Title: Entanglement distillation in tensor networks

Speakers: Takato Mori

Series: Perimeter Institute Quantum Discussions

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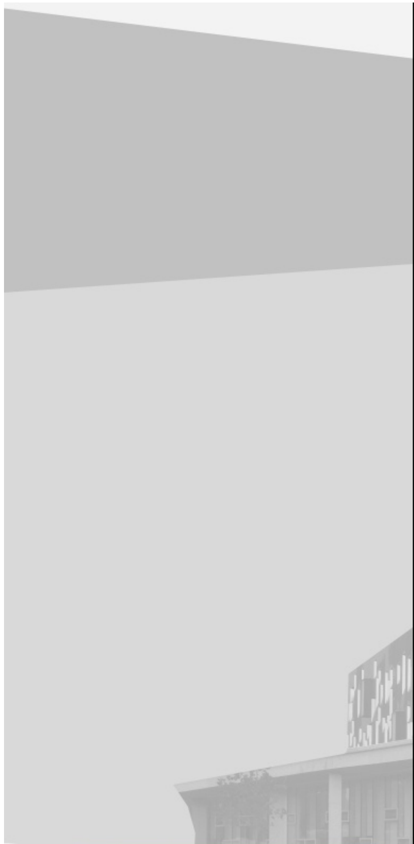
Abstract: Tensor network provides a geometric representation of quantum many-body wave functions. Inspired by holography, we discuss a geometric realization of (one-shot) entanglement distillation for tensor networks including the multi-scale entanglement renormalization ansatz and matrix product states. We evaluate the trace distances between the 'distilled' states and EPR states step by step and see a trend of distillation. If time permits, I will mention a possible field theoretic generalization of this geometric distillation.

Zoom link: https://pitp.zoom.us/j/98545776462?pwd=b1Z3ZENNRWVITlNOZG1GdzJaMmN1Zz09



Entanglement distillation in tensor networks

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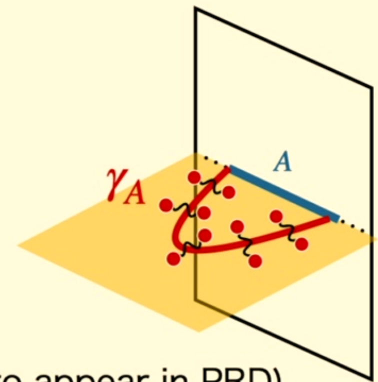
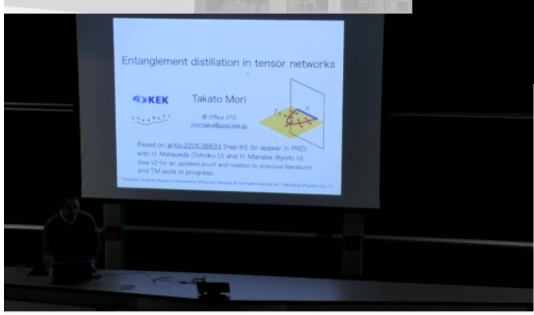


Based on arXiv:2205.06633 [hep-th] (to appear in PRD)
with H. Matsueda (Tohoku U) and H. Manabe (Kyoto U)
(See v2 for an updated proof and relation to previous literature)
and TM work in progress

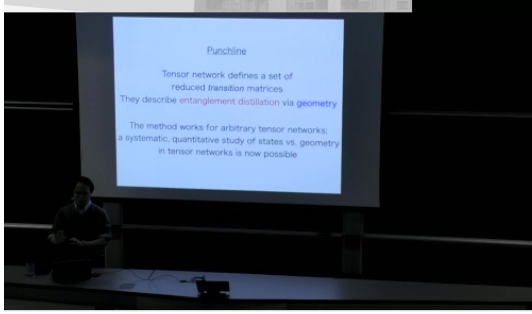
Perimeter Institute Quantum Discussions (PiQuDos) Seminar @ Perimeter Institute for Theoretical Physics, Oct. 12



Punchline

Tensor network defines a set of
reduced *transition* matrices
They describe entanglement distillation via geometry

The method works for arbitrary tensor networks;
a systematic, quantitative study of states vs. geometry
in tensor networks is now possible



Outline

1. Introduction — AdS/CFT, tensor networks as toy models
2. Entanglement distillation in MERA
3. Numerical results for random MERA
4. Entanglement distillation in MPS
5. Entanglement distillation in higher dimensions
6. Quantum circuit implementation
7. Summary

(Items 2–4: 2205.06633; items 5–6: work in progress)

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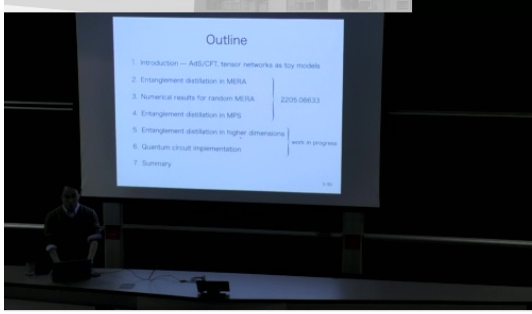
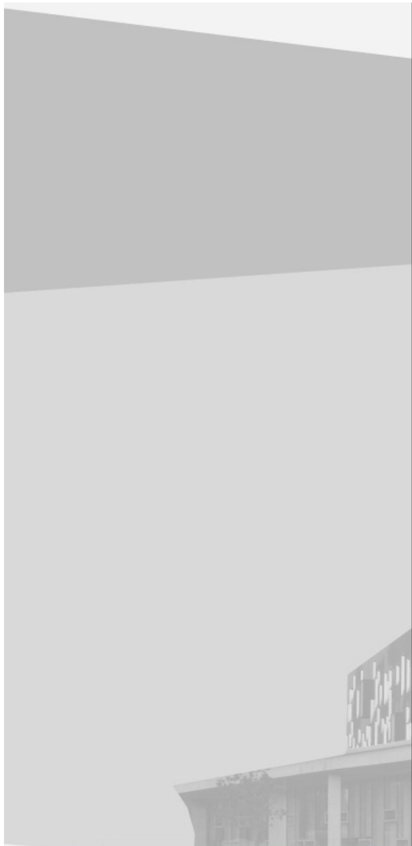




Motivation: Understanding AdS/CFT from entanglement

(d+1)-dim. AdS spacetime ⇆ d-dim. quantum field theory (CFT) [Maldacena]

