Title: Quantum Field Theory II - Lecture 221115

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Collection: Quantum Field Theory II (2022/2023)

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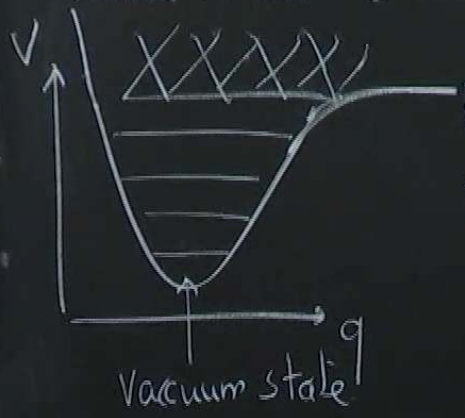
Real time $t \longrightarrow$ "imaginary" time



E. e.values of H

$E \geq E_0$

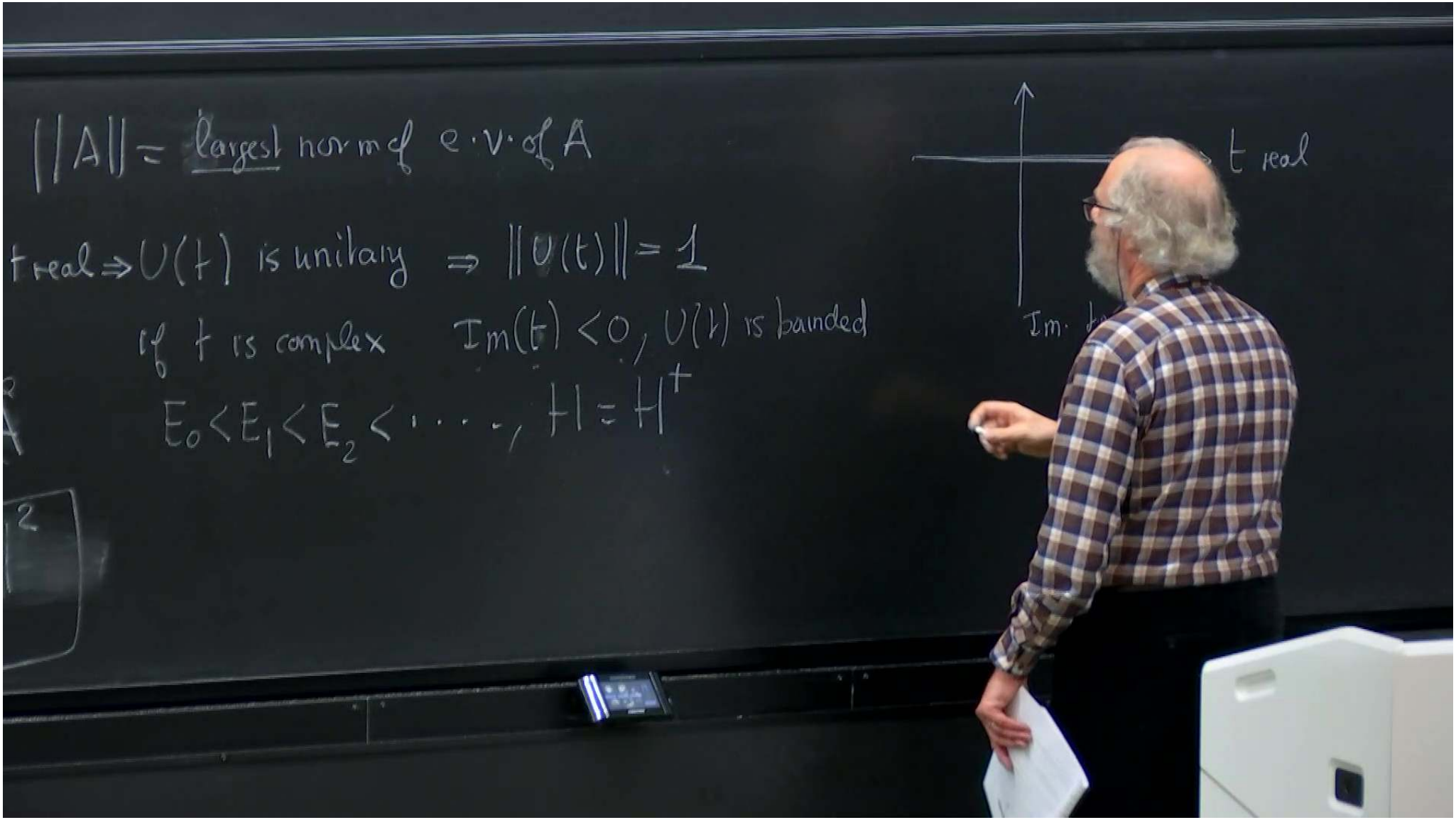
