Title: Microstates of a 2d Black Hole in string theory

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Series: Quantum Fields and Strings

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Abstract: We analyse models of Matrix Quantum Mechanics in the double scaling limit that contain non-singlet states. The finite temperature partition function of such systems contains non-trivial winding modes (vortices) and is expressed in terms of a group theoretic sum over representations. We then focus on the model of Kazakov-Kostov-Kutasov when the first winding mode is dominant. In the limit of large representations (continuous Young diagrams), and depending on the values of the parameters of the model such as the compactification radius and the string coupling, the dual geometric background corresponds either to that of a long string (winding mode) condensate or a 2d (non-supersymmetric) semi-classical Black Hole competing with the thermal linear dilaton background. In the matrix model we are free to tune these parameters and explore various regimes of this phase diagram. Our construction allows us to identify the origin of the microstates of the long string condensate/2d Black Hole arising from the non trivial representations.

Zoom Link: https://pitp.zoom.us/j/95764320439?pwd=L1E1cEREM29ORjZhK21WdFN0RVgyQT09

Plan of the talk

- Introduction
  - Matrix Quantum Mechanics
  - $c = 1$ Liouville theory
  - Correspondence between MQM and $c = 1$ Liouville
- Main Part
  - 2D black hole and connection to the WZW model and Sine Liouville
  - The corresponding MQM model
  - Partition function in the Grand- Canonical ensemble connection with integrable hierarchies and representations
  - Computation of the Free energy for different Saddles

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  - Phase Diagram

- Summary and Future Directions

- MQM (gauged) is a $0+1$ dimensional quantum mechanical theory of $N \times N$ Hermitian matrices $M(t)$ and a non dynamical gauge field $A(t)$.

- The Path Integral is:

$$e^{-iW} = \int \mathcal{D}M \mathcal{D}A \exp\left[-iN \int_{t_{in}}^{t_f} dt \, \mathrm{Tr}\left(\frac{1}{2}(D_t M)^2 + \frac{1}{2}M^2 - \frac{\kappa}{3!}M^3 + \ldots\right)\right]$$

- One can diagonalise M by a unitary transformation $M(t) = U(t)\Lambda(t)U^\dagger(t)$ where $\Lambda(t)$ is diagonal and $U(t)$ unitary

- One then picks up a Jacobian from the path integral measure $(\forall t)$

$$\mathcal{D}M = \mathcal{D}U \prod_{i=1}^{N} d\lambda_i \Delta^2(\Lambda), \quad \Delta(\Lambda) = \prod_{i<j}^{N} (\lambda_i - \lambda_j)$$

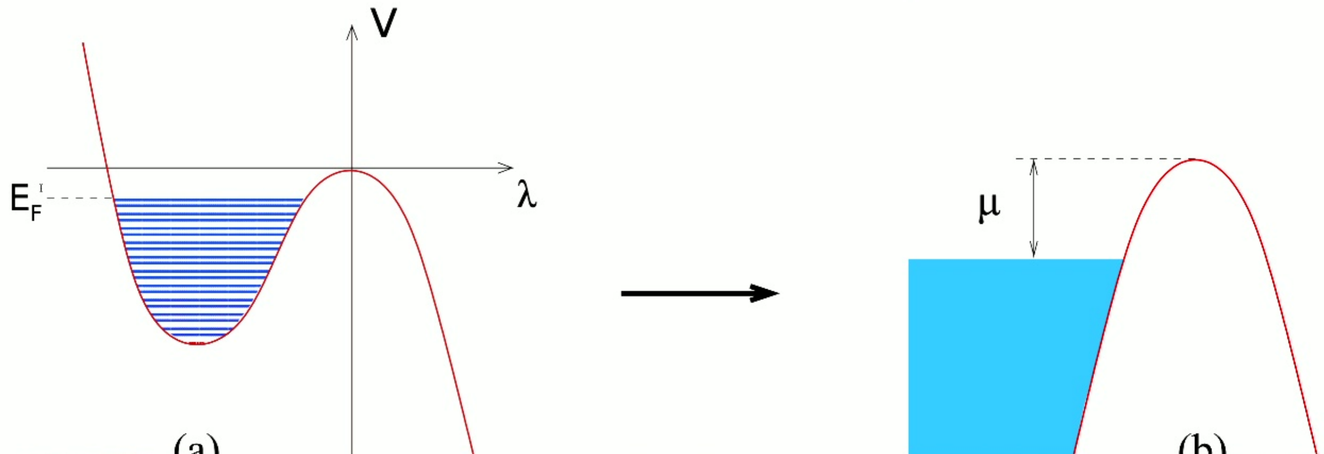
- The Hamiltonian is

$$H = -\frac{1}{2\Delta^2(\lambda)}\frac{d}{d\lambda_i}\Delta^2(\lambda)\frac{d}{d\lambda_i} + \sum_{i<j}\frac{J_{ij}J_{ji}}{(\lambda_i-\lambda_j)^2} + V(\lambda_i) \ ,$$

- J_{ij} are "momenta" conjugate to $SU(N)$ rotations
- Impose the Gauss-law constraint $\delta S/\delta A = i[M,\dot{M}] \sim J = 0$ (*singlet sector projection*)
- Upon rescaling $\lambda \to \frac{\sqrt{N}}{\kappa}\lambda$ and redefining the wavefunction as $\tilde{\Psi}(\lambda) \equiv \Delta(\lambda)\Psi(\lambda)$, the Schrödinger equation now reads

$$\left(-\frac{1}{2}\frac{d^2}{d\lambda_i^2} - \tfrac{1}{2}\lambda_i^2 + \frac{\sqrt{\hbar}}{3!}\lambda_i^3 + \ldots\right)\tilde{\Psi}(\lambda) = \hbar^{-1}E\tilde{\Psi}(\lambda), \qquad \hbar^{-1} = \frac{N}{\kappa^2}$$

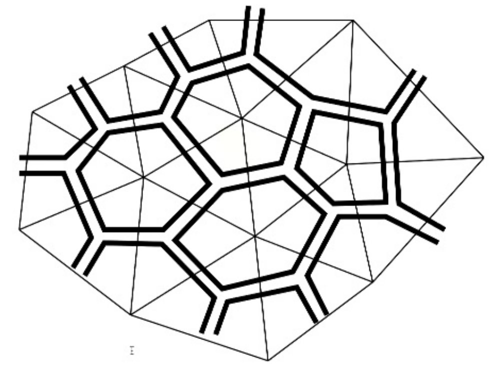
- Consider an initial state where the energy levels are populated up to some Fermi energy E_F below the top of the barrier, and send $\hbar \to 0$, $N \to \infty$, such that $E_F \to 0$
- Enough to focus on the quadratic maximum of the potential. We hold $\mu = -E_F/\hbar$ fixed in the limit
- The result is quantum mechanics of free fermions in an inverted harmonic oscillator potential, with states filled up to $-\mu < 0$
- At this limit the model is perturbatively stable in $1/N \to 0$ expansion



Connection with the quantum gravity path integral

The connection with the QG path integral is through this double scaling limit [Kazakov, Migdal...]. This is *not* the usual 't Hooft limit

- The double scaling limit produces smooth surfaces out of the Matrix fat-graphs while at the same time keeping all higher genera. It is defined by $\hbar, E_F \to 0$ as we discussed, while keeping $\mu \sim g_{st}^{-1}$ fixed



- The QG theory is the $c = 1$ *Liouville theory*. (It can also be interpreted as a 2D critical string theory in a linear dilaton background with a time direction t and a space direction ϕ)

Liouville theory

[Polyakov, David, Distler, Kawai...], Reviews by: [Ginsparg, Nakayama]

Can one make sense of string theory in case the conformal anomaly is not canceled?