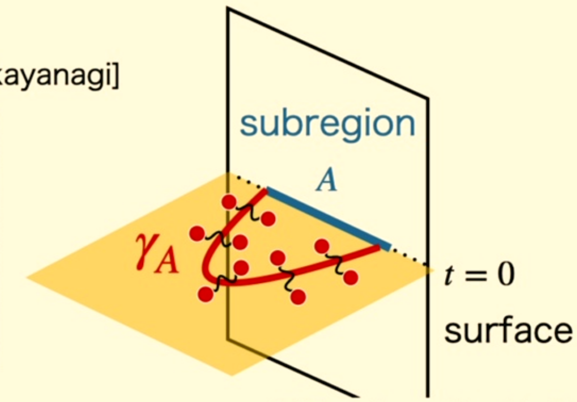
geometry   quantum information

Ryu-Takayanagi formula [Ryu-Takayanagi]

$$S_A = -\mathrm{tr}\rho_A \log \rho_A, \quad \rho_A = \mathrm{tr}_{\bar{A}}\rho$$

$$\parallel$$

$$S_{HEE}(A) = \min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$



cf. [Freedman-Headrick]

🤔 **Beyond AdS (hyperbolic) / CFT (critical)?**

Can we really see EPR pairs across γ_A?

How does the geometry arise from a wave function?

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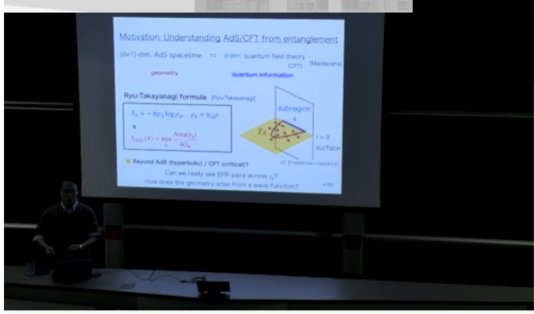
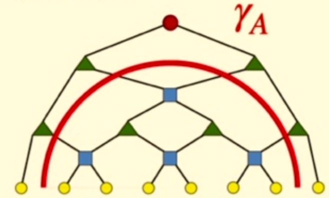


Tensor networks as toy models of holography

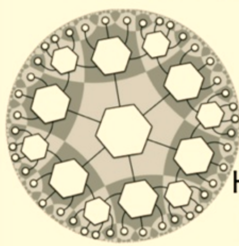
- Tensor networks (=variational wave function) provide a *qualitative* picture [Swingle]

$S_{TN}(A) \lesssim \min_{\gamma_A} (\#\text{ bond cut by } \gamma_A) \times \log \chi$ ~RT formula?

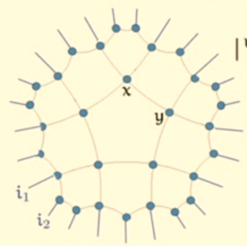
Multi-scale entanglement renormalization ansatz (MERA) [Vidal]



- Some proposals try to mimic holography (esp. RT formula)



Holographic state [Pastawski et al.]



$|\Psi\rangle = \left(\otimes_{\langle x,y \rangle} \langle \mathrm{MES}|_{xy} \right) \left(\otimes_x U_{random} |0_x\rangle \right)$

Random TN with large bond dim. [Hayden et al.] [Qi-Yang][Apel et al.]

CONS: Lack of expressivity; TN state ≠ conformally invariant

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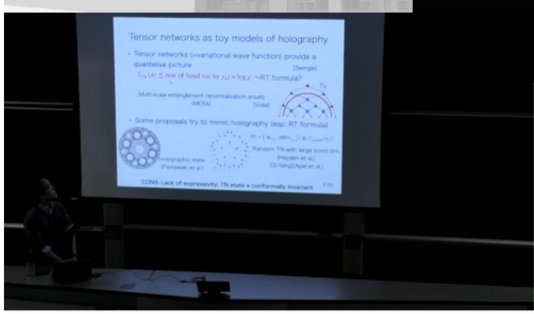
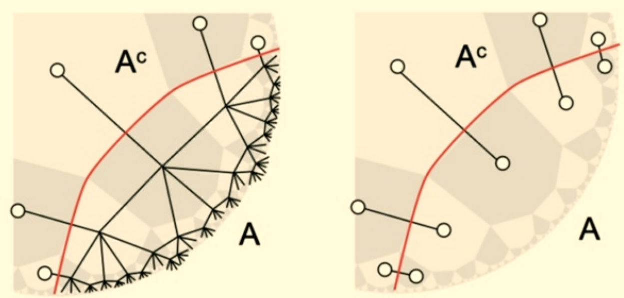


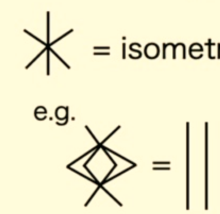
Entanglement distillation in holographic tensor networks

[Pastawski et al.]
(Similar work: [Bao-Penington-Sorce-Wall], [Lin-Sun-Sun])

- A holographic state (or isometric tensor networks in general) is known to geometrize entanglement distillation
  = extracting S_A bits of EPR pairs from the state
- Removing tensors, we obtain EPR pairs across the minimal surface $\because S_A(V|\Psi\rangle) = S_A(|\Psi\rangle)$



$$i \text{ —— } j = \frac{\delta_{ij}}{\sqrt{2}} |i\rangle |j\rangle$$



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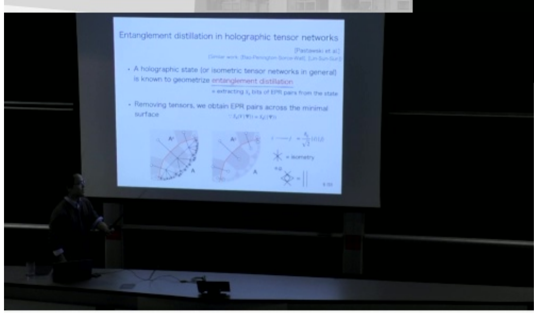

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$$V_A : \mathscr{H}_A \to \mathscr{H}_{\gamma_A}, \quad W_{\bar{A}} : \mathscr{H}_{\bar{A}} \to \mathscr{H}_{\gamma_{\bar{A}}}$$

$$S_A(|\Psi\rangle) = n \log 2$$

Isometries $V_A V_A^\dagger = \mathbf{1}_{\gamma_A}, \quad W_{\bar{A}} W_{\bar{A}}^\dagger = \mathbf{1}_{\gamma_{\bar{A}}}$

$$|\Psi\rangle_{A\bar{A}} = (V_A \otimes W_{\bar{A}}) \left(\frac{1}{\sqrt{2}} \sum_{i=0}^{1} |i\rangle_{\gamma_A} \otimes |i\rangle_{\gamma_{\bar{A}}} \right)^{\otimes n} \in \mathscr{H}_A \otimes \mathscr{H}_{\bar{A}}$$

$$\mapsto \left(\frac{1}{\sqrt{2}} \sum_{i=0}^{1} |i\rangle_{\gamma_A} \otimes |i\rangle_{\gamma_{\bar{A}}} \right)^{\otimes n} \in \mathscr{H}_{\gamma_A} \otimes \mathscr{H}_{\gamma_{\bar{A}}}$$