norm of the operator A

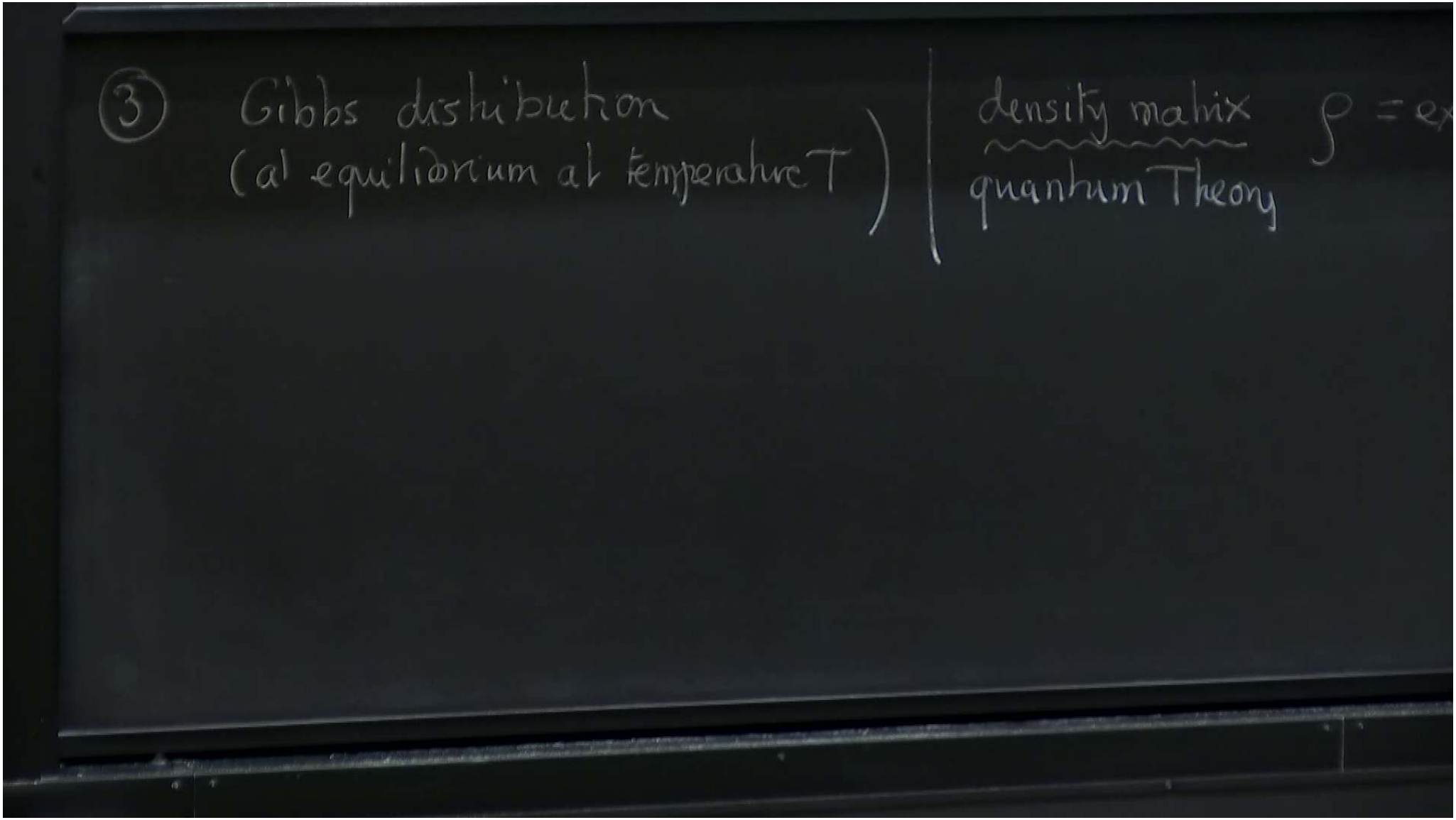
time complex: $U(t)$ not unitary

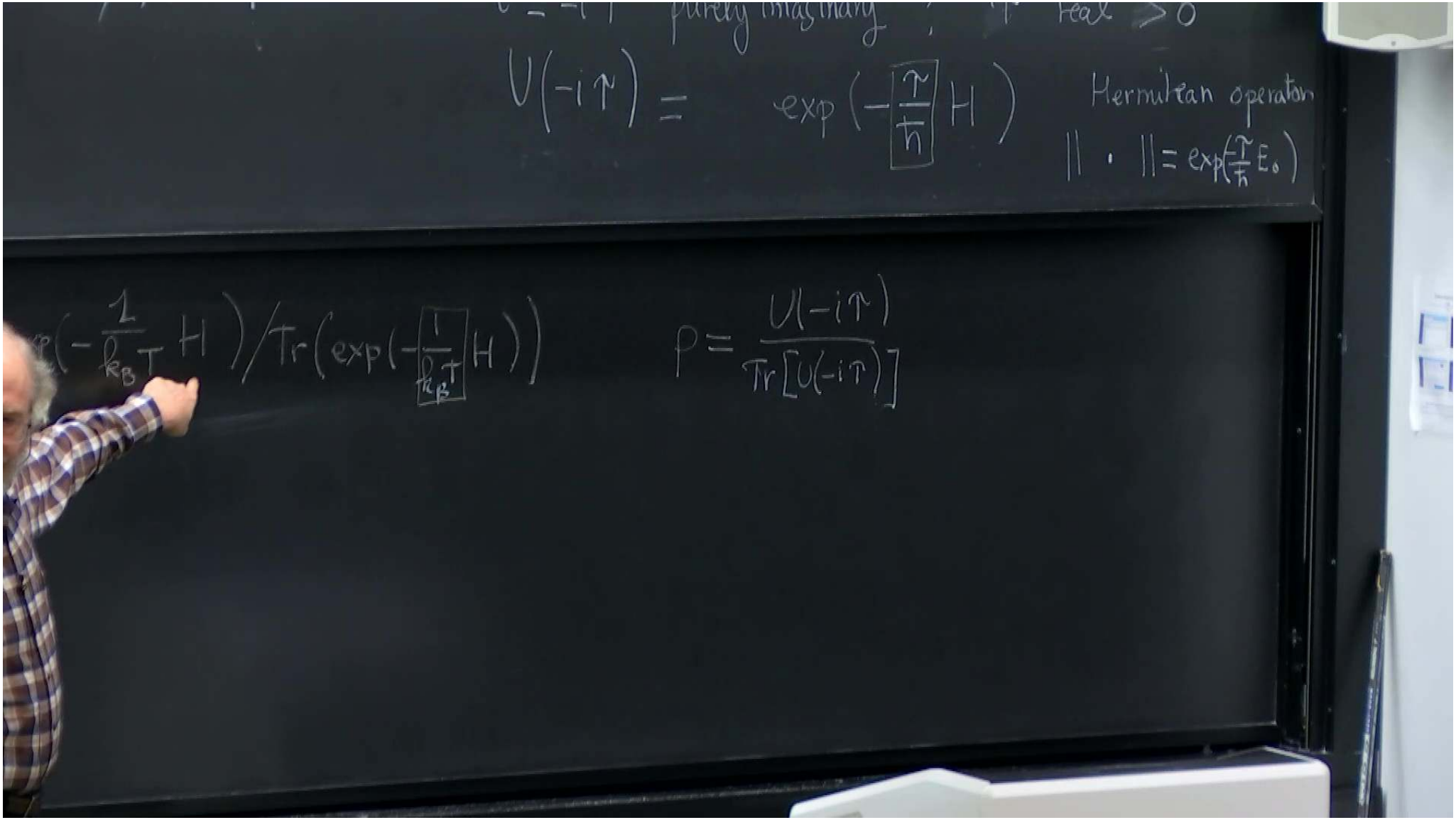
Bounded operators $\boxed{\langle\psi|A^\dagger A|\psi\rangle \doteq \|A\|^2}$

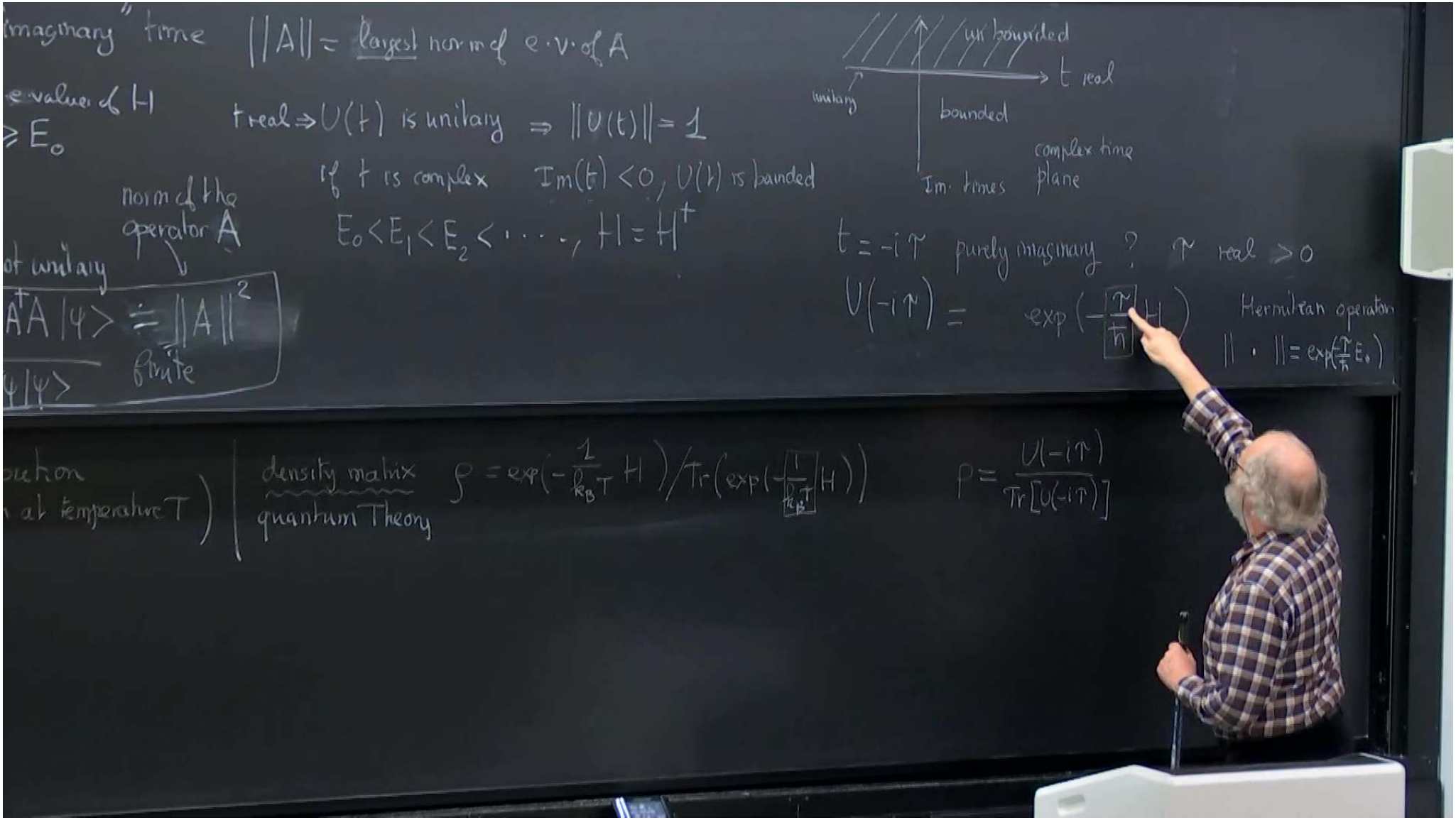
$\boxed{\sup_{|\psi\rangle} \frac{\langle\psi|A^\dagger A|\psi\rangle}{\langle\psi|\psi\rangle} \text{ finite}}$

