Title: Quantum Field Theory II - Lecture 221115

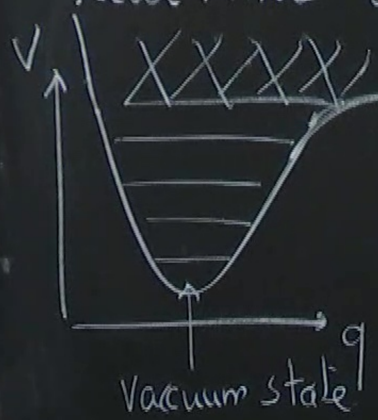
Speakers: Francois David

Collection: Quantum Field Theory II (2022/2023)

Date: November 15, 2022 - 10:45 AM

URL: https://pirsa.org/22110004

Real time $t \longrightarrow$ "imaginary" time



E . e.value of H

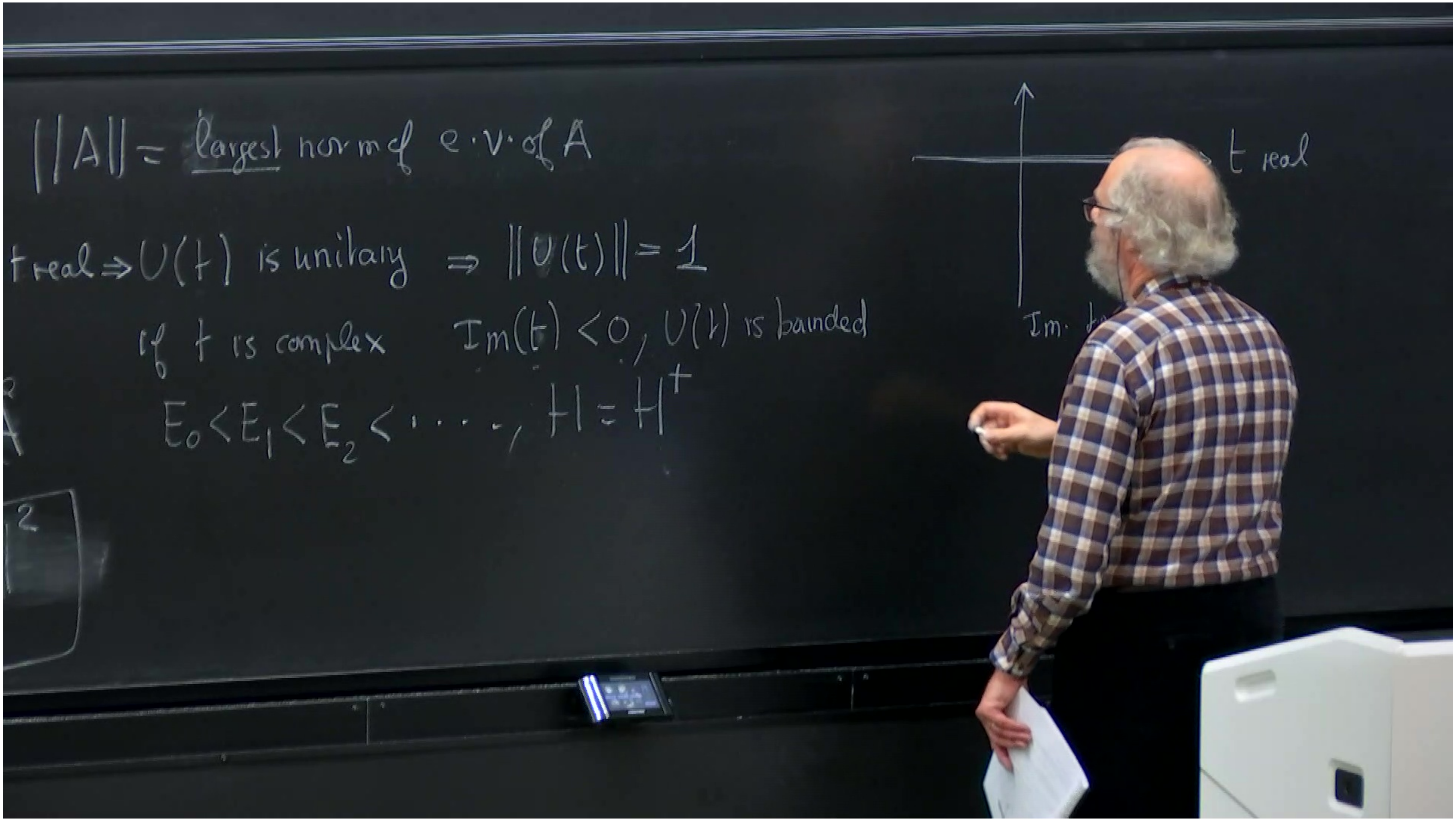
$E \geqslant E_0$

norm of the operator A

time complex: $U(t)$ not unitary

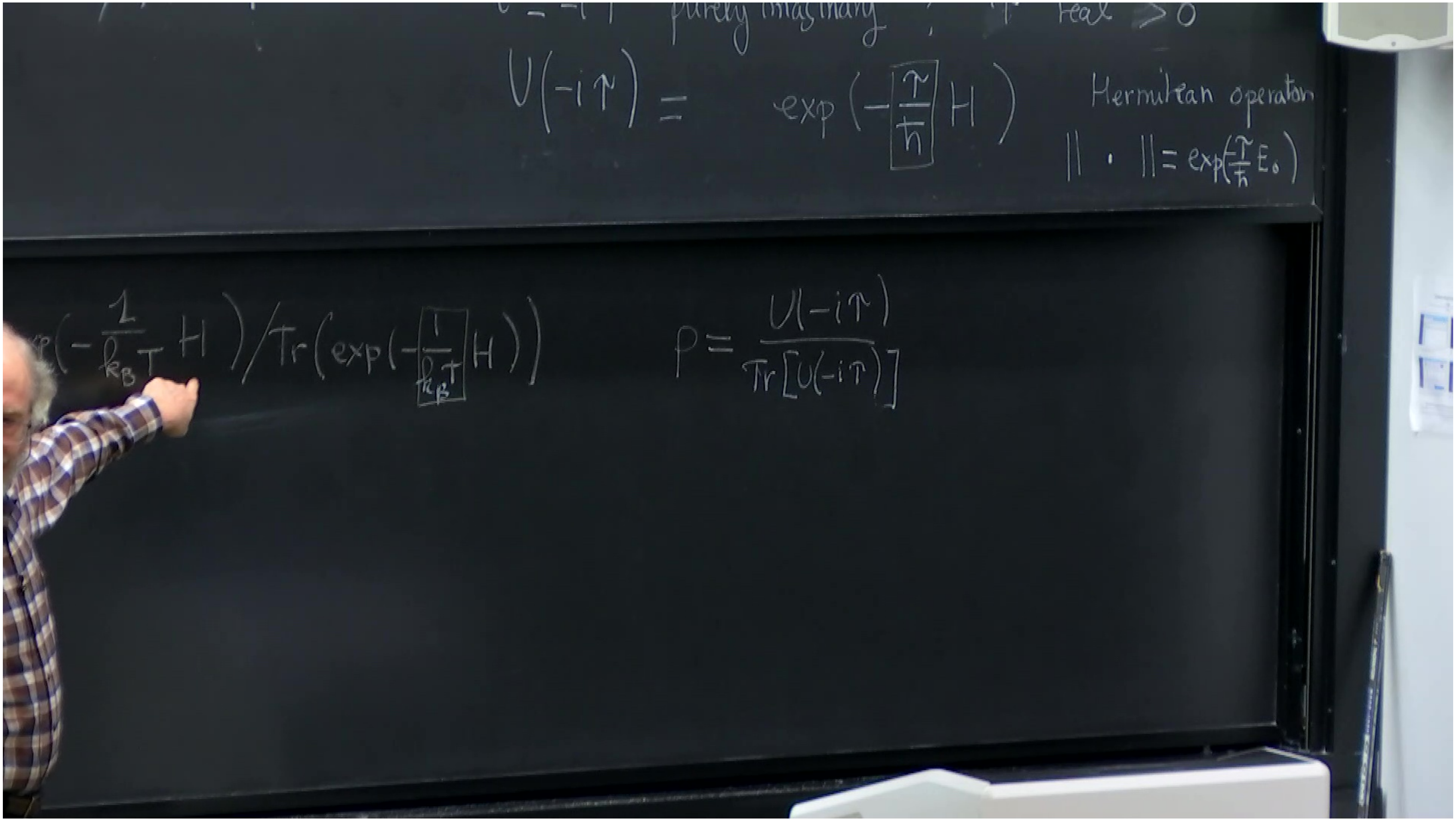
Bounded operators $\langle \psi | A^{\dagger} A | \psi \rangle \doteq \|A\|^2$ finite

