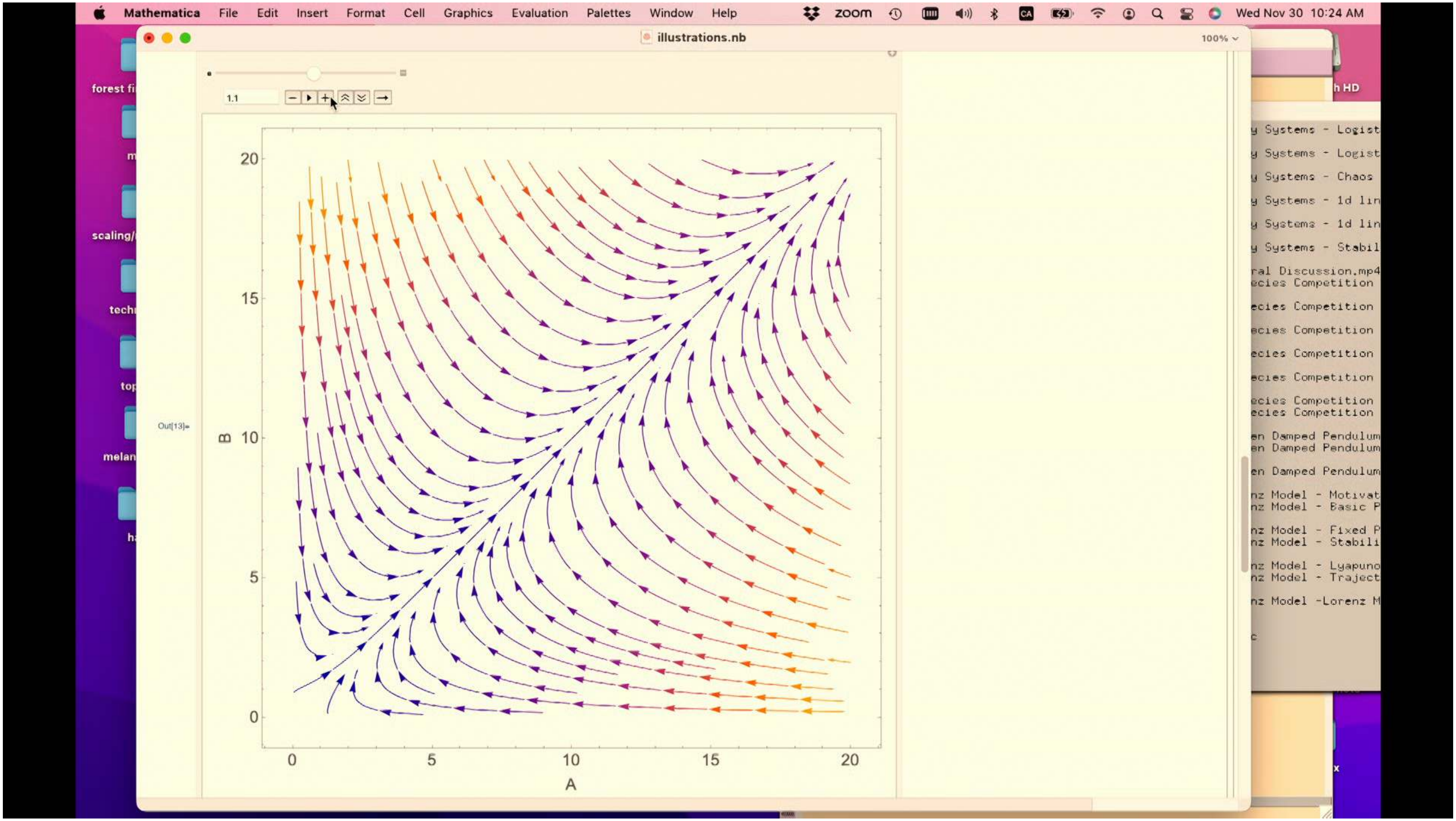


B
10
$\dot{A}=0$
$\dot{B}=0$
1
1
10 A
$\varepsilon<1$
$\dot{B}=0$
10
$\dot{A}=0$
10
$\varepsilon>1$