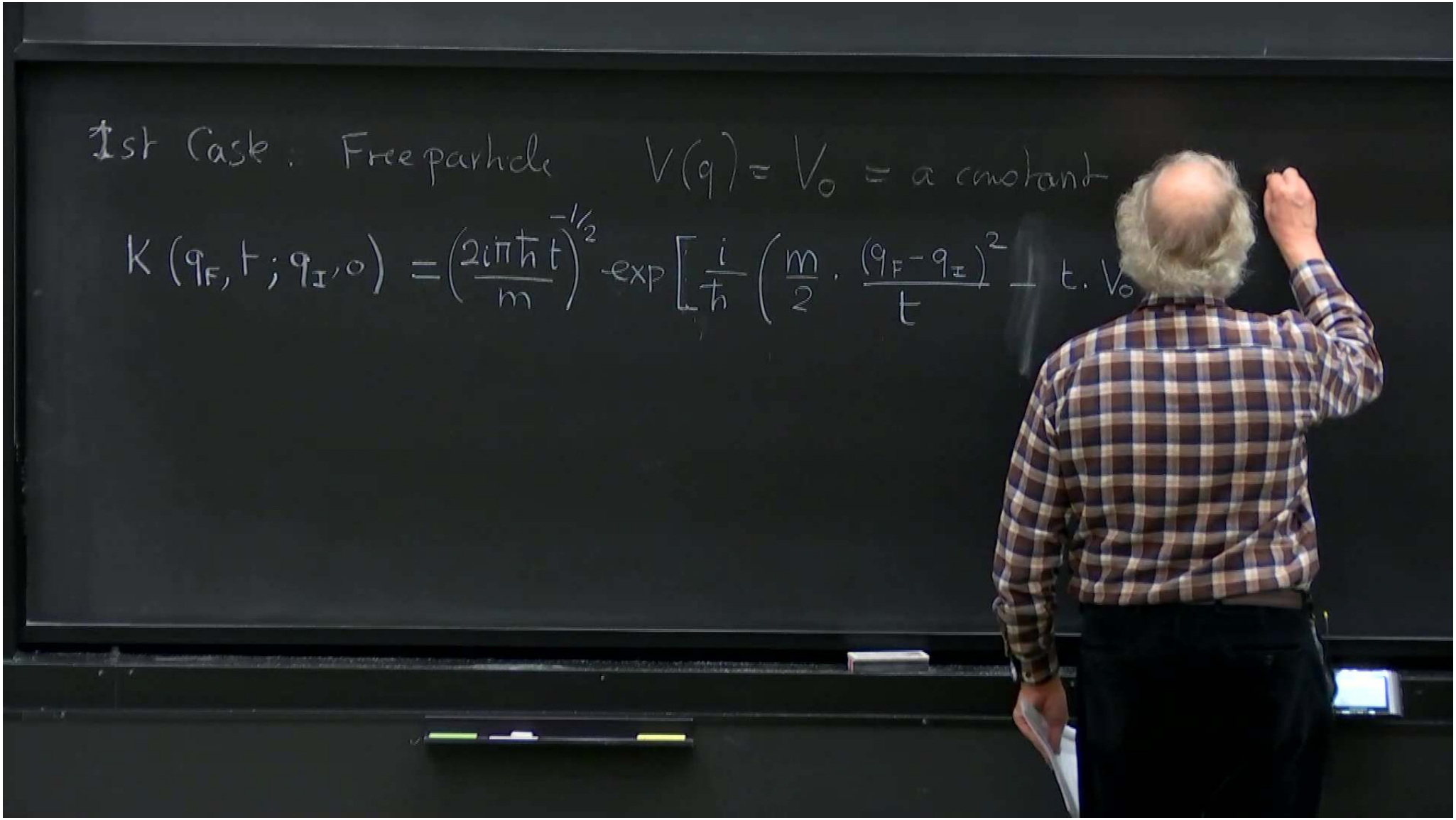
② Definition

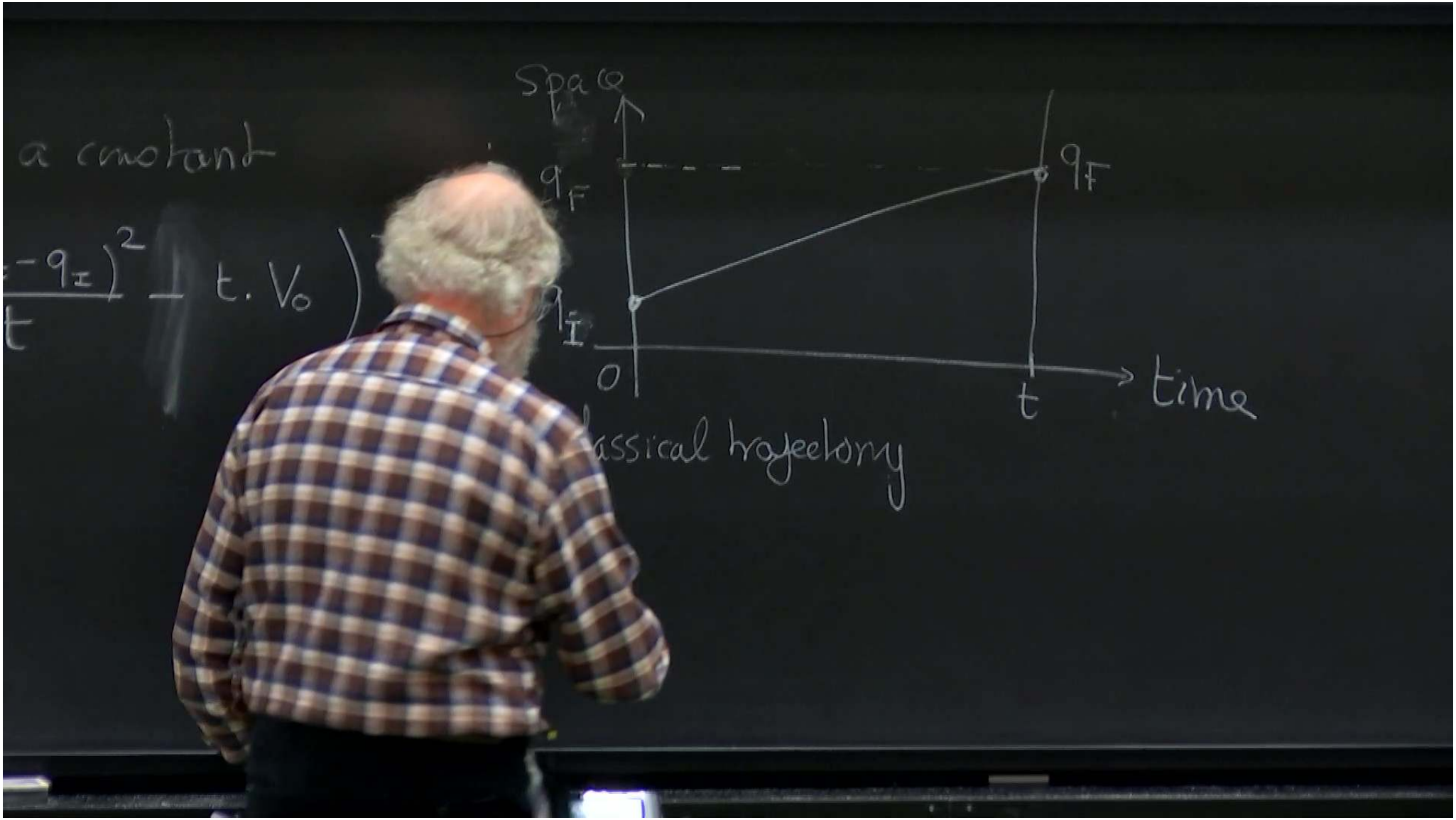

||A|| = largest norm of e.v. of A
t real ⇒ U(t) is unitary ⇒ ||U(t)|| = 1
if t is complex Im(t) < 0, U(t) is banded
E0 < E1 < E2 < ···, H† = H
t real

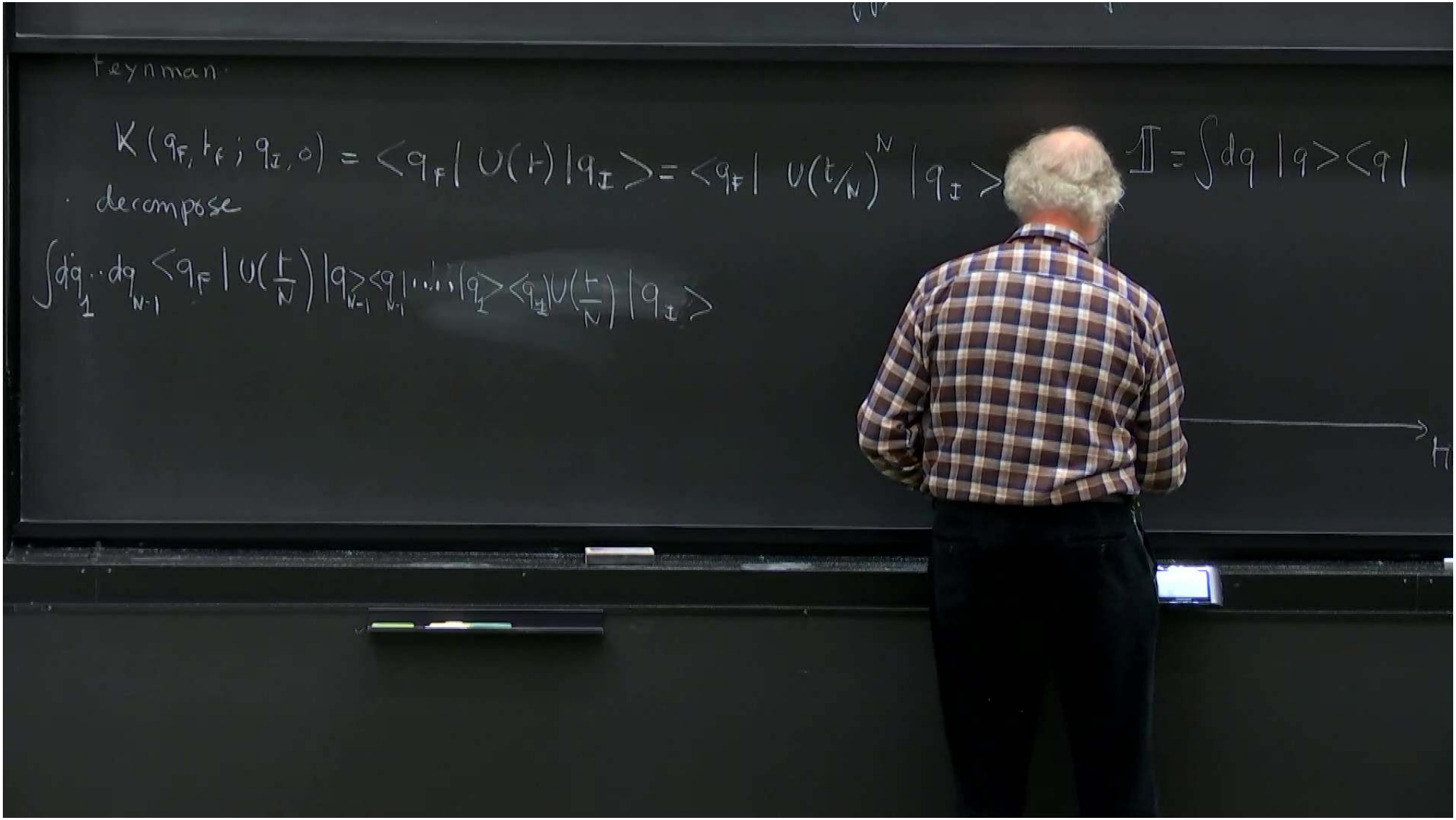

③ Gibbs distribution
(at equilibrium at temperature T)
density matrix
quantum Theory


purely imaginary
Real > 0
$U(-i\tau) = \exp\left(-\frac{\tau}{\hbar} H\right)$
Hermitian operator
$\|\cdot\| = \exp\left(\frac{\tau}{\hbar} E_0\right)$
$\left(-\frac{1}{k_B T} H\right) / \mathrm{Tr}\left(\exp\left(-\frac{1}{k_B T} H\right)\right)$
$\rho = \frac{U(-i\tau)}{\mathrm{Tr}[U(-i\tau)]}$


imaginary" time
||A|| = largest norm of e.v. of A
e.value of H
≥ E₀
t real ⇒ U(t) is unitary ⇒ ||U(t)|| = 1
if t is complex Im(t) < 0, U(t) is bounded
E₀ < E₁ < E₂ < ...., H = H†
norm of the operator A
ot unitary
A†A |ψ> ≐ ||A||²
finite
ψ|ψ>
un bounded
t real
unitary
bounded
complex time plane
Im. times
t = -iτ purely imaginary ? τ real > 0
U(-iτ) = exp(-τ/ħ H)
Hermitian operator
|| · || = exp(-τ/ħ E₀)
ution
at temperature T)
density matrix
quantum Theory
ρ = exp(-1/k_B T H) / Tr(exp(-1/k_B T H))
ρ = U(-iτ) / Tr[U(-iτ)]