- Gauge fix only the worldsheet diffeos and keep the conformal mode of the metric dynamical
- Note that the measure $\mathcal{D}g$ is not invariant under $g_{ab} \to e^{\rho(\sigma)} g_{ab}$
- Exponentiating the conformal anomaly from the measure, the total action becomes ($\mu = 1, ...d$, conformal gauge $g_{ab} = \hat{g}_{ab} e^{\phi(\sigma)}$)

$$S_{CFT} = \frac{1}{4\pi} \int d^2\sigma \sqrt{\hat{g}} \left[\hat{g}^{ab} (\partial_a X^\mu \partial_b X_\mu + \partial_a \phi \partial_b \phi) + Q\hat{R}\phi + 4\pi\mu e^{2b\phi} \right] + ghosts$$

- This new theory is a "conformal theory" under the simultaneous transformation $g_{ab} \to e^{\rho(\sigma)} g_{ab}$, $\phi(\sigma) \to \phi(\sigma) - \rho(\sigma)$, iff

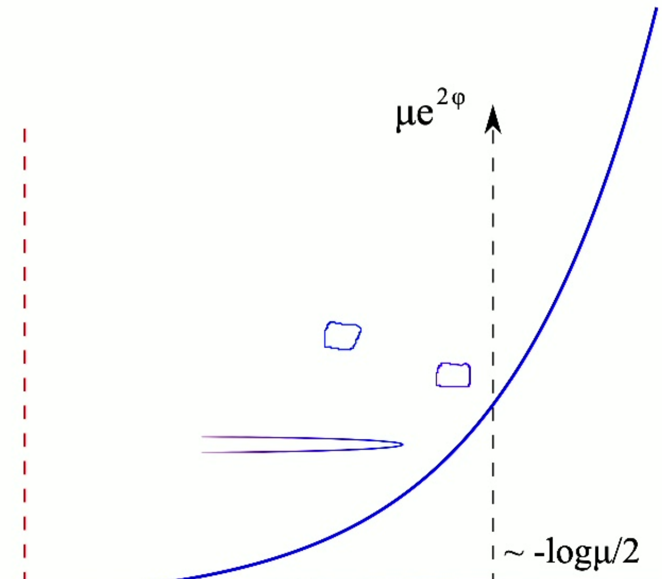
Liouville theory and Long strings

- The Liouville action on a worldsheet with boundaries is

$$S = \int_R d^2z\sqrt{g}\left(\frac{1}{4\pi}g^{ab}\partial_a\phi\partial_b\phi + \frac{1}{4\pi}QR\phi + \mu e^{2b\phi}\right) + \int_{\partial R} dsg^{1/4}\left(\frac{QK\phi}{2\pi} + \mu_B e^{b\phi}\right)$$

K is extrinsic curvature and μ, μ_B the bulk-boundary cosmological constants. For $c_{matter} = 1 \Rightarrow b = 1, Q = 2$ and $\mu_B = \sqrt{\mu}\cosh(\pi\sigma)$

- Closed strings see the bulk Liouville wall
- Open strings have their endpoints pinned near the weak coupling region
- A very energetic open string can stretch a lot (large σ), before it scatters back
- The ZZ boundary state ($D0$ brane anchored at large ϕ)

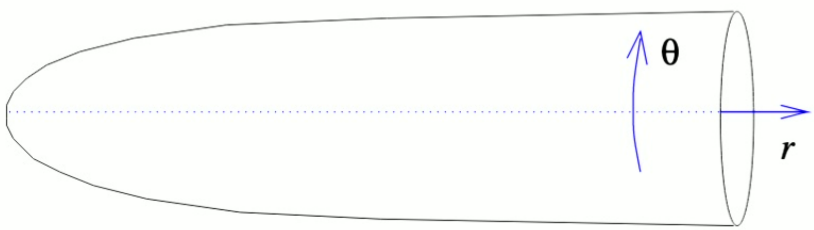


Main Part

Euclidean 2d black hole

Elitzur-Forge-Rabinovici, Mandal-Sengupta-Wadia

- String theory effective action: $S = \int d^2x e^{-2\Phi}(R - 4\nabla\Phi^2 - \frac{8}{\alpha'})$
- $2-D$ Cigar solution: $e^{\Phi} = \frac{e^{\Phi_{tip}}}{\cosh^2 r}$, $ds^2 \sim \left(dr^2 + \tanh^2 r d\theta^2\right)$



- $g_{st} = e^{\Phi}$ is a parameter of the solution
- The weak string coupling region is at the "boundary of the cigar"
- The strong coupling region is at the tip
- It has a fixed temperature

- It is the near horizon limit of higher dimensional black holes at large D, Emparan-Grumiller-Tanabe

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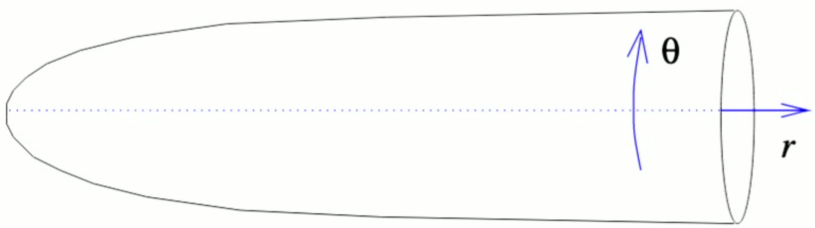

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- Contradiction: From Liouville we find a black hole solution but it does not exist in the gauged MQM
- This gravity solution is at string scale, so the gravity description cannot be trusted

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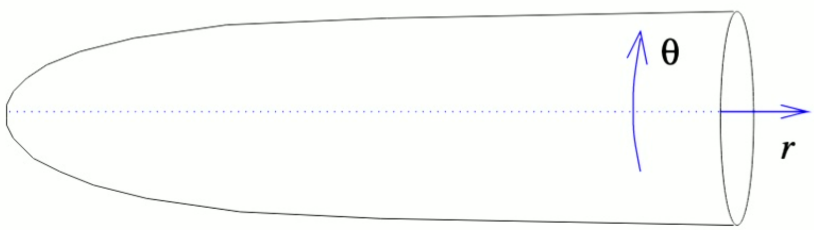

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- Wrong to assume that the exact in α' black hole solution $(SL(2)/U(1))$ satisfies the same boundary conditions as the linear dilaton background
- The metric and the dilaton of the Black hole solution do obey the same boundary conditions as the linear dilaton background
- The issue emanates from the string winding modes close to the tip of the cigar
- The winding is not conserved $\Rightarrow$ The model has an expectation value for the winding modes that decreases exponentially as we go closer to the boundary but not fast enough to be a normalisable deformation of the original background (when $k < 3$)
- This is a peculiarity of $2-D$ (small radius and large dilaton gradient)
- The fields scale with $\frac{1}{g_{st}}$ and close to the boundary $g_{st} \ll 1$, so they can diverge
- Thus we deform the linear dilaton background with non-normalisable