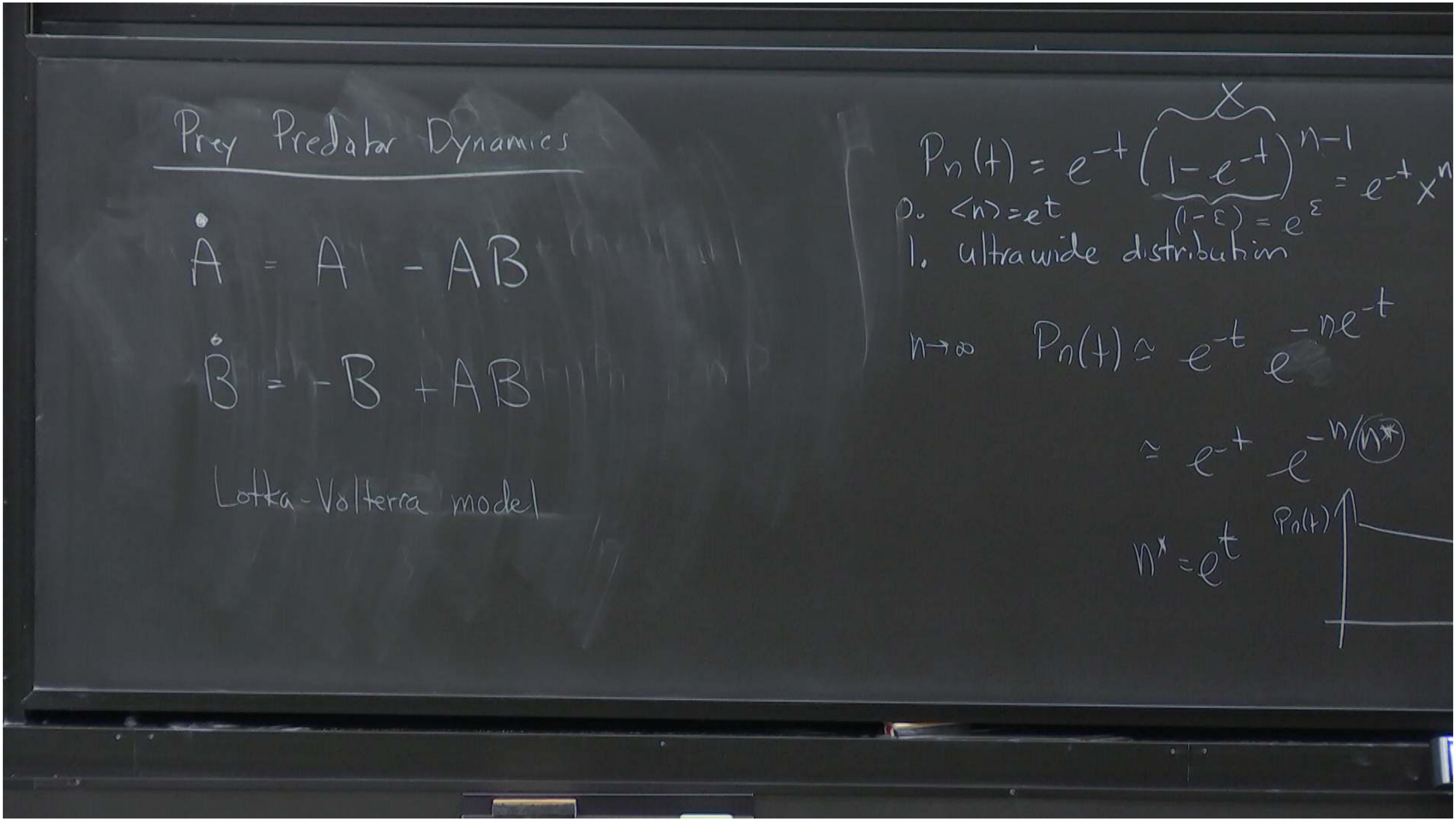

illustrations.nb
0.1
B
A
0
5
10
15
20

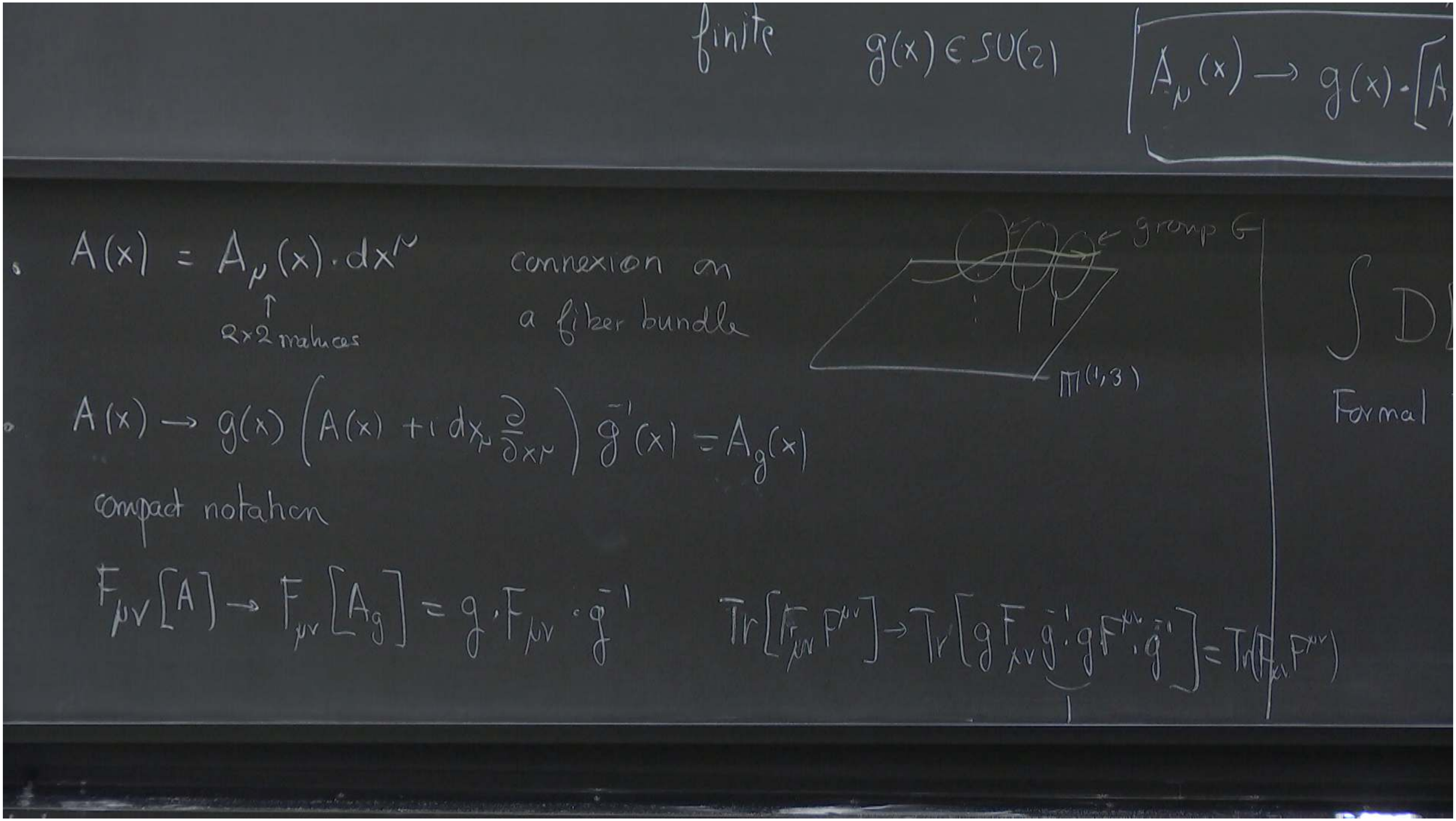

Mathematica File Edit Insert Format Cell Graphics Evaluation Palettes Window Help
zoom
Wed Nov 30 10:24 AM
illustrations.nb
100%
0.5
Out[13]=
B
A
0
5
10
15
20
y Systems - Logist
y Systems - Logist
y Systems - Chaos
y Systems - 1d lin
y Systems - 1d lin
y Systems - Stabil
ral Discussion.mp4
ecies Competition
ecies Competition
ecies Competition
ecies Competition
ecies Competition
ecies Competition
ecies Competition
en Damped Pendulum
en Damped Pendulum
en Damped Pendulum
nz Model - Motivat
nz Model - Basic P
nz Model - Fixed P
nz Model - Stabili
nz Model - Lyapuno
nz Model - Traject
nz Model -Lorenz M