1st Case: Free particle V(q) = V0 = a constant
K(qF, t; qI, 0) = (2iπħt/m)^(-1/2) exp[i/ħ (m/2 · (qF - qI)^2/t - t · V0


a constant
Space
q_F
q_F
q_I
0
t
time
assical trajectory


feynman
$K(q_F, t_F; q_I, 0) = \langle q_F | U(t) | q_I \rangle = \langle q_F | U(t/N)^N | q_I \rangle$
$\mathbb{1} = \int dq \, |q\rangle\langle q|$
· decompose
$\int dq_1 \cdots dq_{N-1} \langle q_F | U(\frac{t}{N}) | q_{N-1} \rangle \langle q_{N-1} | \cdots | q_1 \rangle \langle q_1 | U(\frac{t}{N}) | q_I \rangle$

classical trajectory

constant velocity $\dot{q} = \frac{q_F - q_I}{t}$

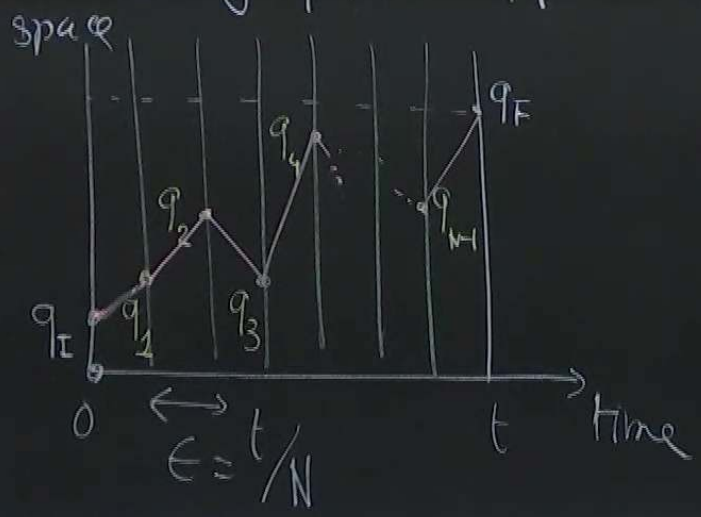
classical

$A = \int_{t_I}^{t_F} dt\, \mathcal{L}(t)$ ← in general not constant if $V(q) =$

$|I\rangle$

$\mathbb{1} = \int dq\, |q\rangle\langle q|$

: discrete trajectory $\underbrace{q_0 = q_I}_{\text{Fixed}} \to \underbrace{q_1 \to q_2 \cdots q_{N-1}}_{\text{free}} \cdots \to \underbrace{q_N = q_F}_{\text{fixed}}$



$$\iint\!\!\int \prod_1^{N-1} dq_i \left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2} \exp\left(\frac{i}{\hbar}\sum_{i=1}^{N}\frac{m}{2}(q_i - q_{i-1})^2\right.$$

classical trajectory

constant velocity $\dot{q} = \frac{q_F - q_I}{t}$

classical

$A = \int_{t_I}^{t_F} dt\, \mathcal{L}(t)$ ← in general not constant if $V(q) =$

$\mathbb{1} = \int dq\, |q\rangle\langle q|$; discrete trajectory $\underbrace{q_0 = q_I}_{\text{Fixed}} \to \underbrace{q_1 \to q_2 \cdots q_{N-1}}_{\text{Free}} \to \underbrace{q_N = q_F}_{\text{fixed}}$

$I\rangle$

space

q_I q_1 q_2 q_3 q_4 0 $\epsilon = t$

$$\iint \prod_{1}^{N-1} dq_i \left[\left(\frac{2\pi i\hbar}{m}\epsilon\right)^{-1/2}\right]^N \exp\left[\frac{i}{\hbar}\sum_{i=1}^{N}\left[\frac{m}{2}\left(\frac{q_i - q_{i-1}}{\epsilon}\right)^2 - V_0\right]\right]$$

$N-1$ N intervals time intervals

$= \int \mathcal{D}[q$

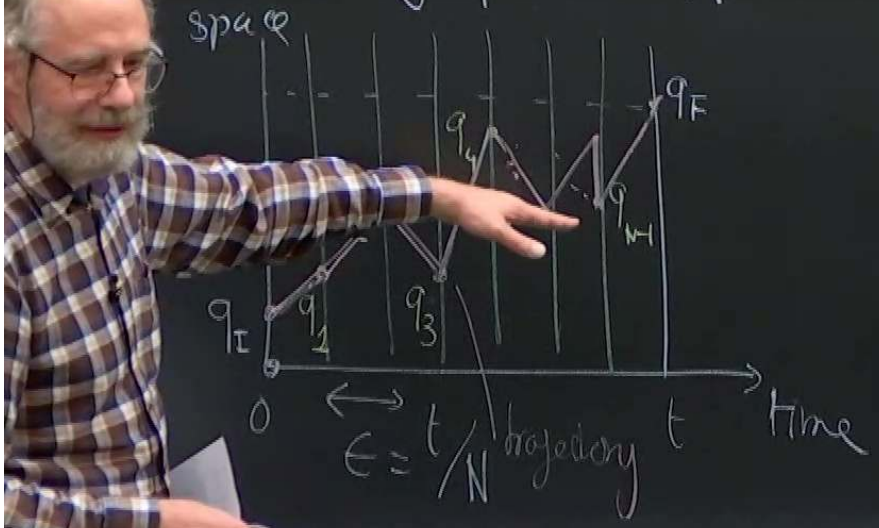
classical trajectory

constant velocity $\dot{q} = \frac{q_F - q_I}{t}$

classical

$A = \int_{t_I}^{t_F} dt\, \mathcal{L}(t)$ ← in general not constant if $V(q) =$

$\mathbb{1} = \int dq\, |q\rangle\langle q|$; discrete trajectory $\underbrace{q_0 = q_I}_{\text{Fixed}} \to q_1 \to q_2 \cdots \underbrace{q_{N-1}}_{\text{free}} \to \underbrace{q_N = q_F}_{\text{fixed}}$