$\dim \mathscr{H}_A$	$\dim \mathscr{H}_{\bar{A}}$
IV	IV
$\dim \mathscr{H}_{\gamma_A}$	$\dim \mathscr{H}_{\gamma_{\bar{A}}}$

Entanglement spectrum remains same $\left(\mathscr{H}_{\gamma_A} = \mathscr{H}_{\gamma_{\bar{A}}}\right)$

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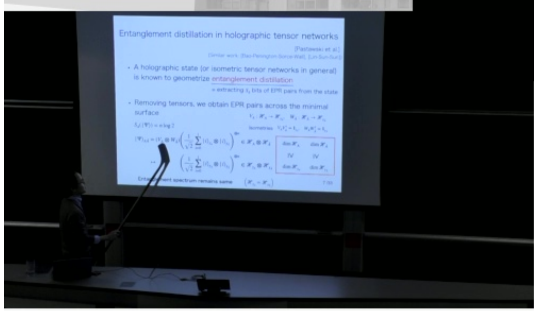

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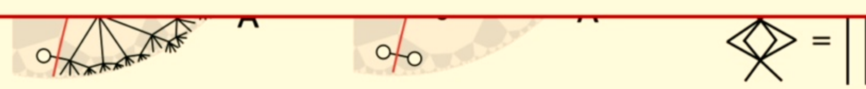
  = extracting S_A bits of EPR pairs from the state
- Removing tensors, we obtain EPR pairs across the minimal surface $\because S_A(V|\Psi\rangle) = S_A(|\Psi\rangle)$

So far this manipulation is limited to isometric TNs.

Geometrization of entanglement distillation in other types of TNs?

➜ Clarify the operational role of internal d.o.f. after optimization!



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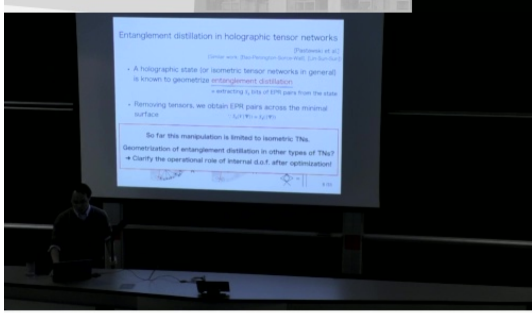

Entanglement distillation in tensor networks

The important aspect for entanglement distillation is

1. Conservation of entanglement (entropy)

2. Extracting strongly entangled pairs

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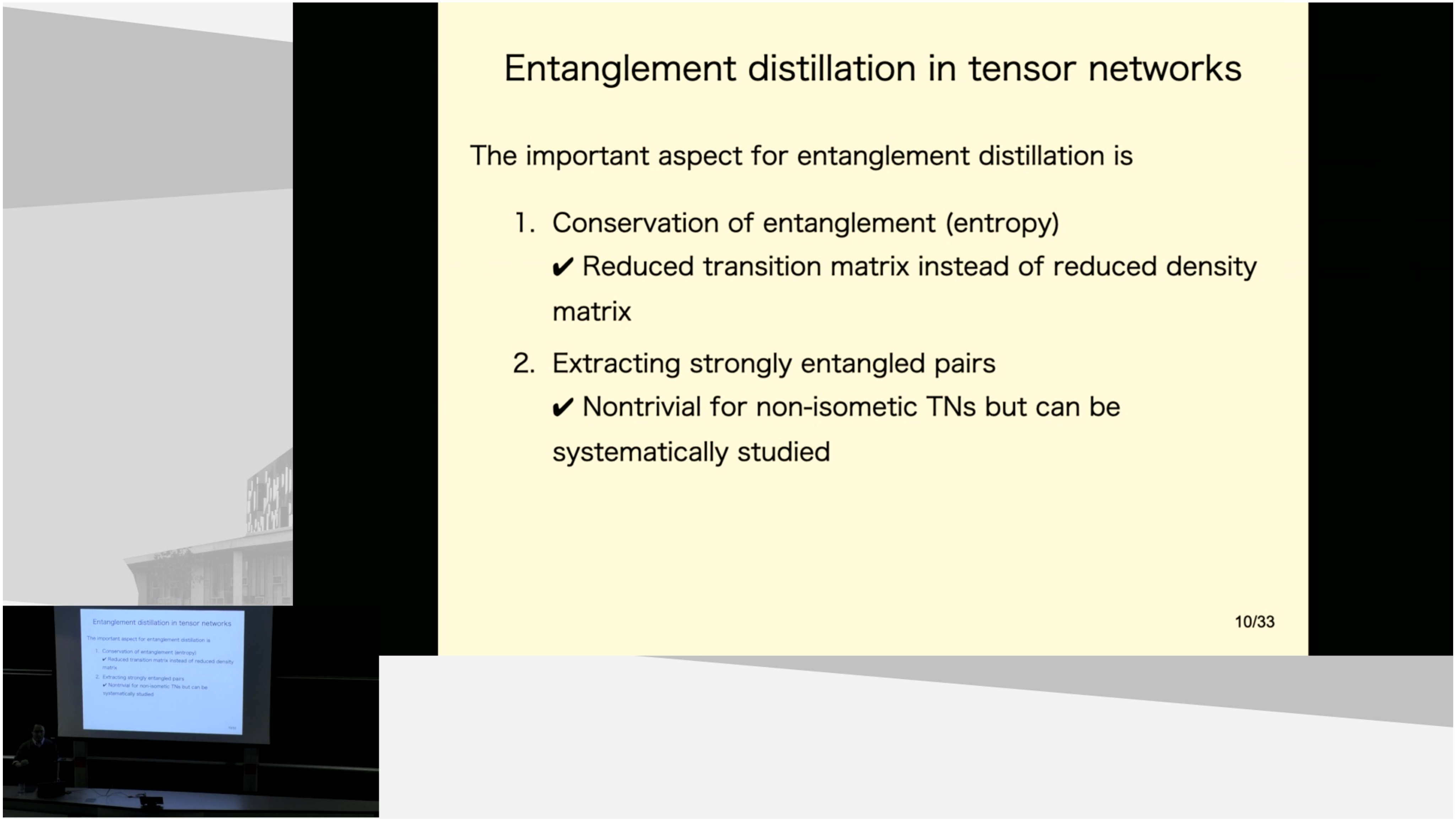
Entanglement distillation in tensor networks
The important aspect for entanglement distillation is
1. Conservation of entanglement (entropy)
✔ Reduced transition matrix instead of reduced density matrix
2. Extracting strongly entangled pairs
✔ Nontrivial for non-isometic TNs but can be systematically studied
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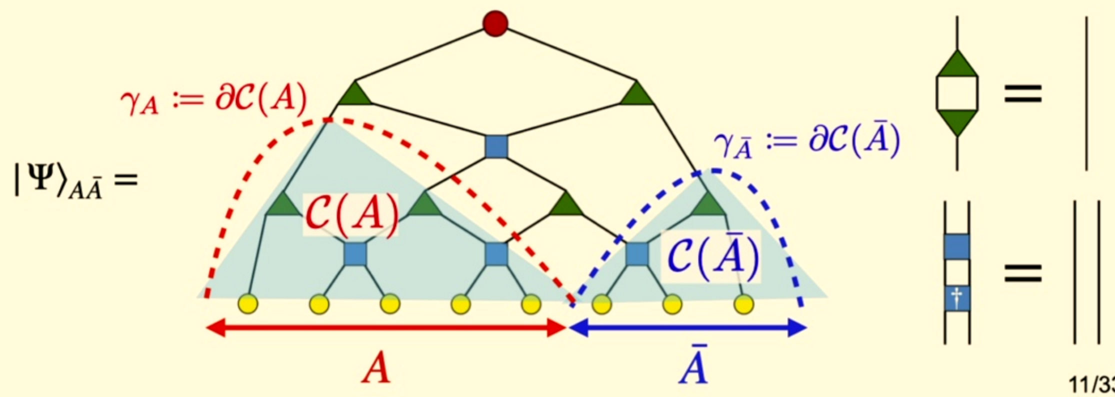
Multi-scale entanglement renormalization ansatz (MERA)