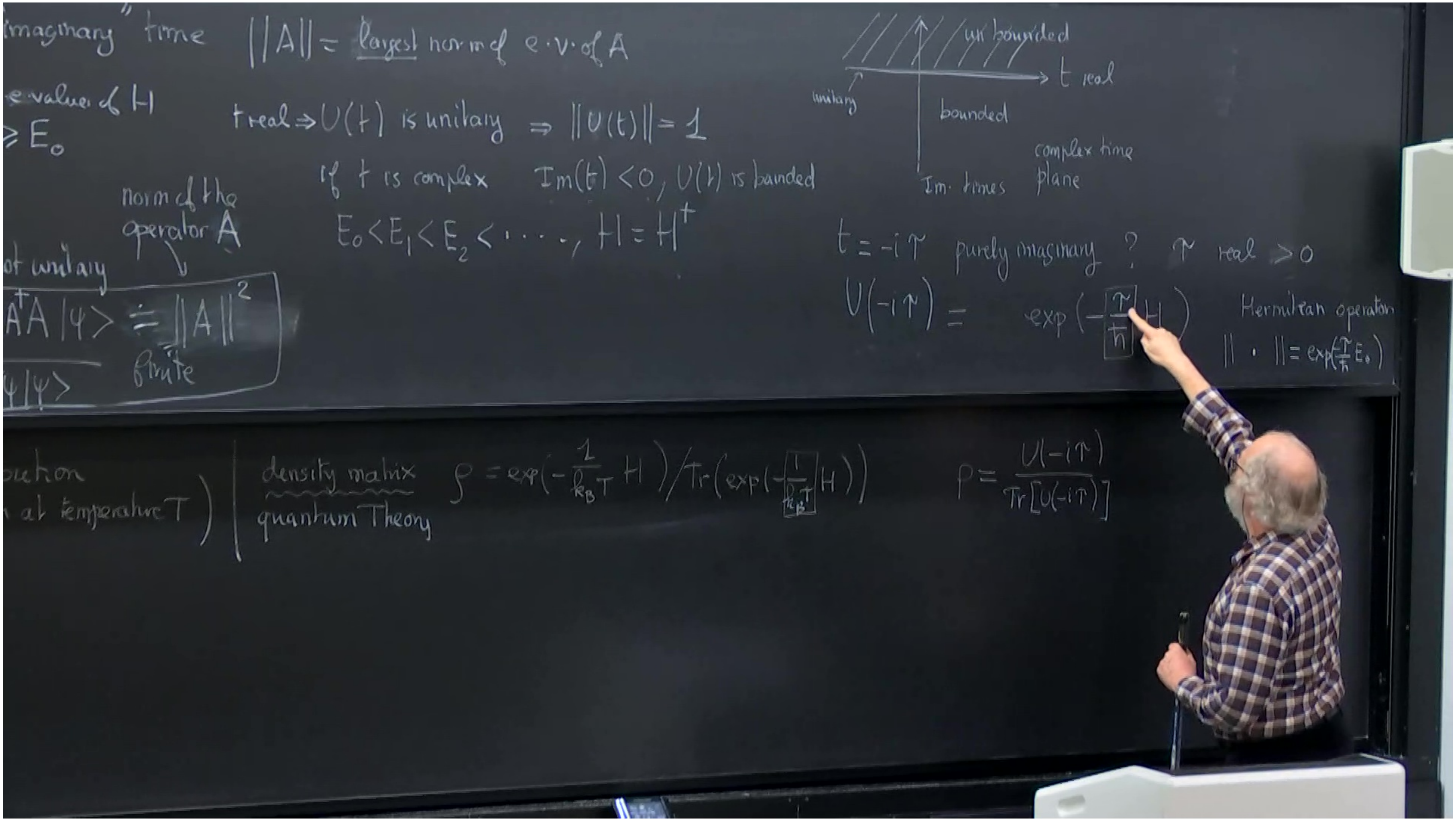
② Definition $\displaystyle \sup_{|\psi\rangle} \frac{\langle \psi | A^{\dagger} A | \psi \rangle}{\langle \psi | \psi \rangle}$

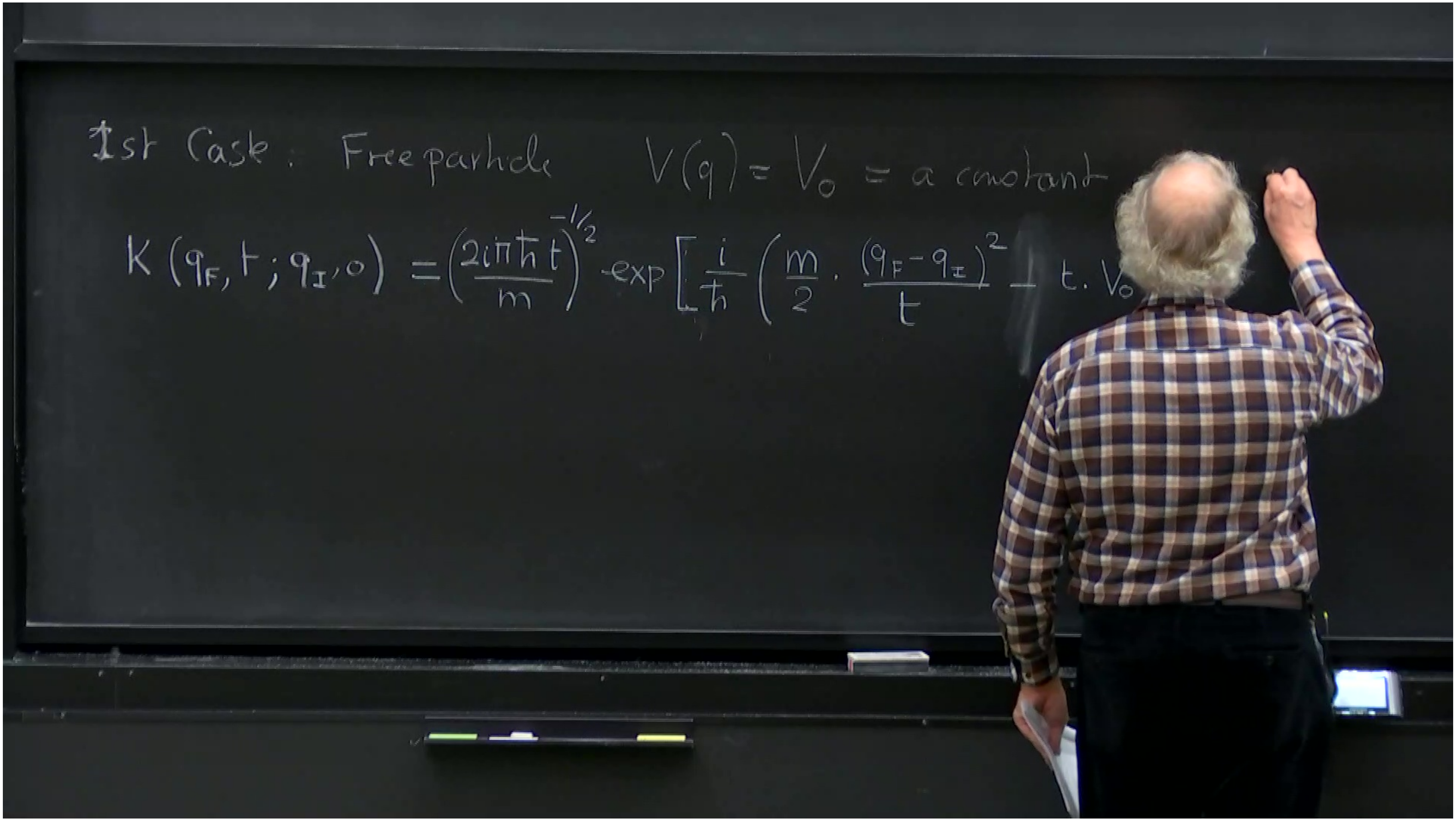

||A|| = largest norm of e.v. of A
t real ⇒ U(t) is unitary ⇒ ||U(t)|| = 1
if t is complex Im(t) < 0, U(t) is bounded
E₀ < E₁ < E₂ < ....., H = H⁺
t real
Im.

