Title: Statistical Physics - Lecture 221128

Speakers:

Collection: Statistical Physics (2022/2023)

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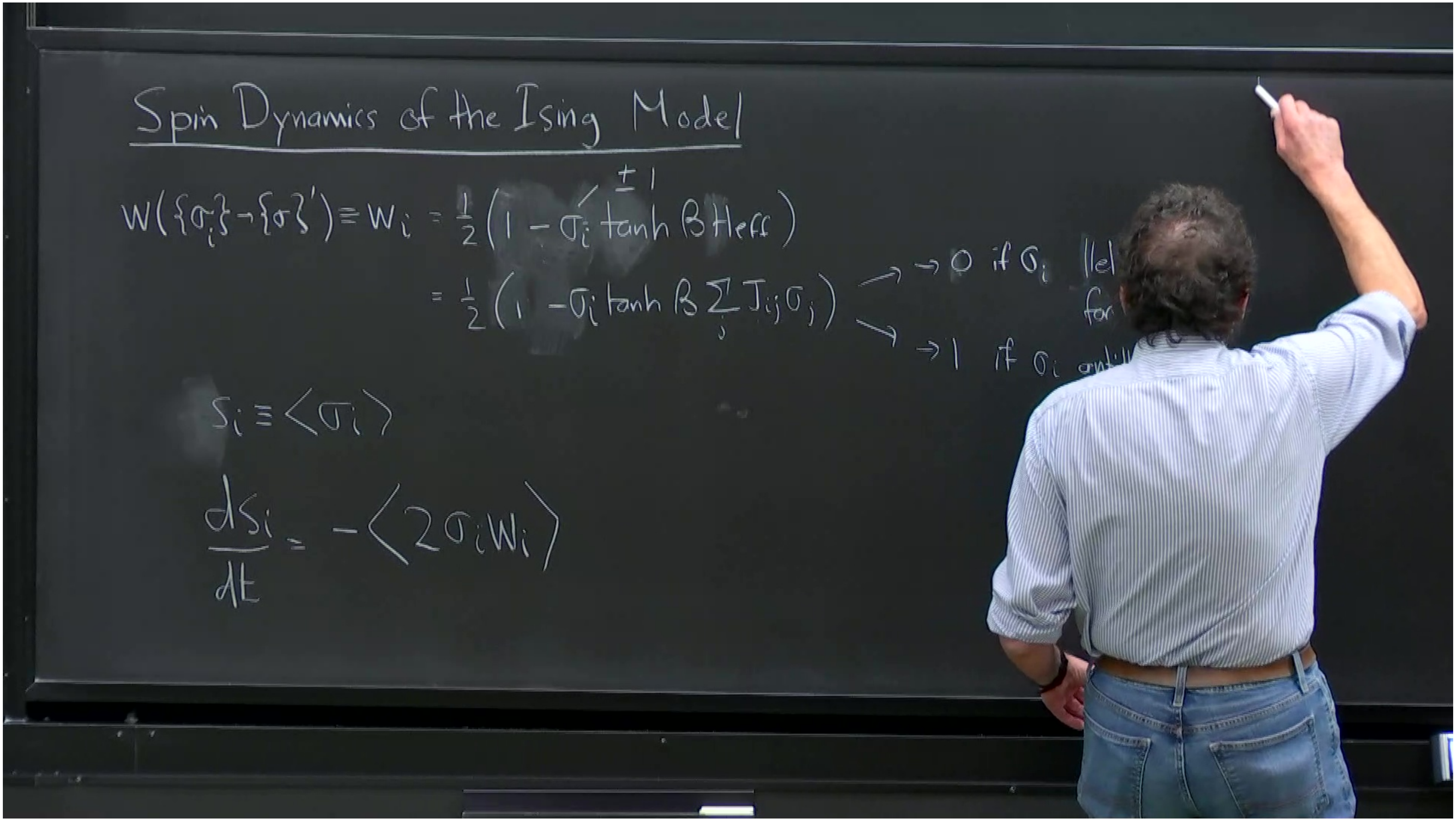
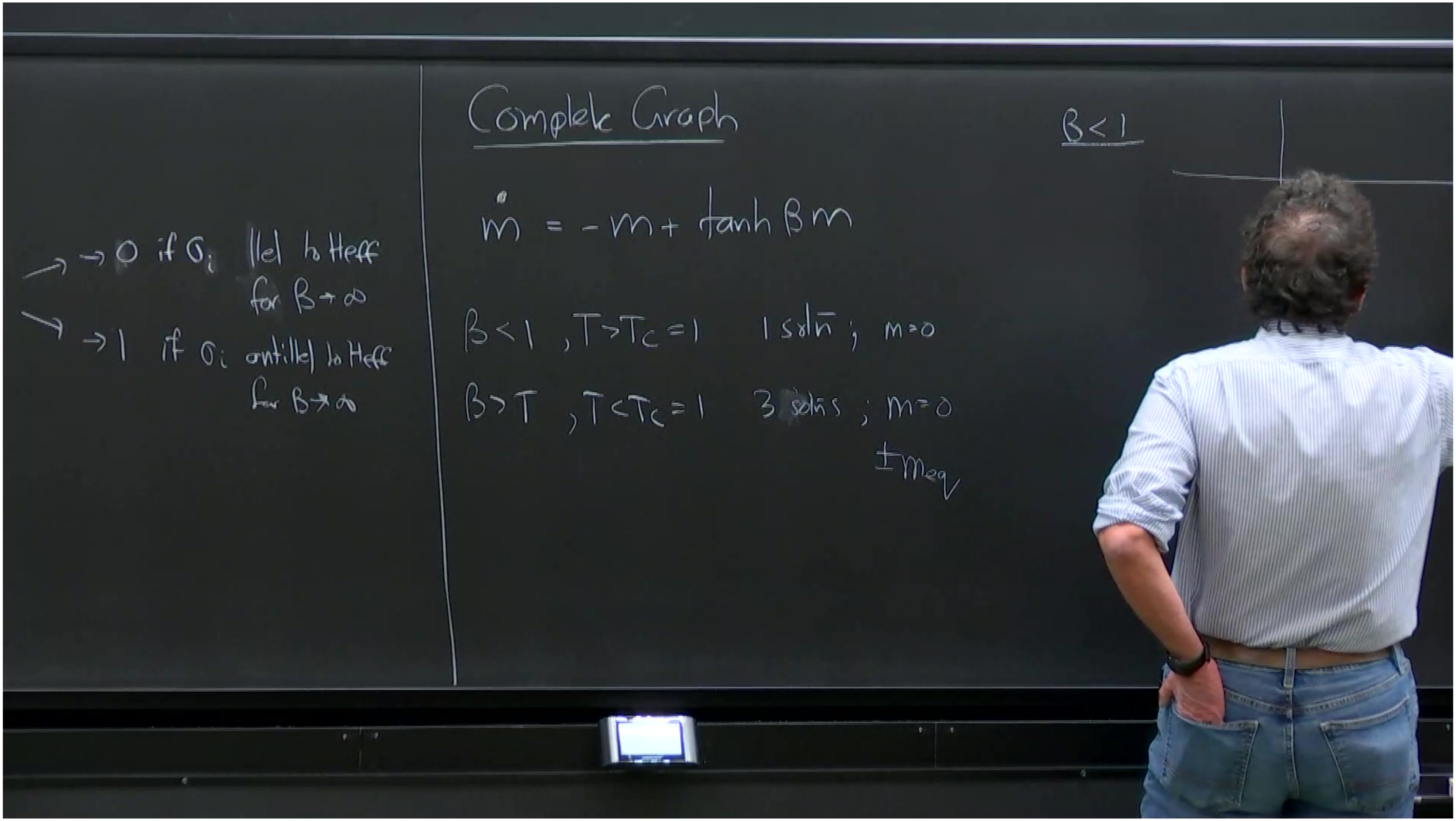

Spin Dynamics of the Ising Model
W({σ_i}→{σ}') ≡ W_i = ½(1 − σ_i tanh βH_eff)
±1
= ½(1 − σ_i tanh β Σ_j J_ij σ_j)
→ 0 if σ_i ||el
for
→ 1 if σ_i anti||
s_i ≡ ⟨σ_i⟩
ds_i/dt = −⟨2σ_i W_i⟩

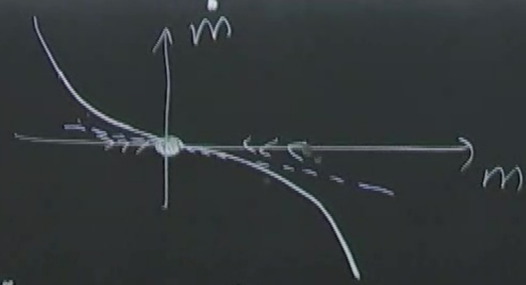

→ 0 if σi ||el to Heff
for β → ∞
→ 1 if σi antillel to Heff
for β → ∞
Complete Graph
ṁ = − m + tanh βm
β < 1 , T > Tc = 1   1 soln ; m = 0
β > 1 , T < Tc = 1   3 solns ; m = 0
±meq
β < 1

$m=0$

$; \; m=0$

$\pm m_{eq}$

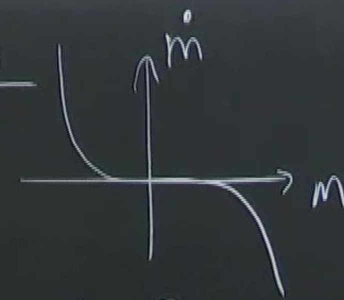
$\underline{\beta<1}$



$$\dot{m} \simeq -m + \beta m$$
$$= -m(1-\beta)$$

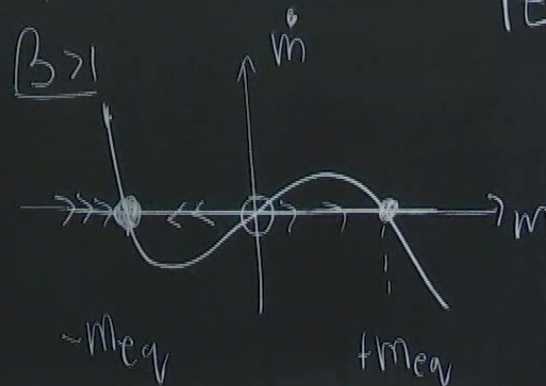
$$\rightarrow m(t) \simeq e^{-t/\tau} \quad ; \tau = \frac{1}{1-\beta}$$

