Title: Quantum Field Theory II - Lecture 221128

Speakers:

Collection: Quantum Field Theory II (2022/2023)

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URL: https://pirsa.org/22110010

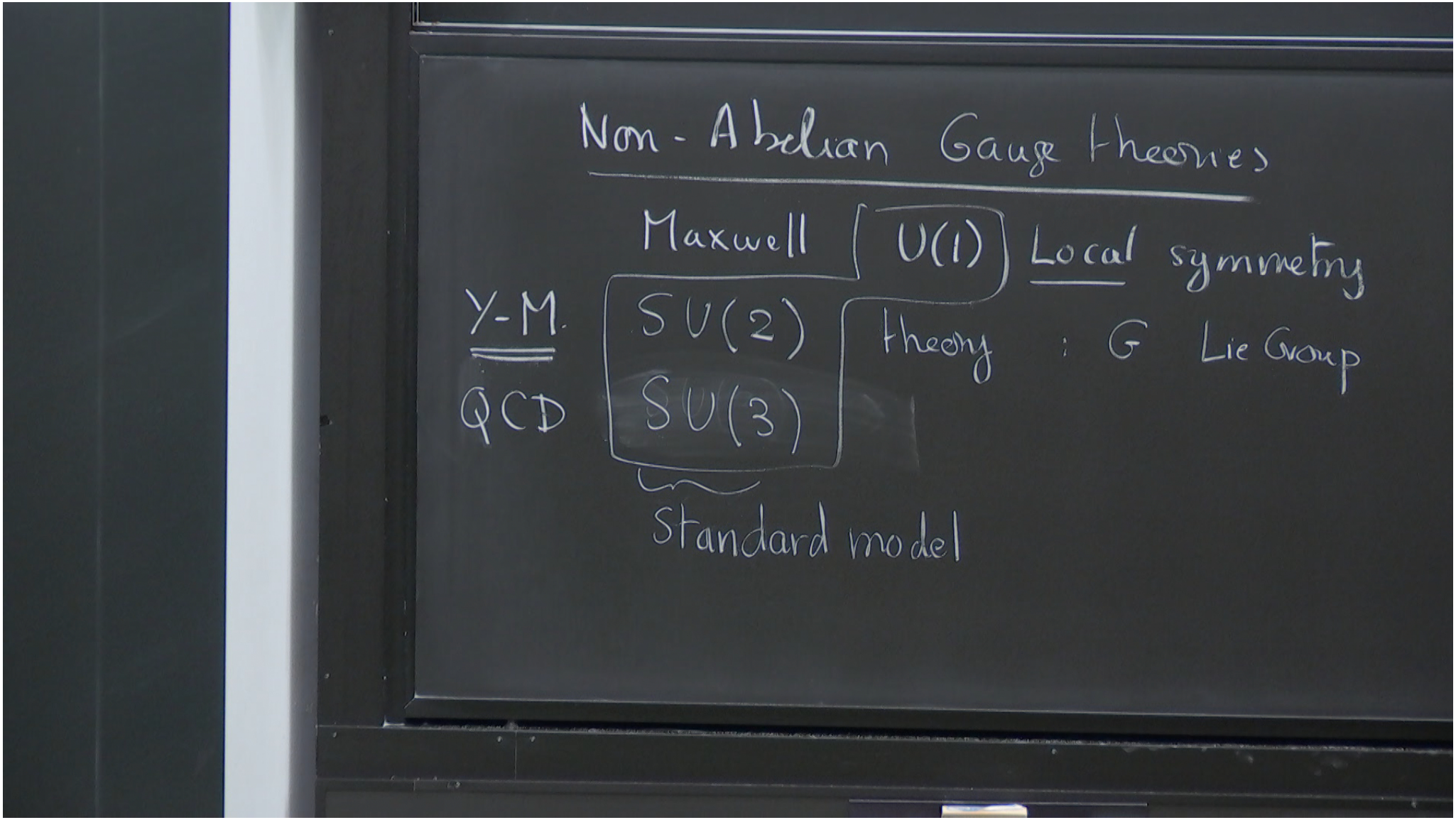
Non-Abelian Gauge theories

Maxwell U(1) Local symmetry

Y-M. SU(2) theory : G Lie Group

QCD SU(3)

St


Non-Abelian Gauge theories
Maxwell
U(1)
Local symmetry
Y-M.
QCD
SU(2)
SU(3)
theory : G Lie Group
Standard model

eories

symmetry

Lie Group

$A_\mu(x)$ Vector Potential $U(1)$ electric charge

$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$ Gauge Transformation

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ Field Strength

$\hookrightarrow \vec{E}, \vec{B}$ Maxwell Equat.

electric charge

sformation

2 charges (quantum numbers)

$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ 2 component complex field

$\bar{\phi} = \begin{pmatrix} \bar{\phi}_1 \\ \bar{\phi}_2 \end{pmatrix}$

Action

$S[\phi] = \int d^4x \; \frac{1}{2}(\partial_\mu \bar{\phi} \cdot \partial_\mu \phi + \frac{m^2}{2} \phi \bar{\phi}$

$SU(2)$ $\phi \to U \cdot \phi$ U unitary matrix 2×2

$\Rightarrow$ currents associated to $SU(2)$ symmetry

$\bar{\phi}.\phi = \bar{\phi}_1\phi_1 + \bar{\phi}_2\phi_2$ 3 currents

electric charge

sformation

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$SU(2)$ $\phi \to U \cdot \phi$ U unitary matrix 2×2

$\Rightarrow$ currents associated to $SU(2)$ symmetry

$\bar{\phi}_1 \phi = \bar{\phi}_1 \phi_1 + \bar{\phi}_2 \phi_2$ 3 currents $\Leftarrow$ 3 generators

$SU(2)$ 3 dim $\Rightarrow$ Lie Algebra $su(2)$

3 generators $a = 1, 2, 3$ $\quad t_a = \frac{1}{2}\sigma_a$ Pauli Matrices

3 currents $J^a_\mu = \frac{i}{2}\left(\bar{\phi}\, t_a\, \partial_\mu \phi - \partial_\mu \bar{\phi}\, t_a\, \phi\right)$ conserved currents

$J_\mu = J^a_\mu\, t_a$ $\quad 2\times 2$ matrix $\in$ Lie$(SU(2))$ $\quad$ Traceless matrix anhisym.