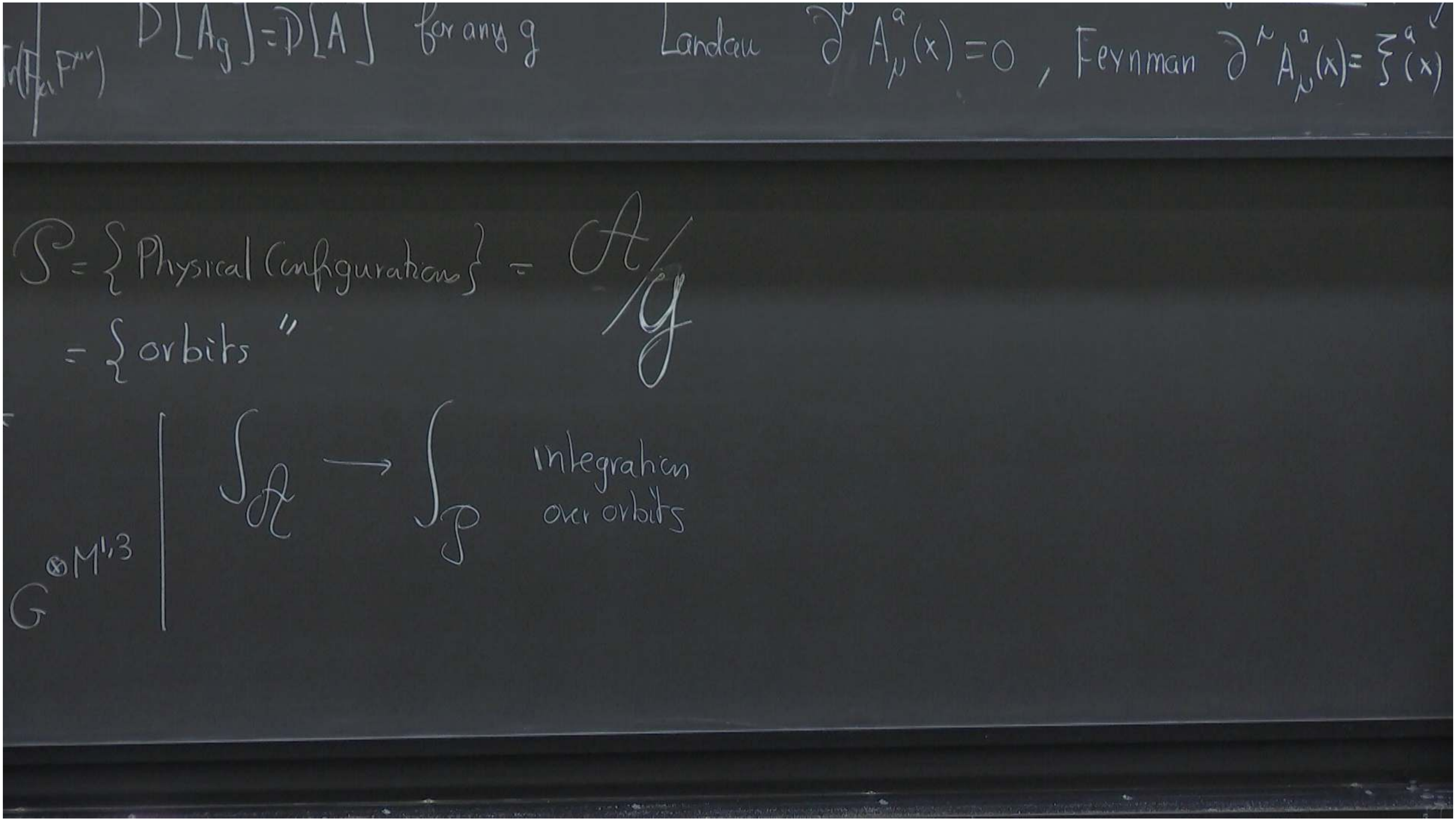

illustrations.nb
0.9
B
A
0
5
10
15
20

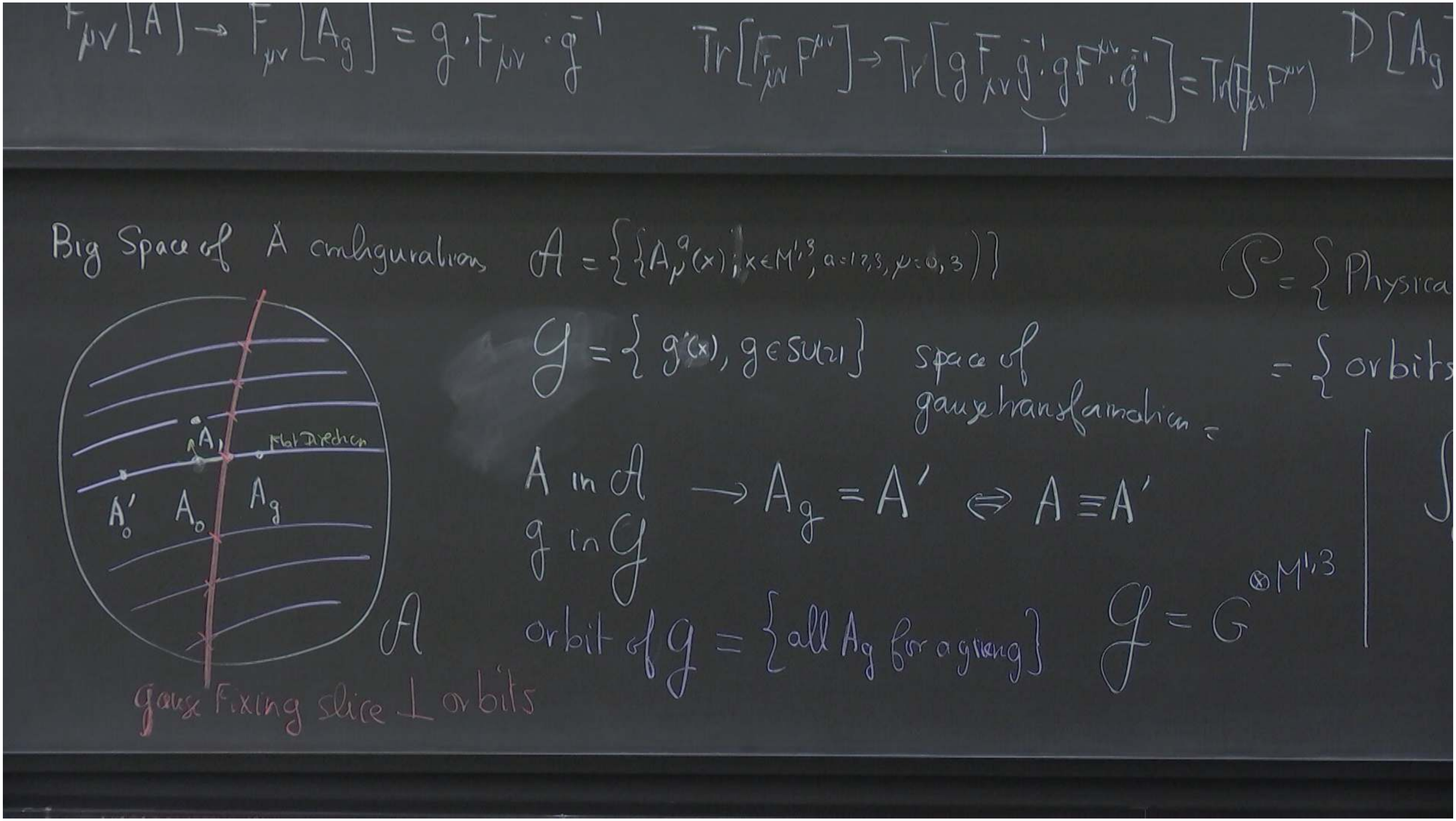


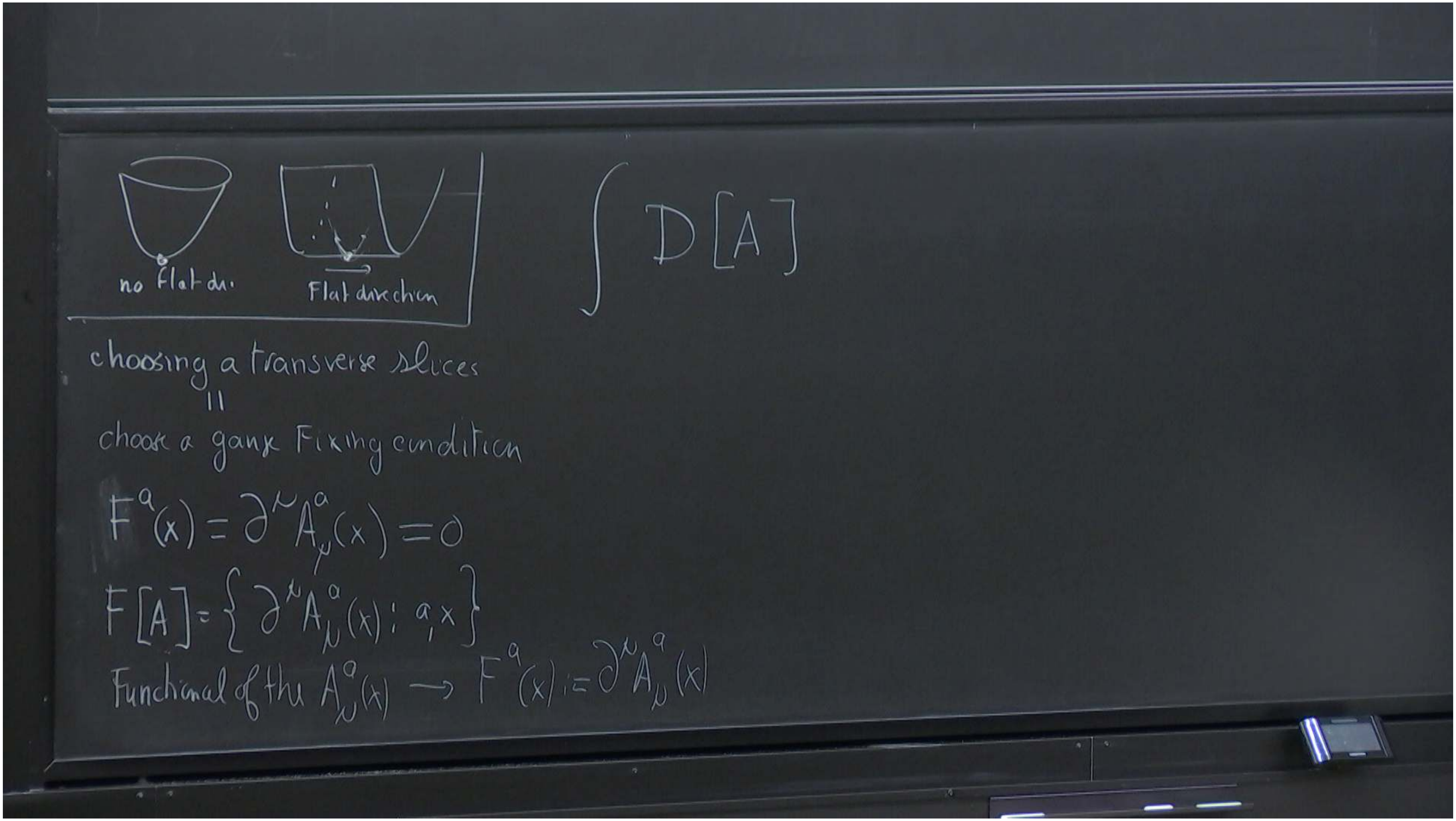
illustrations.nb
Out[13]=
20
15
10
5
0
B
0
5
10
15
20
A

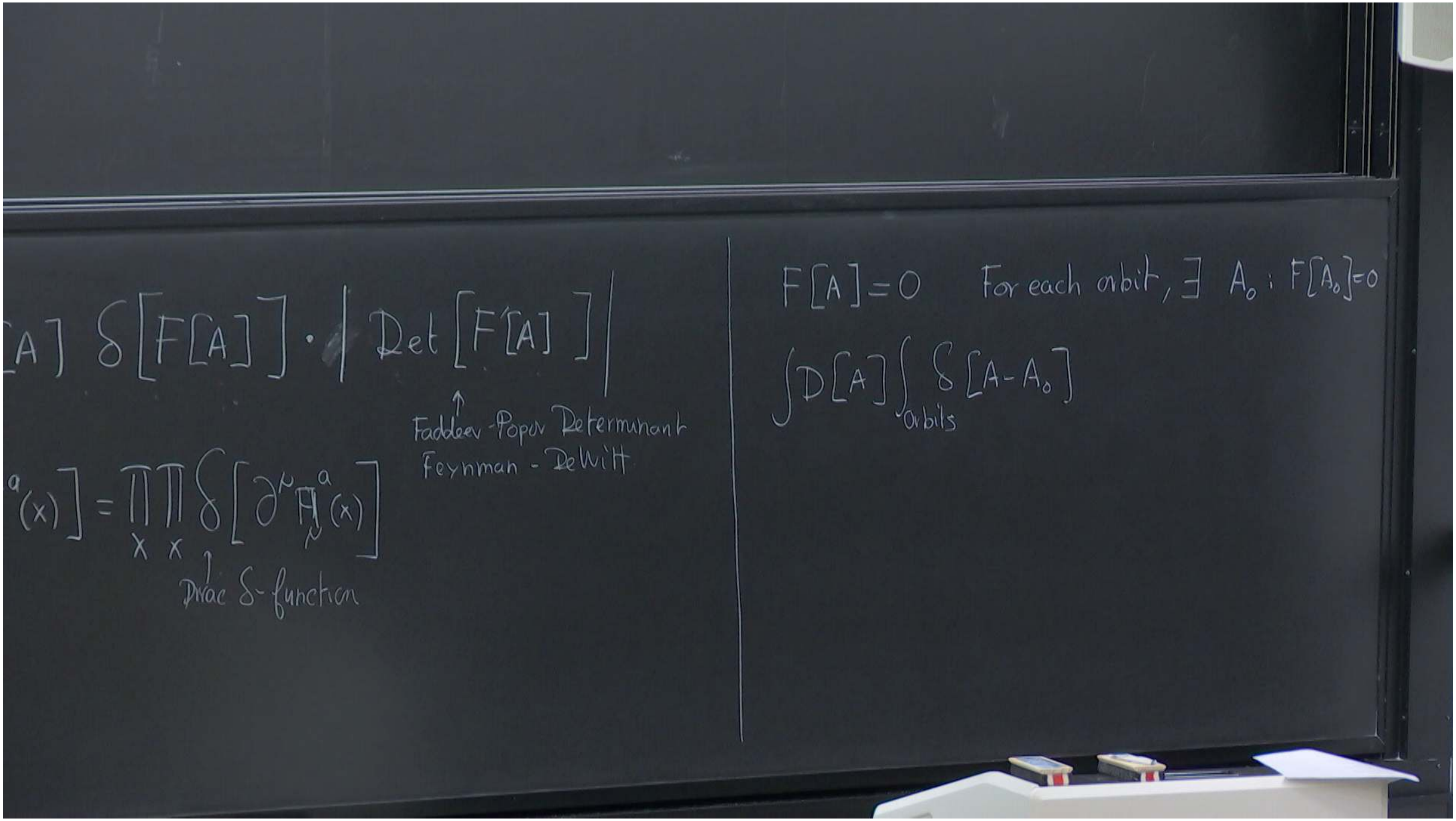

Prey Predator Dynamics
$\dot{A} = A - AB$
$\dot{B} = -B + AB$
Lotka-Volterra model
$P_n(t) = e^{-t}(1-e^{-t})^{n-1} = e^{-t}x^n$
0. $\langle n \rangle = e^t$
$(1-\varepsilon) = e^{\varepsilon}$
1. ultrawide distribution
$n \to \infty$ $P_n(t) \simeq e^{-t} e^{-ne^{-t}}$
$\simeq e^{-t} e^{-n/n^*}$
$n^* = e^t$
$P_n(t)$