[Vidal]

- MERA has minimal bond cut surface(s) $\gamma_* = \min(\gamma_A, \gamma_{\bar{A}})$
- What we want to see: A state on $\gamma_* \overset{?}{=}$ EPR pairs
- Need to provide a method to properly define a state on a bond cut surface γ



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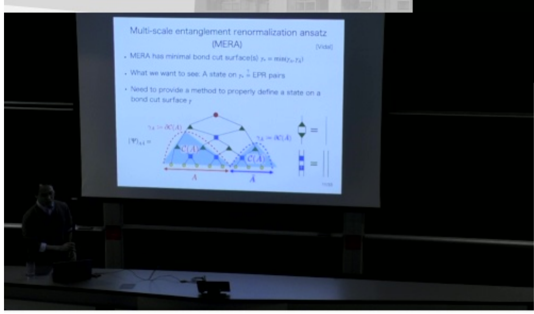
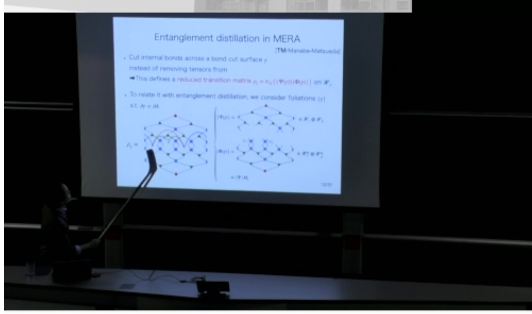


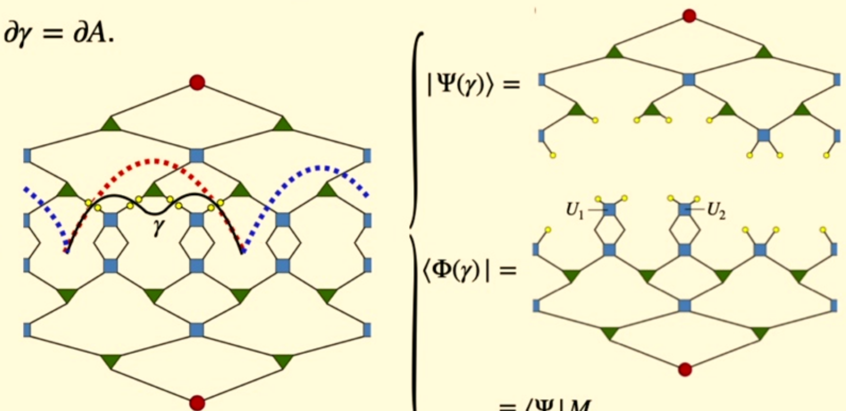


Entanglement distillation in MERA

[**TM**-Manabe-Matsueda]

- Cut internal bonds across a bond cut surface γ instead of removing tensors from
  ➡This defines a reduced transition matrix $\rho_\gamma = \mathrm{tr}_{\bar{A}}\left(|\Psi(\gamma)\rangle\langle\Phi(\gamma)|\right)$ on $\mathscr{H}_\gamma$
- To relate it with entanglement distillation, we consider foliations $\{\gamma\}$ s.t. $\partial\gamma = \partial A$.

$$\rho_\gamma = \left\{ \begin{array}{l} |\Psi(\gamma)\rangle = \quad \in \mathscr{H}_\gamma \otimes \mathscr{H}_{\bar{A}} \\ \langle\Phi(\gamma)| = \quad \in \mathscr{H}^*_\gamma \otimes \mathscr{H}^*_{\bar{A}} \\ \equiv \langle\Psi| M_\gamma \end{array} \right.$$



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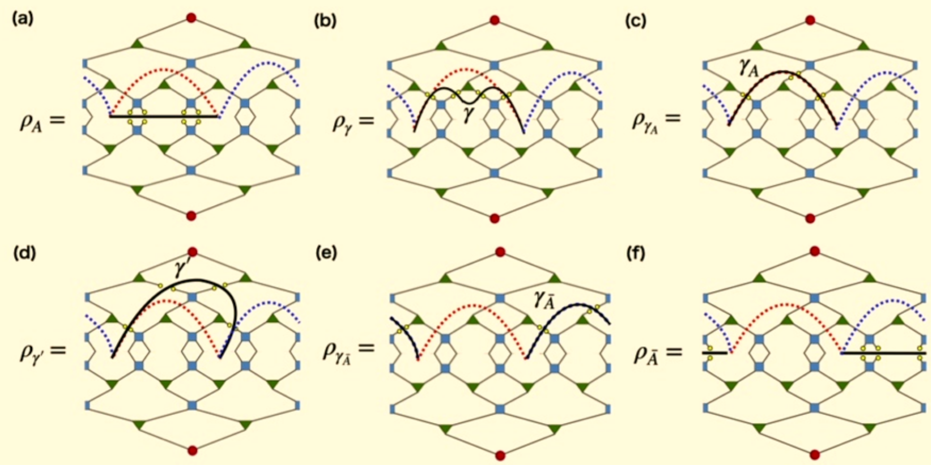
Entanglement distillation in MERA