$(\phi, \bar{\phi}) \in$ Fund$(SU(2))$

- 3 vector fields $A^a_\mu(x)$ $a = 1, 2, 3$ $\rightarrow A_\mu(x) = A^a_\mu(x) \cdot t_a$

  2×2 matrix in $Lie(SU(2))$

  Multiplet

- $SU(2)$ is non abelian $\neq U(1)\times U(1)$
- Covariant derivatives

$A_\mu(x) \to A_\mu(x) + \partial_\mu \alpha(x)$ Gauge Transformation

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ Field Strength

↳ $\vec{E}$, $\vec{B}$ Maxwell Equat.

covariant derivative

$D_\mu \Phi = \partial_\mu \Phi - i A_\mu \cdot \Phi$

Φ

SU

$\Rightarrow$ C

Φ, ϕ

- Covariant derivatives

values

conserved currents

hix

$M \cdot U^{-1} = M'$

- $\underline{3}$ vector fields $A^a_\mu(x)$ $a = 1, 2, 3$ $\rightarrow A_\mu(x) = A^a_\mu(x) \cdot t_a$

  2×2 matrix in $Lie(SU(2))$

  Multiplet

- $SU(2)$ is non abelian $\neq U(1)\times U(1)$

- Covariant derivatives

$\phi_F \in Fund.(SU(2))$

2-vector

$$\partial_\mu \phi_F - i A_\mu \cdot \phi_F = D_\mu \phi_F$$

$$\partial_\mu \phi_F - i A^a_\mu \cdot t_a \cdot \phi_F$$

2×2 matrix

ex

$(x) \cdot t_a$

Lie(SU(2))

explicitely $\phi = (\phi_i)$ $i = 1, 2$

$t_a = \frac{1}{2}\sigma_a$ $a = 1, 3$

$(D_\mu \phi)_i = \partial_\mu \phi_i - i A^a_\mu \frac{1}{2} (\sigma_a)_{ij} \cdot \phi_j$

Pauli matrices

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$

$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

hermitean

Traceless 2×2

↓

3 indep components

real fields

fields $A^a_\mu(x) \to A_\mu(x) = A^a_\mu(x) \cdot t_a$

2, 3

2×2 matrix in Lie(SU(2))

Multiplet

explicitely $\phi =$

$t_a =$

non abelian $\neq U(1) \times U(1)$

$(D_\mu \phi)_i = \partial_\mu \phi_i - i A^a_\mu$

t derivatives

and (SU(2)) $\partial_\mu \phi_F - i A_\mu \cdot \phi_F = D_\mu \phi_F$

$\partial_\mu \phi_F - i A^a_\mu \cdot t_a \cdot \phi_F$

↑

2×2 matrix

3 generators $a = 1, 2, 3$ $\quad t_a = \frac{1}{2}\sigma_a$ Pauli Matrices

$a = 1, 2, 3$

$J^a_\mu = \frac{i}{2}\left(\bar{\phi}\cdot t_a \cdot \partial_\mu \phi - \partial_\mu \bar{\phi}\, t_a \cdot \phi\right)$ conserved currents

$J_\mu = J^a_\mu \cdot t_a$ 2×2 matrix $\in$ Lie$(SU(2))$ Traceless matrix anhisym.

$(\phi, \bar{\phi}) \in$ Fund$(SU(2))$

Adj Repr. $M \to U \cdot M \cdot U^{-1} = M'$

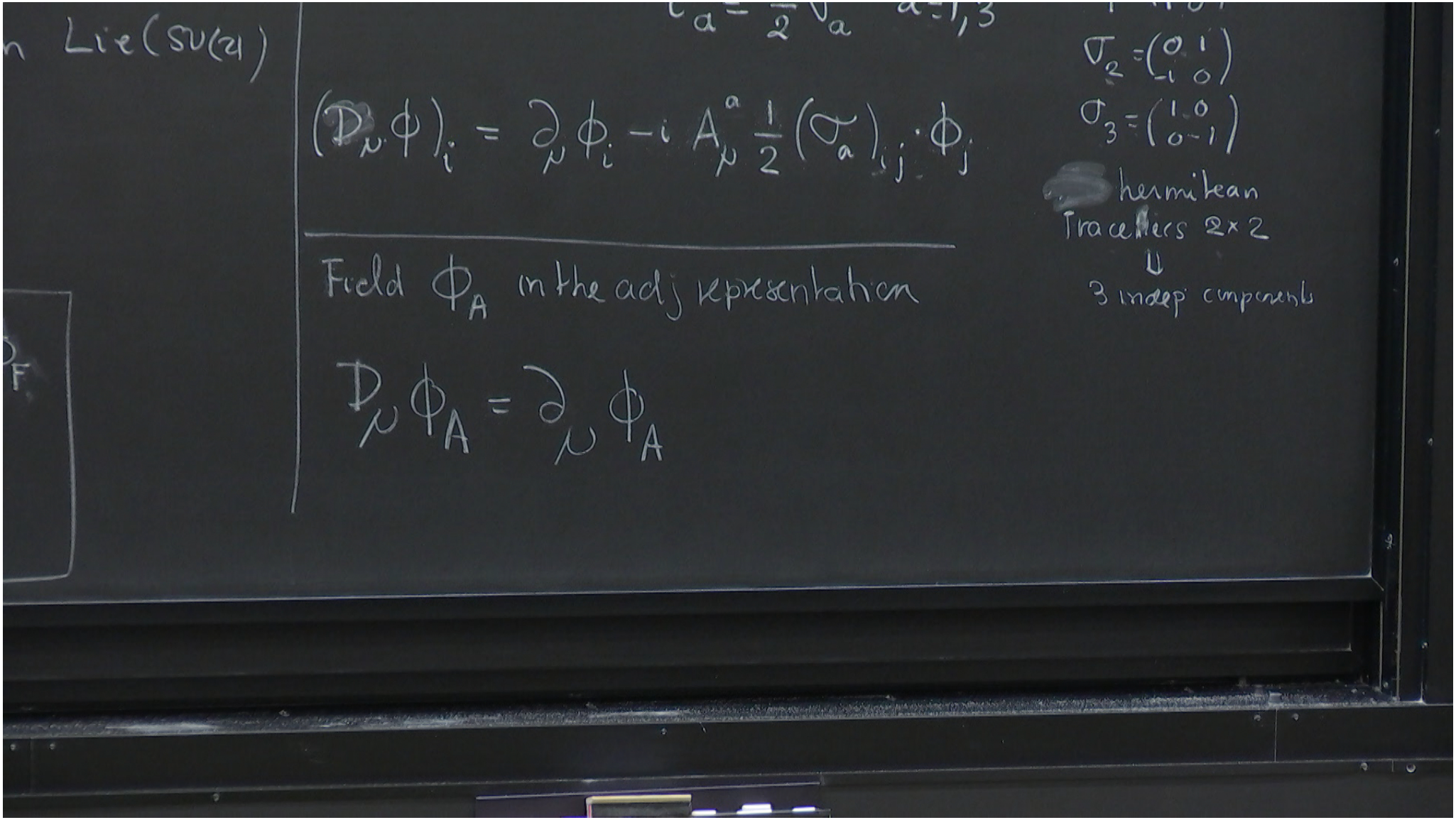
$U = 1 + i\epsilon S + \ldots$ $\quad M \to M + i\epsilon [S, M] + \ldots$

• $SU(2)$ in non abelia

• Covariant derivati

$\phi_F \in$ Fund$(SU(2))$

2-vector


Lie(SU(2))
$t_a = \frac{1}{2}\sigma_a \quad a = 1,3$
$\sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$
$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$(D_\mu \phi)_i = \partial_\mu \phi_i - i A^a_\mu \frac{1}{2} (\sigma_a)_{ij} \cdot \phi_j$
hermitean
Traceless 2×2
3 indep components
Field ϕ_A in the adj representation
$D_\mu \phi_A = \partial_\mu \phi_A$

$A^a_\mu(x) \cdot t_a$

in Lie(SU(2))

ϕ_F

explicitely $\quad \phi = (\phi_i) \quad i = 1, 2$

$t_a = \frac{1}{2}\sigma_a \quad a = 1, 3$

$$(D_\nu \phi)_i = \partial_\nu \phi_i - i A^a_\mu \frac{1}{2} (\sigma_a)_{ij} \cdot \phi_j$$