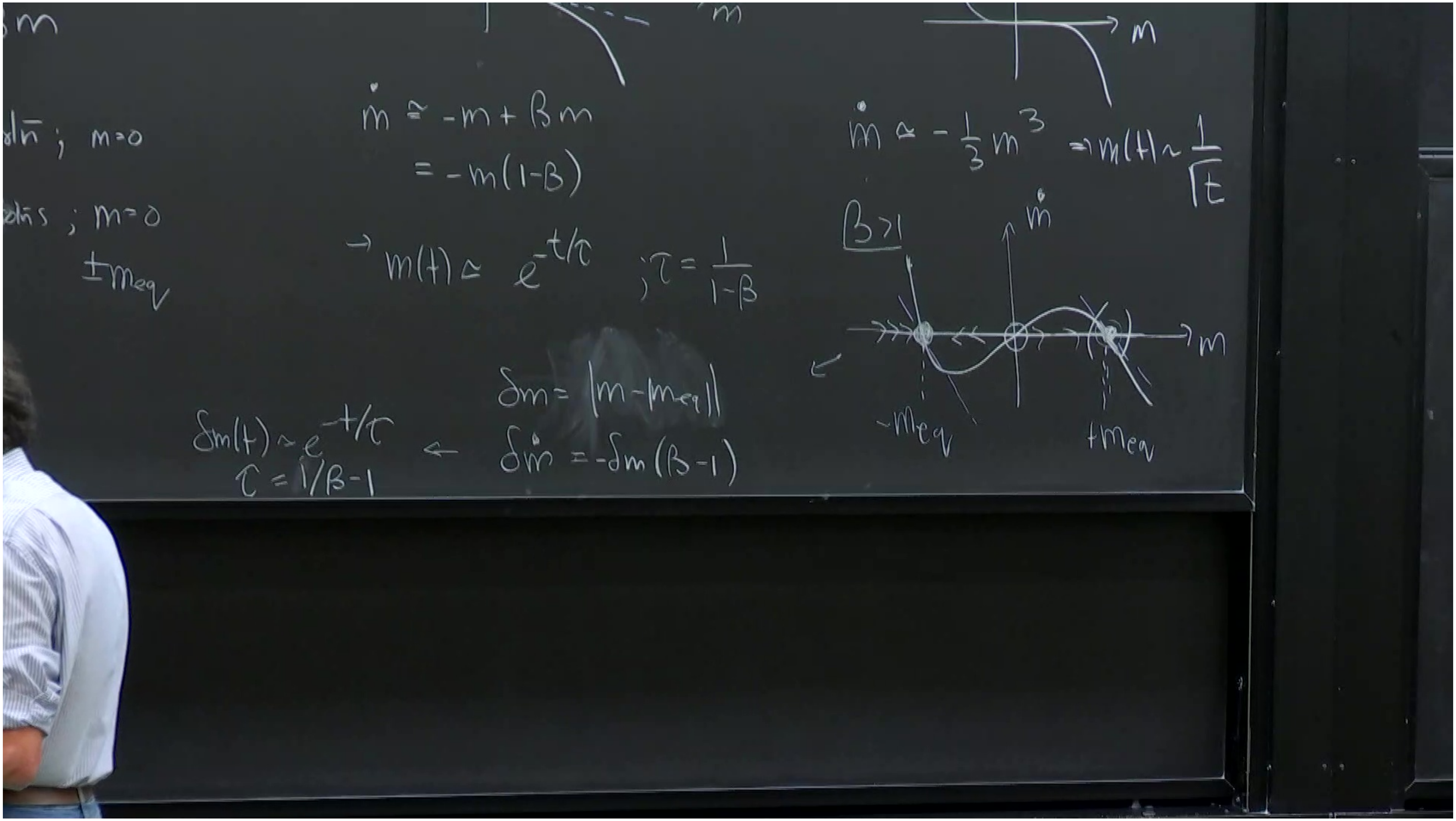
$\underline{\beta=1}$

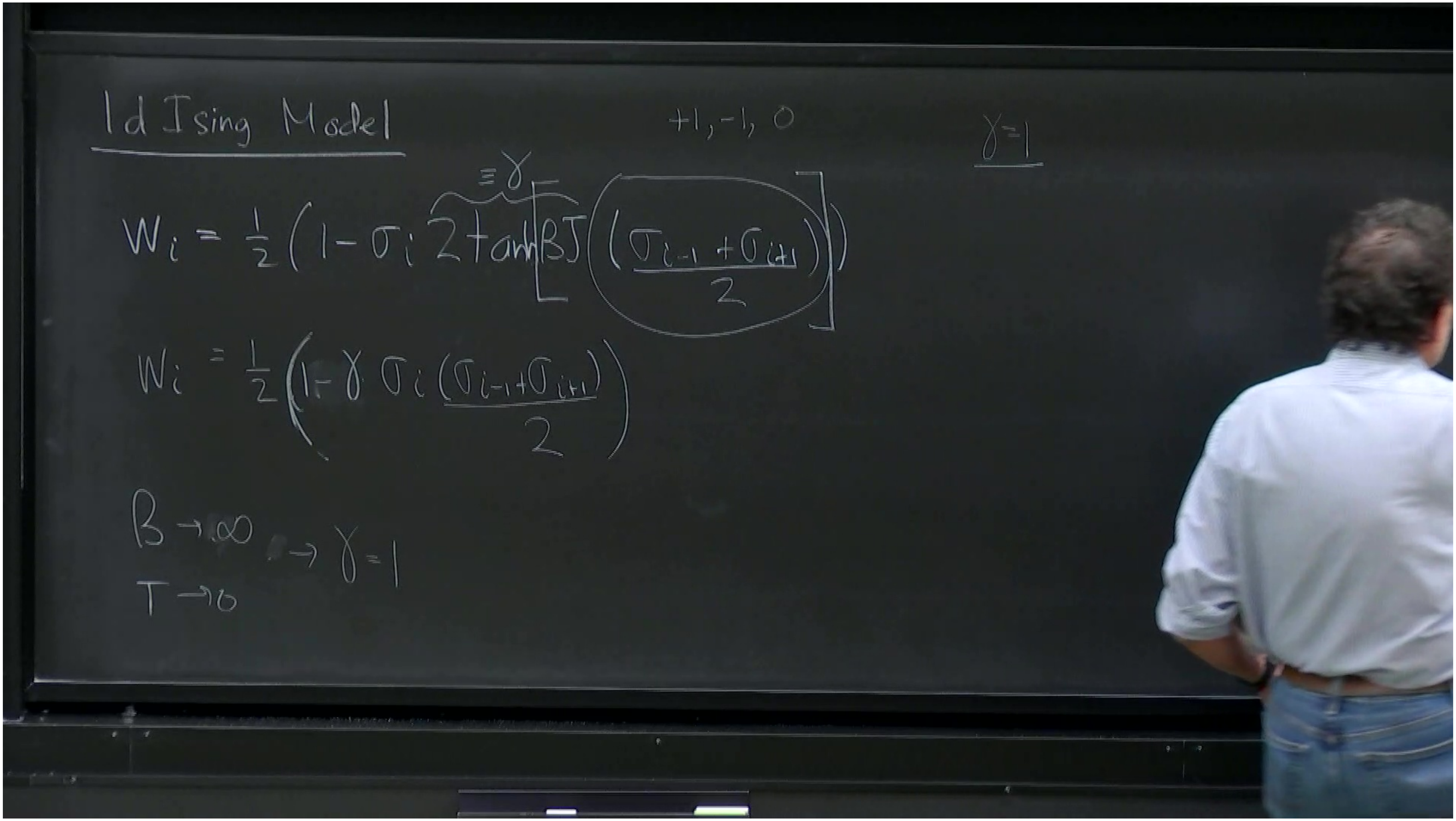


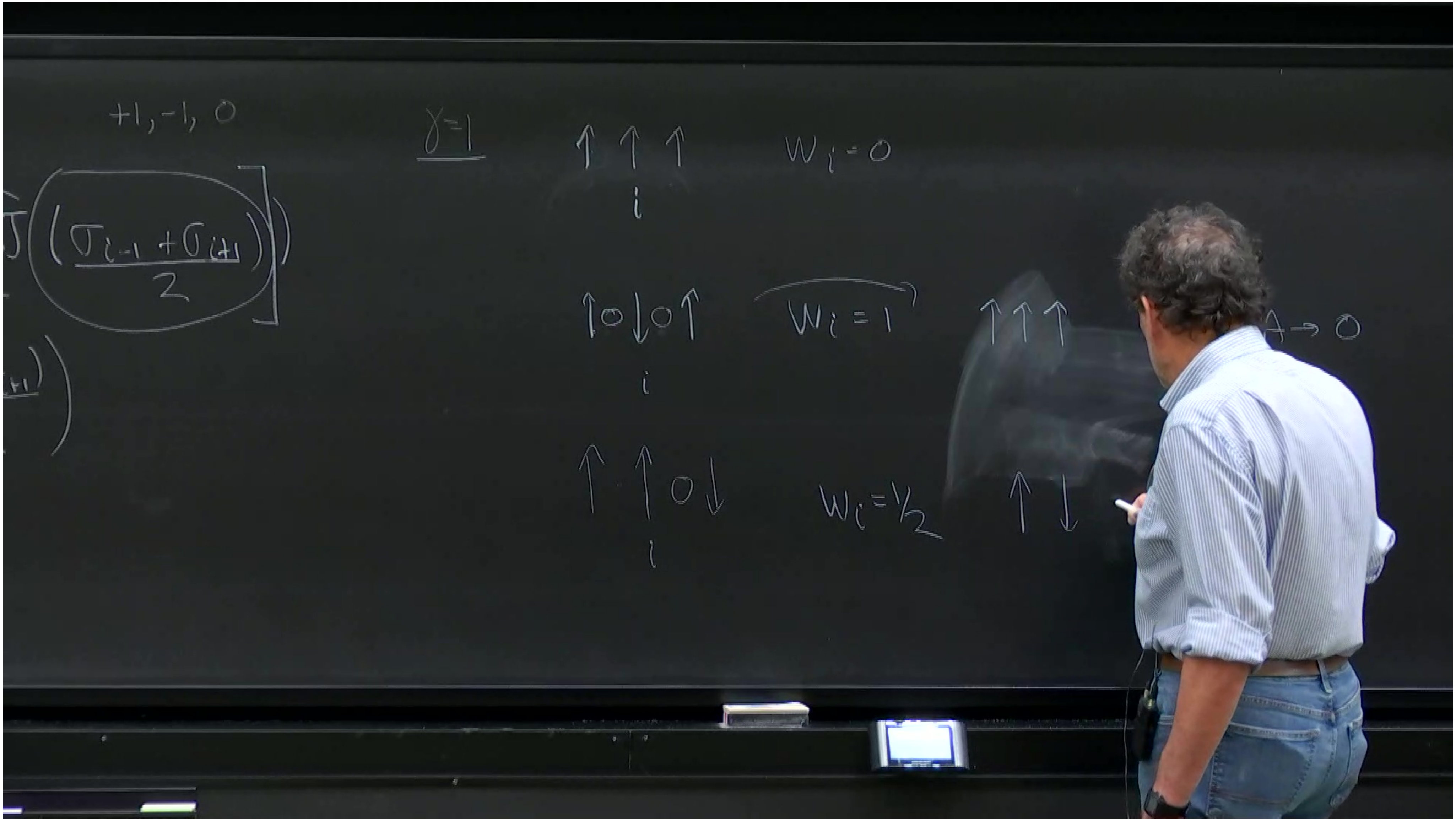
$$\dot{m} \simeq -\frac{1}{3} m^3 \Rightarrow m(t) \sim \frac{1}{\sqrt{t}}$$

$\underline{\beta>1}$




m
m = 0
m = 0
± m_eq
ṁ ≃ -m + βm
= -m(1-β)
→ m(t) ≃ e^{-t/τ} ; τ = 1/(1-β)
δm = |m - m_eq|
δṁ = -δm (β-1)
δm(t) ~ e^{-t/τ}
τ = 1/β-1
ṁ ≃ -1/3 m^3 ⇒ m(t) ~ 1/√t
β>1
ṁ
m
-m_eq
+m_eq


1d Ising Model
+1, -1, 0
$\gamma = 1$
$\equiv \gamma$
$W_i = \frac{1}{2}\left(1 - \sigma_i \tanh\left[\beta J\left(\frac{\sigma_{i-1} + \sigma_{i+1}}{2}\right)\right]\right)$
$W_i = \frac{1}{2}\left(1 - \gamma\, \sigma_i \frac{(\sigma_{i-1} + \sigma_{i+1})}{2}\right)$
$\beta \to \infty \to \gamma = 1$
$T \to 0$