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- The fields scale with $\frac{1}{g_{st}}$ and close to the boundary $g_{st} \ll 1$, so they can diverge
- Thus we deform the linear dilaton background with non-normalisable operators, this corresponds to adding sources at infinity for the winding modes

$$M = 4e^{-2\Phi_0}, \qquad T_h = \frac{1}{2\pi\sqrt{k-2}}$$

- There are ambiguities in the thermodynamics of both the exact coset background and the gravity solution
- For the coset $T_a = \frac{1}{2\pi\sqrt{k}}$ and different subtraction schemes give very different results for the various thermodynamic quantities
- This is also to be expected due to the fact that the background is a string scale background
- Nevertheless one can safely make some qualitative estimates, $\Rightarrow$ the entropy is expected to scale as

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FZZ Duality and Black-Hole string transition

- FZZ duality: coset CFT is dual to the *Sine-Liouville theory*, `Fateev-Zamolodchikov`[2]
- Sine-Liouville: $L = \frac{1}{4\pi}\left((\partial x)^2 + (\partial \phi)^2 + Q\hat{R}\phi + \xi e^{b\phi}\cos R\,(x_L - x_R)\right)$
- Matches with the coset for radius of x to be $R = \sqrt{k}$ and

$$c_{\text{cigar}} = c_{SL} = 2 + 6Q^2\,, \quad Q^2 = \frac{1}{k-2}\,, \quad b = \sqrt{k-2} = \frac{1}{Q}$$

- The asymptotic weakly coupled region is $\phi \to -\infty$
- The strongly coupled region is for $\phi \to \infty$ near the potential wall
- The duality is a strong-weak duality
- For small radii, the black hole is better described in terms of a

- In SL theory we can define the following winding SL-operators

$$\mathcal{T}^{-}_{\pm R} = e^{\pm iR(X_L - X_R)} e^{(Q - |Q-1/Q|)\phi}, \quad \mathcal{T}^{+}_{\pm R} = e^{\pm iR(X_L - X_R)} e^{(Q + |Q-1/Q|)\phi}$$

- These are all operators of dimension $(1,1)$ and hence marginal
- The upperscript sign refers to the two possibilities of SL-dressing
- The $(-)$ case corresponds to a non-normalisable operator whose wavefunction grows at weak coupling $\phi \to -\infty$ and creates a local-disturbance on the worldsheet
- The $(+)$ case is a normalisable operator that creates a macroscopic loop on the worldsheet (supported at strong coupling $\phi \to +\infty$)

- Deform the linear dilaton background using conformal perturbation theory via including in the Lagrantzian the first winding terms

$$L_w = \frac{1}{4\pi}\xi e^{(2-R)\phi}\cos R\left(x_L - x_R\right) \sim \mathcal{T}^{-}_{+R} + \mathcal{T}^{-}_{-R}$$

and try to approach the $\xi \to \infty$, $\mu \to 0$ region of parameters (SL-point)

- This perturbation agrees with the SL term for $R = 3/2$, when $Q = 2$, so one is describing the same system at this point

- It makes sense for $R < 2$, so that the perturbation of the Lagrangian does not blow up in the asymptotic weakly coupled region and is a relevant deformation

- The KPZ-DDK scaling analysis of the free energy indicates that the two independent scaling ratios are g_s/μ and $g_s/\xi^{2/(2-R)}$

Our Matrix Model

Similar models by [Minahan, Polychronakos, Gaiotto, Dorey, Tong...]

- Consider the (gauged) MQM action with the $N \times N$ Hermitian matrices $M(t)$ and $A(t)$ (a non dynamical gauge field)

$$S = \int dt \mathrm{Tr} \left(\frac{1}{2} (D_t M)^2 - V(M) \right) , \quad V(M) \sim -M^2 \text{ double scaling limit}$$

- Describe open strings between N-ZZ and N_f-FZZT branes $\Rightarrow$ Extend by adding $N_f \times N$ (anti)- fundamental fields $\chi_{\alpha i}, \psi_{\alpha i}$. In 1-d they can be either fermions or bosons

$$S_f = \int dt \sum_{\alpha}^{N_f} \mathrm{Tr} \left(i\psi^\dagger_\alpha D_t \psi_\alpha - m_\alpha \psi^\dagger_\alpha \psi_\alpha + i\chi^\dagger_\alpha D^*_t \chi_\alpha - m_\alpha \chi^\dagger_\alpha \chi_\alpha \right) ,$$

$$: i[M\,, \dot{M}]_{ij} :=: J_{ij} := -k\delta_{ij} + \sum_{\alpha}^{N_f} \left[\psi^\dagger_{\alpha j}\psi_{\alpha i} - \chi^\dagger_{\alpha i}\chi_{\alpha j}\right]$$

- The Hamiltonian then is

$$\hat{H} = \sum_i^N -\tfrac{1}{2}\frac{\partial^2}{\partial\lambda_i^2} + V(\lambda_i) + \tfrac{1}{2}\sum_{i\neq j}\frac{J_{ij}J_{ji}}{(\lambda_i-\lambda_j)^2} + \sum_{i,\alpha}^{N,N_f} m_\alpha \psi^\dagger_{\alpha i}\psi_{\alpha i} + m_\alpha \chi^\dagger_{\alpha i}\chi_{\alpha i}$$

- The fundamentals thus "feed" non-trivial representations
- This model can be written as a spin Calogero model [Polychronakos] using $\Psi^\dagger_{\tilde{\alpha} i} = (\psi^\dagger_{\alpha i}, \chi_{\alpha i})$, $S_i^{\tilde{A}} = \Psi^\dagger_{\tilde{\alpha} i} T^{\tilde{A}}_{\tilde{\alpha}\tilde{\beta}} \Psi_{\tilde{\beta} i}$ and $\tilde{k} = k \mp N_f$

$$\hat{H}_C = \tfrac{1}{2}\sum_{i\neq j}\frac{\tilde{k}(2N_f \pm \tilde{k})/2N_f \pm 2S_i^{\tilde{A}}S_j^{\tilde{A}}}{(\lambda_i-\lambda_j)^2} + \sum_{i\tilde{A}} B^{\tilde{A}} S_i^{\tilde{A}}$$

with $T^{\tilde{A}}$ the $SU(2N_f)$ generators and the "magnetic" field

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Limit of 2-d Black hole Matrix Model

Model of [Kazakov, Kostov, Kutasov]

- The canonical partition function can be computed to be

$$Z_N^{(N_f)} \sim \int_{U(N)} \mathcal{D}U \, \det U^{-k} \frac{\exp\left[\sum_{l=1} \frac{(-1)^{l+1}}{l} \sum_{\alpha}^{N_f} e^{-\beta m_\alpha} \left(\mathrm{Tr}\, U^l + \mathrm{Tr}\, U^{-l}\right)\right]}{\exp\left(\sum_l \frac{q^l}{l} \mathrm{Tr}\, U^l \mathrm{Tr}\, (U^{-1})^l\right)}$$

- Take a double scaling limit (assuming $m_\alpha = m$)

$$N_f \to \infty, \quad m \to \infty, \quad \text{with} \quad N_f e^{-\beta m} = \tilde{t}, \quad \text{finite}$$