Field ϕ_A in the adj representation

$$D_\nu \phi_A = \partial_\nu \phi_A - i [A_\mu, \phi_A]$$

Pauli matrices

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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hermitean

Traceless 2×2

$\Downarrow$

3 indep components

real fields

$\rightarrow A_\mu(x) = A_\mu^a(x) \cdot t_a$

2×2 matrix in $Lie(SU(2))$

Multiplet

$U(1)$

explicitely $\quad \phi = (\phi_i) \quad i = 1, 2$

$t_a = \frac{1}{2}\sigma_a \quad a = 1, 3$

$(D_\mu \phi)_i = \partial_\mu \phi_i - i A_\mu^a \frac{1}{2} (\sigma_a)_{ij} \cdot \phi_j$

Pauli matrices

$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

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hermitean

Traceless 2×2

$\Downarrow$

3 indep components

$-i A_\mu \phi_F = D_\mu \phi_F$

$-i A_\mu^a \cdot t_a \phi_F$

$\uparrow$

2×2 matrix

Field Φ_A in the adj representation

$$D_\mu \Phi_A = \partial_\mu \Phi_A - i [A_\mu, \Phi_A]$$

$-i A_\mu \cdot t_a \phi_F$

↑ 2×2 matrix

$$D_\mu \phi_A^a = \partial_\mu \phi_A^a - i \left[A_\mu^b t_b, \Phi_A^c t_c \right]^a$$

$$= \partial_\mu \phi_A^a - i A_\mu^b \phi_A^c \left[t_b, t_c \right]^a$$

$$= \partial_\mu \phi_A^a - i A_\mu^b \phi_A^c f_{abc}$$

↑ structure const of Lie(SU(2))

For SU(2) $f_{abc} = \epsilon_{abc}$ ← Levi Civita antisym. tensor

Gauge Transformation : (Local) For SU(2) Fields

2-vector $\quad \phi_F(x) \longrightarrow \phi_F(x) + i\ \alpha(x) \cdot \phi_F(x)$

2×2 Herm Traceless $\quad \phi_A(x) \longrightarrow \phi_A(x) + i\,[\alpha, \phi_A]$

" $\quad A_\mu(x) \longrightarrow A_\mu(x) + D_\mu \alpha(x) = A_\mu(x) + \partial_\mu \alpha(x) - i\,[A_\mu, \alpha]$

when $\alpha(x)$ gauge transf ge
↳ in the lie Alg SU(2)

$\alpha(x) = \alpha^a(x)\, t_a$

2×2 m

↳ Real function of x

new
SU(2

$$D_\mu \Phi_A^a = \partial_\mu \Phi_A^a - i \left[A_\mu^b t_b, \Phi_A^c t_c \right]^a$$

$$= \partial_\mu \Phi_A^a - i A_\mu^b \Phi_A^c \left[t_b, t_c \right]^a$$

$$= \partial_\mu \Phi_A^a - i A_\mu^b \Phi_A^c f_{abc}$$

f_{abc} ← structure const of Lie(SU(2))

For SU(2) $f_{abc} = \epsilon_{abc}$ ← Levi Civita antisym. tensor

$\Phi_A(x) = \Phi_A^a(x) t_a$ ← real component field of Φ_A

Gauge Transformation : (Local) For SU(2) Fields

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when $\alpha(x)$ gauge transf g
↳ in the lie Alg SU(2)

$\alpha(x) = \alpha^a(x)\, t_a$

↳ real function of x

2×2 m

new
SU(2

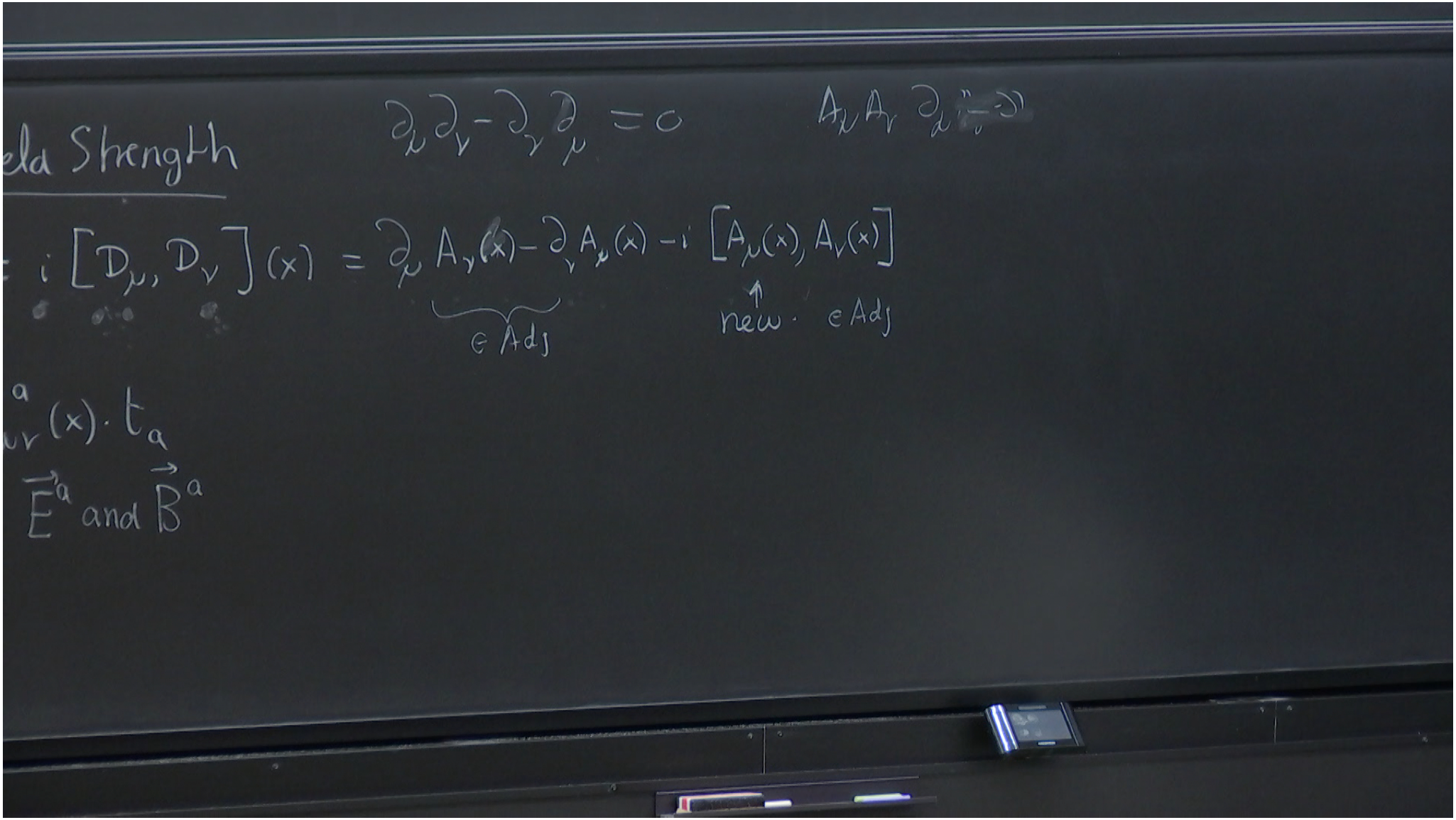
Physics : Field Strength $\quad \partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0$