+1, -1, 0
γ=1
W_i = 0
W_i = 1
W_i = 1/2

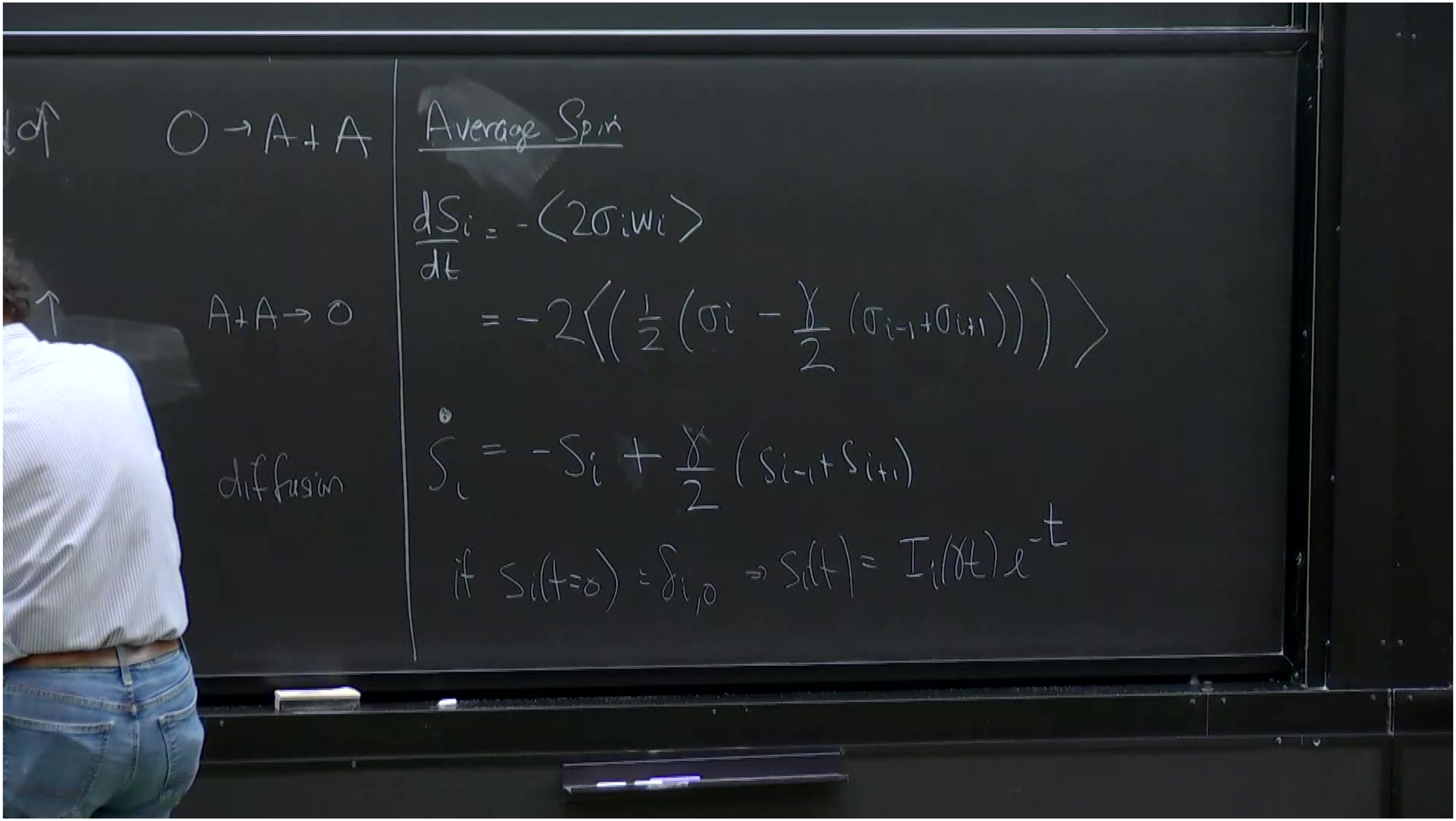
$\cancel{\gamma > 1}$

$\gamma < 1$

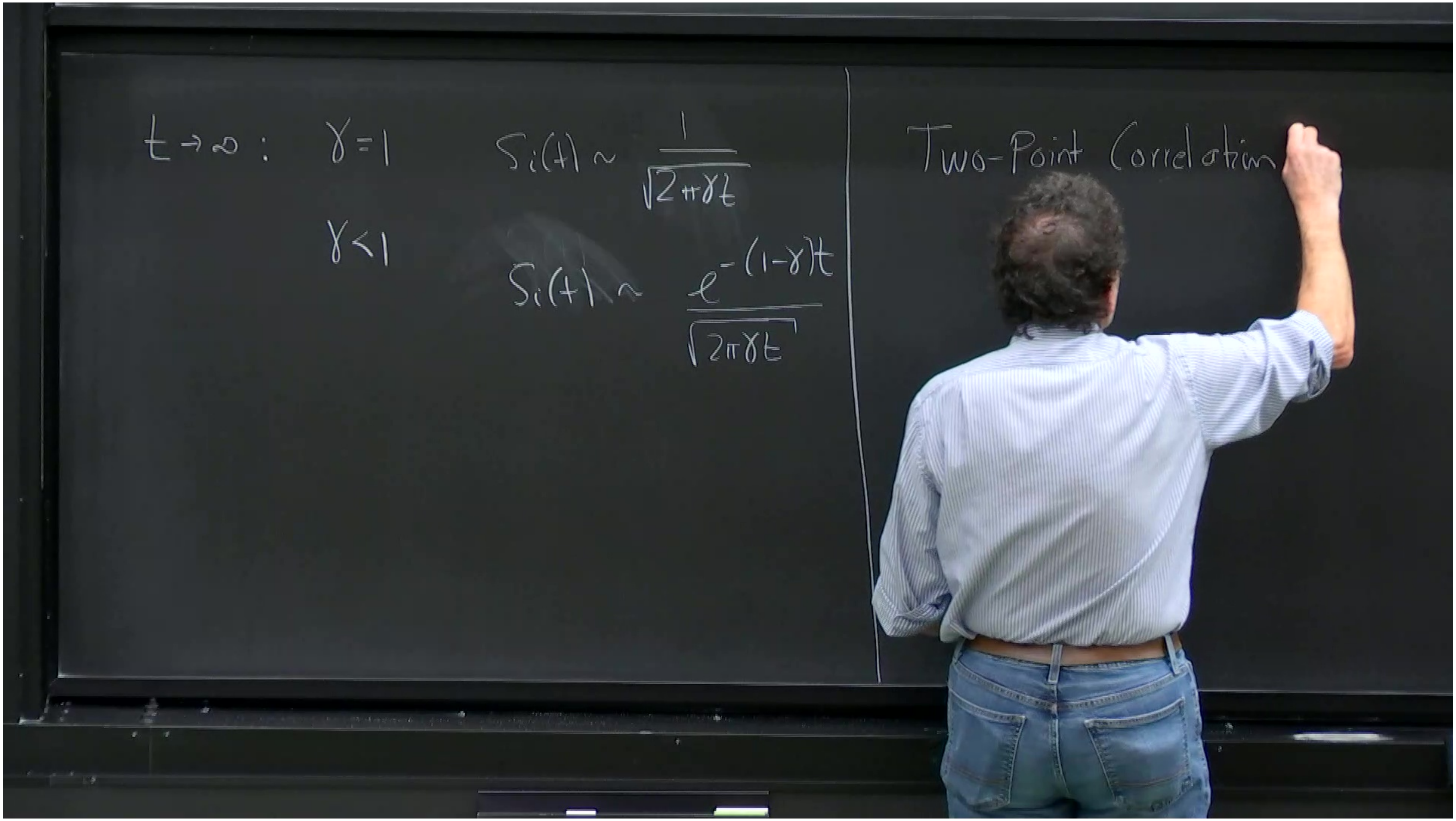
$\underset{i}{\uparrow\uparrow\uparrow}$ $\quad w_i \gtrsim 0 \quad$ $\uparrow 0 \downarrow 0 \uparrow$ $\quad 0 \to A + A$

$\underset{i}{\uparrow 0 \downarrow 0 \uparrow}$ $\quad w_i = 1 \quad$ $\uparrow\uparrow\uparrow$ $\quad A + A \to 0$

$\underset{i}{\uparrow\uparrow 0 \downarrow}$ $\quad w_i = 1/2 \quad$ $\uparrow 0 \downarrow \downarrow$ $\quad$ diffusion



$O \to A + A$
$A + A \to O$
diffusion
Average Spin
$\frac{dS_i}{dt} = -\langle 2\sigma_i w_i \rangle$
$= -2\left\langle \left(\frac{1}{2}\left(\sigma_i - \frac{\gamma}{2}(\sigma_{i-1} + \sigma_{i+1}) \right) \right) \right\rangle$
$\dot{S}_i = -S_i + \frac{\gamma}{2}(S_{i-1} + S_{i+1})$
if $S_i(t=0) = \delta_{i,0} \Rightarrow S_i(t) = I_i(\gamma t)\, e^{-t}$


$t \to \infty$: $\gamma = 1$ $S_i(t) \sim \frac{1}{\sqrt{2\pi\gamma t}}$
$\gamma < 1$ $S_i(t) \sim \frac{e^{-(1-\gamma)t}}{\sqrt{2\pi\gamma t}}$
Two-Point Correlation

Two-Point Correlation Fn

$$\boxed{\rho = \frac{1+G_1}{2}}$$

$G_k = \langle \sigma_i \sigma_j \rangle \qquad |i-j| = k$

$$G_1 = \langle \sigma_i \sigma_{i+1} \rangle = \text{prob}(\uparrow\uparrow \text{ or } \downarrow\downarrow) \times (+1) + \text{prob}(\uparrow\downarrow \text{ or } \downarrow\uparrow) \times (-1)$$

$\rho \equiv$ density of domain walls

$$= (1-\rho)(+1) + \rho(-1)$$

$$= 1 - 2\rho$$

Two-Spin Correlation Fn

$$G_k = \langle \sigma_i \sigma_j \rangle \qquad |i-j| = k$$

$$G_1 = \langle \sigma_i \sigma_{i+1} \rangle = \text{prob}(\uparrow\uparrow \text{ or } \downarrow\downarrow) \times (+1)$$

$$+ \text{prob}(\uparrow \circ \downarrow \text{ or } \downarrow \circ \uparrow) \times (-1)$$