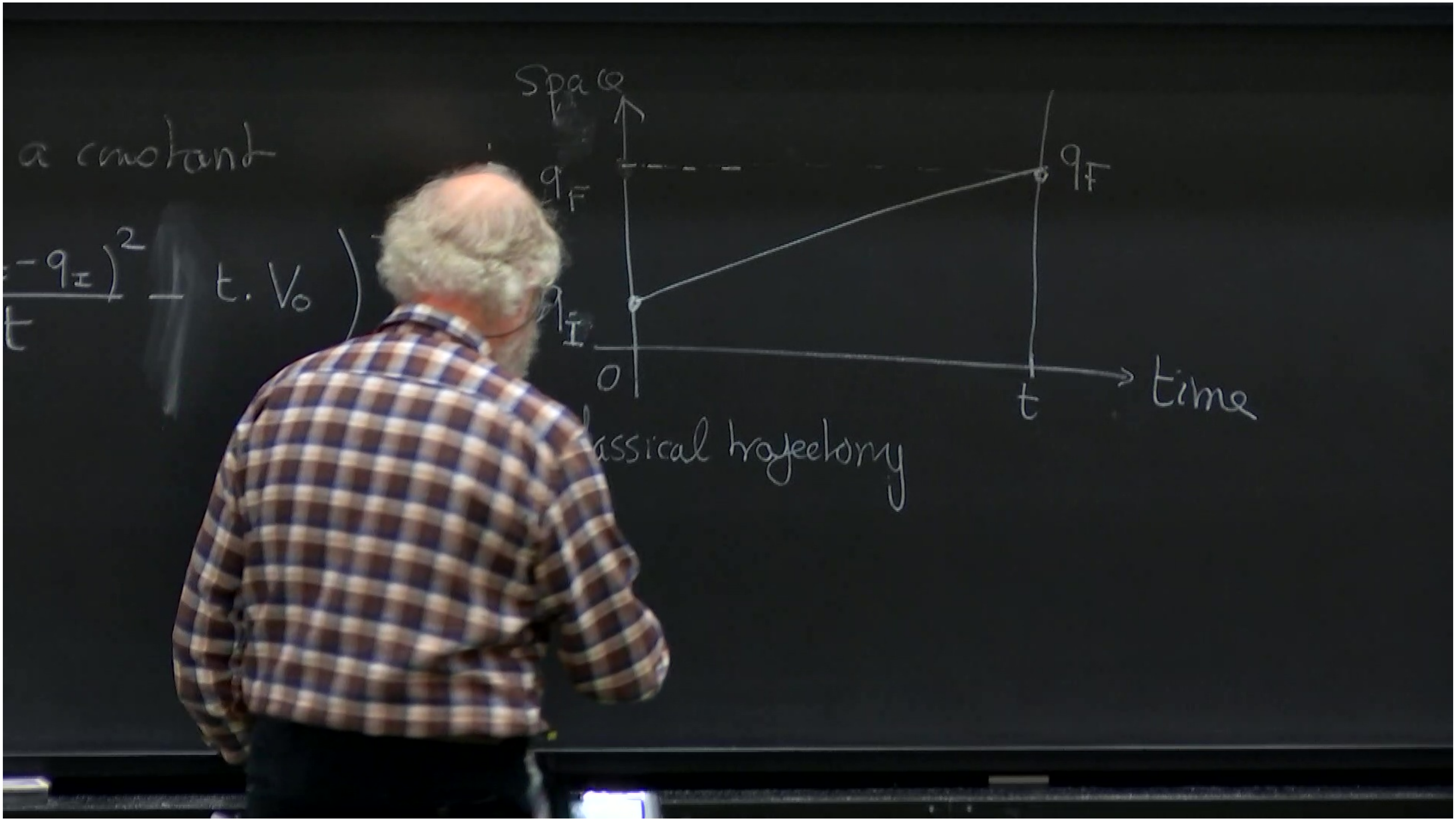
3 Gibbs distribution
(at equilibrium at temperature T)

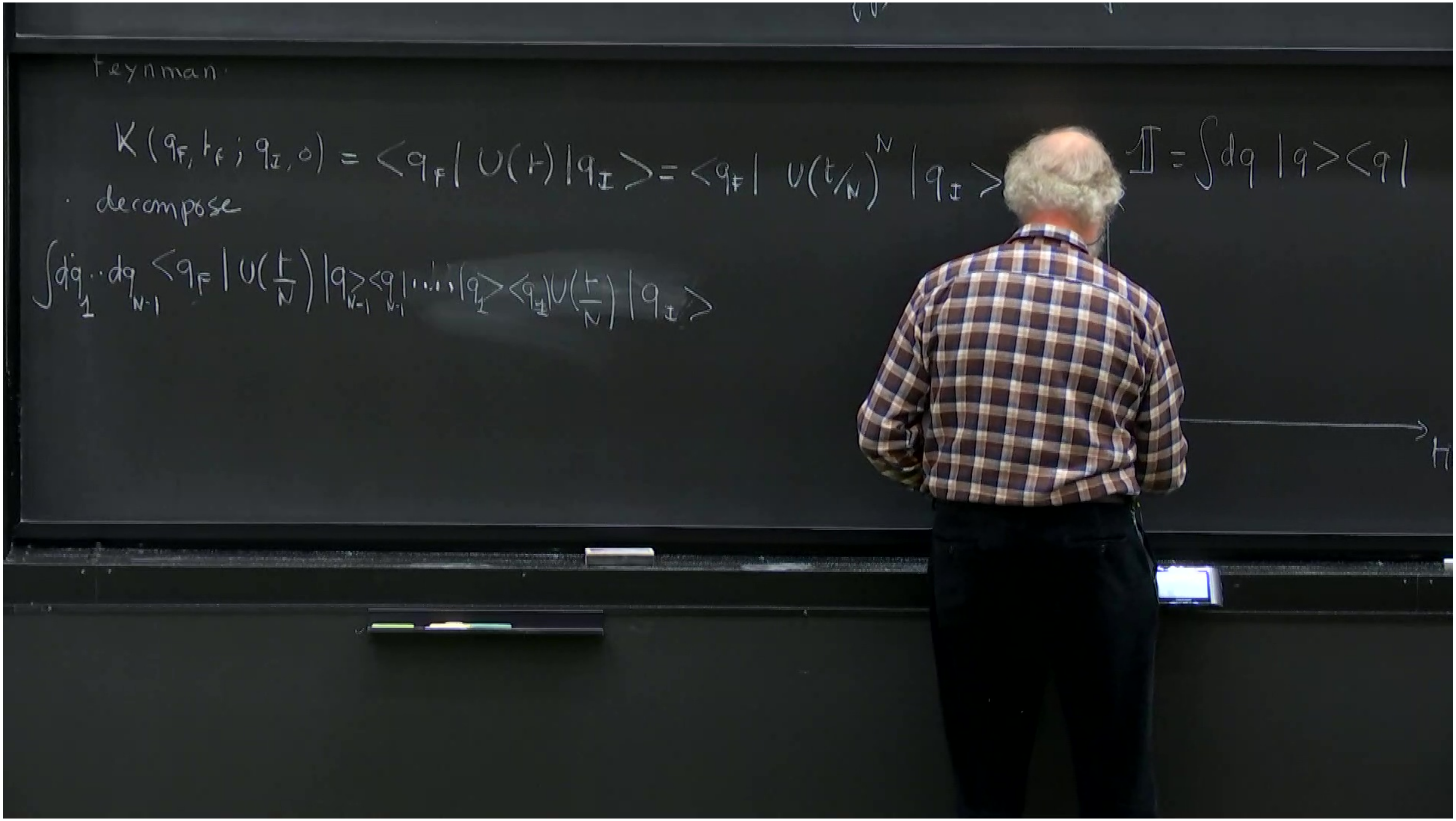
density matrix $\rho = ex$
quantum Theory