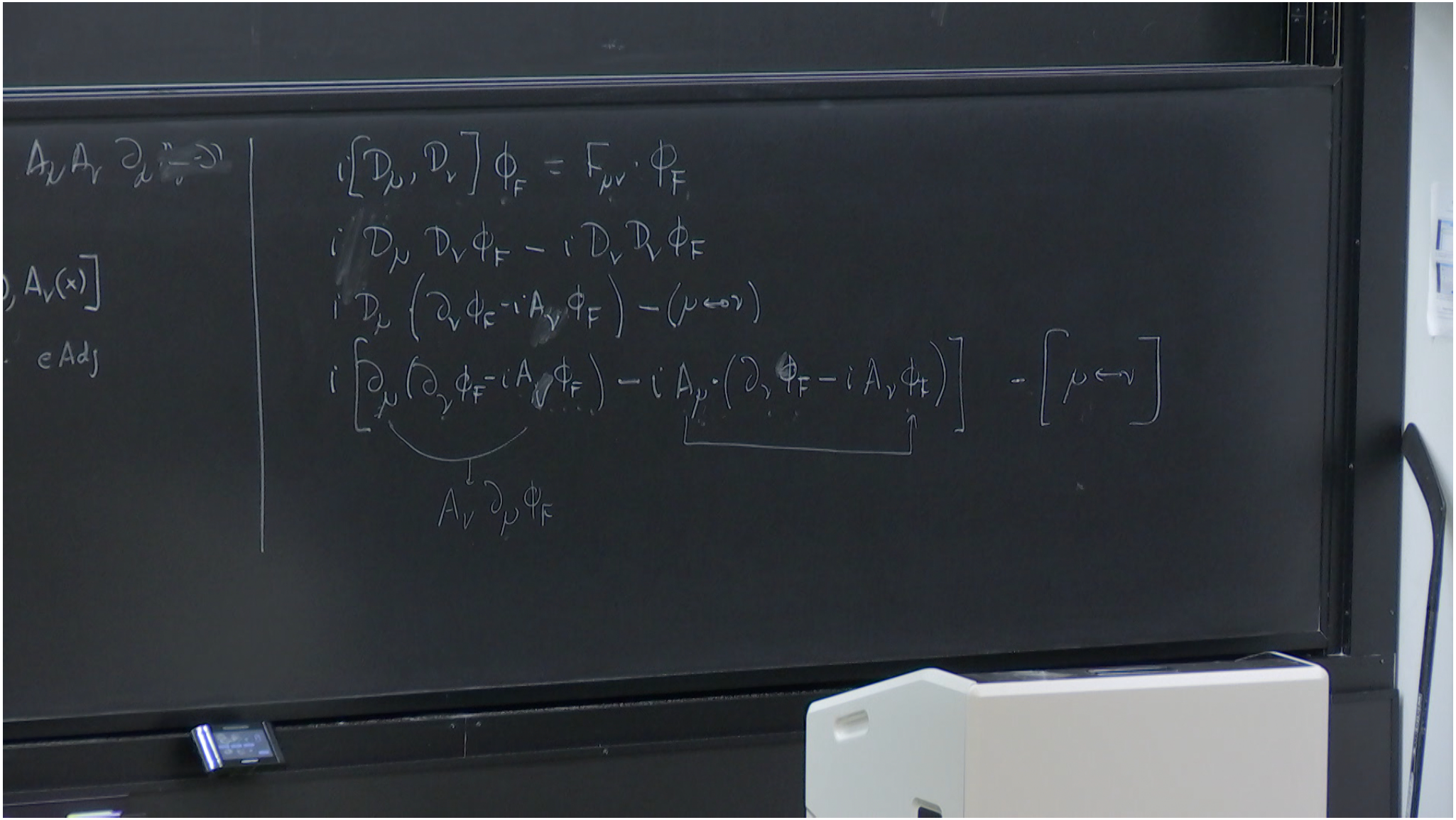
$$F_{\mu\nu}(x) = i\,[D_\mu, D_\nu](x) = \partial_\mu \underbrace{A_\nu(x)}_{\in \text{Adj}} - \partial_\nu A_\mu(x) - i\,[\underbrace{A_\mu(x)}_{\text{new.}\in\text{Adj}}, A_\nu(x)]$$