$\rho \equiv$ density of domain walls

$$= (1-\rho)(+1) + \rho(-1)$$

$$= 1 - 2\rho$$

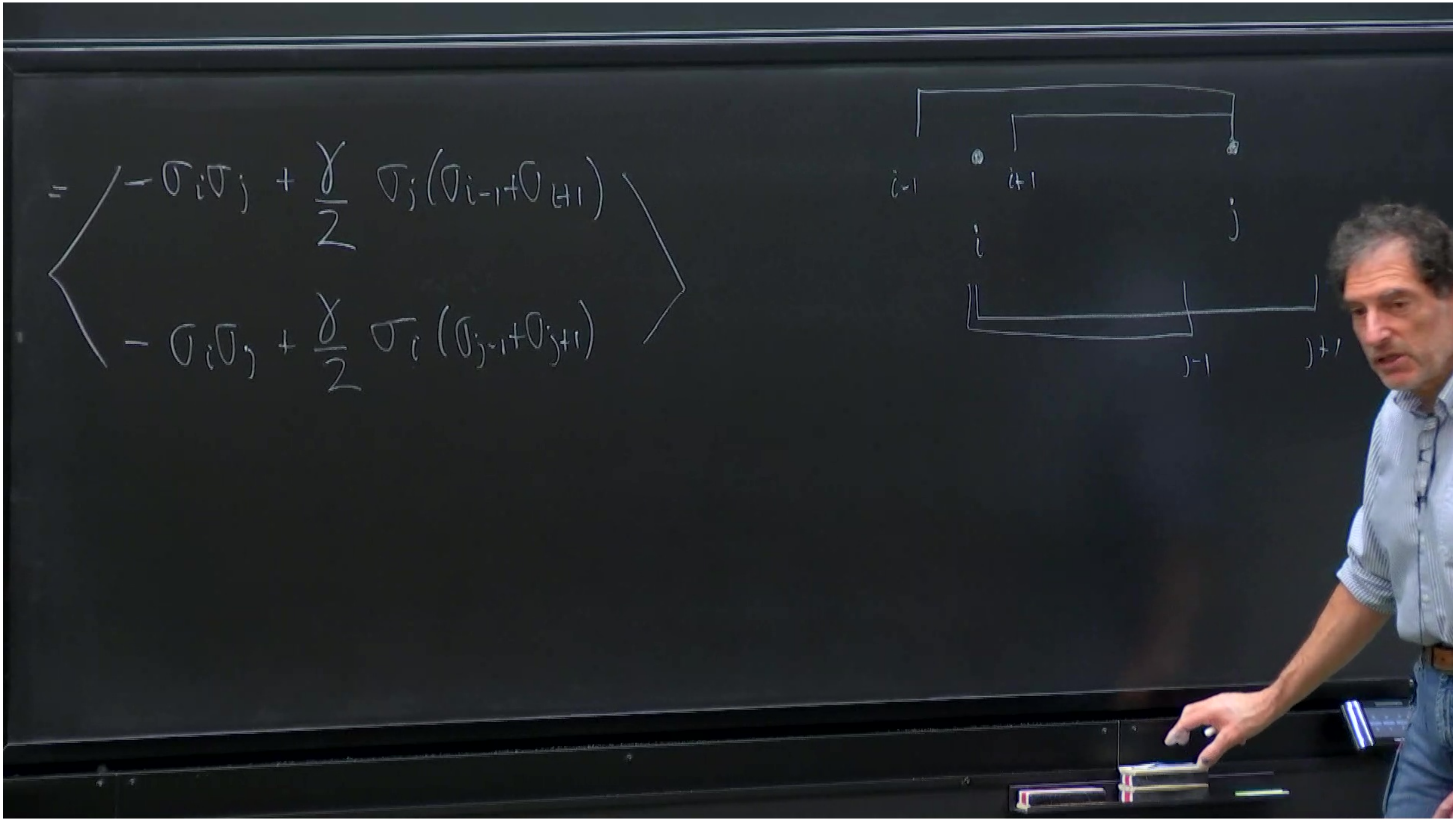
$$\boxed{\rho = \frac{1+G_1}{2}}$$

$$\frac{\partial G_k}{dt} = -\langle 2\, \sigma_i \sigma_j \, (W$$

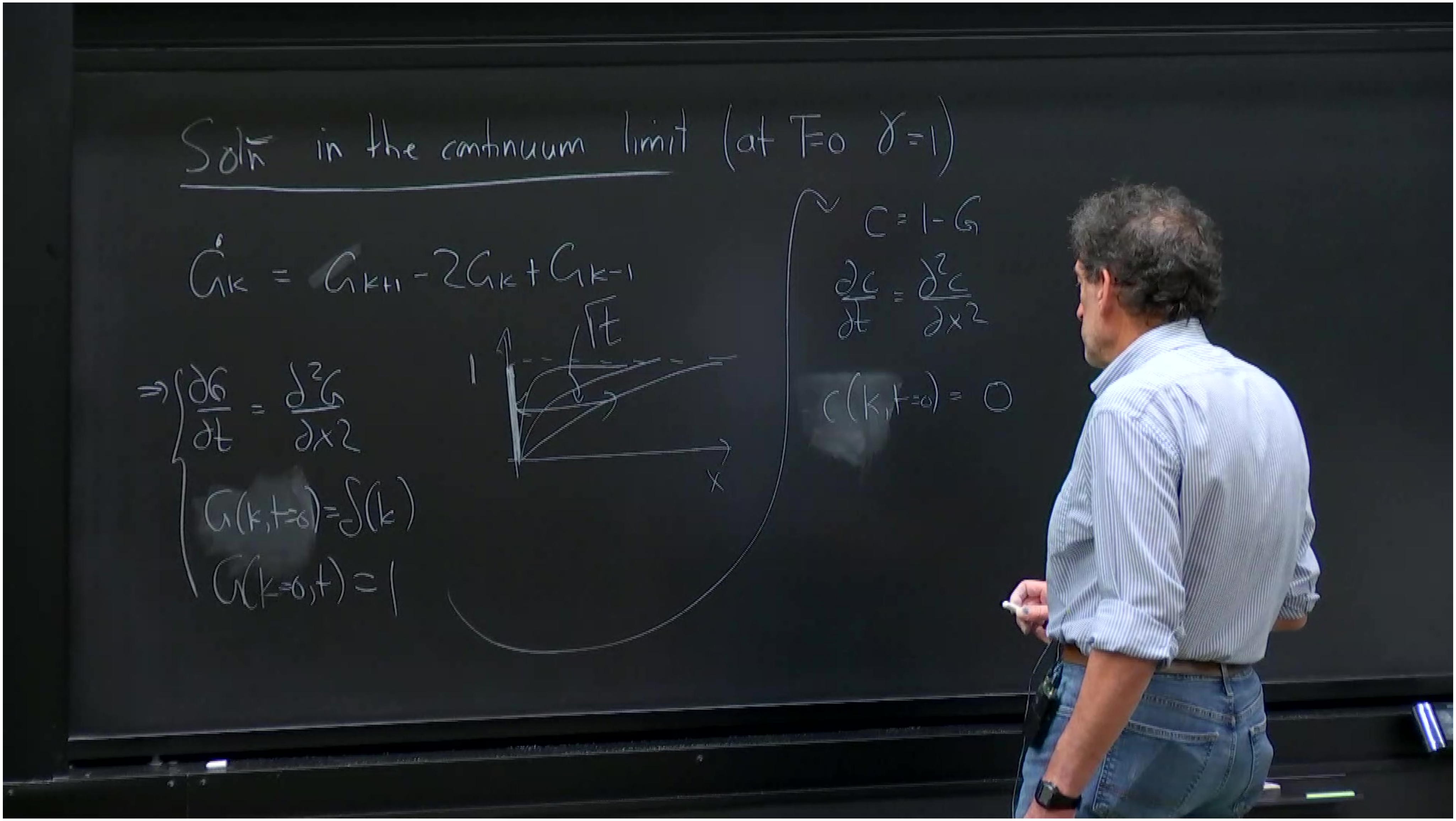
$$\boxed{\rho = \frac{1+G_1}{2}}$$

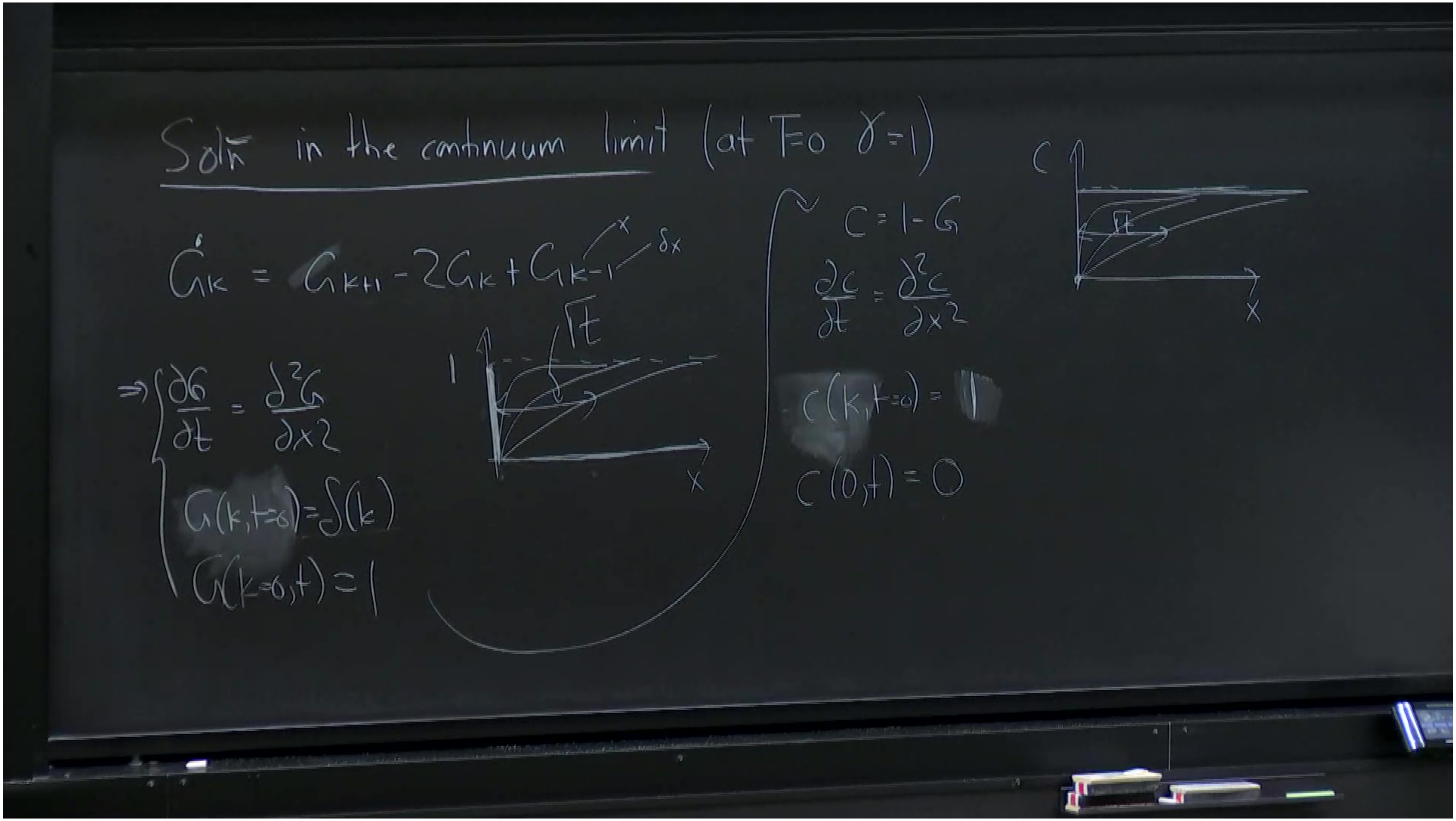
$$\frac{\partial G_k}{dt} = -\left\langle 2\,\sigma_i\sigma_j\,(w_i + w_j)\right\rangle$$

$$= -\left\langle 2\sigma_i\sigma_j \left[\frac{1}{2}\left(1 - \frac{\gamma}{2}\sigma_i(\sigma_{i-1}+\sigma_{i+1})\right) + \frac{1}{2}\left(1 - \frac{\gamma}{2}\sigma_j(\sigma_{j-1}+\sigma_{j+1})\right)\right]\right\rangle$$

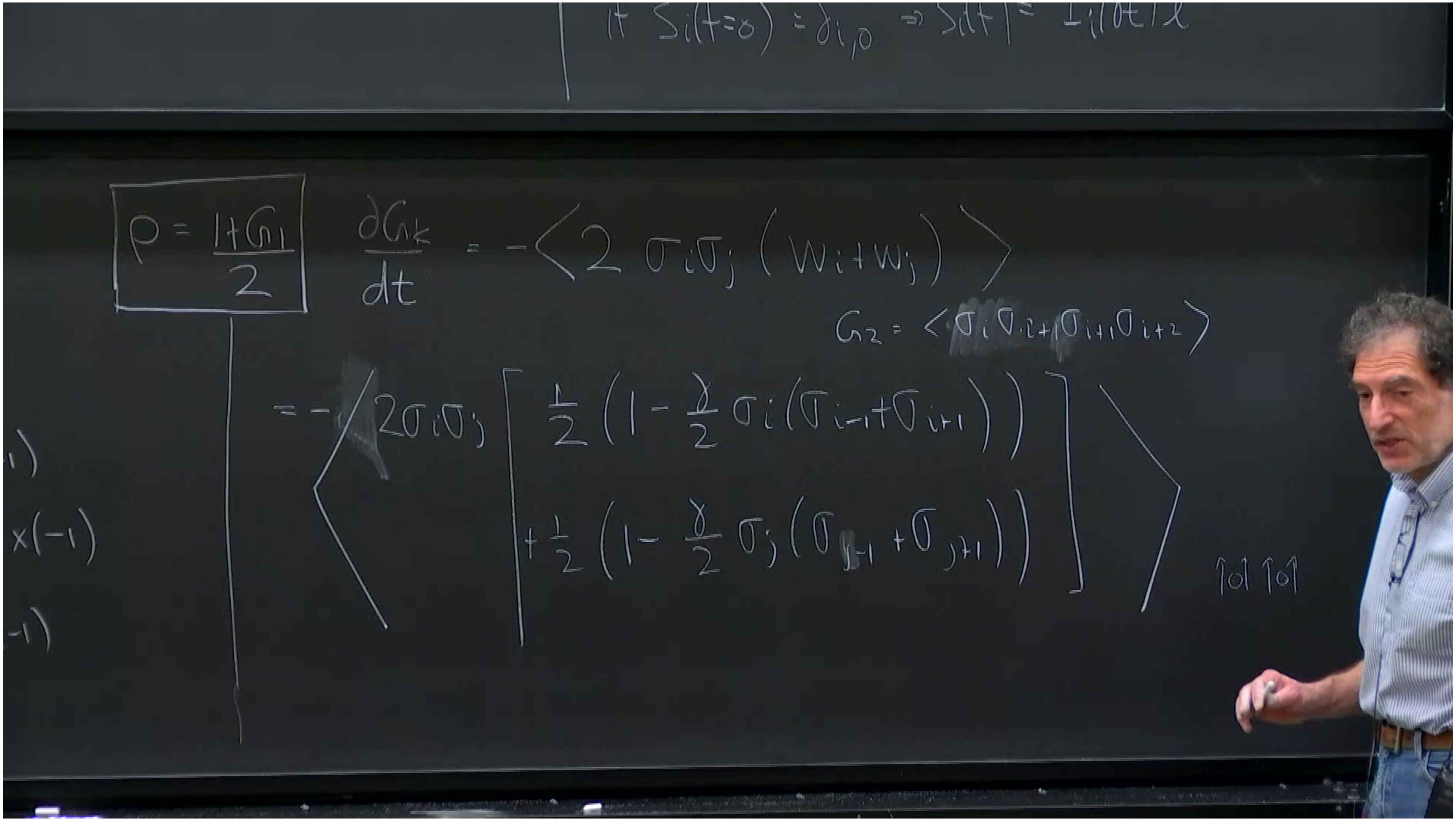


$= \left\langle -\sigma_i\sigma_j + \frac{\gamma}{2}\sigma_j(\sigma_{i-1}+\sigma_{i+1}) \right.$
$\left. -\sigma_i\sigma_j + \frac{\gamma}{2}\sigma_i(\sigma_{j-1}+\sigma_{j+1}) \right\rangle$
i-1
i+1
i
j
j-1
j+1