finite
$g(x) \in SU(2)$
$A_\mu(x) \longrightarrow g(x)\cdot[A$
$A(x) = A_\mu(x)\cdot dx^\mu$
2×2 matrices
connexion on a fiber bundle
group G
$M^{(1,3)}$
$\int D[$
Formal
$A(x) \longrightarrow g(x)\left(A(x) + i\, dx_\mu \frac{\partial}{\partial x^\mu}\right) g^{-1}(x) = A_g(x)$
compact notation
$F_{\mu\nu}[A] \rightarrow F_{\mu\nu}[A_g] = g\cdot F_{\mu\nu}\cdot g^{-1}$
$Tr[F_{\mu\nu}F^{\mu\nu}] \rightarrow Tr[g F_{\lambda\nu} g^{-1}\cdot g F^{\lambda\nu}\cdot g^{-1}] = Tr(F_{\mu\nu}F^{\mu\nu})$


D[A_g] = D[A] for any g
Landau ∂^μ A^a_μ(x) = 0 , Feynman ∂^μ A^a_μ(x) = ξ^a(x)
P = { Physical Configurations } = A/G
= { orbits }
∫_A → ∫_P integration over orbits
G^⊗M^{1,3}


Big Space of A configurations
gauge transformation
Flat Direction
gauge Fixing slice ⊥ orbits
space of
orbit of
all A_g for a given g
Physica
orbits


no flat dir.
Flat direction
$\int D[A]$
choosing a transverse slices
=
choose a gauge fixing condition
$F^a(x) = \partial^\mu A^a_\mu(x) = 0$
$F[A] = \{ \partial^\mu A^a_\mu(x) ; a, x \}$
Functional of the $A^a_\mu(x) \to F^a(x) := \partial^\mu A^a_\mu(x)$

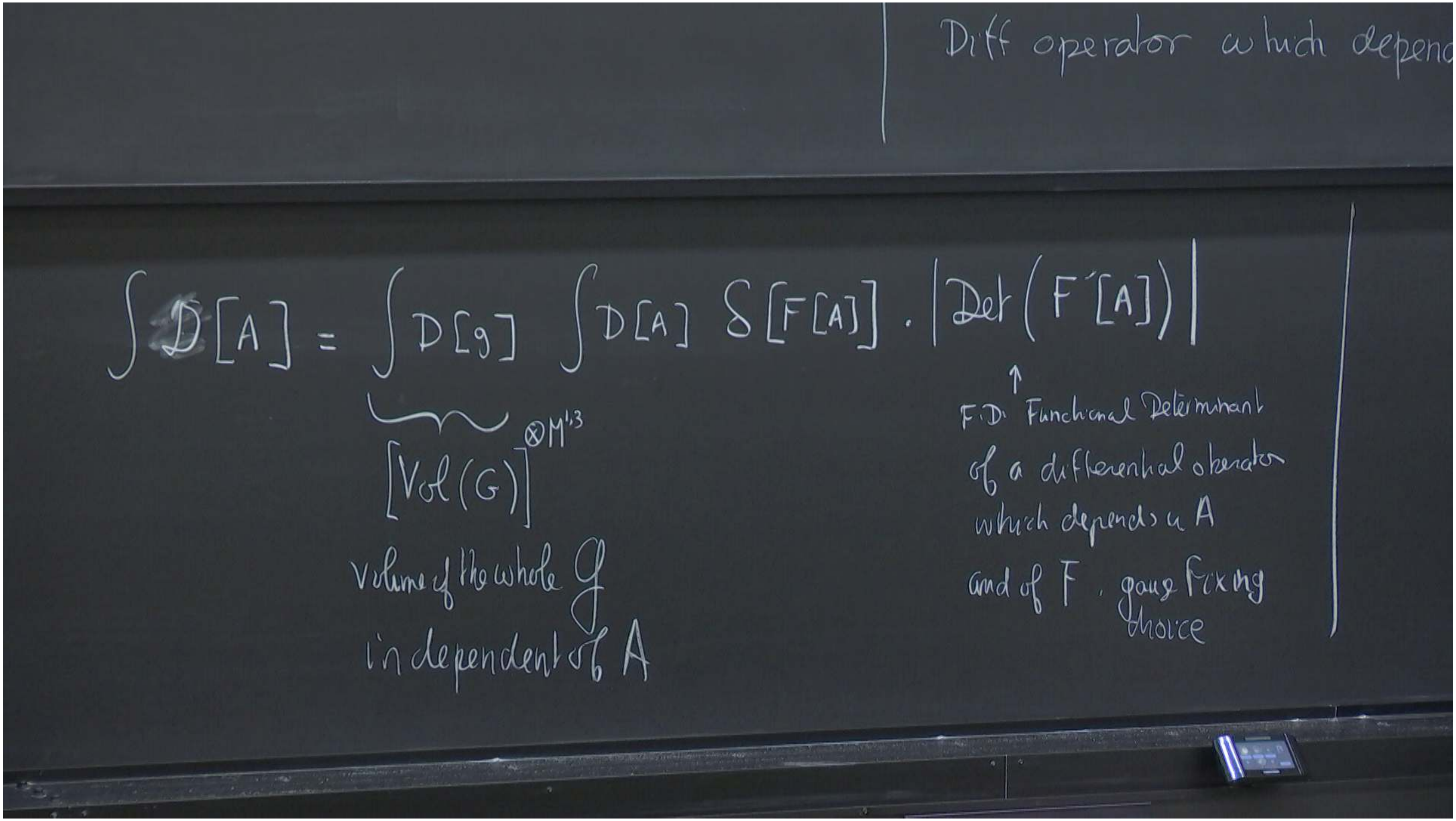

[A] δ[F[A]] · | Det[F'[A]] |
Faddeev-Popov Determinant
Feynman - DeWitt
a(x)] = ∏ ∏ δ[∂^μ A^a_μ(x)]
x x
Dirac δ-function
F[A]=0
For each orbit, ∃ A_0 : F[A_0]=0
∫D[A] ∫_Orbits δ[A-A_0]