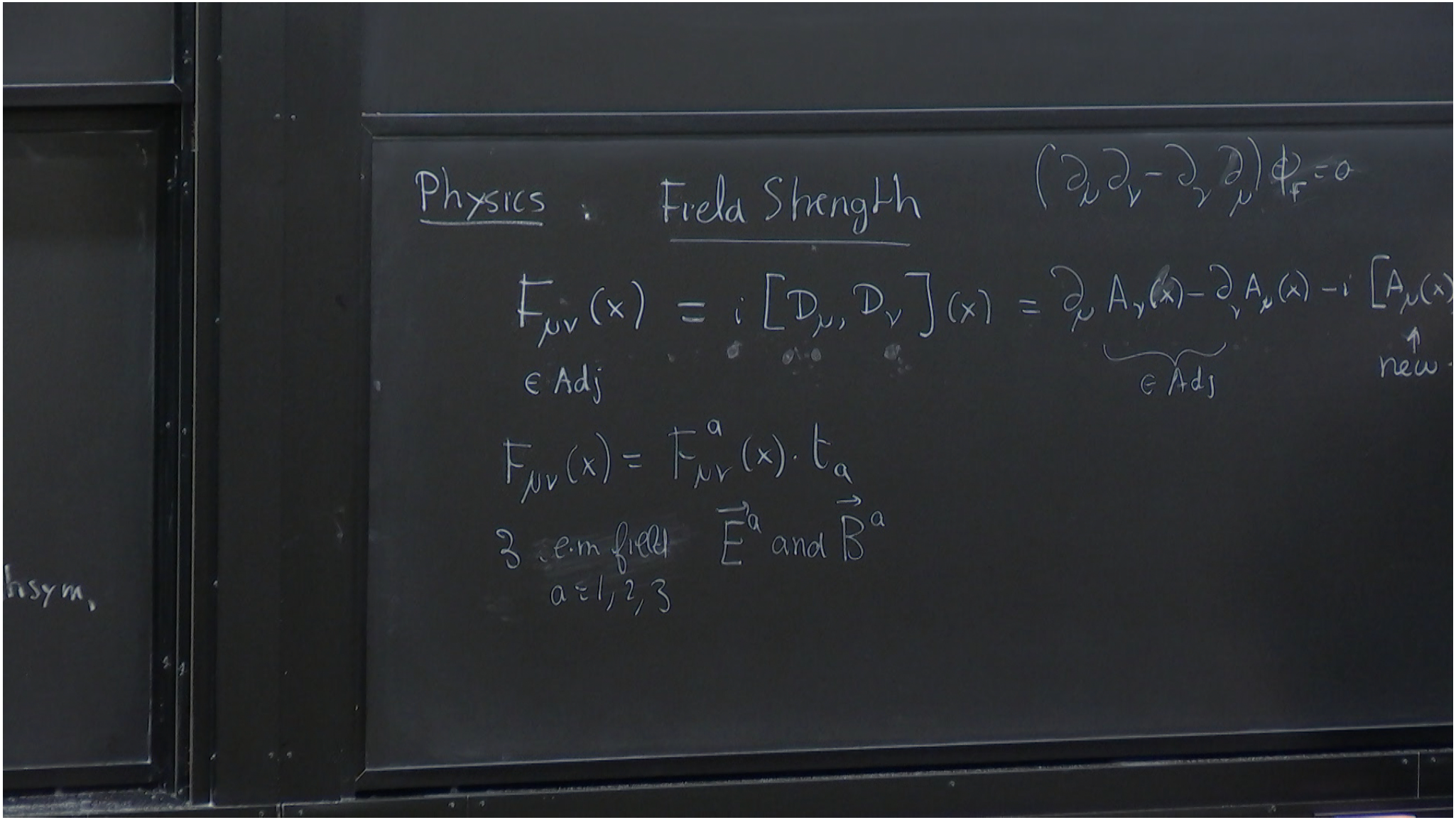
$\in$ Adj

$$F_{\mu\nu}(x) = F^a_{\mu\nu}(x) \cdot t_a$$

3 component


eld Strength
$\partial_\mu \partial_\nu - \partial_\nu \partial_\mu = 0$
$A_\mu A_\nu \partial_\mu$
$i[D_\mu, D_\nu](x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i[A_\mu(x), A_\nu(x)]$
$\in Adj$
new. $\in Adj$
a
$_{\mu\nu}(x) \cdot t_a$
$\vec{E}^a$ and $\vec{B}^a$


$A_\nu A_\nu \partial_\mu \to$
$A_\nu(x)]$
$\in Adj$
$i[D_\mu, D_\nu]\phi_F = F_{\mu\nu}\cdot\phi_F$
$i D_\mu D_\nu \phi_F - i D_\nu D_\nu \phi_F$
$i D_\mu (\partial_\nu \phi_F - i A_\nu \phi_F) - (\mu \leftrightarrow \nu)$
$i[\partial_\mu(\partial_\nu \phi_F - i A_\nu \phi_F) - i A_\mu \cdot (\partial_\nu \phi_F - i A_\nu \phi_F)] - [\mu \leftrightarrow \nu]$
$A_\nu \partial_\mu \phi_F$


Physics : Field Strength
$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\,\phi_F = 0$
$F_{\mu\nu}(x) = i\,[D_\mu, D_\nu](x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x) - i\,[A_\mu(x)$
$\in$ Adj
$\in$ Adj
new
$F_{\mu\nu}(x) = F^a_{\mu\nu}(x) \cdot t_a$
3 e.m field $\vec{E}^a$ and $\vec{B}^a$
$a = 1, 2, 3$
hsym.

$+i\epsilon[S,M]+\dots$

2×2 matrix

(2) Fields α infinitesimal gauge transf.

$x)$

when $\alpha(x)$ gauge transf generator
↑ in the lie Alg $SU(2)$

$\alpha(x) = \alpha^a(x) t_a$ 2×2 matrix

↑ real function of x

$\mu(x) + \partial_\mu \alpha(x) - i[A_\mu, \alpha]$ new: $SU(2)$ nonabelian