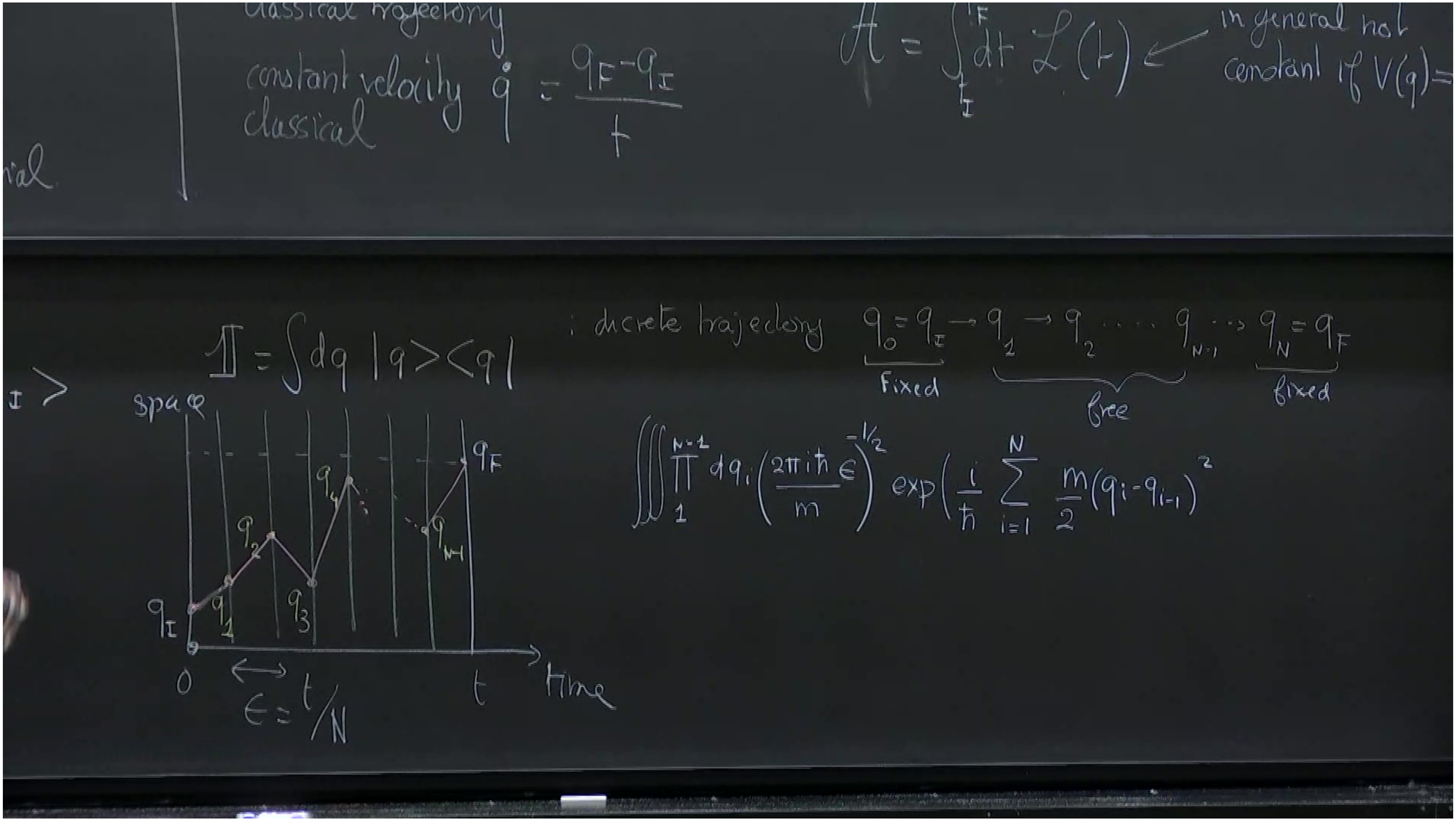

purely imaginary ; real > 0
$U(-i\tau) = \exp\left(-\boxed{\frac{\tau}{\hbar}} H\right)$
Hermitean operator
$\| \cdot \| = \exp\left(-\frac{\tau}{\hbar} E_0\right)$
$\exp\left(-\frac{1}{k_B T} H\right) / \mathrm{Tr}\left(\exp\left(-\boxed{\frac{1}{k_B T}} H\right)\right)$
$\rho = \frac{U(-i\tau)}{\mathrm{Tr}[U(-i\tau)]}$