[**TM**-Manabe-Matsueda]

- Changing the location of foliations γ, we obtain a family of states on various bond cut surfaces (e.g. ρ_A for $\gamma = A$, ρ_{γ_*} for $\gamma = \gamma_*$)
- Entanglement conservation w.r.t. $\forall\gamma$: $S(\rho_\gamma) = S(\rho_A)$, where $S(\rho_\gamma)$ is the pseudo entropy [Nakata-Takayanagi et al.] $S(\rho_\gamma) = -\mathrm{tr}\rho_\gamma \log \rho_\gamma$



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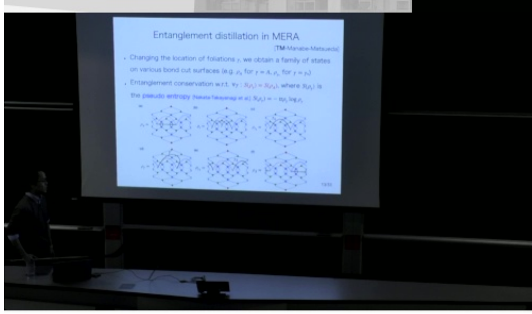

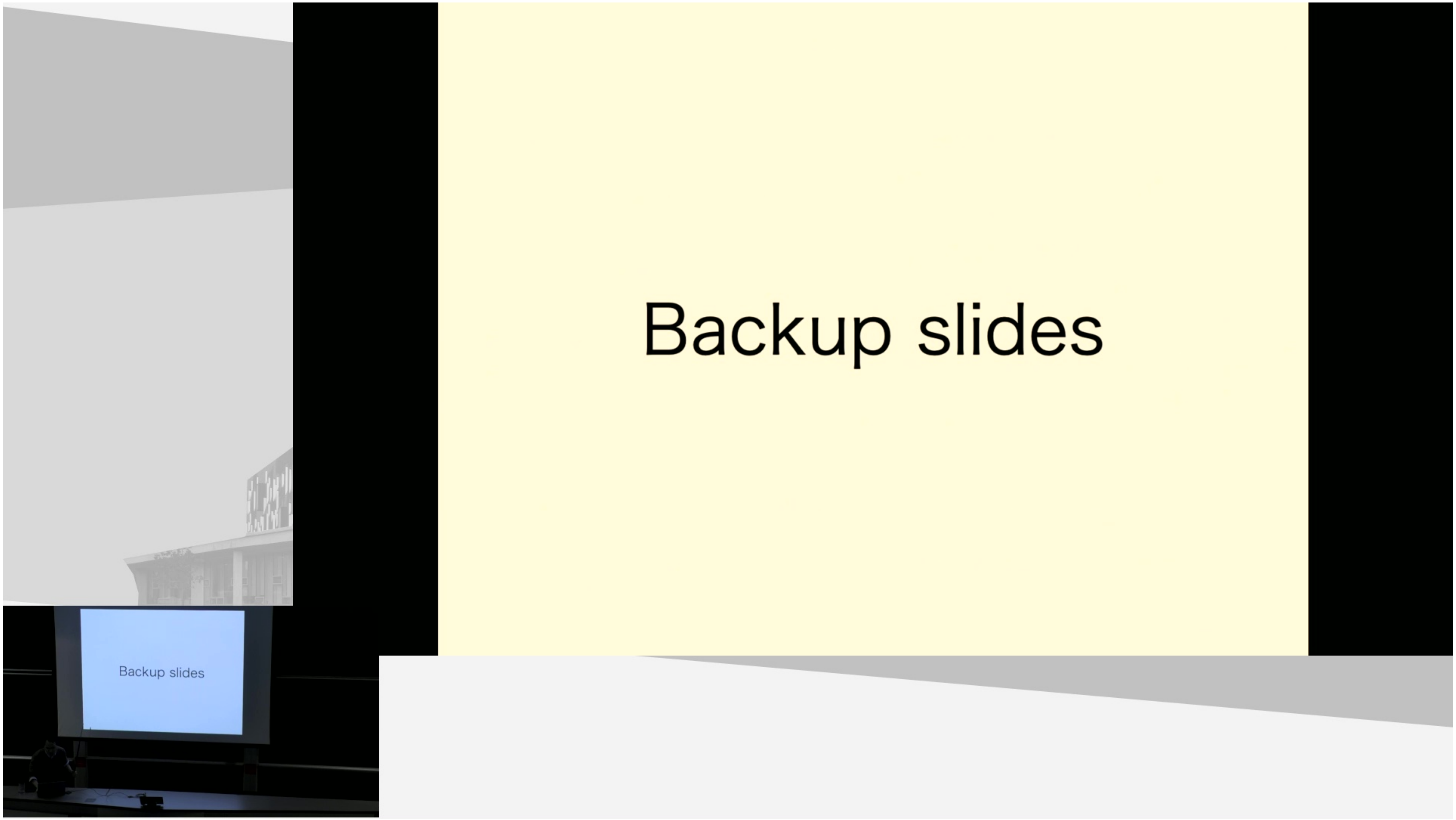
Backup slides

Operational interpretation of pseudo entropy

[Nakata-Takayanagi et al.]

When the reduced transition matrices for A and $\bar{A}$ both are positive, semi-definite, and Hermitian,
the operational interpretation of the pseudo entropy

$$S_A\left(\mathcal{T}^{\psi_1|\psi_2} = \frac{|\psi_1\rangle\langle\psi_2|}{\langle\psi_2|\psi_1\rangle}\right) = -\operatorname{Tr}\left[\mathcal{T}^{\psi_1|\psi_2}\log\mathcal{T}^{\psi_1|\psi_2}\right]$$ is

the average number of EPR pairs that can be distilled from an intermediate state $|n\rangle$ during the transition $\langle\psi_2|\psi_1\rangle = \sum_n \langle\psi_2|n\rangle\langle n|\psi_1\rangle$ in the i.i.d. limit.