- Limit of "heavy quarks" / strong magnetic field (Calogero picture)
- The only surviving winding modes in this case: $\exp\left(\tilde{t}\mathrm{Tr}U + \tilde{t}\mathrm{Tr}U^\dagger\right)$, are identical to those studied in the matrix model conjectured to describe the physics of the Euclidean 2-d black hole ($SL(2,R)/U(1)$ coset)
- Advance of our approach $\Rightarrow$ the couplings $\tilde{t}$ are in terms of Liouville

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- One can also add the Chern-Simons term $S_{CS} = k \int dt \mathrm{Tr} A$ $\Rightarrow$ Related to the addition of k-units of flux in the dual 2D string theory sourced by the FZZT's

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Grand-Canonical Ensemble

- The matrix model grand canonical free energy is computed from the canonical one by

$$\mathcal{Z}_{MQM}(\mu, R; \tilde{t}) = \sum_{N=0}^{\infty} e^{\beta\mu N} \langle e^{\sum_n \tilde{t}_n \left(\mathrm{Tr} U^n + \mathrm{Tr} U^{\dagger\, n}\right)} \rangle_U \,, \qquad \beta = 2\pi R$$

- The relation between the Liouville couplings t_n and the matrix model couplings $\tilde{t}_n$ is $t_n = \frac{\tilde{t}_n}{2i \sin n\pi R}$
- This relation is derived upon realising the matrix model grand canonical partition function as a τ-function of the general Toda hierarchy

$$\mathcal{Z}(\tilde{t}_+, \tilde{t}_-; \mu + ik) \equiv \tau_k(t_+, t_-; \mu) = \langle k | e^{J_+(t_+)} \mathbf{G} e^{-J_-(t_-)} | k \rangle$$

- t_+, t_- are the Miwa time variables and $J_\pm(t_\pm)$ are the currents generating the “time” flows couplings $\tilde{t}_n$ is $t_n = \frac{\tilde{t}_n}{2i \sin n\pi R}$

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- The $GL(\infty)$ element/operator $\mathbf{G} = \mathbf{G}(\mu, R)$ can be expressed as a bilinear of free fermion operators: $\mathbf{G} = \exp \sum_{m,n \in \mathbb{Z}+\frac{1}{2}} b_{mn} \psi_m \psi_n^*$
- k corresponds to the overall vacuum $U(1)$ "charge" of the τ-function

Partitions and Representations

Expanding the τ-function in terms of representations

- Under T-duality the τ function is a generating function for the reflection amplitudes (`Dijkgraaf-Moore-Plesser`)
- Then $\mathbf{G}(\mu, R) \equiv \mathbf{S}(\mu, R) \Rightarrow$ Incorporates all the MQM dynamics
- Using the following expansion

$$
\begin{aligned}
e^{J_-(t_-)}|k\rangle &= \sum_\lambda s_\lambda(t_-)|\lambda; k\rangle \\
\langle k|e^{J_+(t_+)} &= \sum_\lambda s_\lambda(t_+)\langle \lambda; k|
\end{aligned}
$$

- $|\lambda; k\rangle$ corresponds to a representation/partition λ created by acting with fermions on the vacuum of charge k

$d(\lambda)$

- We can expand the τ function as a statistical sum in terms of transition amplitudes between different representations

- The MQM dynamics are diagonal in the representation basis so that

$$\tau_k(t_+, t_-; \mu) = \sum_{\lambda} s_\lambda(t_+) s_\lambda(t_-)(-1)^{|\lambda|}\langle \lambda; k|\mathbf{G}|\lambda; k\rangle$$

- The summation over λ is over all possible Young diagrams that describe the different partitions/representations

- This representation theoretic expansion for the τ function will allow us to give a meaning to the microstates comprising the long string condensate/black hole background

- We can expand the τ function as a statistical sum in terms of transition amplitudes between different representations

- The MQM dynamics are diagonal in the representation basis so that

$$\tau_k(t_+, t_-; \mu) = \sum_\lambda s_\lambda(t_+) s_\lambda(t_-) (-1)^{|\lambda|} \langle \lambda; k | \mathbf{G} | \lambda; k \rangle$$

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- Take a double scaling limit (assuming $m_\alpha = m$)

$$N_f \to \infty, \quad m \to \infty, \quad \text{with} \quad N_f e^{-\beta m} = \tilde{t}, \quad \text{finite}$$

- Limit of "heavy quarks" / strong magnetic field (Calogero picture)
- The only surviving winding modes in this case: $\exp\left(\tilde{t}\mathrm{Tr}U + \tilde{t}\mathrm{Tr}U^\dagger\right)$, are identical to those studied in the matrix model conjectured to describe the physics of the Euclidean 2-d black hole ($SL(2,R)/U(1)$ coset)
- Advance of our approach $\Rightarrow$ the couplings $\tilde{t}$ are in terms of Liouville theory/Matrix model parameters N_f, $\sigma = 2m$.

$$\tilde{t} = N_f \mu^{R/2} \mu_B^{-R}$$

- The canonical partition function can be computed to be

$$Z_N^{(N_f)} \sim \int_{U(N)} \mathcal{D}U \, \det U^{-k} \frac{\exp\left[\sum_{l=1} \frac{(-1)^{l+1}}{l} \sum_{\alpha}^{N_f} e^{-\beta m_\alpha} \left(\mathrm{Tr}\, U^l + \mathrm{Tr}\, U^{-l}\right)\right]}{\exp\left(\sum_l \frac{q^l}{l} \mathrm{Tr}\, U^l \mathrm{Tr}\, (U^{-1})^l\right)}$$

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- The matrix model grand canonical free energy is computed from the canonical one by

$$\mathcal{Z}_{MQM}(\mu, R; \tilde{t}) = \sum_{N=0}^{\infty} e^{\beta\mu N} \langle e^{\sum_n \tilde{t}_n (\mathrm{Tr} U^n + \mathrm{Tr} U^{\dagger\, n})} \rangle_U \,, \qquad \beta = 2\pi R$$

- The relation between the Liouville couplings t_n and the matrix model couplings $\tilde{t}_n$ is $t_n = \frac{\tilde{t}_n}{2i \sin n\pi R}$
- This relation is derived upon realising the matrix model grand canonical partition function as a τ-function of the general Toda hierarchy

$$\mathcal{Z}(\tilde{t}_+, \tilde{t}_-; \mu + ik) \equiv \tau_k(t_+, t_-; \mu) = \langle k | e^{J_+(t_+)} \mathbf{G} e^{-J_-(t_-)} | k \rangle$$

- t_+, t_- are the Miwa time variables and $J_\pm(t_\pm)$ are the currents generating the "time" flows
- The $GL(\infty)$ element/operator $\mathbf{G} = \mathbf{G}(\mu, R)$ can be expressed as a

- This representation theoretic expansion for the τ function will allow us to give a meaning to the microstates comprising the long string condensate/black hole background

Measures in the space of representations

Okounkov-Borodin-Olshanski

- The general measure that weights the representations/partitions λ is the Schur-measure

$$\mathfrak{M}_\lambda(t) = \frac{1}{Z} s_\lambda(t_+) s_\lambda(t_-) \,, \qquad Z_0 = \sum s_\lambda(t_+) s_\lambda(t_-) = e^{\sum_{k>0} k t_k^+ t_k^-}$$