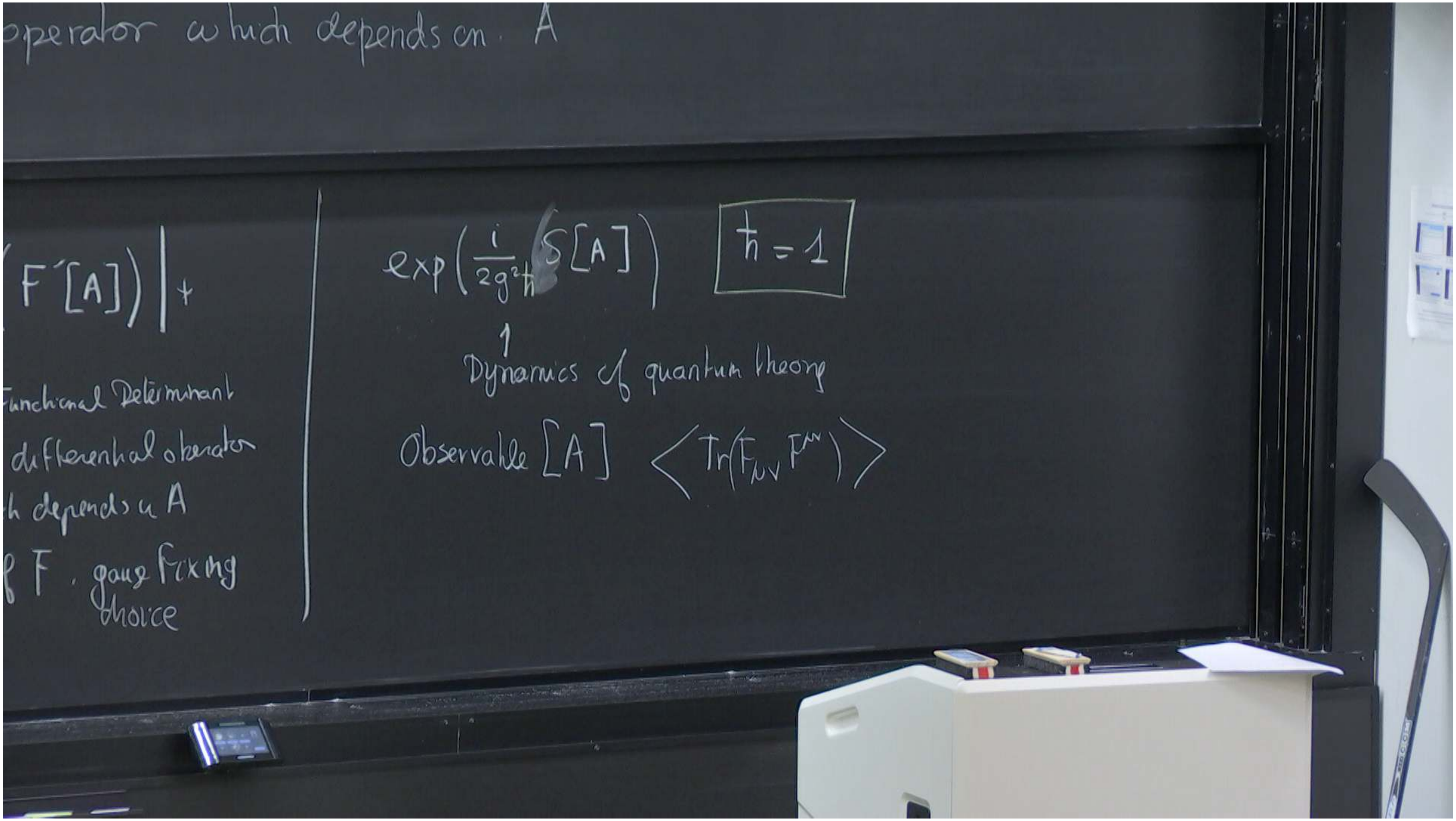
$$A_\mu(x) = A_\mu^a(x)\cdot t_a, \quad \alpha(x) = \alpha^a(x)\, t_a$$

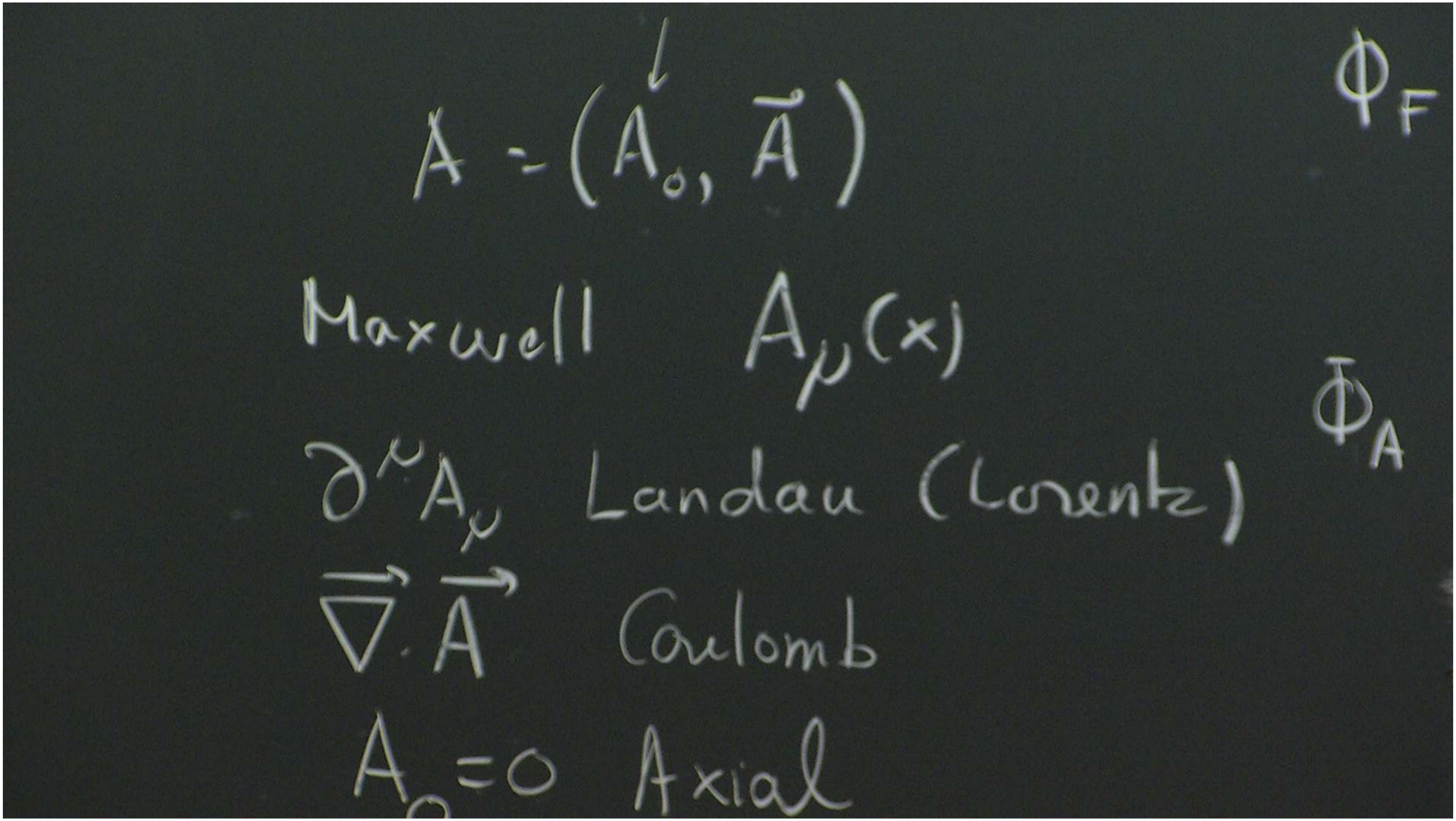
$$[A_\mu, \alpha] = [A_\mu, \alpha]^a\, t_a = A_\mu^a \cdot \alpha^b \cdot i\epsilon_{abc} \cdot t_c$$