Soln in the continuum limit (at T=0 γ=1)
Ġk = Gk+1 − 2Gk + Gk−1
∂G/∂t = ∂²G/∂x²
G(k,t=0) = δ(k)
G(k=0,t) = 1
√t
1
x
C = 1 − G
∂c/∂t = ∂²c/∂x²
c(k,t=0) = 0



Soln in the continuum limit (at T=0 γ=1)
$\dot{G}_k = G_{k+1} - 2G_k + G_{k-1}$
x
δx
$\Rightarrow \begin{cases} \frac{\partial G}{\partial t} = \frac{\partial^2 G}{\partial x^2} \\ G(k, t=0) = \delta(k) \\ G(k=0, t) = 1 \end{cases}$
1
$\sqrt{t}$
x
$C = 1 - G$
$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}$
$c(k, t=0) = 1$
$c(0, t) = 0$
C
$\sqrt{t}$
x


$\rho = \frac{1+G_1}{2}$
$\frac{\partial G_k}{dt} = -\langle 2\sigma_i\sigma_j(w_i+w_j)\rangle$
$G_2 = \langle \sigma_i\sigma_{i+1}\sigma_{i+1}\sigma_{i+2}\rangle$
$= -\langle 2\sigma_i\sigma_j\left[\frac{1}{2}\left(1-\frac{\gamma}{2}\sigma_i(\sigma_{i-1}+\sigma_{i+1})\right) + \frac{1}{2}\left(1-\frac{\gamma}{2}\sigma_j(\sigma_{j-1}+\sigma_{j+1})\right)\right]\rangle$

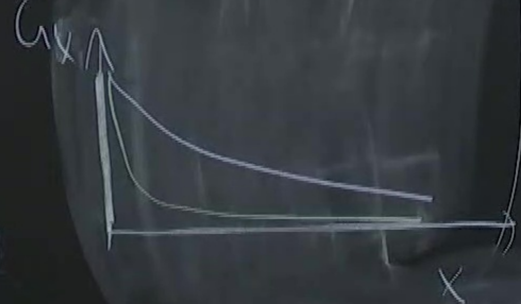
Soln in the continuum limit (at $T=0$ $\gamma=1$)

$$\dot{G}_k = G_{k+1} - 2G_k + G_{k-1}$$

x

δx

$$\Rightarrow \begin{cases} \dfrac{\partial G}{\partial t} = \dfrac{\partial^2 G}{\partial x^2} \\ G(k, t=0) = \delta(k) \\ G(k=0, t) = 1 \end{cases}$$

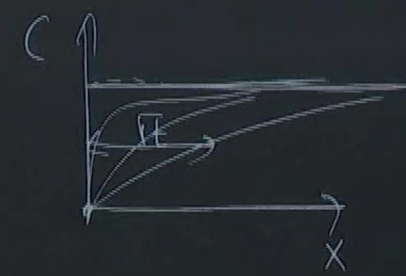


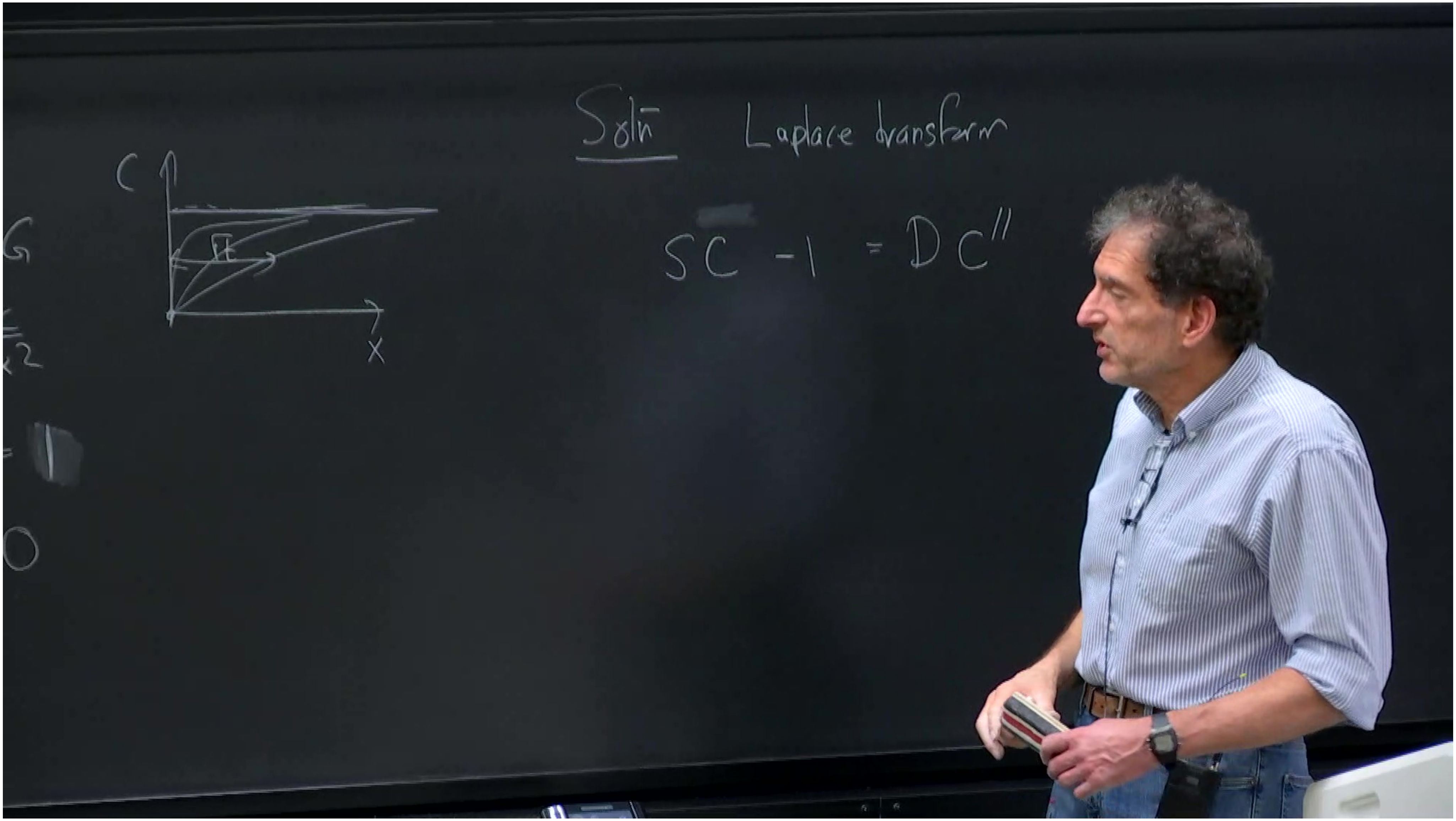
$$C = 1 - G$$

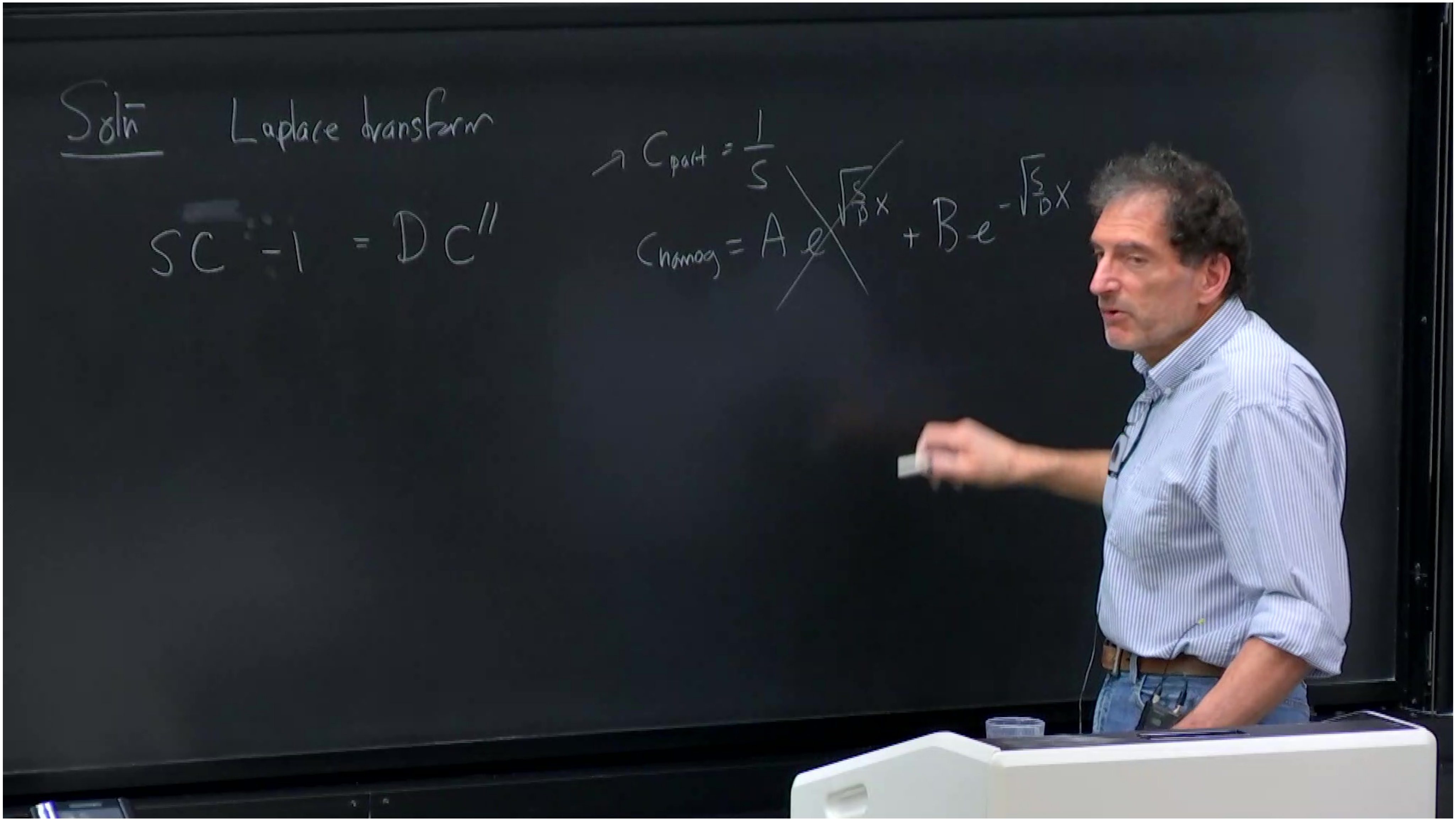
$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial x^2}$$