The proof is same as entanglement entropy but insert projection operator in the transition amplitude to calculate the probability of distilling an entangled state.

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Entanglement distillation in MERA

[**TM**-Manabe-Matsueda]

- Changing the location of foliations γ, we obtain a family of states on various bond cut surfaces (e.g. ρ_A for $\gamma = A$, ρ_{γ_*} for $\gamma = \gamma_*$)

- Entanglement conservation w.r.t. $\forall \gamma$: $S(\rho_\gamma) = S(\rho_A)$, where $S(\rho_\gamma)$ is the pseudo entropy $S(\rho_\gamma) = -\mathrm{tr}\rho_\gamma \log \rho_\gamma$

  ⬅ This is owing to the common eigenvalue distribution between ρ_γ and ρ_A ; the same entanglement spectrum!

- Furthermore, when M_γ is isometric, we can show $\langle \Phi(\gamma) | = \langle \Psi(\gamma) |$

  ⇒ For isometric TNs, we obtain EPR pairs across the minimal bond cut surface (i.e. $\rho_{\gamma_*} \propto \mathbf{1}$) when $\gamma = \gamma_*$

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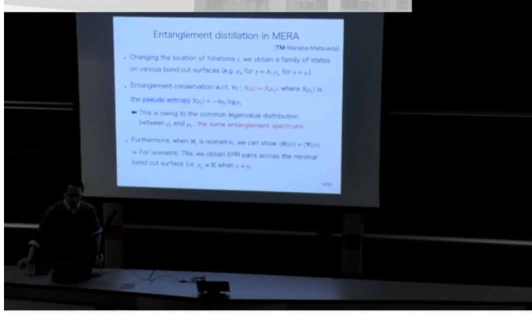

Entanglement distillation in MERA

[**TM**-Manabe-Matsueda]

- To go beyond the isometric case, we need to define a state from a reduced transition matrix.

We use the purification technique (a.k.a. channel-state duality)

$$|\rho_\gamma^{1/2}\rangle \equiv \mathcal{N}_\gamma \sqrt{\dim \mathscr{H}_\gamma}(\rho_\gamma^{1/2} \otimes \mathbf{1})|\mathrm{EPR}_\gamma\rangle,$$

where $\mathcal{N}_\gamma = \left[\mathrm{tr}(\rho_\gamma^{\dagger\,1/2}\rho_\gamma^{1/2})\right]^{-1/2}$ and $|\mathrm{EPR}_\gamma\rangle = (\dim \mathscr{H}_\gamma)^{-1/2} \sum_{i=1}^{\dim \mathscr{H}_\gamma} |i\rangle \otimes |i\rangle$.

This will be regarded as a geometrically distilled state up to γ by TN.

Now we can make a quantitative comparison w.r.t. EPR pairs!

➡ Trace distance from the EPR pair: $D_\gamma \equiv \sqrt{1 - \left|\langle \mathrm{EPR}_\gamma | \rho_\gamma^{1/2}\rangle\right|^2}$

(related to Rényi-1/2 entropy $\left|\langle \mathrm{EPR}_\gamma | \rho_\gamma^{1/2}\rangle\right|^2 = \frac{\mathcal{N}_\gamma^2}{\dim \mathscr{H}_\gamma} e^{S_{1/2}}$)

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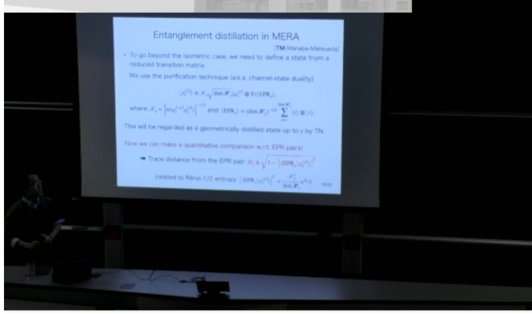
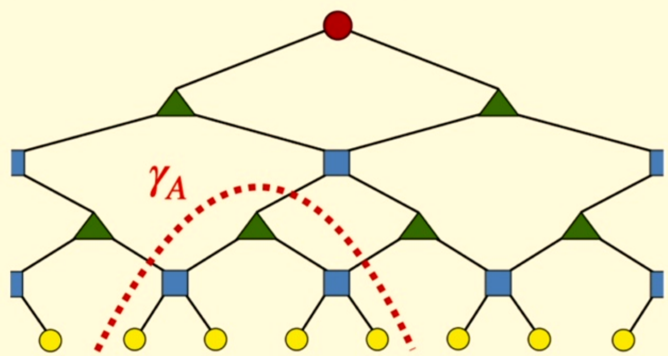


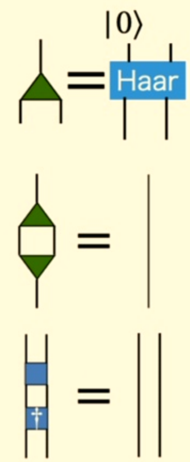
Random MERA

- Random TN is expected to reproduce RT formula in the large bond dimension limit [Hayden et al.]

🤔 Why is the minimal bond cut surface is special?
(RT formula only tells us about entanglement entropy)

🤔 The large bond dimension limit is essential?





(All tensors are Haar random)