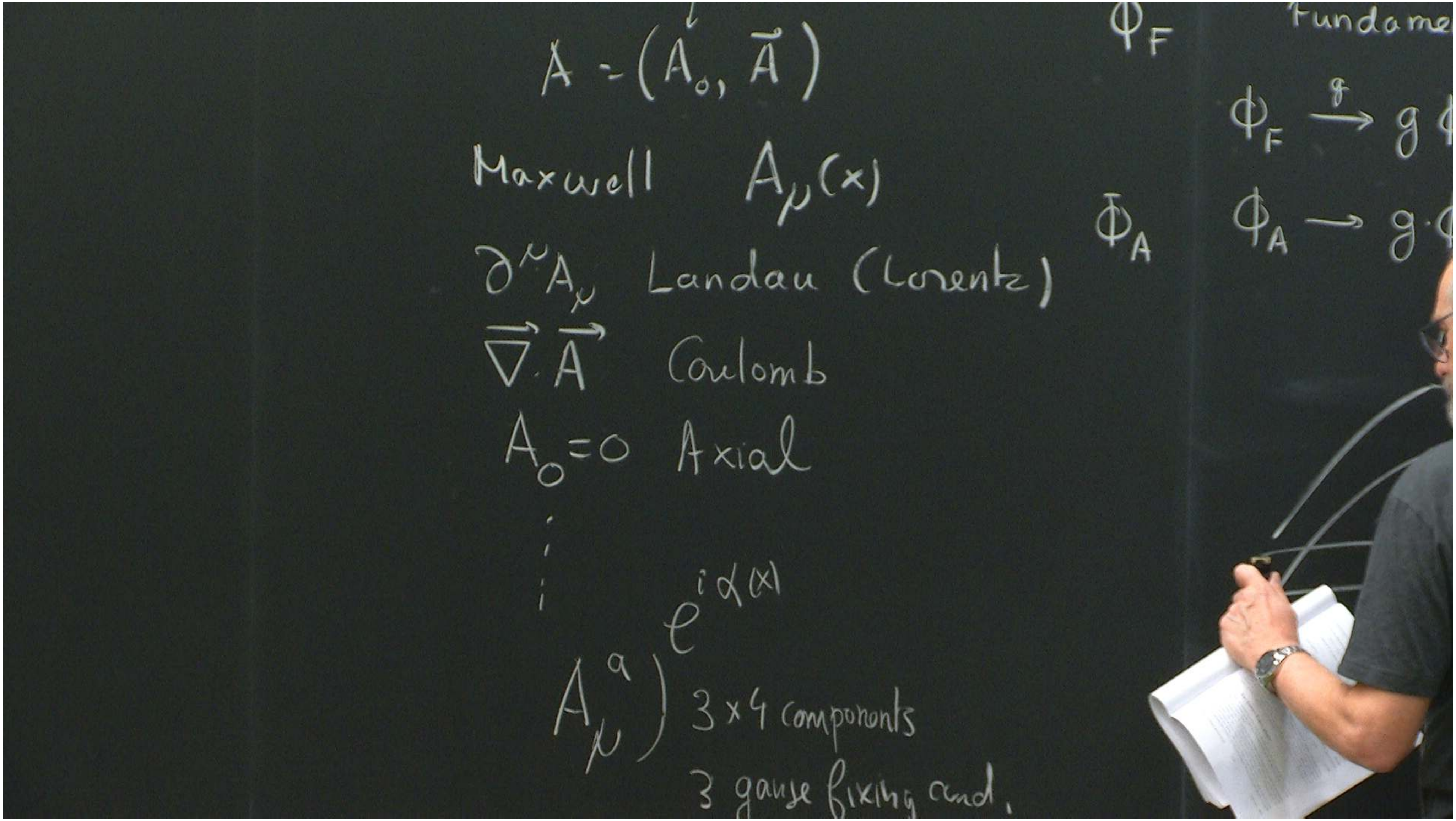
$$\partial^\mu [A_\mu(x), \alpha(x)] = \partial^\mu \left(A_\mu^a(x) \cdot \alpha^b(x)\right) \cdot i\epsilon_{abc}\, t_c$$

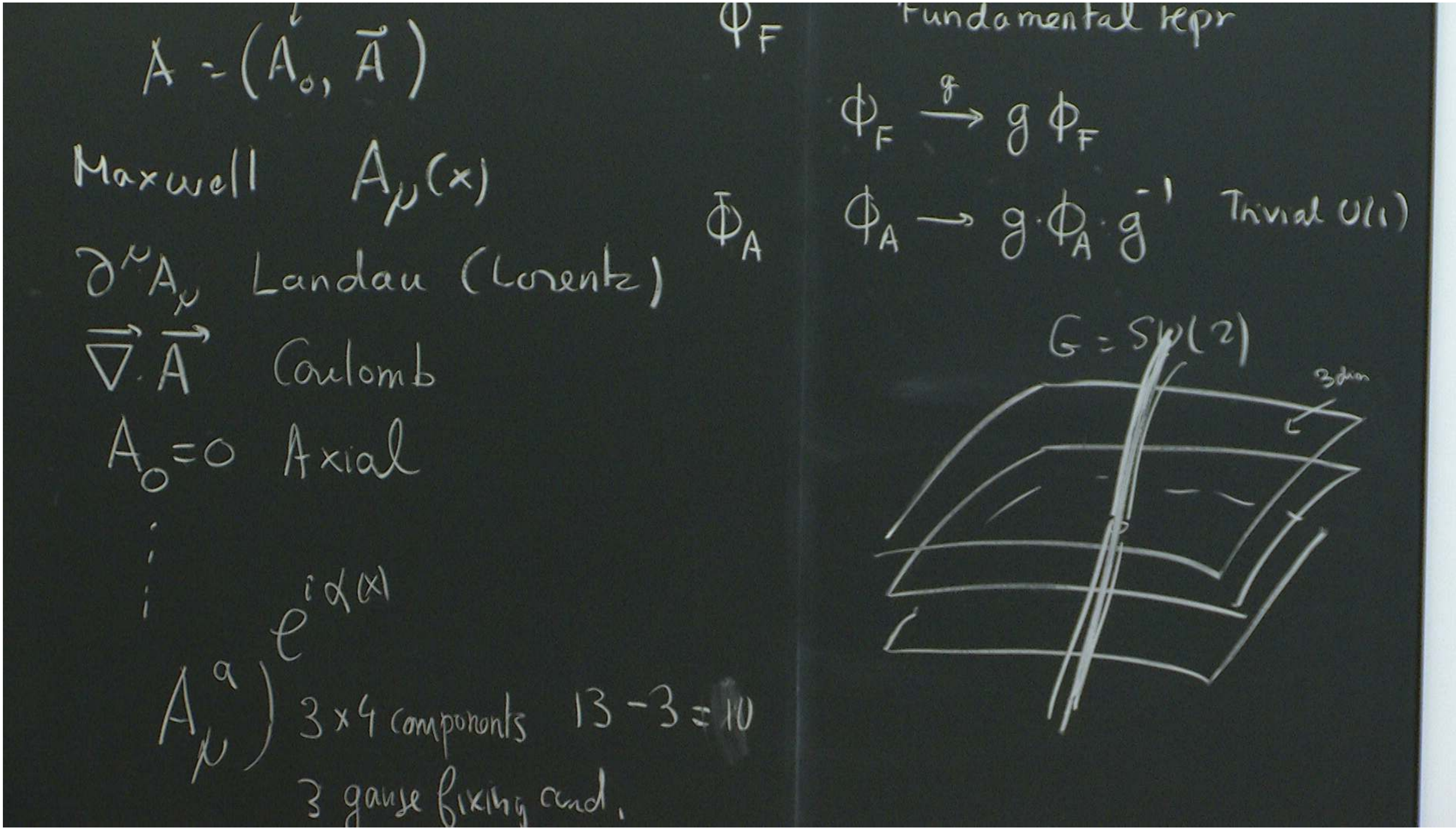
$$= \left[\partial^\mu A_\mu^a(x)\right] \alpha^b(x) + A_\mu^a(x) \cdot \left(\partial^\mu \alpha^b(x)\right)$$