$$c(k, t=0) = 1$$

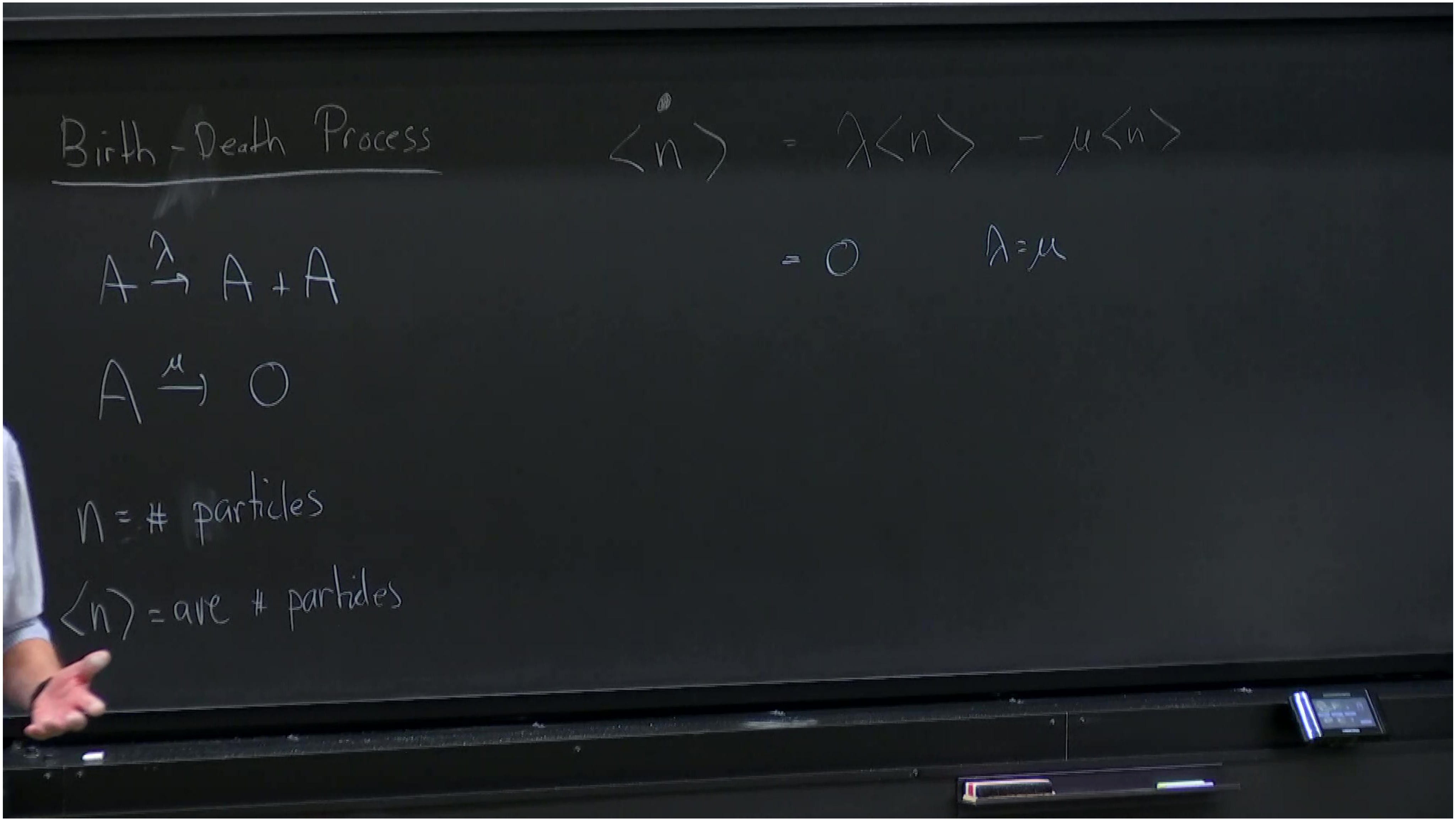
$$c(0, t) = 0$$

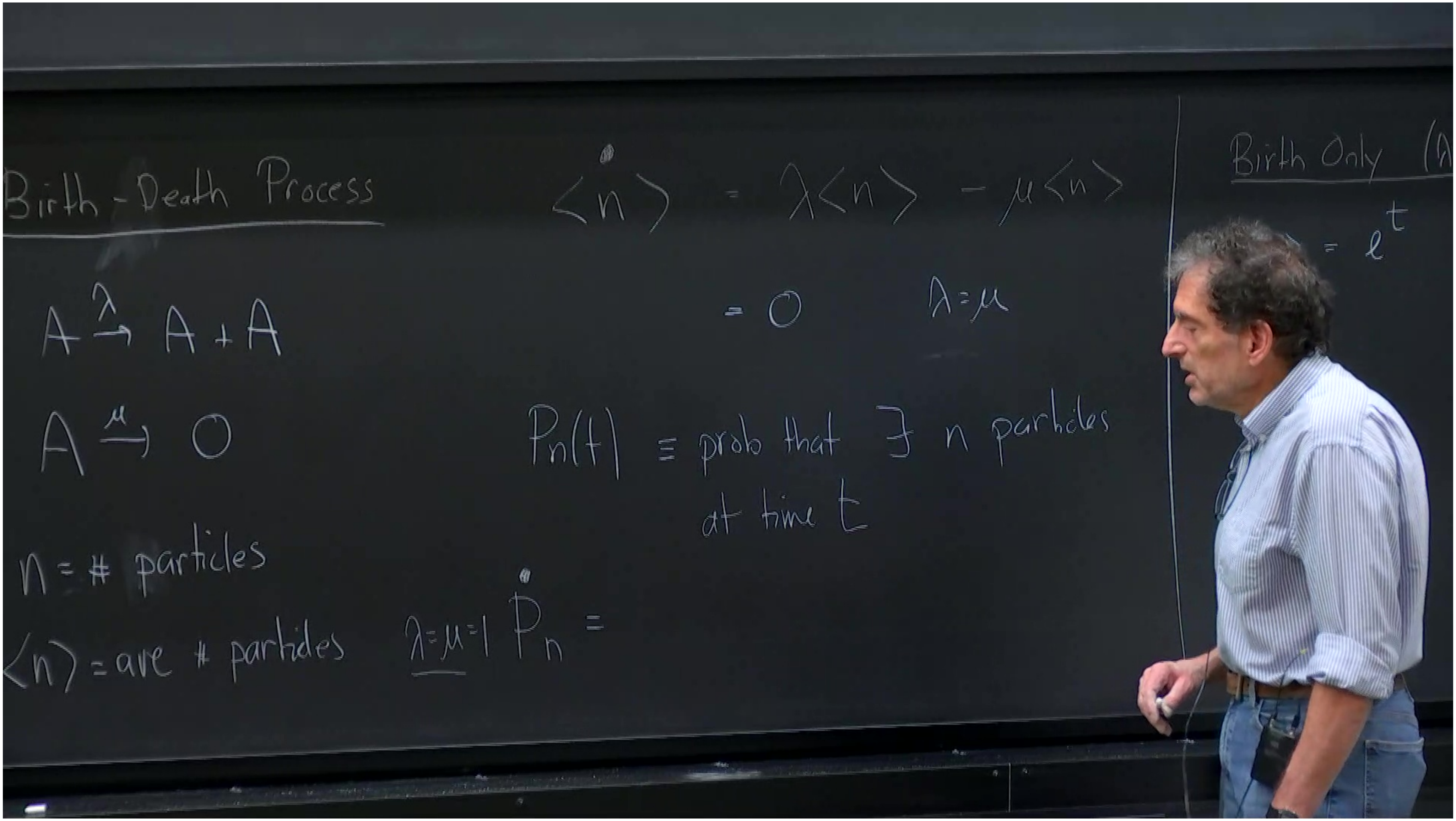


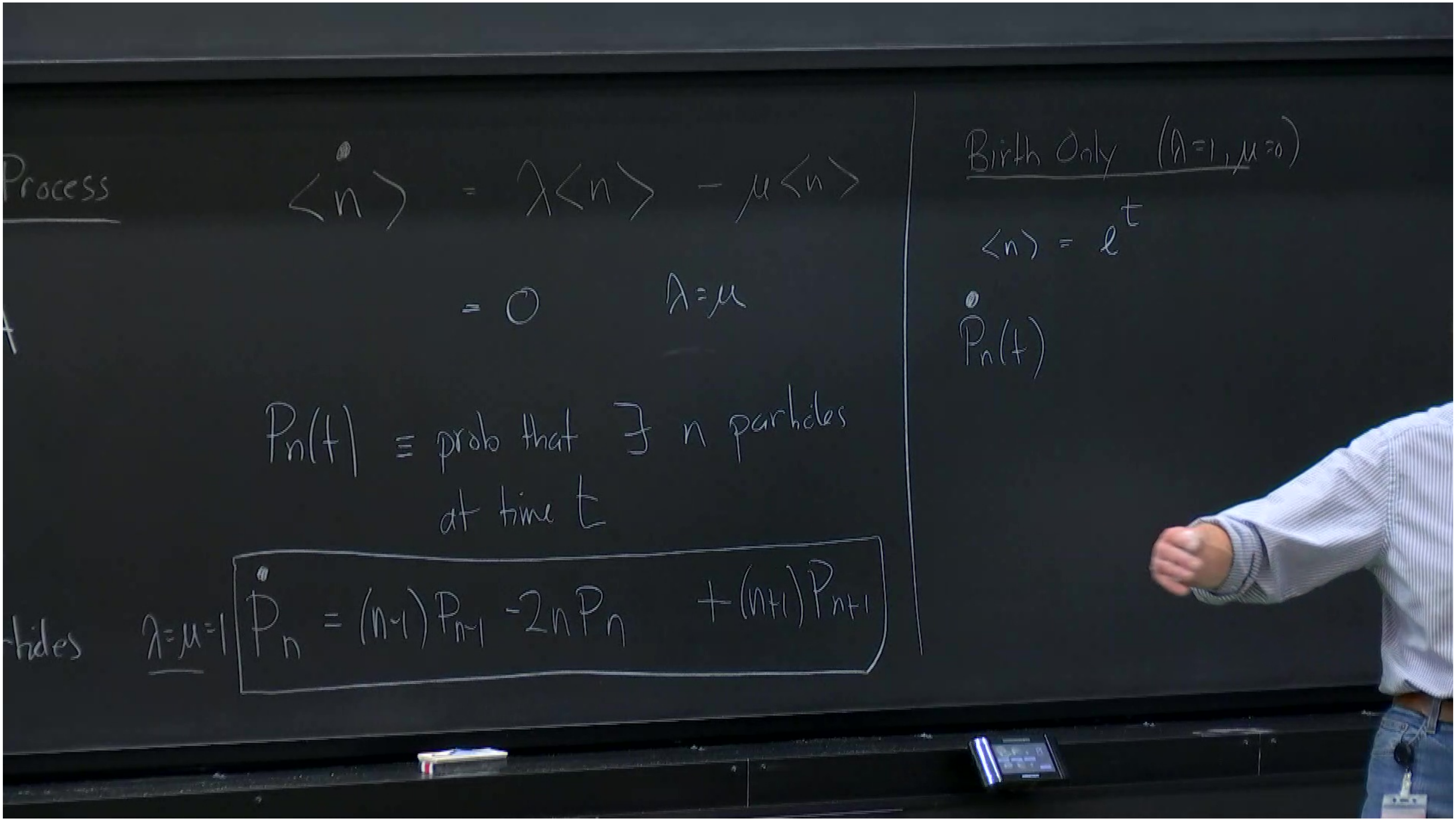

Soln Laplace transform
sC − 1 = DC''
C
x



Sol$^{\underline{n}}$ Laplace transform
$sC - 1 = DC''$
$C_{part} = \frac{1}{s}$
$C_{homog} = \cancel{A e^{\sqrt{\frac{s}{D}} x}} + B e^{-\sqrt{\frac{s}{D}} x}$


Birth - Death Process
$A \xrightarrow{\lambda} A + A$
$A \xrightarrow{\mu} 0$
n = # particles
$\langle n \rangle$ = ave # particles
$\dot{\langle n \rangle} = \lambda \langle n \rangle - \mu \langle n \rangle$
= 0
$\lambda = \mu$


Birth - Death Process
$\langle \dot{n} \rangle = \lambda \langle n \rangle - \mu \langle n \rangle$
$= 0$
$\lambda = \mu$
$A \xrightarrow{\lambda} A + A$
$A \xrightarrow{\mu} 0$
$P_n(t) \equiv$ prob that $\exists$ n particles at time t
n = # particles
$\langle n \rangle$ = ave # particles
$\lambda = \mu = 1$ $\dot{P}_n =$
Birth Only
$= e^t$


Process
$\dot{\langle n \rangle} = \lambda \langle n \rangle - \mu \langle n \rangle$
$= 0 \quad \lambda = \mu$
$P_n(t) \equiv$ prob that $\exists$ n particles at time t
rticles
$\lambda = \mu = 1$
$\dot{P}_n = (n-1)P_{n-1} - 2nP_n + (n+1)P_{n+1}$
Birth Only ($\lambda = 1, \mu = 0$)
$\langle n \rangle = e^t$
$\dot{P}_n(t)$