$D_\mu \phi_A^a = \partial_\mu \phi_A^a - i[A_\mu^b t$

$= \partial_\mu \phi_A^a - i A_\mu^b \phi$

$= \partial_\mu \phi_A^a - i A_\mu^b \phi$

For $SU(2)$ $f_{abc} = \epsilon$

$\phi_A(x) = \phi_A^a(x) t_a$ ← real compo fields

- $F_{\mu\nu}$ transforms covariantly under a gauge transformation

$$F_{\mu\nu}(x) \to F_{\mu\nu}(x) + i\left[\alpha(x), F_{\mu\nu}(x)\right] + \cdots$$

- Y-M Action Invariant under local gauge transf.

$$S[A] = -\frac{1}{2g^2}\int d^4x \, \mathrm{Tr}\left[F_{\mu\nu} F^{\mu\nu}\right]$$

↑ c.c.

⚠ action is non linear → interaction

g coupling constant "charge of the gauge field"

Gauge Invariance $\Longrightarrow$ interaction

Gauge Field + Matter

ϕ_A in Fund. Representation Scalar charged field

$$S_{matter}[\phi, A] = \int d^4x \, \frac{1}{2}(\overline{D_\mu \phi} \cdot D_\mu \phi) + \frac{1}{2} m^2 \bar{\phi} \cdot \phi + \frac{\lambda}{8}(\bar{\phi} \cdot \phi)^2$$

↑ depends on A

↑ normalization

$$D_\mu \phi = \partial_\mu \phi - i A_\mu \cdot \phi$$

$$\bar{\phi} \cdot A \cdot \partial_\mu \phi$$

$$\bar{\psi} \cdot (i \not{D} - m) \cdot \psi$$

Gauge Field + Matter

ϕ_A in Fund. Representation Scalar charged field

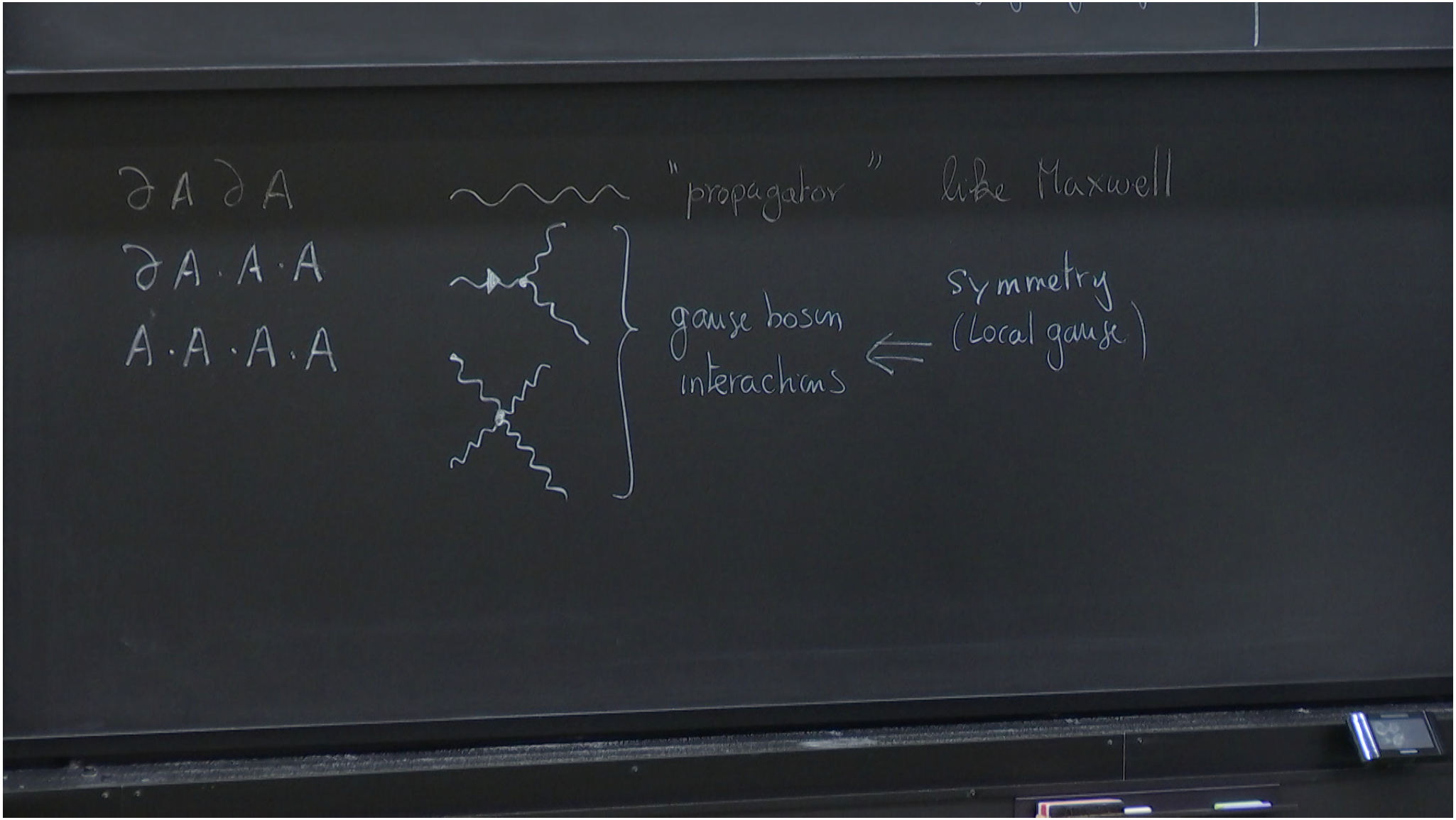
$$S_{matter}[\phi, A] = \int d^4x \; \frac{1}{2}\left(\overline{D_\mu \phi} \cdot D_\mu \phi\right) + \frac{1}{2} m^2 \bar{\phi}\cdot\phi + \frac{\lambda}{8}(\bar{\phi}\cdot\phi)^2$$