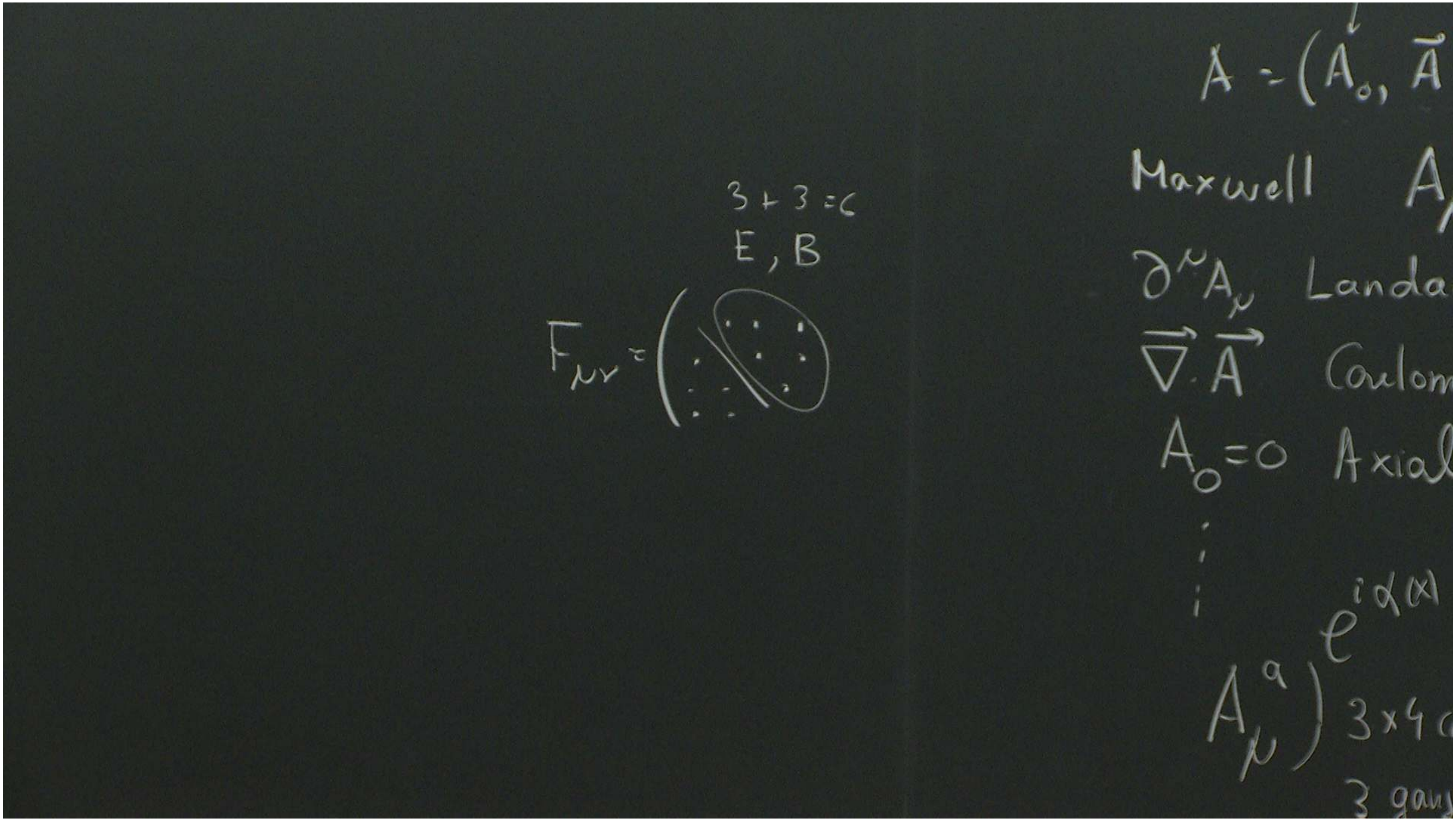

Diff operator which depend
$\int \mathcal{D}[A] = \int D[g] \int D[A]\ \delta[F[A]] \cdot |\mathrm{Det}(F'[A])|$
$[\mathrm{Vol}(G)]^{\otimes M^{1,3}}$
volume of the whole $\mathcal{G}$
independent of A
F.D. Functional Determinant
of a differential operator
which depends on A
and of F, gauge fixing choice


operator which depends on A
F'[A]) | +
Functional Determinant
differential operator
h depends on A
F . gauge fixing choice
exp(i/(2g²ħ) S[A])
ħ = 1
Dynamics of quantum theory
Observable [A]
⟨Tr(F_{μν} F^{μν})⟩


$A = (A_0, \vec{A})$
Φ_F
Maxwell $A_\mu(x)$
Φ_A
$\partial^\mu A_\mu$ Landau (Lorentz)
$\vec{\nabla} \cdot \vec{A}$ Coulomb
$A_0 = 0$ Axial


A = (A_0, A⃗)
Maxwell A_μ(x)
∂^μ A_μ Landau (Lorentz)
∇⃗·A⃗ Coulomb
A_0 = 0 Axial
e^{iα(x)}
A_μ^a) 3 × 4 components
3 gauge fixing cond.
Φ_F
Fundame
Φ_F → g Φ
Φ_A
Φ_A → g·Φ


A = (A0, A)
Maxwell Aμ(x)
∂μAμ Landau (Lorentz)
∇·A Coulomb
A0=0 Axial
e^iα(x)
Aμa) 3×4 components 13−3=10
3 gauge fixing cond.
ΦF
ΦA
Fundamental repr
ΦF → gΦF
ΦA → g·ΦA·g^-1 Trivial U(1)
G = SU(2)
3 dim


3 + 3 = 6
E, B
Maxwell
Landa
Coulom
Axial
3 gau