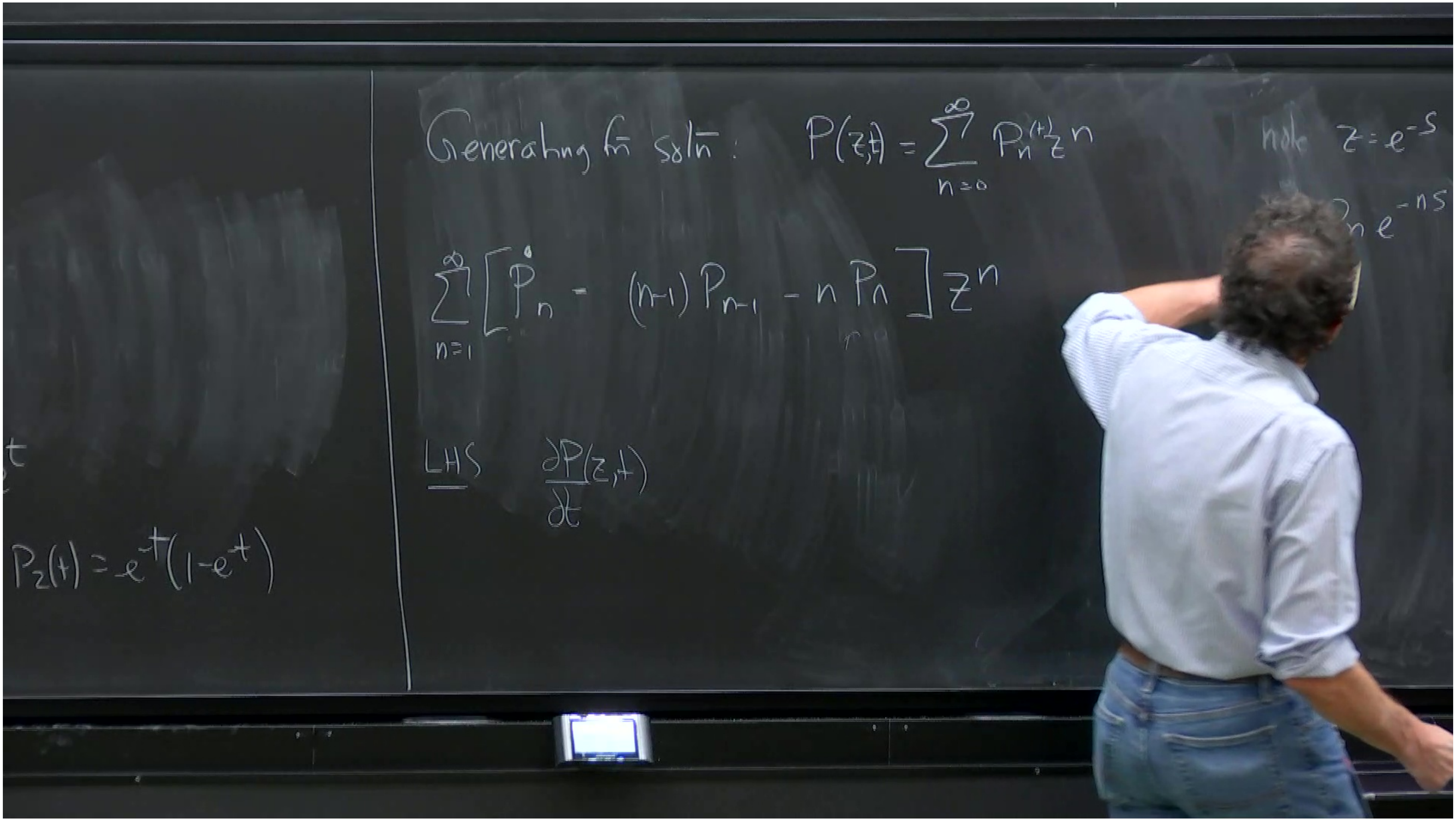
Solve sequentially

$$\dot{P}_1 = -P_1 \quad \to P_1(t) = P_1(0)\, e^{-t} = e^{-t}$$

$$\dot{P}_2 = P_1 - 2P_2 \quad \Rightarrow \left(\dot{P}_2 + 2P_2 = P_1\right) e^{2t}$$

$$= \dot{\left(P_2 e^{2t}\right)} = P_1 e^{2t} = e^{t}$$

$$P_2 e^{2t} = e^{t} - 1 \quad \to P_2(t) = e^{-t}\left(1 - e^{-t}\right)$$


Generating fn soln: $P(z,t) = \sum_{n=0}^{\infty} P_n(t) z^n$
note $z = e^{-s}$
$P_n e^{-ns}$
$\sum_{n=1}^{\infty} \left[\dot{P}_n - (n-1) P_{n-1} - n P_n \right] z^n$
LHS $\frac{\partial P(z,t)}{\partial t}$
$P_2(t) = e^{-t}(1-e^{-t})$

sl'n: $P(z,t) = \sum_{n=1}^{\infty} P_n^{(t)} z^n$

$) P_{n-1} - n P_n \,] z^n$

$= (z^2 - z) \frac{\partial P}{\partial z}$

note

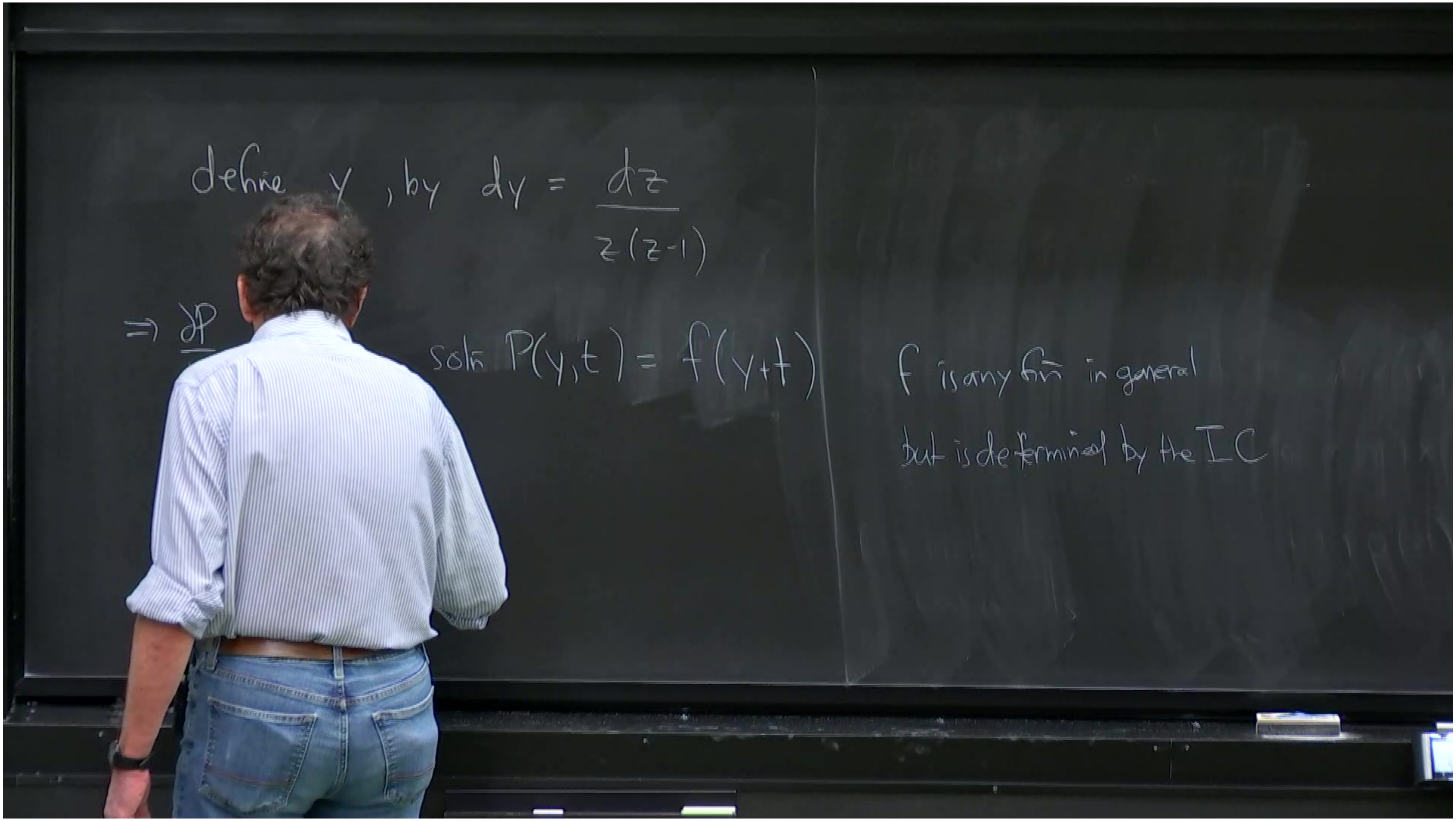
2nd term on right

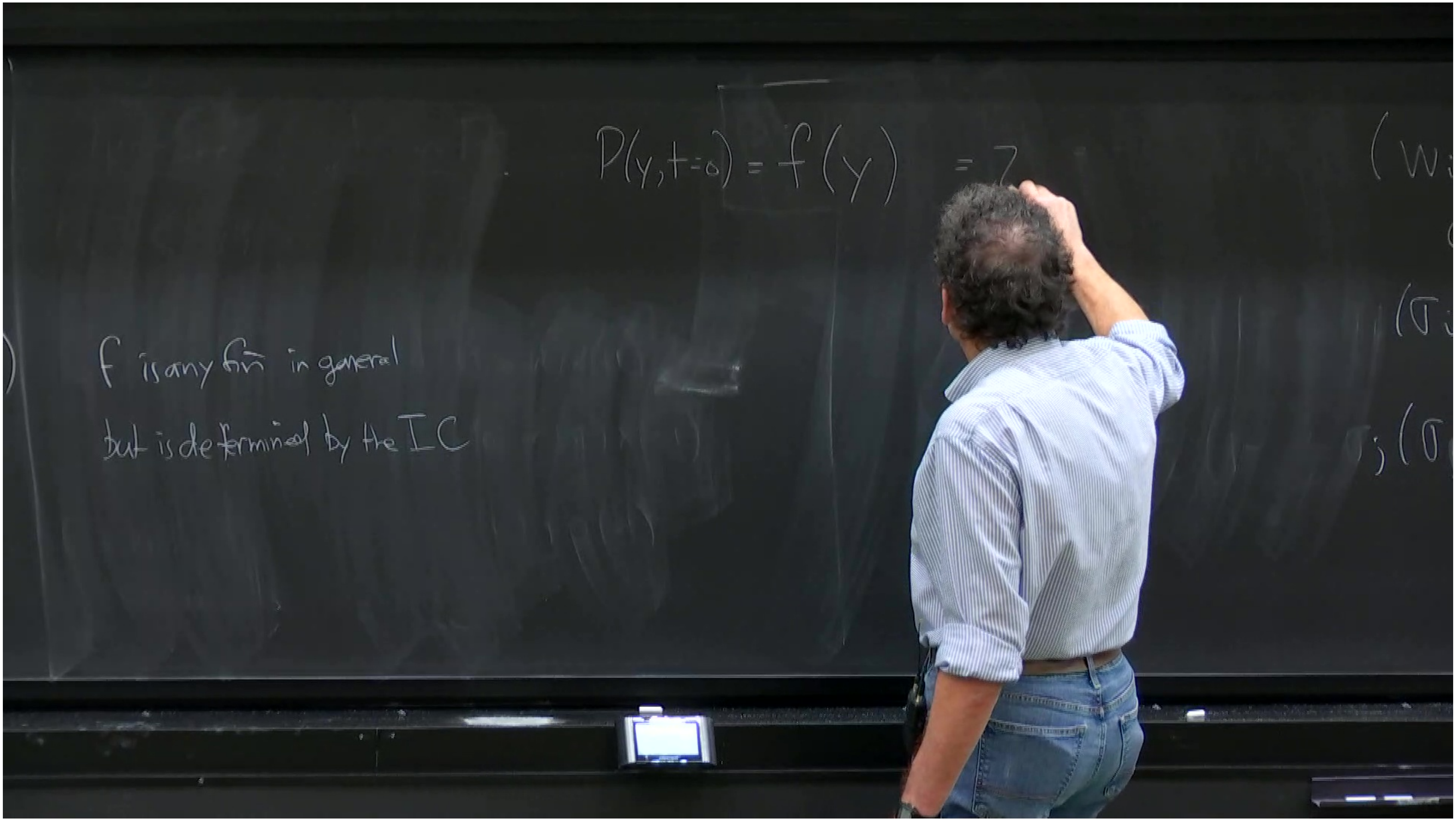
$$\sum_{n=1}^{\infty} n P_n z^n = z \frac{d}{dz} \sum_{n=1}^{\infty} P_n z^n$$