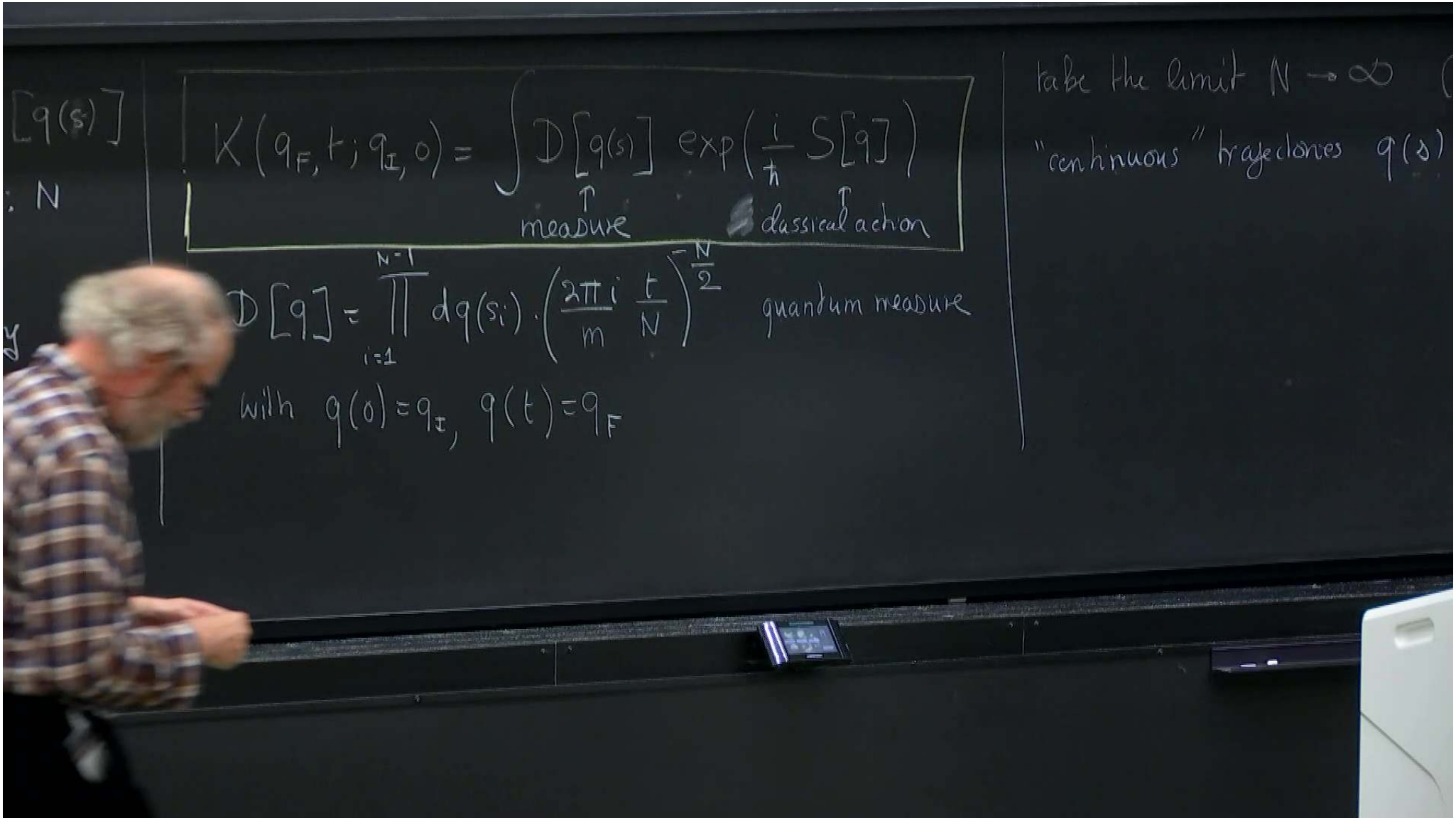


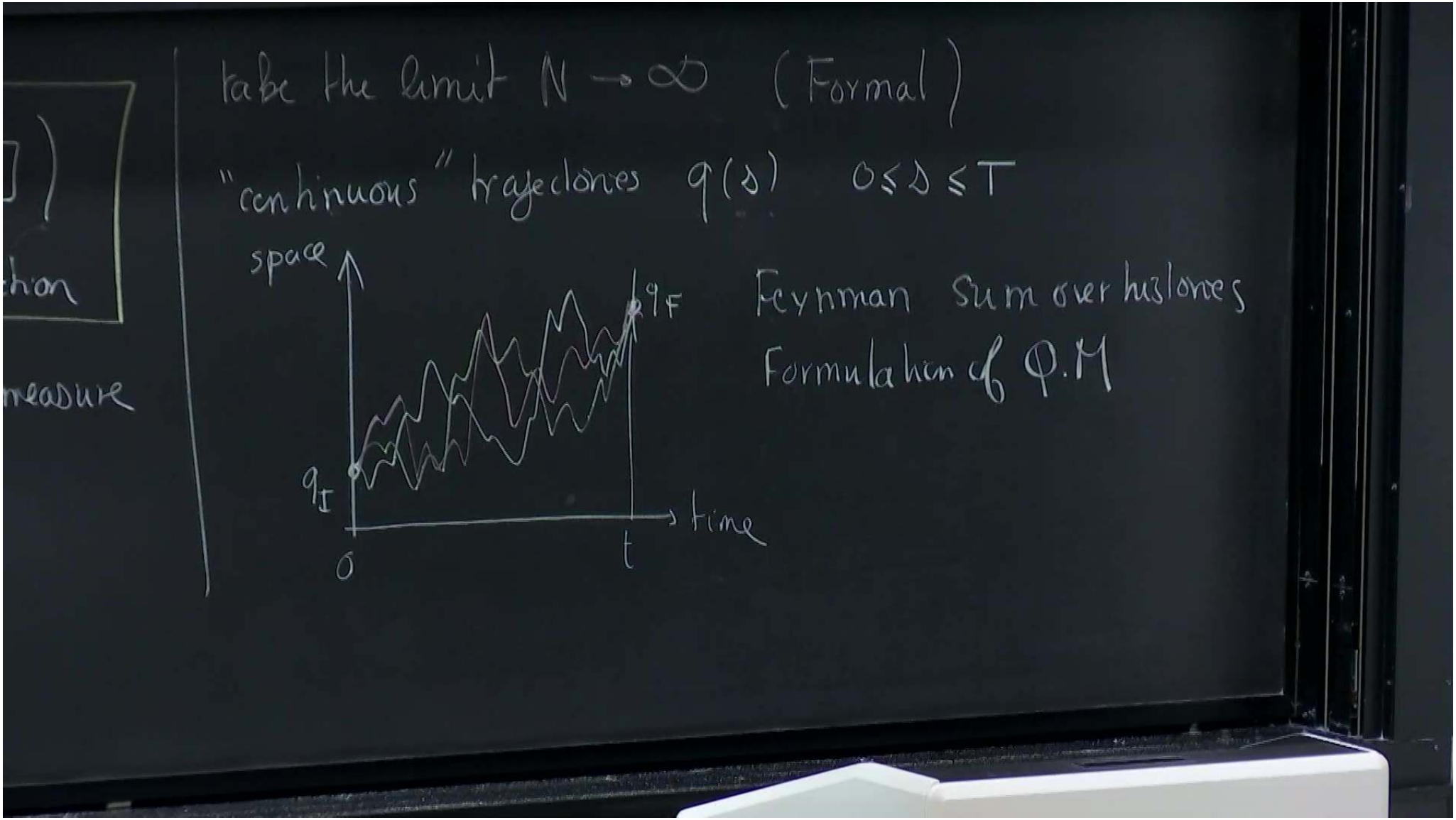
$$\int \prod_{1}^{N-1} dq_i \left[\left(\frac{2\pi i\hbar\epsilon}{m}\right)^{-1/2}\right]^N \exp\left[\frac{i}{\hbar}\sum_{i=1}^{N}\left[\frac{m}{2}\left(\frac{q_i - q_{i-1}}{\epsilon}\right)^2 - V_0\right]\right]$$

$N-1$ integrals; N intervals; time intervals.

$$\int \mathcal{D}[q(t)]\, \exp\left(\frac{i}{\hbar} S[q]\right)$$

measure


$[q(s)]$
$: N$
$K(q_F, t; q_I, 0) = \int \mathcal{D}[q(s)] \exp\left(\frac{i}{\hbar} S[q]\right)$
measure
classical action
$\mathcal{D}[q] = \prod_{i=1}^{N-1} dq(s_i) \cdot \left(\frac{2\pi i}{m} \frac{t}{N}\right)^{-\frac{N}{2}}$ quantum measure
with $q(0) = q_I$, $q(t) = q_F$
take the limit $N \to \infty$
"continuous" trajectories $q(s)$


take the limit N → ∞ (Formal)
"continuous" trajectories q(s) 0 ≤ s ≤ T
space
q_F
q_I
time
0
t
Feynman Sum over histories
Formulation of Q.M
measure

$q(\tau_N) = q_F$

(6) Same game for imaginary time ? $U(-i\tau) := U_E(\tau)$ E goes for "Euclidean" $\Rightarrow$ QFT