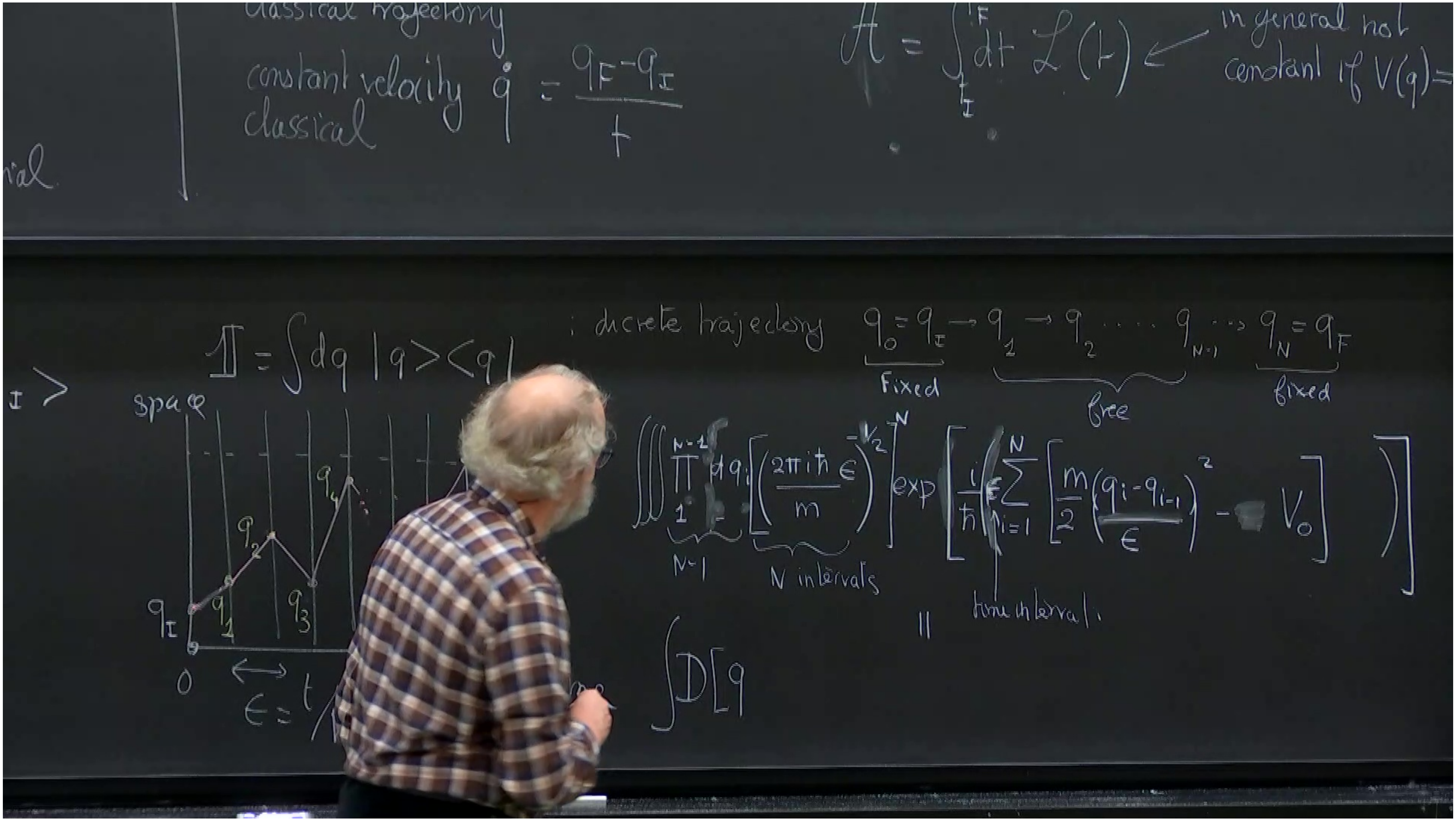

"imaginary" time
||A|| = largest norm of e.v. of A
e value of H
≥ E0
t real ⇒ U(t) is unitary ⇒ ||U(t)|| = 1
if t is complex Im(t) < 0, U(t) is bounded
norm of the operator A
E0 < E1 < E2 < ...., H = H†
ot unitary
A†A |ψ> ≐ ||A||²
finite
ψ|ψ>
unbounded
t real
unitary
bounded
complex time plane
Im. times
t = -iτ purely imaginary ? τ real > 0
U(-iτ) = exp(-τ/ħ H)
Hermitian operator
|| · || = exp(-τ/ħ E0)
ution
at temperature T)
density matrix
quantum Theory
ρ = exp(-1/(kB T) H) / Tr(exp(-1/(kB T) H))
ρ = U(-iτ) / Tr[U(-iτ)]

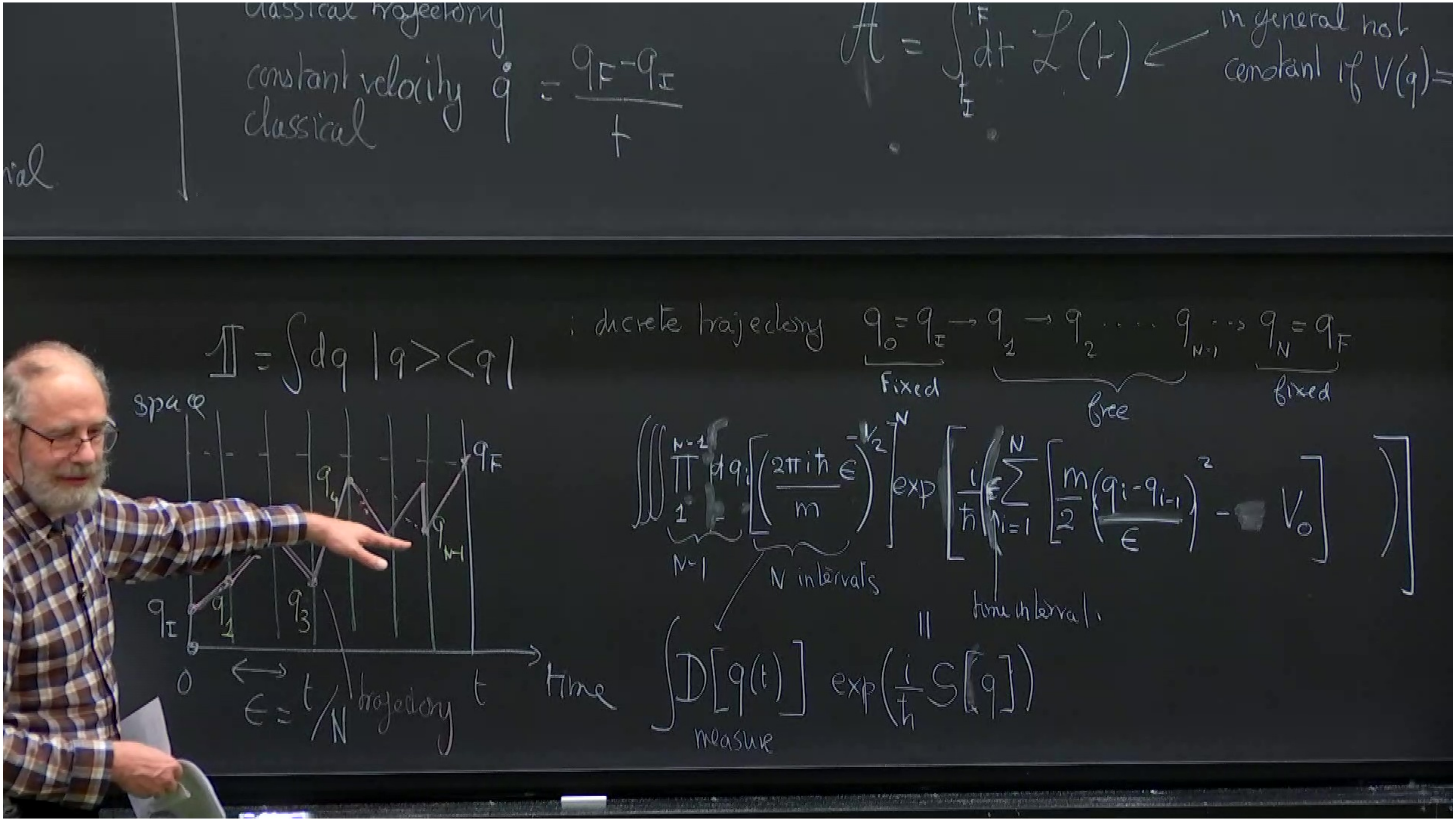

1st Case : Free particle $V(q) = V_0$ = a constant
$K(q_F, t; q_I, 0) = \left(\frac{2i\pi\hbar t}{m}\right)^{-1/2} \exp\left[\frac{i}{\hbar}\left(\frac{m}{2} \cdot \frac{(q_F - q_I)^2}{t} - t \cdot V_0\right.\right.$