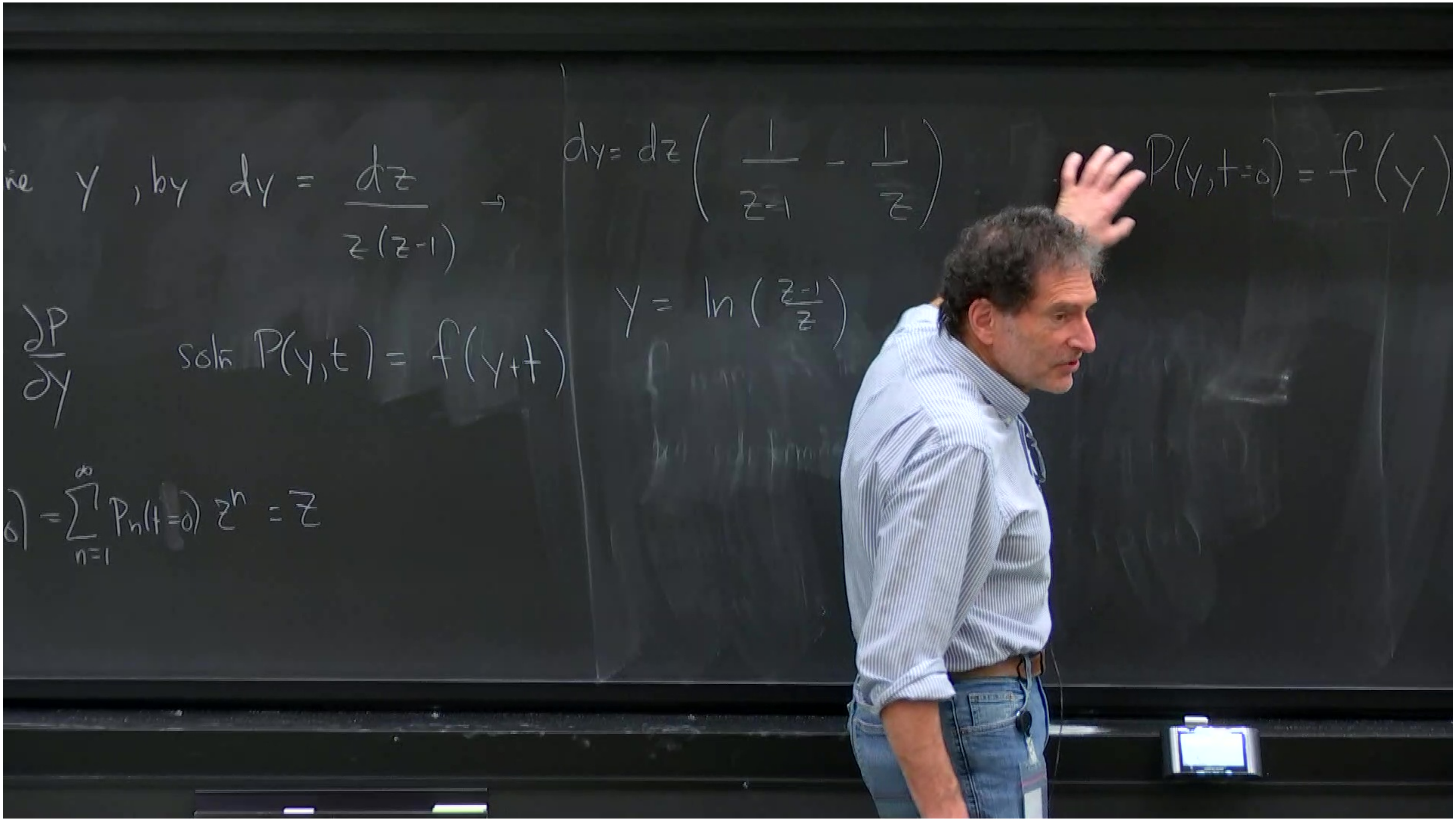
1st term on the right

$$z \sum_{n=1}^{\infty} (n-1) P_{n-1} z^{\overbrace{n-1}^{=m}} = z \sum_{\cancel{m=0}\; m=1}^{\infty} m P_m z^m$$

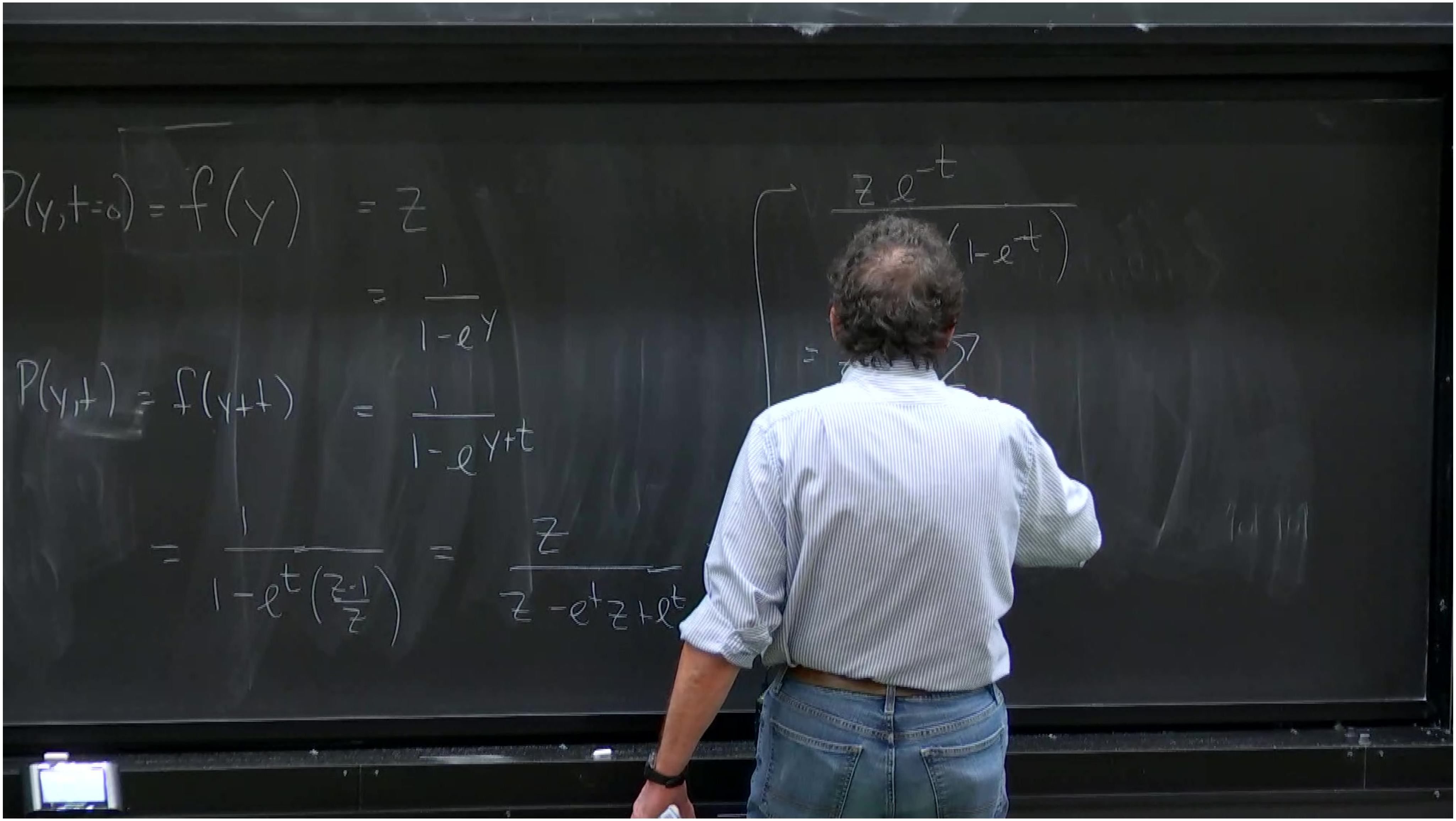
$$= z^2 \frac{\partial P}{\partial z}$$

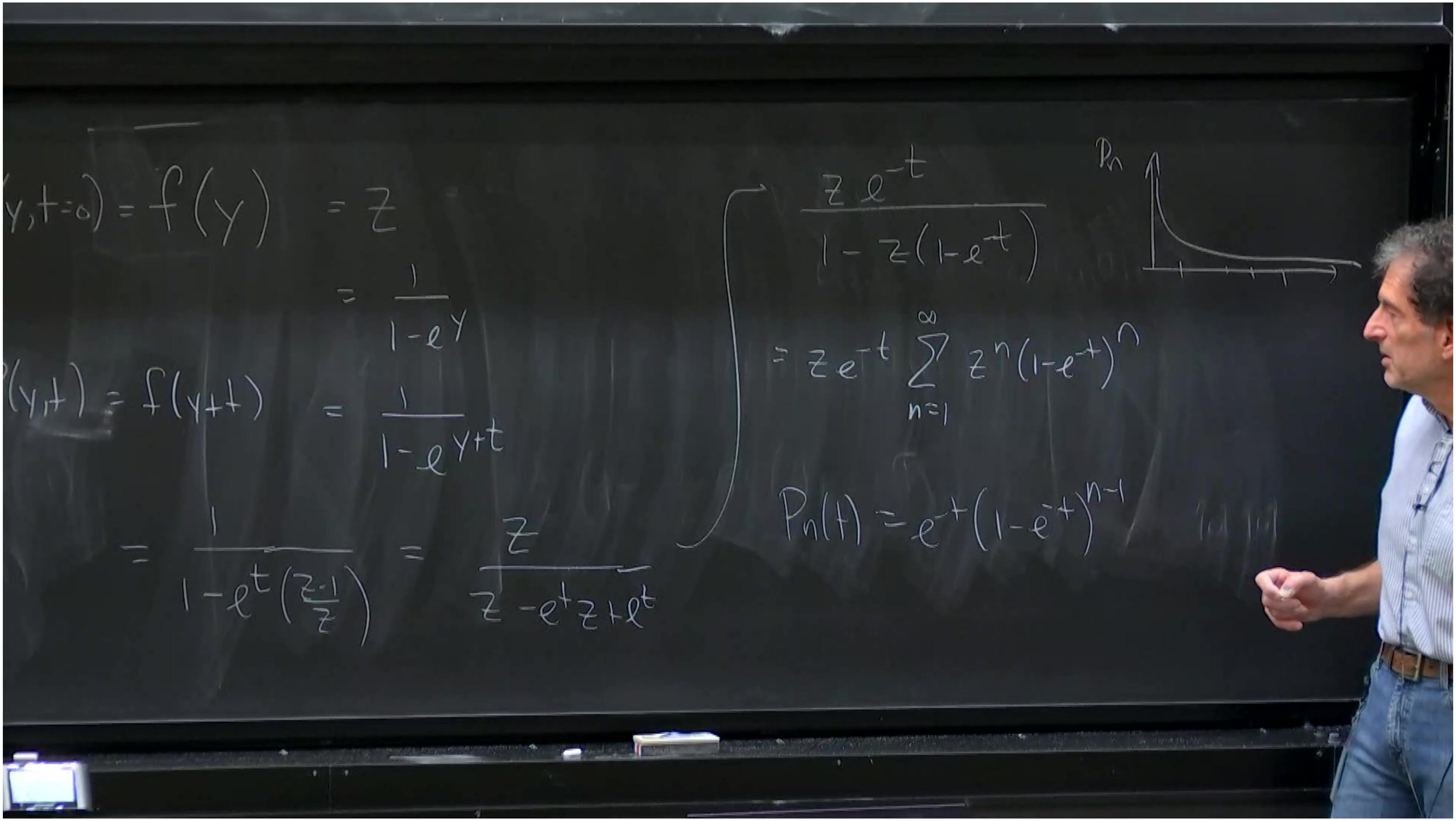

define y, by dy = dz / z(z-1)
⇒ ∂P
soln P(y,t) = f(y+t)
f is any fn in general
but is determined by the IC


P(y,t=0) = f(y) = ?
f is any fn in general
but is determined by the IC



ne y , by dy = dz / z(z-1) →
dy = dz (1/(z-1) - 1/z)
P(y, t=0) = f(y)
y = ln ((z-1)/z)
∂P/∂y
soln P(y,t) = f(y+t)
0) = Σ_{n=1}^∞ P_n(t=0) z^n = Z


$P(y,t=0) = f(y) = z$
$= \frac{1}{1-e^y}$
$P(y,t) = f(y+t) = \frac{1}{1-e^{y+t}}$
$= \frac{1}{1-e^t\left(\frac{z-1}{z}\right)} = \frac{z}{z-e^t z+e^t}$
$\frac{z e^{-t}}{(1-e^{-t})}$


$(y,t=0) = f(y) = z$
$= \frac{1}{1-e^{y}}$
$(y,t) = f(y+t) = \frac{1}{1-e^{y+t}}$
$= \frac{1}{1-e^{t}\left(\frac{z-1}{z}\right)} = \frac{z}{z-e^{t}z+e^{t}}$
$\frac{z e^{-t}}{1-z(1-e^{-t})}$
$= z e^{-t} \sum_{n=1}^{\infty} z^{n}(1-e^{-t})^{n}$
$P_n(t) = e^{-t}(1-e^{-t})^{n-1}$
P_n