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- The expected average size of the partition with respect to the Schur measure is

$$\langle |\lambda| \rangle_{\mathfrak{M}} = \sum_k k^2 t_k^+ t_k^-$$

- It becomes very large, for large deformations/time variables

$$\mathfrak{M}_\lambda(\zeta) = e^{\cdots} \zeta^{\cdots} \left(\frac{\cdots}{|\lambda|!} \right) = e^{\cdots} \sum_{n=0} \frac{\cdots}{n!} M_n(\lambda)\delta(|\lambda| - n)$$

- $M_n(\lambda)$ is called the Plancherel measure on partitions of n

$$M_n(\lambda) = \frac{(\dim \lambda)^2}{n!}, \quad |\lambda| = n$$

- As the size of the partitions goes to infinity $n \to \infty$, the Plancherel measure exhibits a Cardy-like growth

$$\lim_{n\to\infty} M_n(\lambda) \sim \exp\left(2\sqrt{n}\right)$$

- and concentrates to a universal limiting Young diagram shape the *Vershik-Kerov-Logan-Shepp limiting shape*

$$\boldsymbol{\Omega}(y) = \begin{cases} \frac{2}{\pi}\left(y \arcsin(y/2) + \sqrt{4-y^2}\right), & |y| < 2\,, \\ |y|\,, & |y| > 2 \end{cases}$$

The Partition function at the continuous limit

- We now take the continuous limit for the complete τ function that should also include the reflection amplitude

$$\mathbf{G}_{\lambda\lambda}(\mu, R) = Z_{singlet} \prod_{j=1}^{d} \sqrt{\frac{\Gamma\left(\frac{1}{2} + i\mu R + p_j R\right)}{\Gamma\left(\frac{1}{2} - i\mu R - p_j R\right)}} \sqrt{\frac{\Gamma\left(\frac{1}{2} - i\mu R + q_j R\right)}{\Gamma\left(\frac{1}{2} + i\mu R - q_j R\right)}}$$

- Using the continuum variables

$$y = \frac{j}{d}, \quad q(y) = \frac{q_j}{d}, \qquad x = \frac{i}{d}, \quad p(x) = \frac{p_i}{d}$$

- The corresponding densities of boxes are

$$\mathcal{N}(q) = -\frac{dy}{dq}, \qquad \widetilde{\mathcal{N}}(p) = -\frac{dx}{dp}$$

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- The partition function and free energy are expressed in the following form

$$\mathcal{Z} = e^{-\mathcal{F}} = \int DpDqe^{-d^2 S_{eff}(p,q)}$$

$$\log\left(\frac{p^2}{\xi^2}\right) - R\log\left[R\left(p+i\mu\right)\right] = 2\int ds \frac{\ldots}{p-s} - 2\int ds \frac{\ldots}{p+s}$$

- We then define the resolvent $\omega(q)$

$$\omega(q) = \int_{\text{supp.}} ds \, \frac{\mathcal{N}(q)}{q-s}\,, \qquad \mathcal{N}(q) = -\frac{1}{2\pi i}\left(\omega(q+i\epsilon) - \omega(q-i\epsilon)\right)$$

- Using these properties of the resolvents and $\Omega(z) = \omega(z) + \tilde{\omega}(-z)$

$$\log\left(\frac{q^2}{\xi^2}\right) - R\log\left[R\left(q-i\mu\right)\right] = 2\omega(q) + 2\tilde{\omega}(-q) = 2\Omega(z=q)\,, \quad q-\text{cuts}\,,$$

$$\log\left(\frac{p^2}{\xi^2}\right) - R\log\left[R\left(p+i\mu\right)\right] = 2\tilde{\omega}(p) + 2\omega(-p) = 2\Omega(-z=p)\,, \quad p-\text{cuts}$$

- We study the Sine-Liouville limit ($\mu \to 0$) then only reflection symmetric Young diagrams $\tilde{\mathcal{N}}(z) = \mathcal{N}(z)$ contribute

- In this limit the relative backreaction of the winding modes on the linear dilaton background is strong

- The cuts of the total resolvent $\Omega(z) = \omega(z)$ are symmetrically distributed with respect to $z = 0$ and belong strictly on the real axis

$$\frac{\partial V_{eff.}(u)}{\partial u} = \log\left(\frac{u}{\xi^2}\right) - \frac{R}{2}\log\left(R^2 u\right) = 2\not\Omega(u)$$

- It has physical cuts for $u \geq 0$ and $u = z^2$

- The effective potential defined determines essentially all the physical features of the solutions

- For $R < 2$, the effective potential is stable

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- For $R > 2$ it is unstable and goes to $-\infty$ as $u \to \infty$

$R < 2$ single cut no saturation

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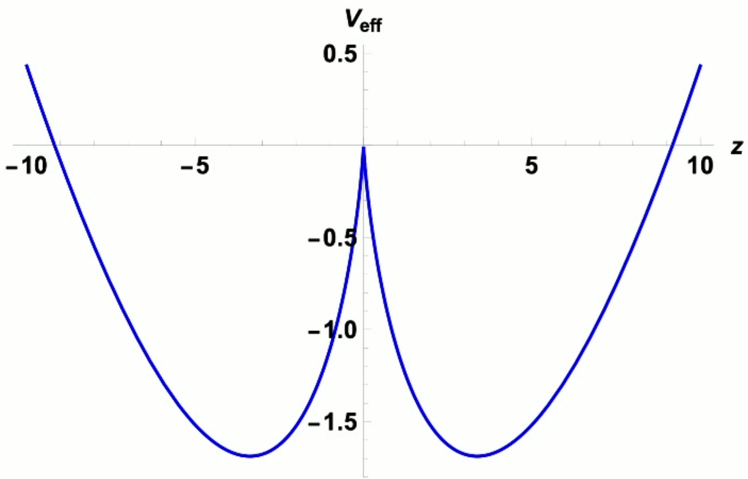

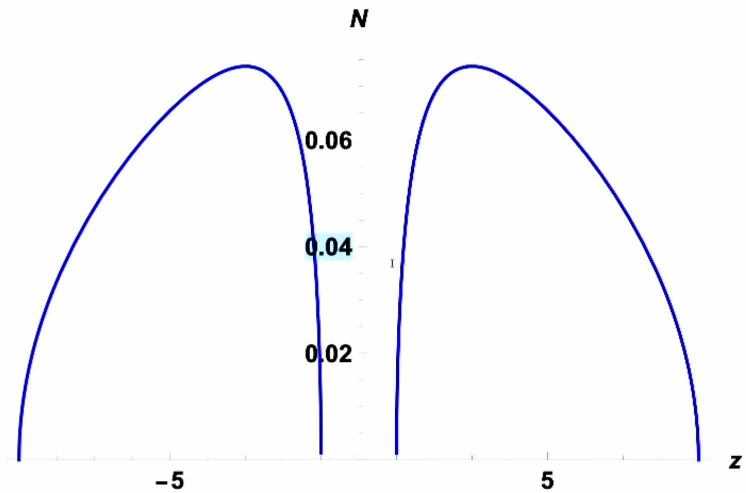

- In the left plot we depict the typical effective potential for large $\xi_{eff} = \xi R^{-R/2}$ (two deep wells)