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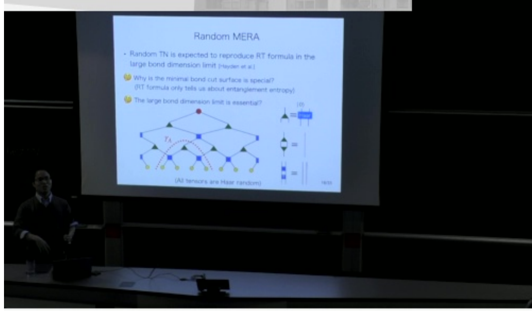
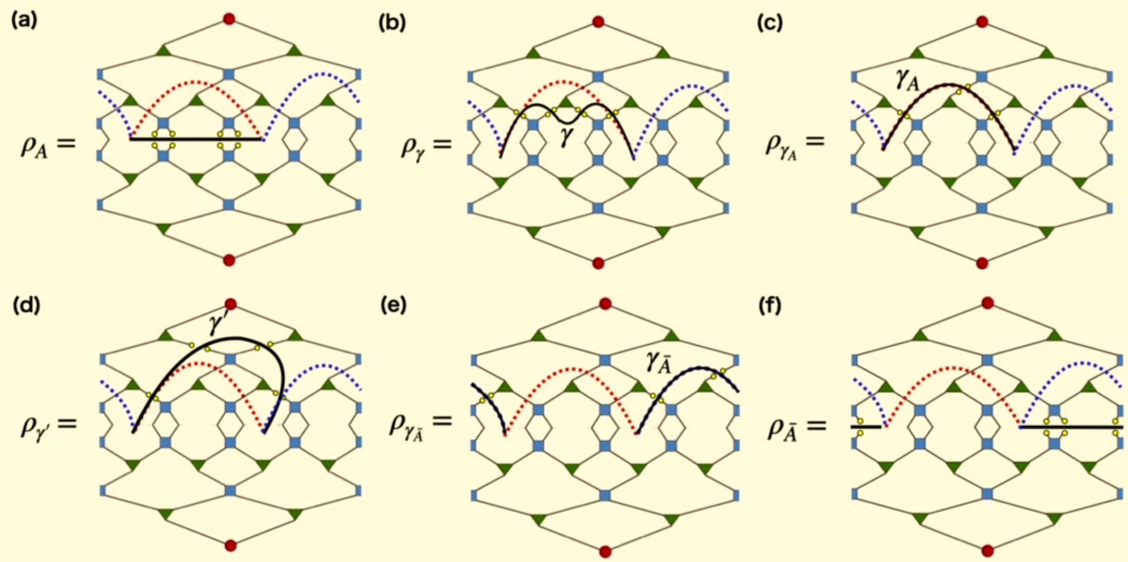


Numerical results for random MERA

[**TM**-Manabe-Matsueda]

Choice of foliations:



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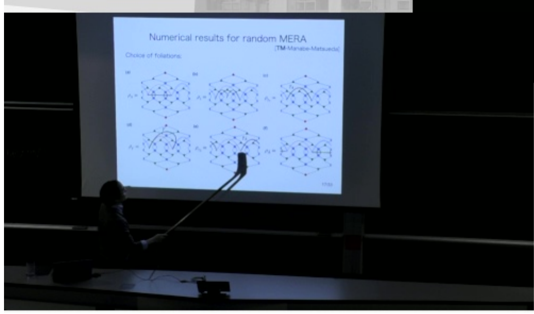
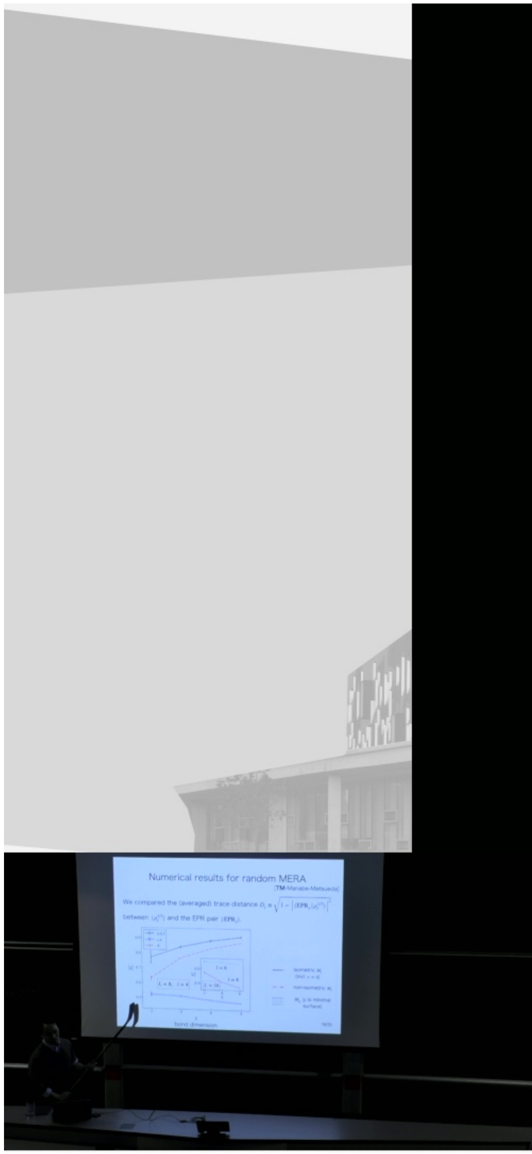


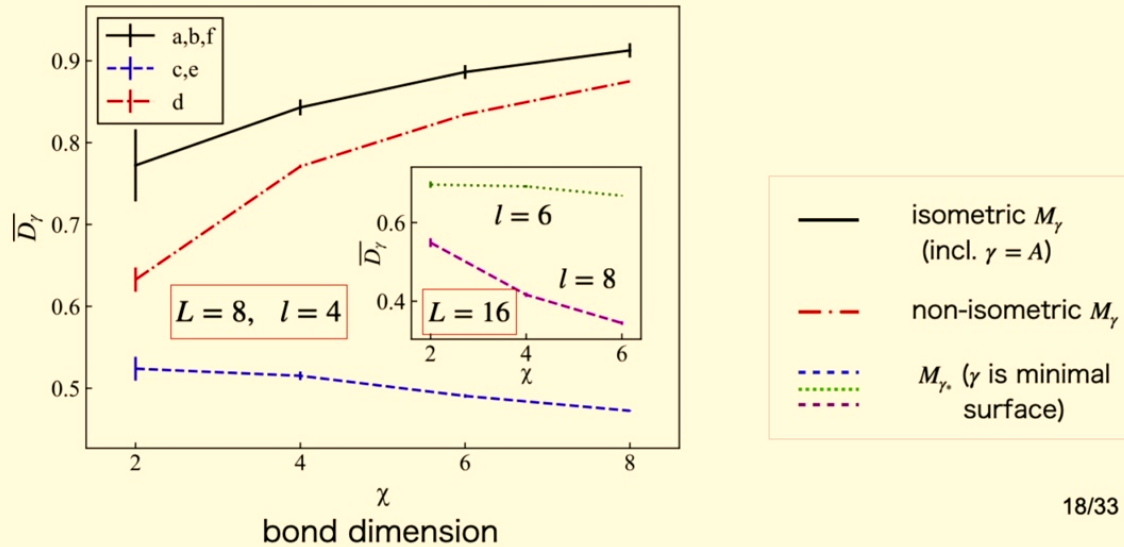


Numerical results for random MERA

[**TM**-Manabe-Matsueda]

We compared the (averaged) trace distance $D_\gamma \equiv \sqrt{1 - \left|\langle \mathrm{EPR}_\gamma | \rho_\gamma^{1/2} \rangle\right|^2}$

between $|\rho_\gamma^{1/2}\rangle$ and the EPR pair $|\mathrm{EPR}_\gamma\rangle$.