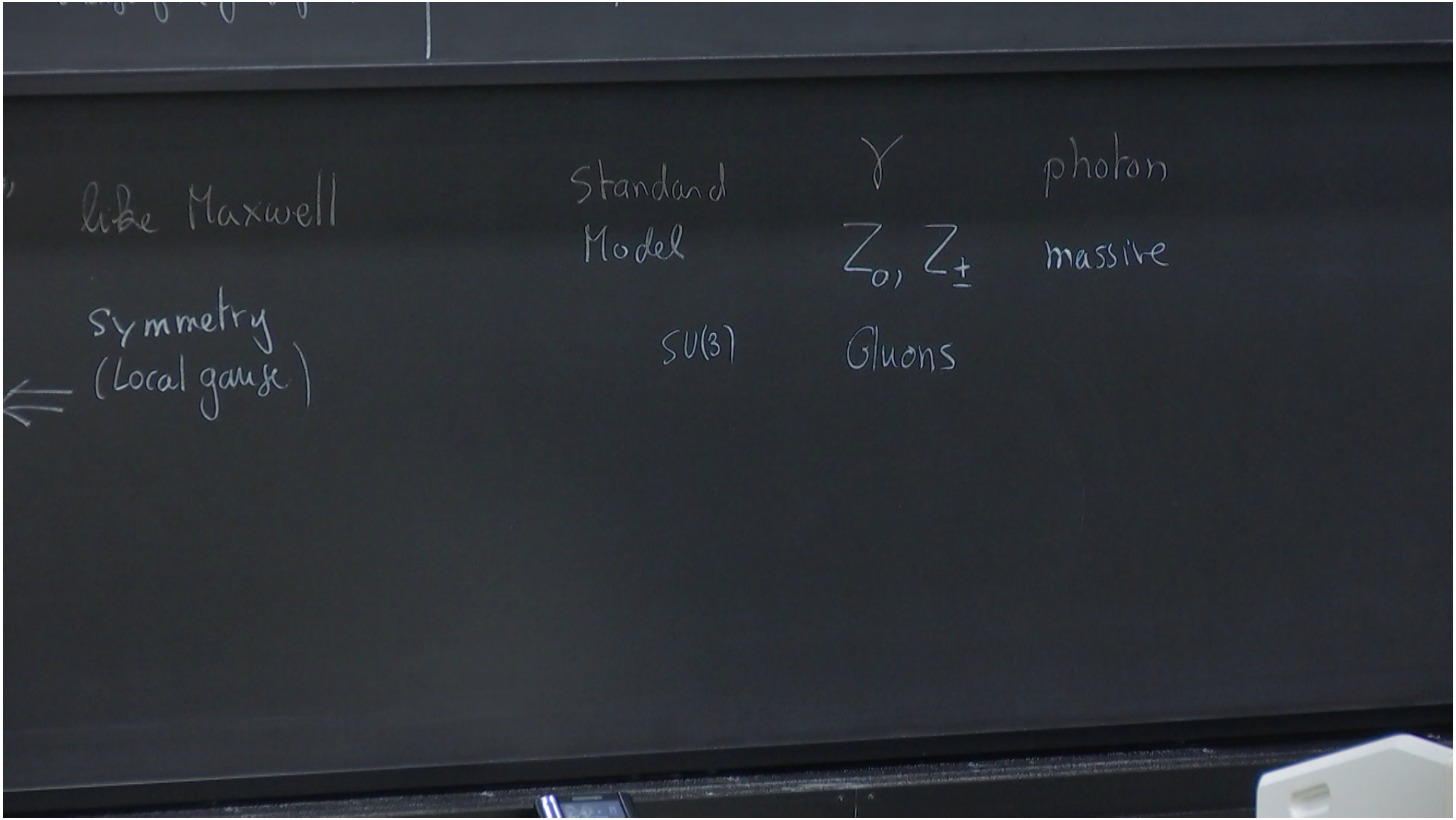
↑ depends on A ↑ normalization

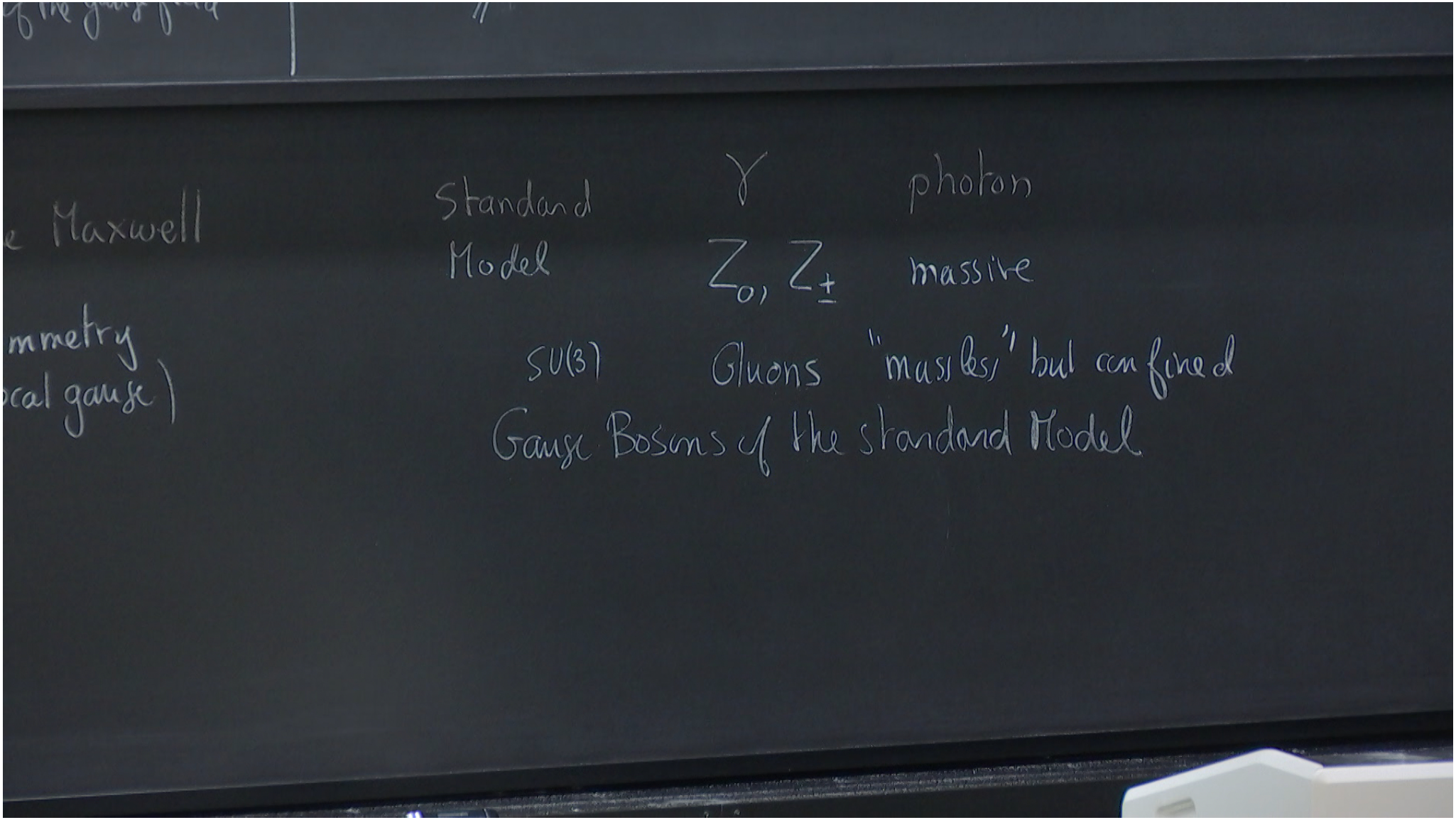
$$D_\mu \phi = \partial_\mu \phi - i A_\mu \cdot \phi$$

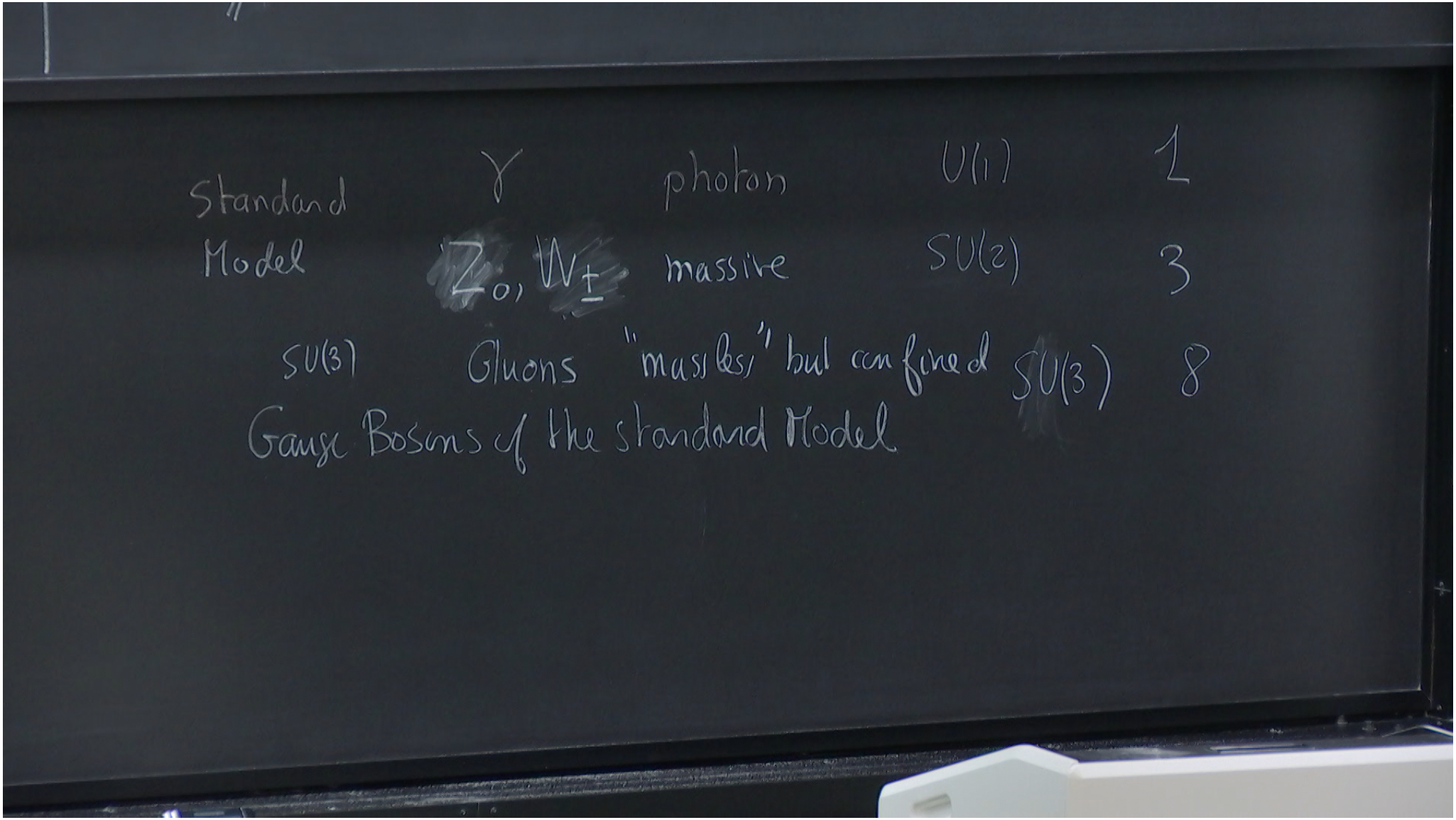
$$\bar{\phi} \cdot A \cdot \partial_\mu \phi$$