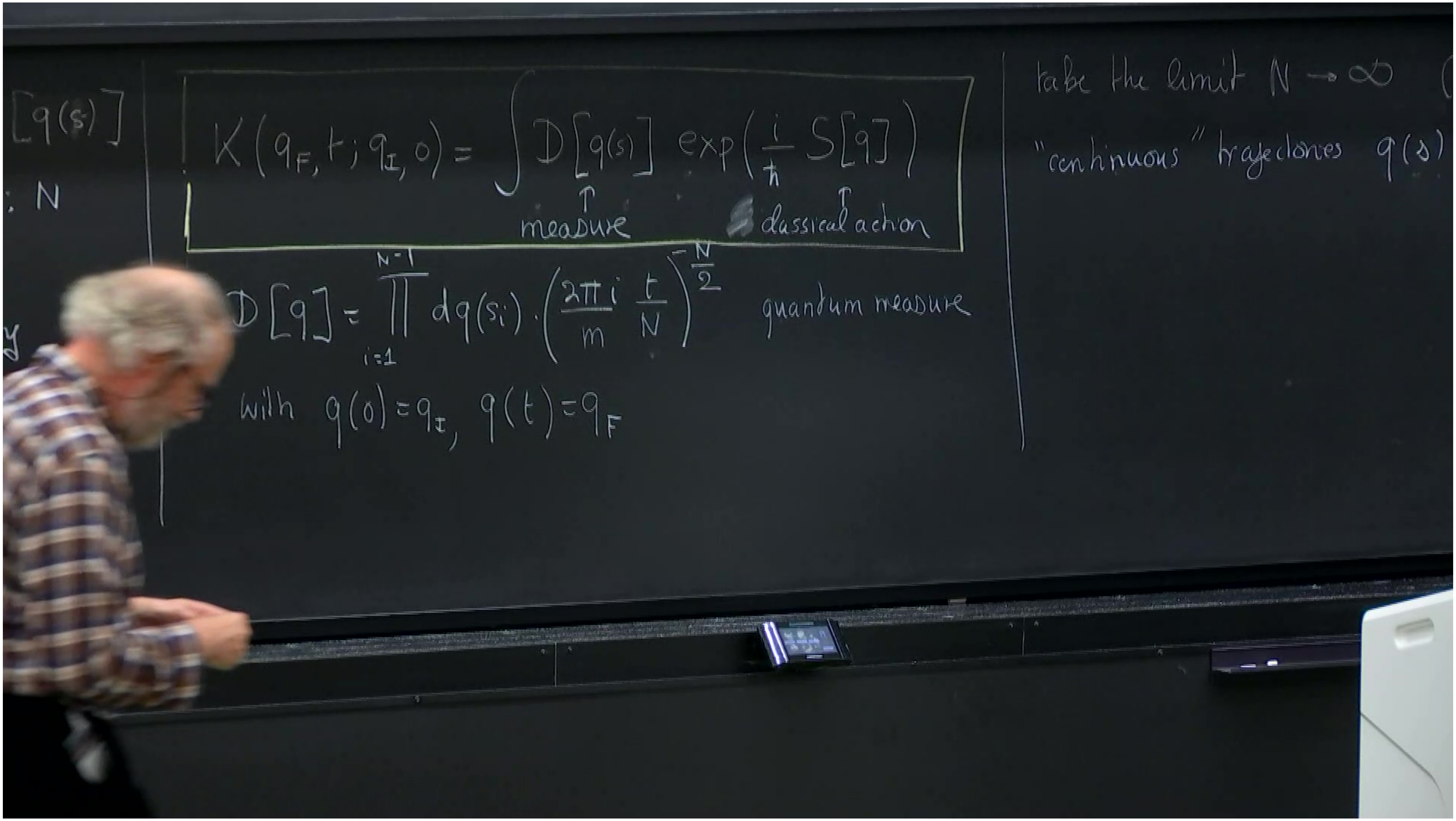

a constant
$-q_I)^2$
t
t. V_0
Space
q_F
q_I
q_F
0
t
time
lassical trajectory

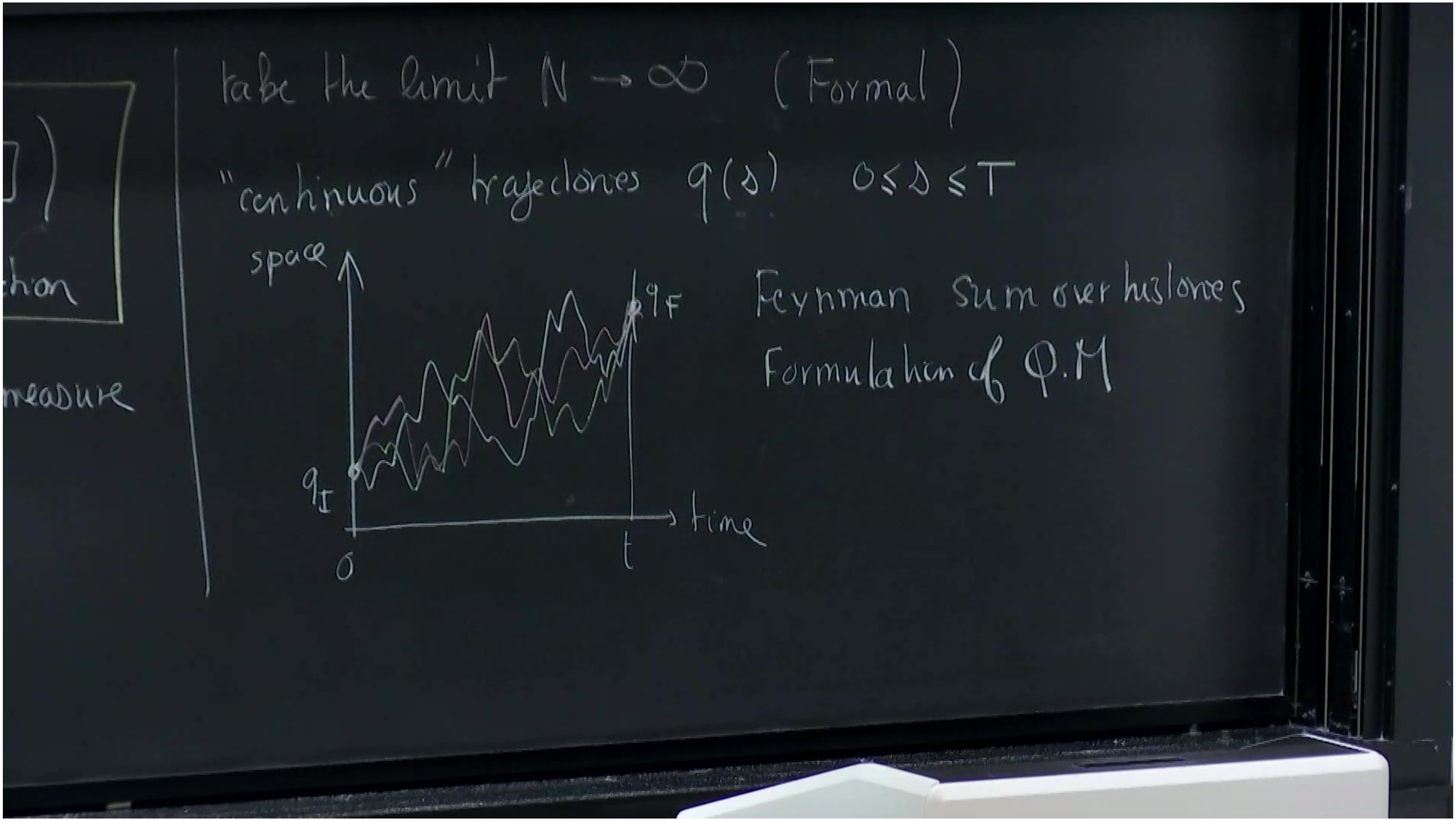

Feynman
$K(q_F, t_F; q_I, 0) = \langle q_F | U(t) | q_I \rangle = \langle q_F | U(t/N)^N | q_I \rangle$
$\mathbb{1} = \int dq \, |q\rangle\langle q|$
decompose
$\int dq_1 \cdots dq_{N-1} \langle q_F | U(\frac{t}{N}) | q_{N-1} \rangle \langle q_{N-1} | \cdots | q_1 \rangle \langle q_1 | U(\frac{t}{N}) | q_I \rangle$


classical trajectory
constant velocity $\dot{q} = \frac{q_F - q_I}{t}$
classical
$A = \int_{t_I}^{t_F} dt\, \mathcal{L}(t)$
in general not constant if $V(q) =$
$\mathbb{1} = \int dq\, |q\rangle\langle q|$
: discrete trajectory $\underbrace{q_0 = q_I}_{\text{Fixed}} \to \underbrace{q_1 \to q_2 \cdots q_{N-1}}_{\text{free}} \cdots \to \underbrace{q_N = q_F}_{\text{fixed}}$
space
q_I q_1 q_2 q_3 q_4 q_{N-1} q_F
0 t time
$\epsilon = t/N$
$\iiint \prod_{1}^{N-1} dq_i \left(\frac{2\pi i \hbar}{m}\epsilon\right)^{-1/2} \exp\left(\frac{i}{\hbar}\sum_{i=1}^{N} \frac{m}{2}(q_i - q_{i-1})^2\right)$


classical trajectory
constant velocity
classical
in general not
constant if V(q)=
discrete trajectory
Fixed
free
fixed
space
exp
N intervals
time interval


classical trajectory
constant velocity $\dot{q} = \frac{q_F - q_I}{t}$
classical
$A = \int_{t_I}^{t_F} dt\, \mathcal{L}(t)$
in general not constant if $V(q) =$
$\mathbb{1} = \int dq\, |q\rangle\langle q|$
: discrete trajectory $q_0 = q_I \to q_1 \to q_2 \cdots q_{N-1} \cdots q_N = q_F$
fixed
free
fixed
space
q_F
q_I
q_1
q_3
q_4
q_{N-1}
0
$\epsilon = t/N$
trajectory
t
time
$\iiint \prod_1^{N-1} dq_i \left[\left(\frac{2\pi i \hbar}{m}\epsilon\right)^{-1/2}\right]^N \exp\left[\frac{i}{\hbar}\epsilon \sum_{i=1}^{N}\left[\frac{m}{2}\left(\frac{q_i - q_{i-1}}{\epsilon}\right)^2 - V_0\right]\right]$
N-1
N intervals
time interval
$\int D[q(t)]\, \exp\left(\frac{i}{\hbar} S[q]\right)$
measure