τ "Euclidean time"

$$K_E(q_F, \tau; q_I, 0) = \left(\frac{2\pi\hbar}{m}\cdot\tau\right)^{-1/2} \exp\left(-\frac{1}{\hbar}\left(\underbrace{\frac{(q_F - q_I)^2}{\tau} + \tau V_0}_{\text{Euclidean Action}}\right)\right)$$

"Euclidean propagator"

E goes for "Euclidean" $\Rightarrow$ QFT

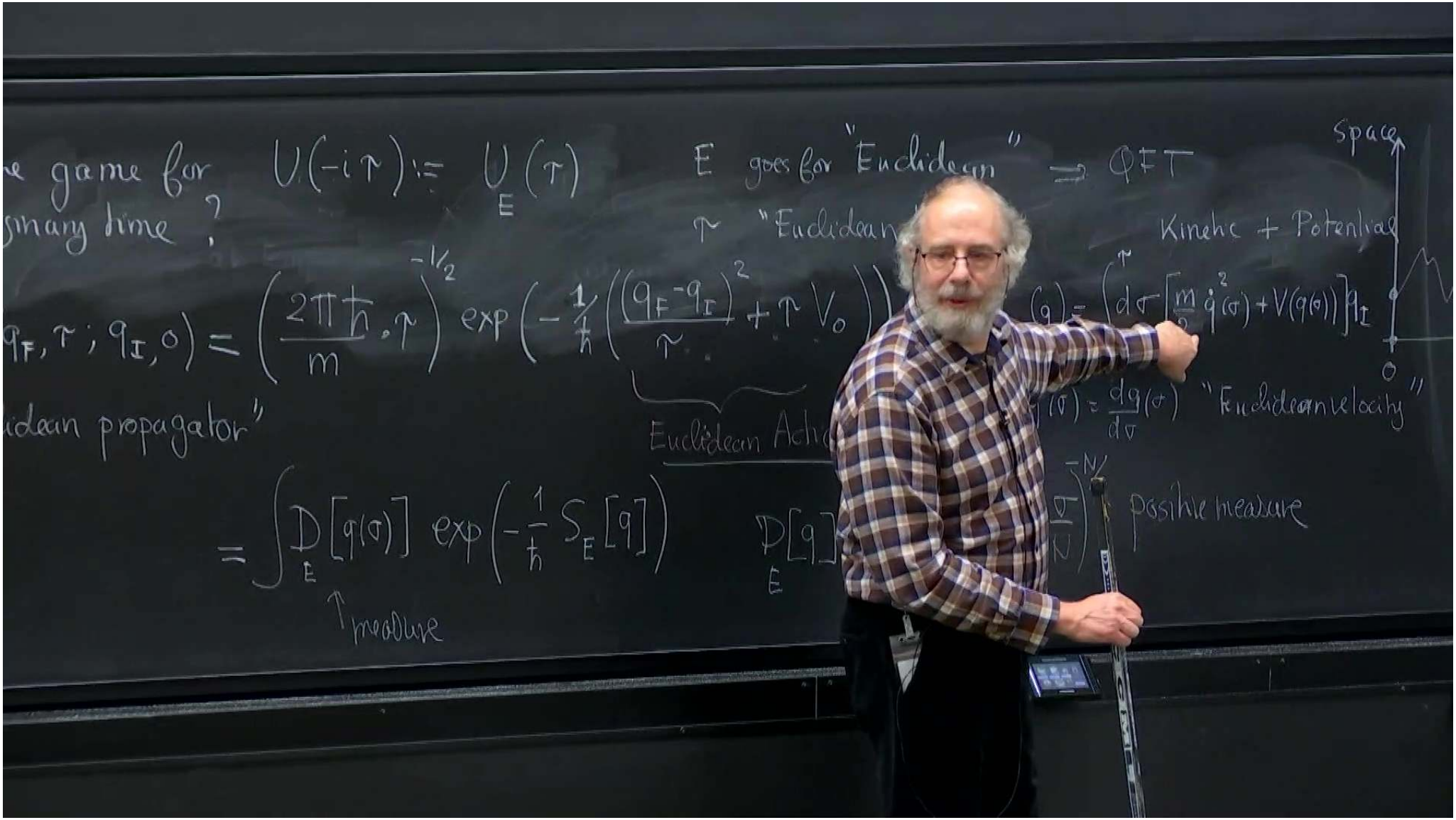
τ "Euclidean time"

$$\left(\frac{(q_F - q_I)^2}{\tau} + \tau V_0\right)$$

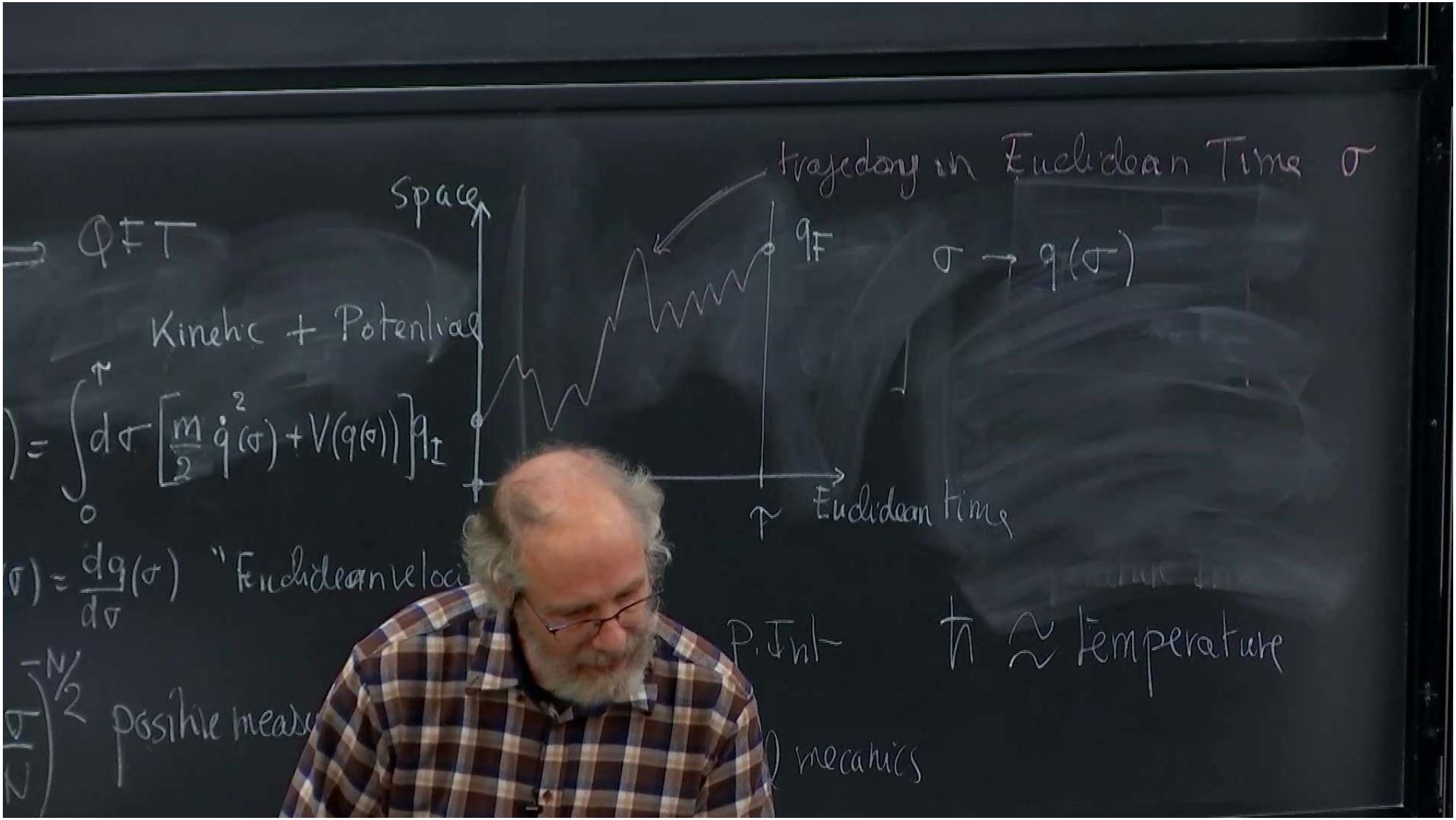
Euclidean Action

$$S_E(q) = \int_0^\tau d\sigma \left[\frac{m}{2}\dot{q}^2(\sigma) + V(q(\sigma))\right]$$

$$\dot{q}(\sigma) = \frac{dq}{d\sigma}(\sigma) \quad \text{"Euclidean velocity"}$$