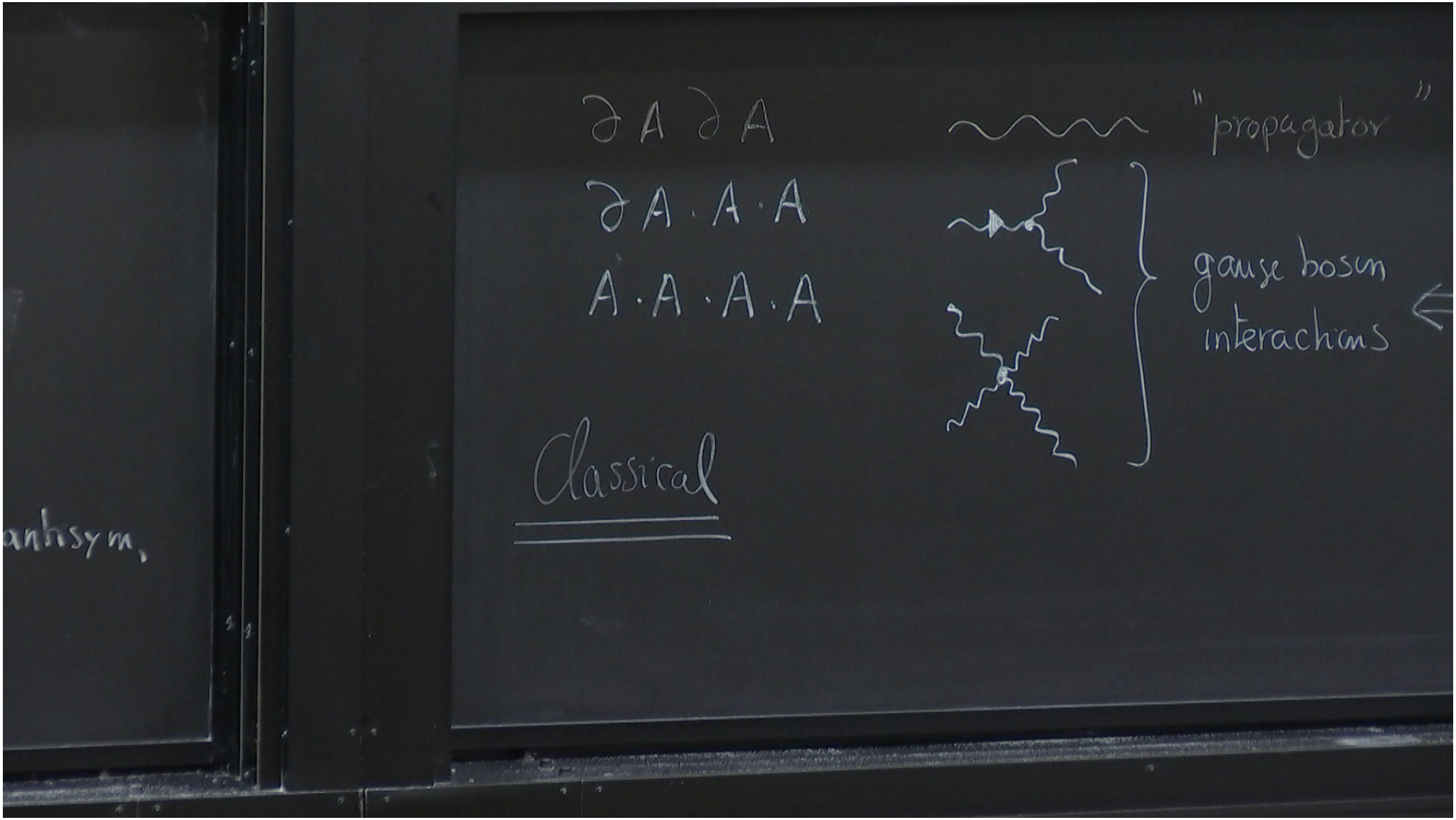
$$\bar{\psi}\, i \not{D} - m \,\psi$$


∂A ∂A
∂A · A · A
A · A · A · A
"propagator"
like Maxwell
gauge bosun interachions
⇐
symmetry
(Local gauge)


like Maxwell
Standard Model
γ
photon
$Z_0, Z_\pm$
massive
Symmetry (Local gauge)
SU(3)
Gluons


Maxwell
mmetry
ocal gauge)
Standard Model
γ photon
$Z_0, Z_\pm$ massive
SU(3) Gluons "massless" but confined
Gauge Bosons of the standard Model


Standard Model
γ photon U(1) 1
$Z_0, W_\pm$ massive SU(2) 3
SU(3) Gluons "massless" but confined SU(3) 8
Gauge Bosons of the standard Model


∂A ∂A
∂A · A · A
A · A · A · A
"propagator"
gauge boson
interactions
Classical
antisym.

$$i\left[A_\mu^b t_b, \Phi_A^c t_c\right]^a$$

$$i A_\mu^b \Phi_A^c \left[t_b, t_c\right]^a$$

$$i A_\mu^b \Phi_A^c F_{abc}$$

↑ structure cct of Lie (SU(2))

$F_{abc} = \epsilon_{abc}$ ← Levi Civita antisym. tensor

real component of Φ_A field

$\partial A \partial A$

$\partial A \cdot A \cdot A$

$A \cdot A \cdot A \cdot A$

Classical

no gauge fixing

Gauge Transformation : (Local) For SU(2) Fields ∝ infinitesima
gauge trans

2-vector $\quad \phi_F(x) \longrightarrow \phi_F(x) + i\,\alpha(x)\cdot\phi_F(x) + O(\alpha^2)$

2×2 Herm Traceless matrices $\quad \phi_A(x) \longrightarrow \phi_A(x) + i\,[\alpha, \phi_A] + O(\alpha^3)$

" $\quad A_\mu(x) \longrightarrow A_\mu(x) + D_\mu \alpha(x) = A_\mu(x) + \partial_\mu \alpha(x) - i\,[A_\mu, \alpha] + O(\alpha^2)$

when $\alpha(x)$ gauge transf g
↳ in the lie Alg SU(2)

$\alpha(x) = \alpha^a(x)\, t_a$ 2×2 m

↳ real function of x

new
SU(

Gauge Invariance $\Longrightarrow$ interaction

(Compact) Simple Lie Group G

g coupling constant "charge of the gauge field"

$\partial A \partial A$ "propagator" like Maxwell

$\partial A \cdot A \cdot A$

$A \cdot A \cdot A \cdot A$



gauge boson interactions $\Longleftarrow$ symmetry (Local gauge)

Classical

no gauge fixing

$SU(3)$ $A_\mu = A_\mu^a t_a$

$[t_a, t_b] = i f_{abc} t_c$

t_a 3×3 matrices Hermitian traceless

3×2 matrices

Standard Model	γ	photon	$U(1)$	1
	$Z_0, W_\pm$	massive	$SU(2)$	3
$SU(3)$	Gluons "massless" but confined		$SU(3)$	8

Gauge Bosons of the standard Model

$U(1)\times U(1)\times U(1)$ $\quad A_\mu^a \quad F_{\mu\nu}^a = \partial_\mu A_\nu^a + \partial_\nu A_\mu^a$

$a = 1,3$

no interactions $\quad -\frac{1}{2e_1^2}\int d^4x\, F_{\mu\nu}^1 F^{\mu\nu 1} + \frac{1}{2e_2^2}\int d^4x\, F_{\mu\nu}^2 F_{\mu\nu}^2 + \frac{1}{2e_3^2}\int d^4x\, F_{\mu\nu}^3 F_{\mu\nu}^3$

$$\frac{1}{2e^2}\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)^2 + \bar{\psi}\left(i\partial\!\!\!/ + A\!\!\!/ - m\right)\psi$$

$$A_\mu \to e^2 A_\mu$$

$$\frac{e^2}{2}\left(\partial_\mu A_\nu\right)^2 + \bar{\psi}\left(i\partial\!\!\!/ + e^2 A\!\!\!/ - m\right)\psi$$

charge of Dirac particle

in Quantization

$$\int \mathcal{D}[A]\, \exp\left(i S[A, \cdots]\right)$$