$K(q_F, t; q_I, 0) = \int D[q(s)] \exp\left(\frac{i}{\hbar} S[q]\right)$
measure
classical action
$D[q] = \prod_{i=1}^{N-1} dq(s_i) \cdot \left(\frac{2\pi i}{m} \frac{t}{N}\right)^{-\frac{N}{2}}$
quantum measure
with $q(0)=q_I$, $q(t)=q_F$
take the limit $N \to \infty$
"continuous" trajectories $q(s)$


take the limit $N \to \infty$ (Formal)
"continuous" trajectories $q(s)$ $0 \leq s \leq T$
space
q_F
q_I
time
0
t
Feynman sum over histories
Formulation of Q.M
tion
measure

$_N = t \quad q(T_N) = q_F$

(6) Same game for imaginary time? $U(-i\tau) := U_E(\tau)$ E goes for "Euclidean" $\Rightarrow$ QFT

τ "Euclidean time"

$$K_E(q_F, \tau; q_I, 0) = \left(\frac{2\pi\hbar}{m}\cdot\tau\right)^{-1/2} \exp\left(-\frac{1}{\hbar}\underbrace{\left(\frac{(q_F - q_I)^2}{\tau} + \tau V_0\right)}_{\text{Euclidean Action}}\right)$$

"Euclidean propagator"

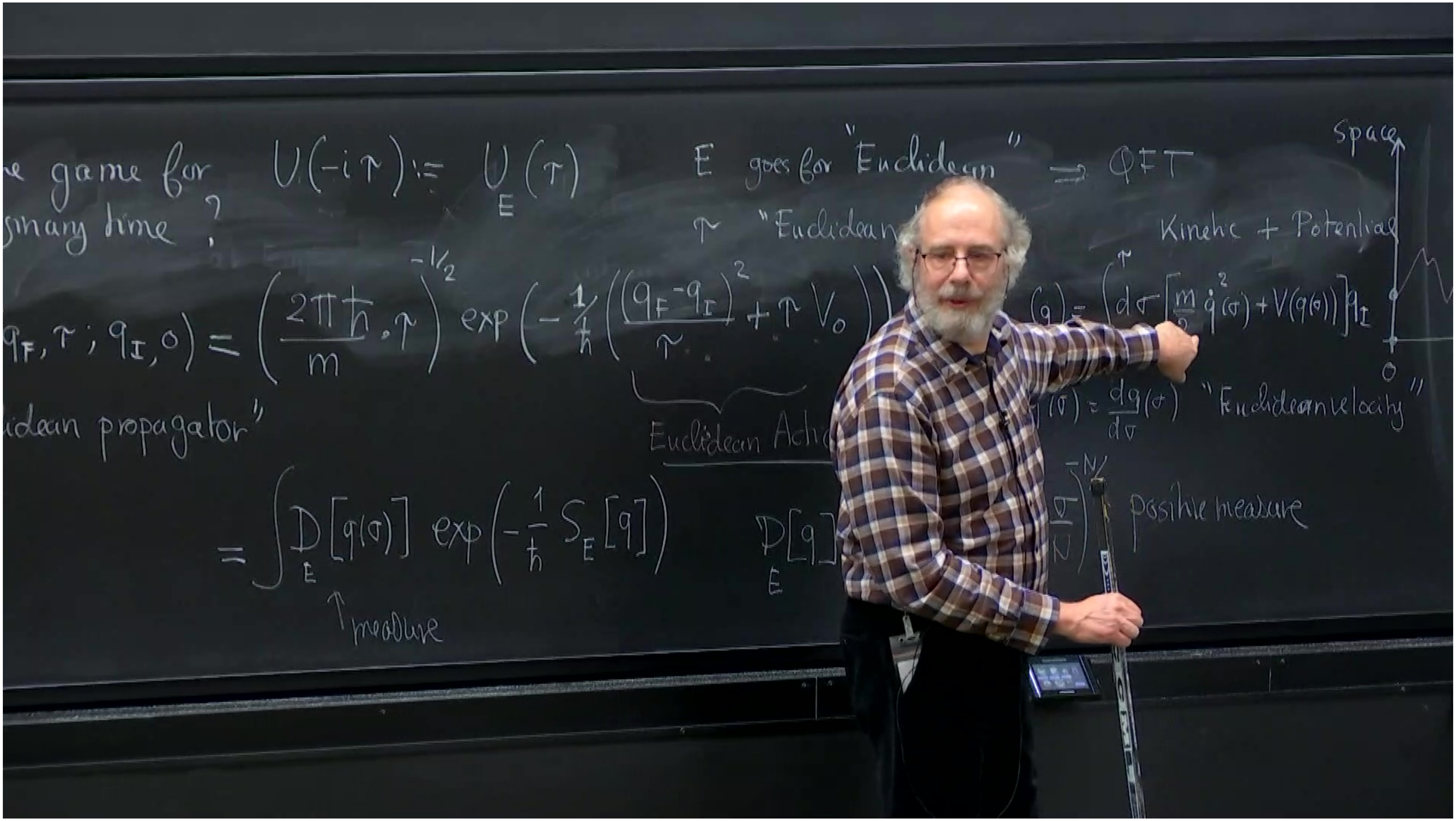
E goes for "Euclidean" $\Longrightarrow$ QFT

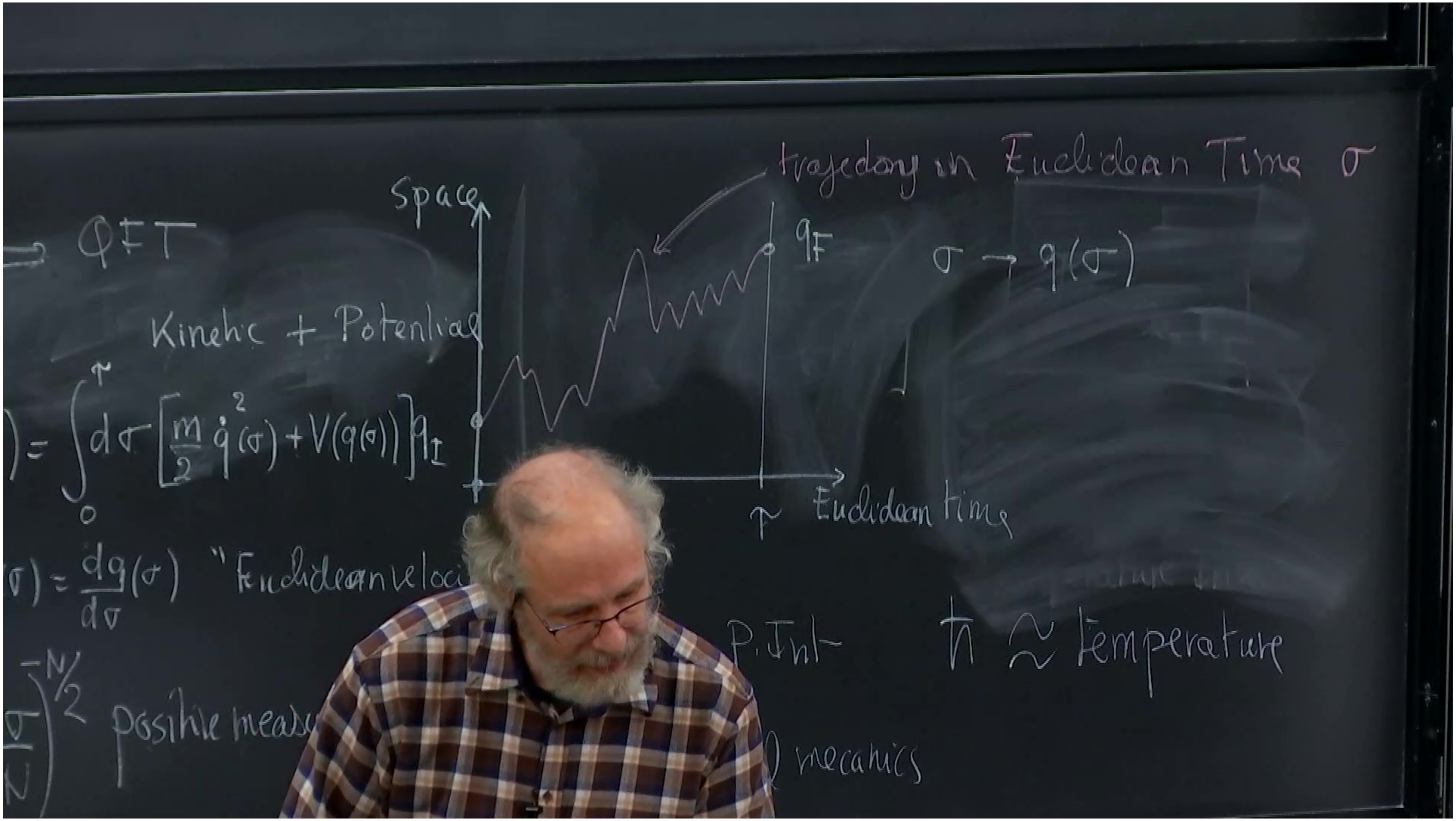
τ "Euclidean time"