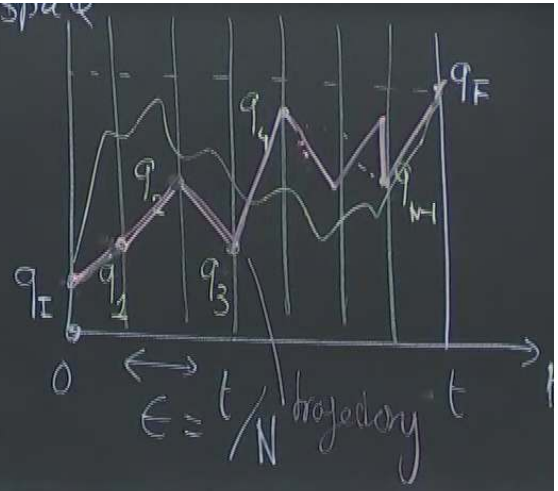
e game for
ginary time ?
U(-iτ) := U_E(τ)
E goes for "Euclidean" ⇒ QFT
τ "Euclidean
Kinetic + Potential
Space
q_F, τ ; q_I, 0) = (2πħ/m · τ)^{-1/2} exp(-1/ħ ((q_F - q_I)^2/τ + τ V_0))
idean propagator"
Euclidean Acti
"Euclidean velocity"
= ∫ D_E[q(σ)] exp(-1/ħ S_E[q])
measure
positive measure


trajectory in Euclidean Time σ
Space
QFT
Kinetic + Potential
q_F
$\sigma \to q(\sigma)$
$= \int_0^\tau d\sigma \left[\frac{m}{2}\dot{q}^2(\sigma) + V(q(\sigma))\right] q_I$
Euclidean time
$\frac{dq}{d\sigma}(\sigma)$ "Euclidean veloci
P. Int
$\hbar \approx$ temperature
positive meas
mecanics

decompose

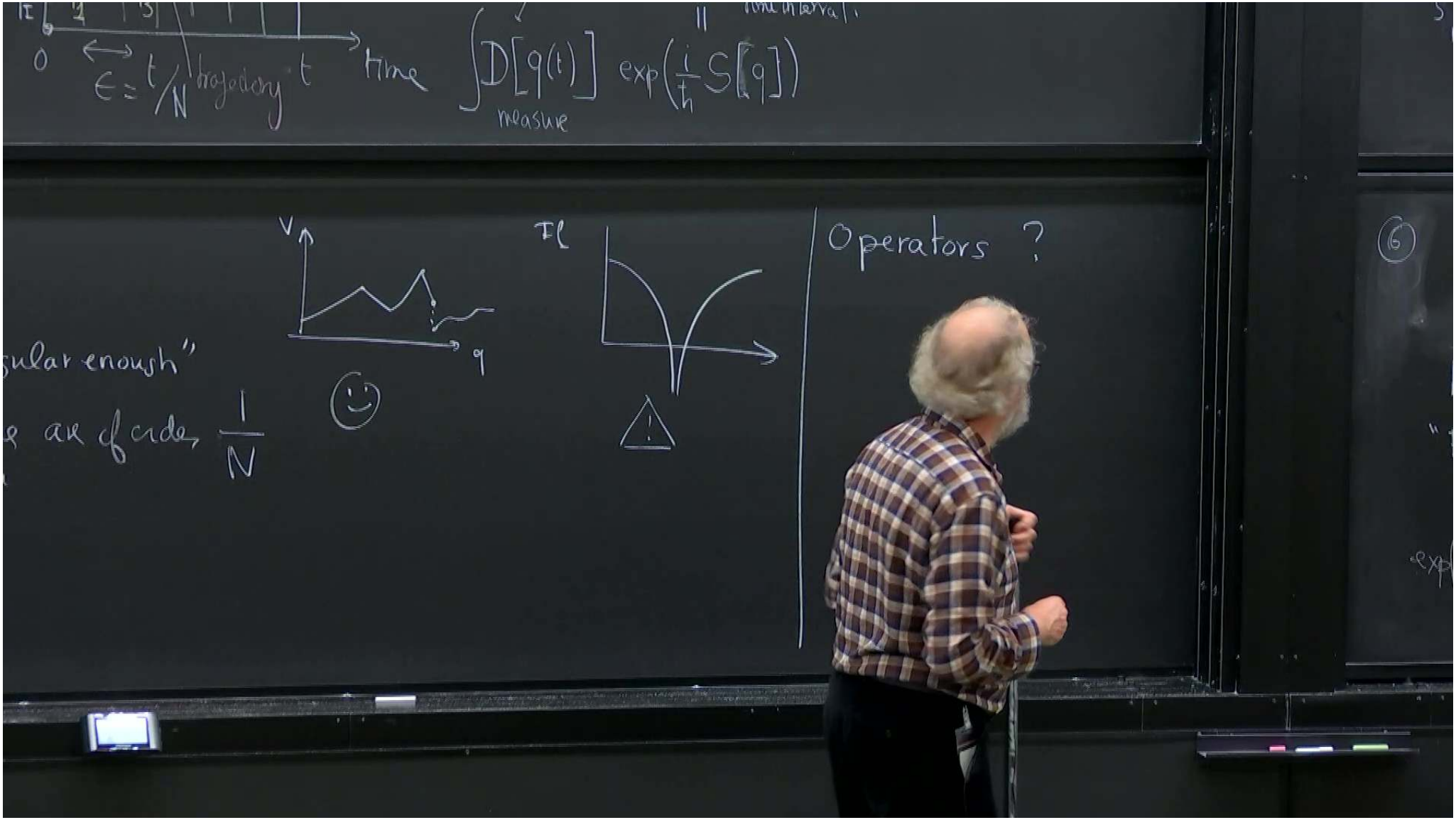
$$\int dq_1 \cdots dq_{N-1} \underbrace{\langle q_F | U(\tfrac{t}{N}) | q \rangle_{N-1} \langle q | \cdots | q \rangle \langle q_1 | U(\tfrac{t}{N}) | q_I \rangle}$$

formula for K



- $V(q) = V_0 = \text{constant}$ no external forces
- ☺ This <u>works</u> when $V(q) \neq \text{constant}$

  But only in the limit $N \to \infty$; correction to the one of order $\frac{1}{N}$ above calculation


0
$\epsilon = t/N$
trajectory
t
time
$\int \mathcal{D}[q(t)]$
measure
$\exp\left(\frac{i}{\hbar} S[q]\right)$
V
q
Operators ?
$\frac{1}{N}$