- In the right one the typical density of boxes that has support on two

$$\frac{2\Omega(u)}{2-R} = \log\left(\frac{u+ab-\sqrt{(u-a^2)(u-b^2)}}{(a+b)R^{-\frac{R}{2-R}}\xi^{\frac{2}{2-R}}}\right)$$

- The density of boxes is $\mathcal{N}(z) = \frac{2-R}{2\pi}\cos^{-1}\frac{z^2+ab}{|z|(a+b)}\,, \qquad |z| \in (a\,,\,b)$
- We can fix the edges of the support a, b in terms of the physical parameters ξ, R
- By demanding vanishing of the leading term in the asymptotic expansion $\Omega(u)|_{u\to\infty}$ and
- By imposing the normalisation condition $\int_{\mathcal{C}_i}\mathcal{N}(z)dz = \frac{1}{2}$ for $\mathcal{N}(z)$
- The result is in terms of Elliptic Functions of the first and the second kind

$$(\pi/(4-2R))^{2-R}R^R \leq \xi^2 \ll (\pi/(2-R))^{2-R}R^R$$

- This is an $O(1)$ number in the regime $1 < R < 2$, this locus is near the phase transition region

- Average size of the partition: $\langle |\lambda| = n \rangle = \frac{1}{2}\xi \frac{\partial \log \mathcal{Z}}{\partial \xi} = -\frac{1}{2}\xi \langle \frac{\partial S_{eff.}}{\partial \xi} \rangle$ thus

- $\langle |\lambda| = n \rangle \simeq \frac{2-R}{4} R^{-\frac{2R}{2-R}} \xi^{\frac{4}{2-R}}$ and $\mathcal{F}_{wide} \simeq -\frac{(2-R)^2}{8} R^{-\frac{2R}{2-R}} \xi^{\frac{4}{2-R}}$

- The scaling of ξ in the free energy (and the property that is vanishes for $R = 2$) coincides with that of `Kazakov-Kostov-Kutasov` when the cut becomes large (near the phase transition region between the single cut and the saturated cut).

ξ, when the potential develops a deep well making the cut narrow

$$\langle |\lambda| = n \rangle \simeq \frac{1}{2} R^{-\frac{R}{2-R}} \xi^{\frac{2}{2-R}}, \quad \mathcal{F}_{narrow} \simeq -\frac{(2-R)}{2} R^{-\frac{R}{2-R}} \xi^{\frac{2}{2-R}}$$

- In the limit of a narrow cut (very large ξ) the behaviour of the free energy changes and starts to scale as the square root of the free energy near the transition region

$R < 2$ Saturated cut

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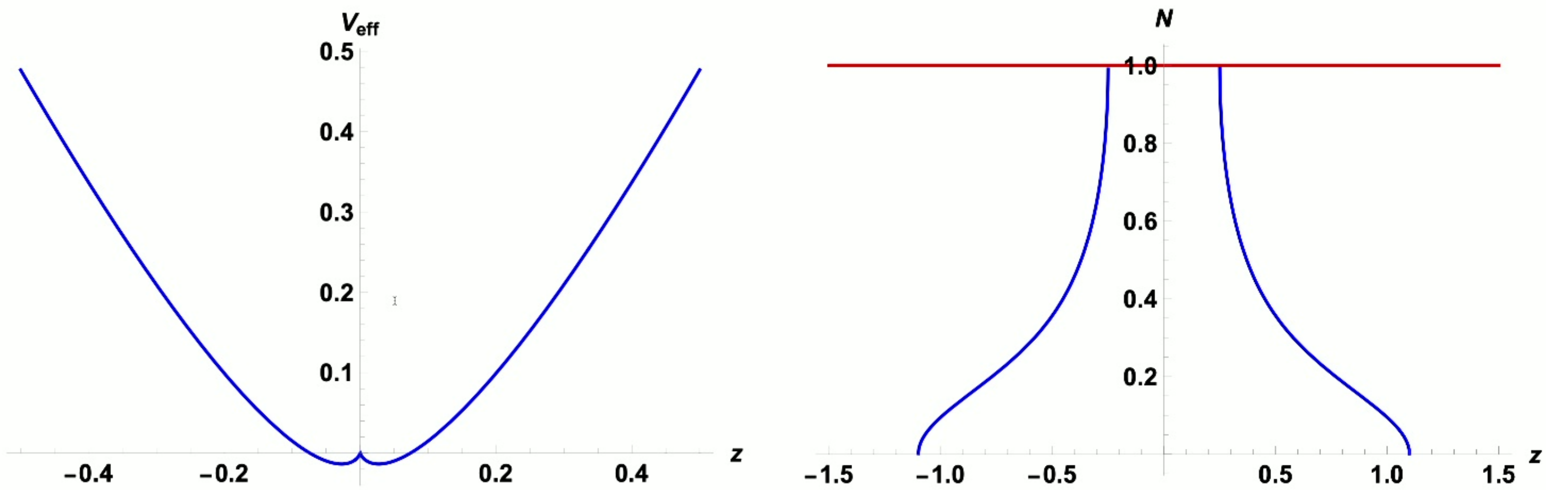

- In the left plot we depict the typical effective potential for small $\xi_{eff} = \xi R^{-R/2}$

$$\mathcal{N}(u) = 1\,, \quad u \in [0, a^2)\,, \qquad \mathcal{N}(u) \neq 0\,, \quad u \in (a^2, b^2)$$

- The resolvent is $\Omega(u) = \log \frac{u}{u-a^2} + \int_{a^2}^{b^2} dv\, \frac{\mathcal{N}(v)}{u-v}$
- The associated density of boxes is now given in the z-variable by

$$\mathcal{N}(z) = \begin{cases} 1\,, & |z| \in [0, a]\,, \\ \frac{2}{\pi} \arccos \sqrt{\frac{z^2-a^2}{b^2-a^2}} - \frac{R+2}{2\pi} \arccos \frac{z^2+ab}{|z|(a+b)}\,, & |z| \in (a, b) \end{cases}$$

- We can fix the edges of the support a, b in terms of the physical parameters ξ, R
- That the leading asymptotic of $\Omega(u)|_{u\to\infty} \to 0$ and
- Imposing the normalisation condition $\frac{1}{2} - a = \int_a^b dz \mathcal{N}(z)$
- The result is in terms of Elliptic Functions of the first and the second kind

- Consistent when $0 < \frac{a}{b} \simeq \frac{1}{4} R^{R/(2-R)} \xi^{-2/(2-R)} - \frac{2-R}{2\pi} \ll 1$
- This condition holds for relatively small ξ, that is the regime of a shallow potential
- This is precisely the opposite bound compared to the one we found for the single cut unsaturated solution (in the wide cut regime), verifying the transition region between the two solutions

$$\langle |\lambda| = n \rangle \simeq \frac{2-R}{4} R^{-\frac{2R}{2-R}} \xi^{\frac{4}{2-R}} , \quad \mathcal{F}_{wide} \simeq -\frac{(2-R)^2}{8} R^{-\frac{2R}{2-R}} \xi^{\frac{4}{2-R}}$$

- We observe that in this phase the leading part of the free energy scales exactly as `Kazakov-Kostov-Kutasov`
- It also exactly coincides with the free energy of the unsaturated phase solution near the transition point (wide-cut) and so do their first derivatives, showing the expected continuous nature of the phase transition