$$\underbrace{\frac{(q_F - q_I)^2}{\tau} + \tau V_0 \Big)\Big)}_{\text{Euclidean Action}}$$

$$S_E(q) = \int_0^{\tau} d\sigma \left[\frac{m}{2} \dot{q}^2(\sigma) + V(q(\sigma)) \right]_{q_I}$$

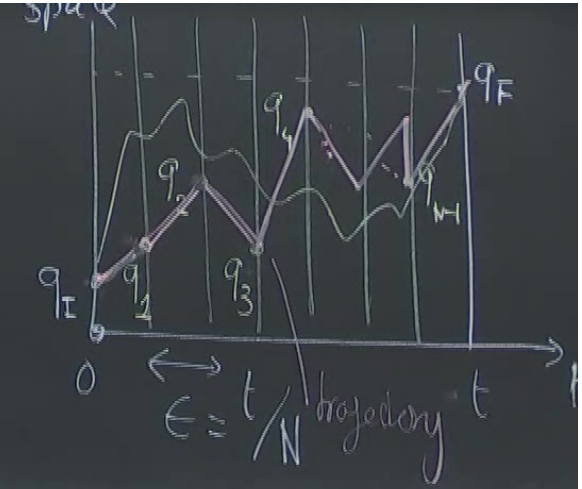
$$\dot{q}(\sigma) = \frac{dq}{d\sigma}(\sigma) \quad \text{"Euclidean velocity"}$$


game for
U(-iτ) := U_E(τ)
E goes for "Euclidean" ⇒ QFT
Space
Kinetic + Potential
propagator"
Euclidean Action
"Euclidean velocity"
positive measure
measure


trajectory in Euclidean Time σ
Space
QFT
Kinetic + Potential
q_F
σ → q(σ)
Euclidean time
P. Int
ħ ≈ temperature
mecanics

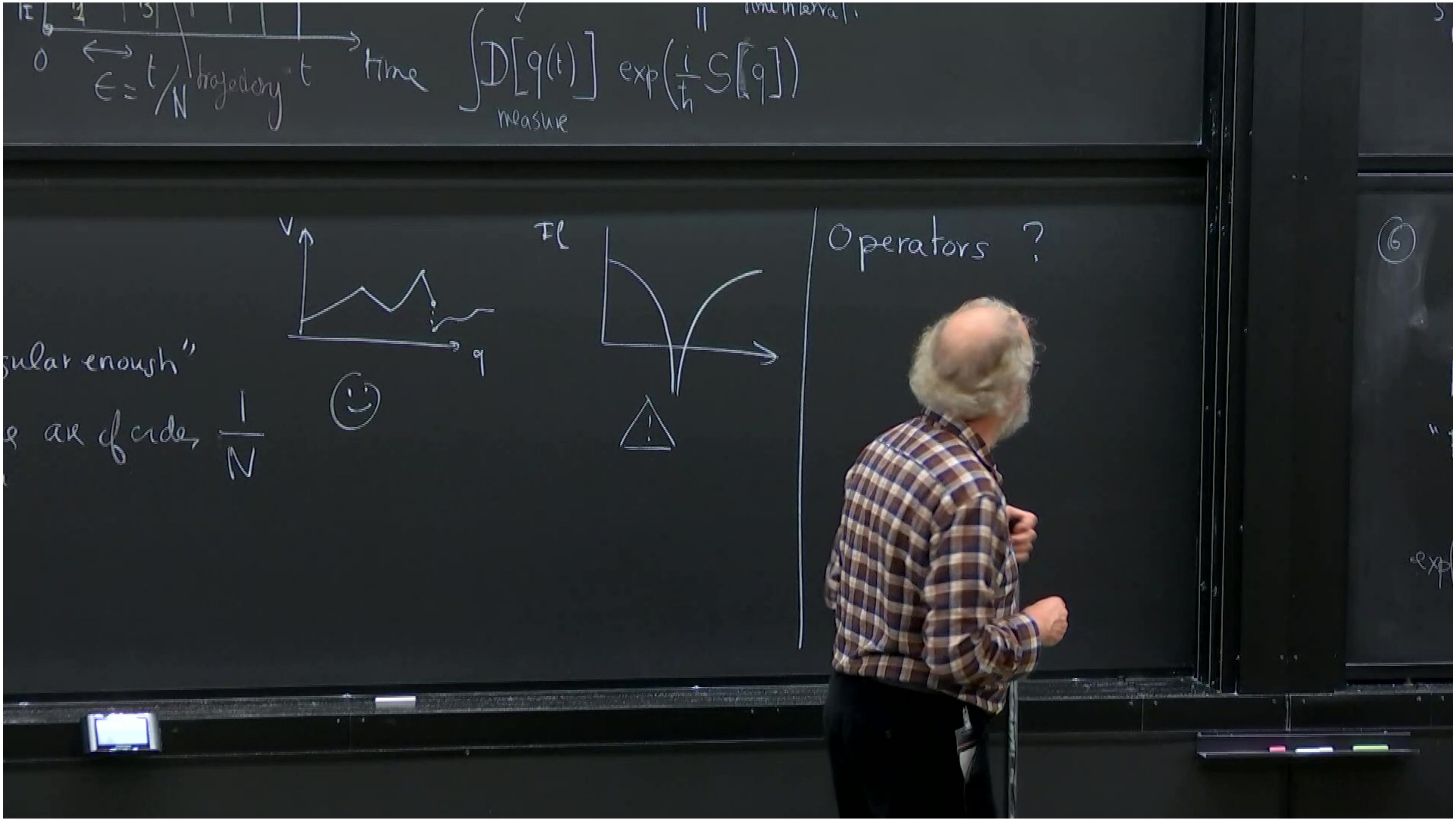
· decompose

$$\int dq_1 \cdots dq_{N-1} \underbrace{\langle q_F | U\left(\tfrac{t}{N}\right) | q_{N-1}\rangle}_{\text{formula for } K} \langle q_{N-1} | \cdots | q_1 \rangle \langle q_1 | U\left(\tfrac{t}{N}\right) | q_I \rangle$$



- $V(q) = V_0 = \text{constant}$ no external forces
- ☺ This works when $V(q) \neq \text{constant}$

  But only in the limit $N \to \infty$! corrections to the above calculation are of order $\frac{1}{N}$


∫D[q(t)] exp(i/ħ S[q])
measure
time
trajectory
ε = t/N
Operators ?
"gular enough"
are of order 1/N