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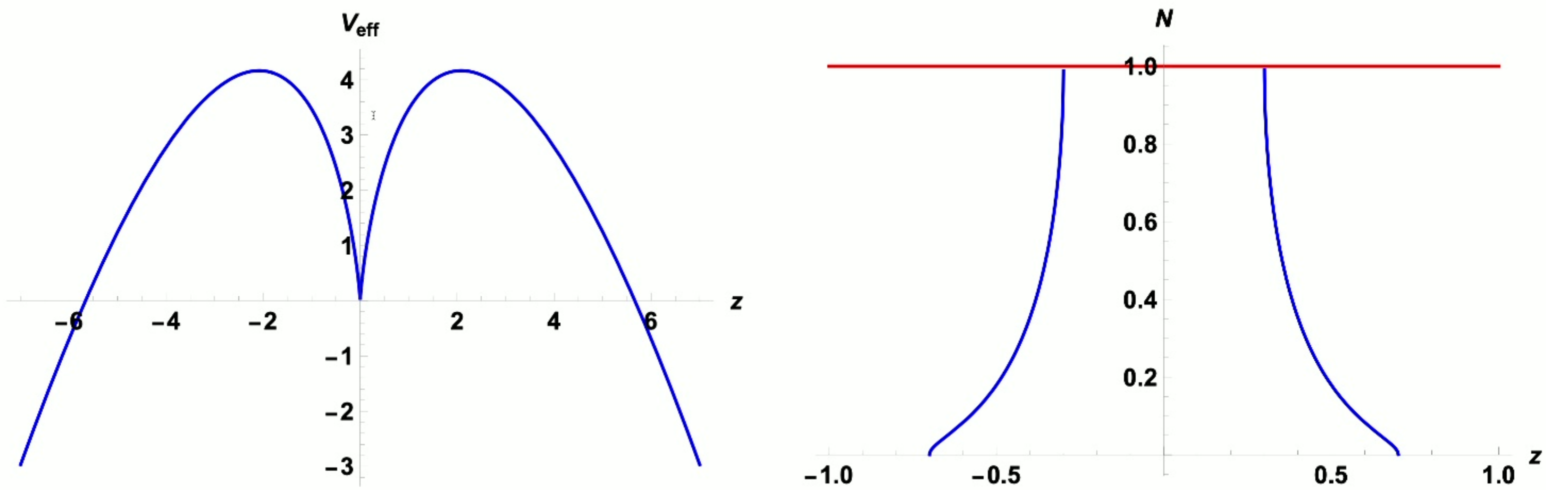
$R > 2$ Unstable Potential Case



- In the left plot we depict the typical effective potential when $R > 2$
- In the right one the typical density of boxes that has a saturation region with two adjacent narrow cuts

• This solution exists for relatively small $\zeta_{eff} = \zeta R$ ..., otherwise the well becomes too shallow to support a metastable solution

$R > 2$ Unstable Potential Case

- The solution without saturation is pathological (both the density of boxes and the resolvent become negative)
- The saturated solution is acceptable as long as (positivity condition for the density of boxes)

$$a < b < \frac{a(R+2)}{R-2} \quad \Rightarrow \quad r_c = \frac{R-2}{R+2} < r = \frac{a}{b} < 1$$

- For a narrow cut $r \sim 1$ which holds as long as $\xi_{eff} \ll 2^{R-3}$, which

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- For a narrow cut $r \simeq 1$ which holds as long as $\xi_{eff} \ll 2^{R-3}$, which makes this approximation more and more natural for very large radii and bad for radii close to $R = 2$
- The opposite regime of the narrow cut, is the limit where the ratio $r = a/b$ approaches the lower critical bound of the positivity condition $r \to r_c$
- This is also the critical limit where the eigenvalues fill the metastable effective potential as much as possible

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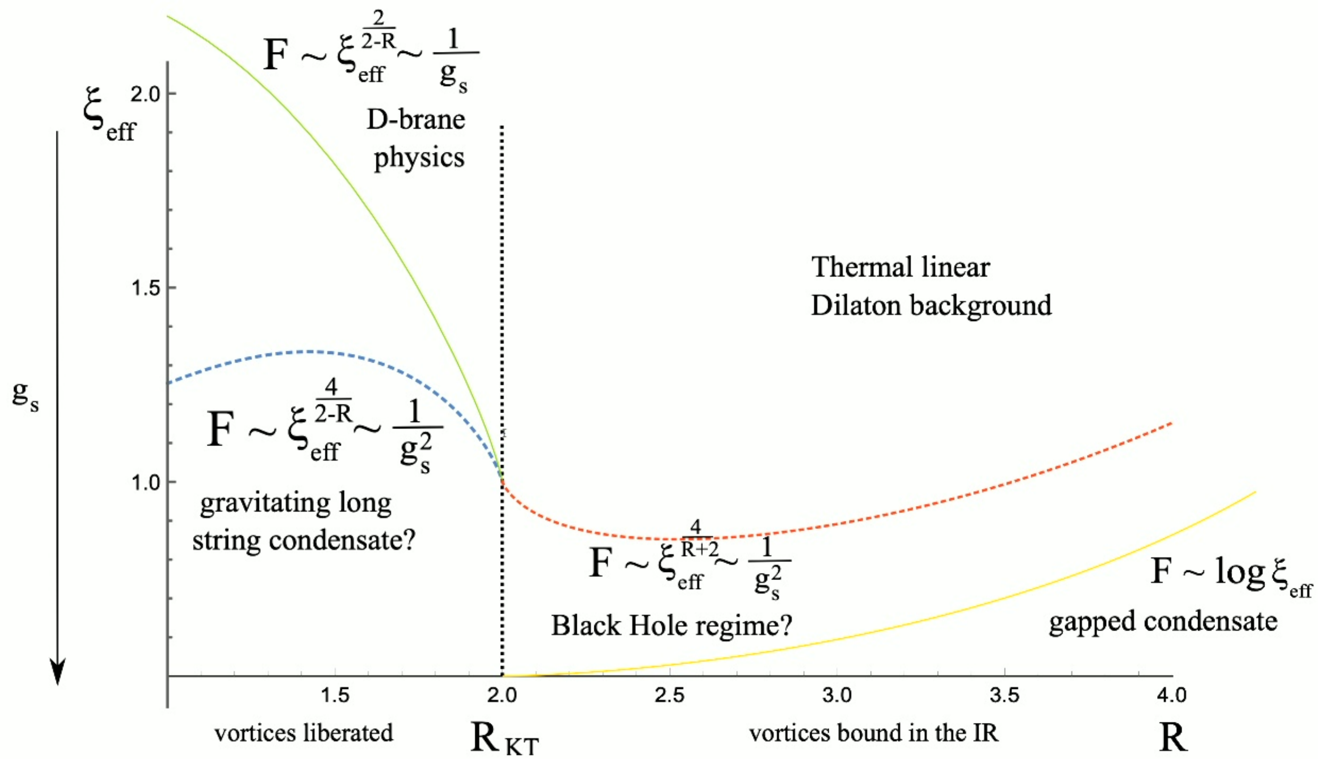
- For the narrow cut: $\langle |\lambda| = n \rangle \simeq \frac{\ldots(1-\epsilon)}{2} \simeq \frac{1}{32}(1 - 2^{3-R}\xi_{eff})$ and is still positive
- The free energy is $\mathcal{F}^{(R>2)}_{narrow} \simeq -\frac{1}{16}\log \xi_{eff} + \frac{1}{16}2^{3-R}\xi_{eff}$
- The leading term is a logarithmic contribution, that could have the interpretation of a "quantum non-singlet" contribution to the entropy
- The situation is more interesting near the critical region, there we find the scaling law

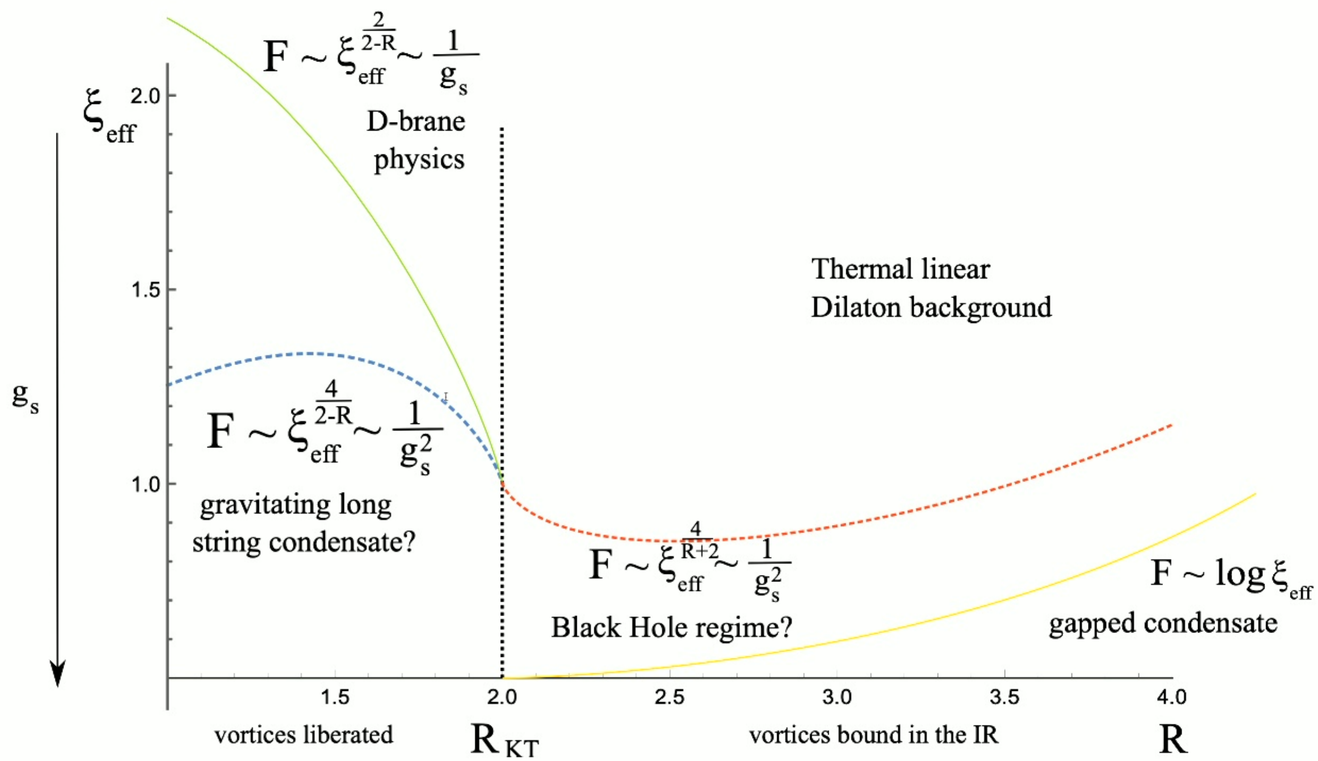
$$\langle |\lambda| = n \rangle_c \sim \xi_{eff}^{\frac{4}{R+2}} \sim \frac{1}{g_s^2} \sim \mathcal{F}$$

- The scaling of the free energy in the critical regime for $R > 2$ reveals an opposite type $(+)$ of dressing for the winding modes, that become normalisable and have support in the strongly coupled region of Liouville
- This scaling means that a black hole can start existing as an excited state in the spectrum of the theory

The phase diagram



- The dashed lines signal phase transitions
- $T_{KT} \sim 1/R_{KT}$ is the Kosterlitz Thouless temperature, above which



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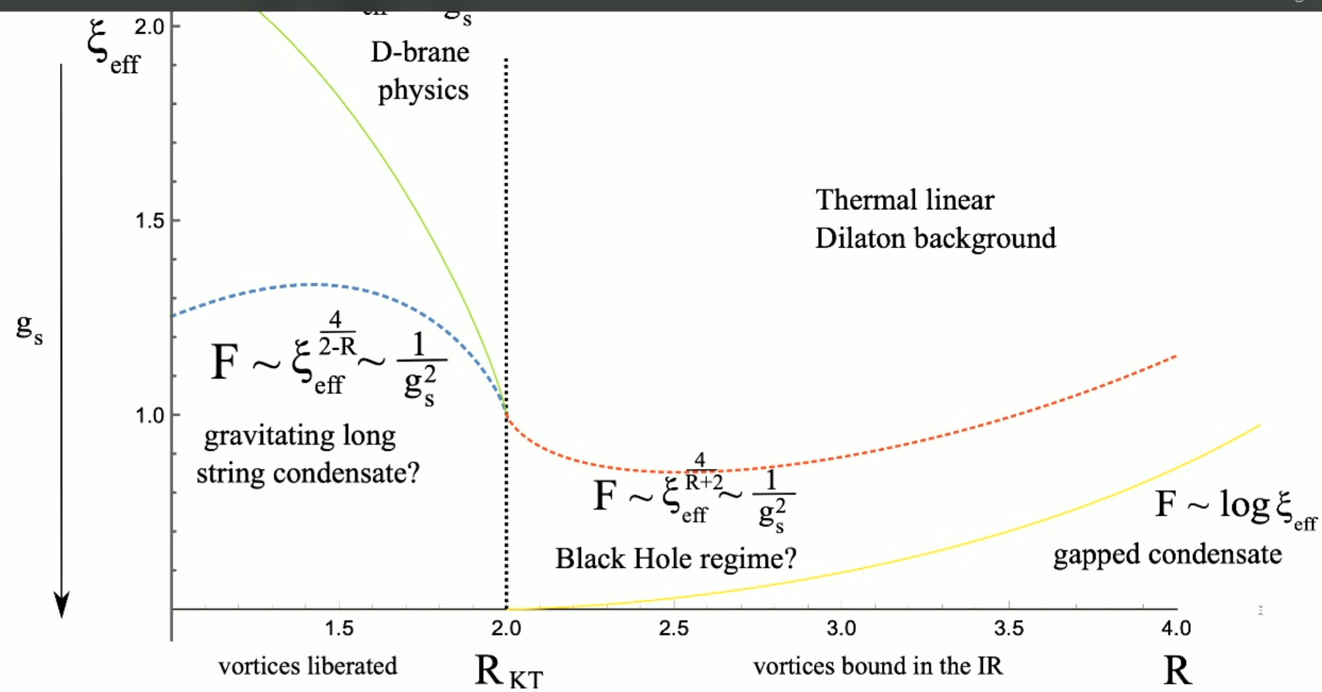

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- The continuous lines are regimes of different behaviour (cross-over)