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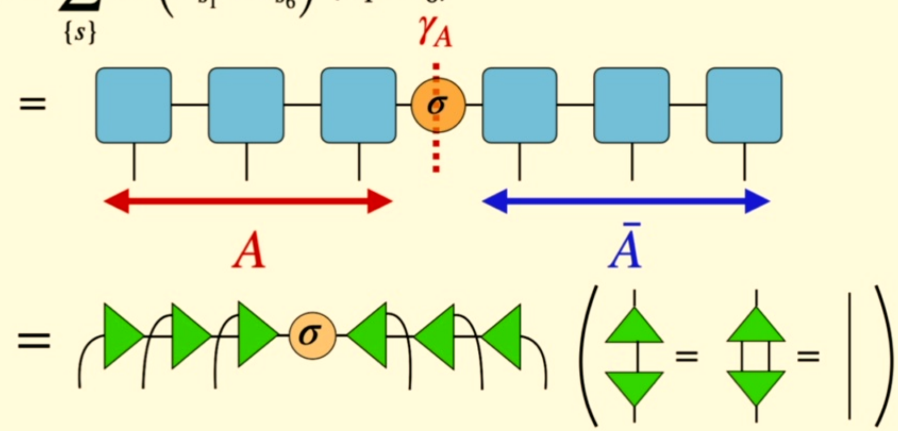
Entanglement distillation in MPS

[**TM**-Manabe-Matsueda]

Similarly consider pushing the foliation towards the minimal bond cut surface in matrix product states (MPS)

Let us focus on the MPS in a mixed canonical form

$$|\Psi\rangle = \sum_{\{s\}} \mathrm{Tr}\left(A_{s_1} \cdots A_{s_6}\right) |s_1 \cdots s_6\rangle$$



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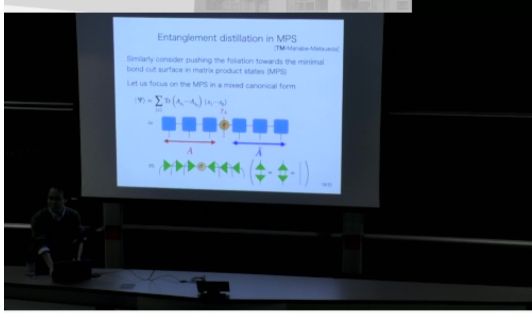
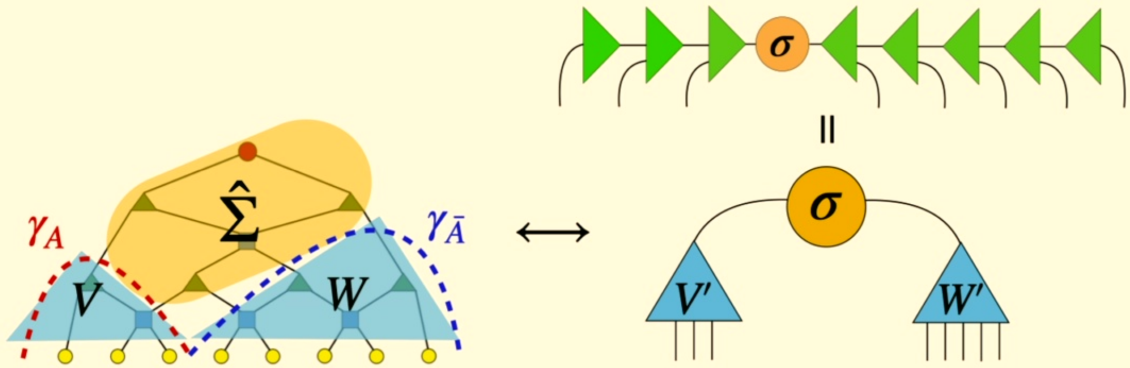


Entanglement distillation in MPS

[**TM**-Manabe-Matsueda]

Note:

An MPS in a mixed canonical form is an analog of MERA (regarding its structure)



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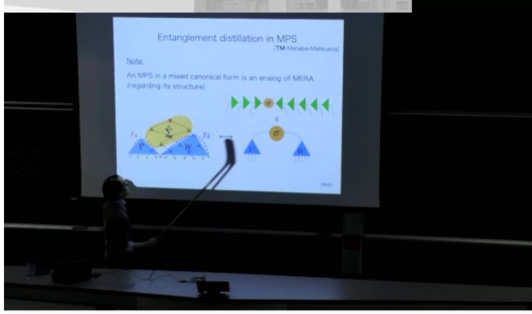
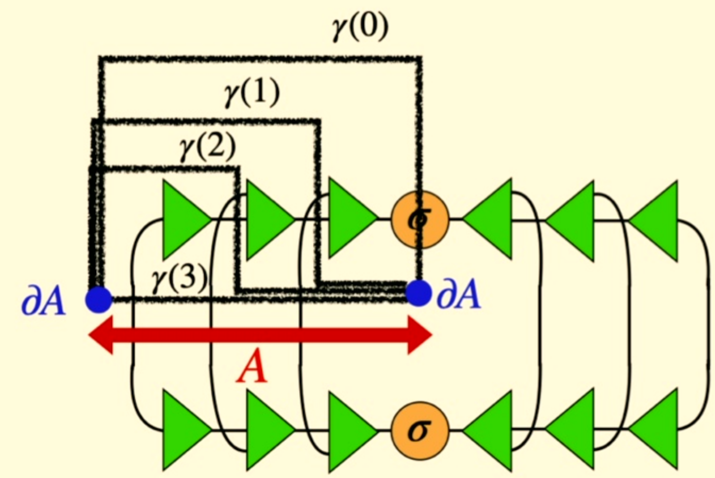


Entanglement distillation in MPS

[**TM**-Manabe-Matsueda]

We can consider the following foliations $\gamma = \gamma(\tau), \quad \tau = 0,1,2,3$

($\tau \sim$ distance from the minimal bond cut surface)



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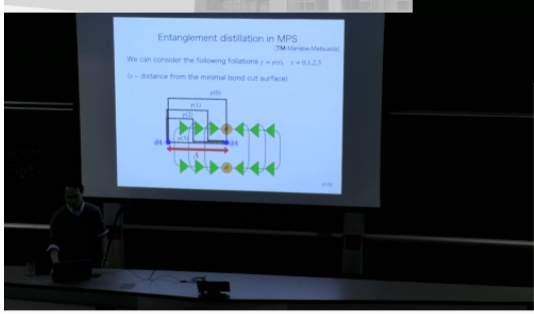
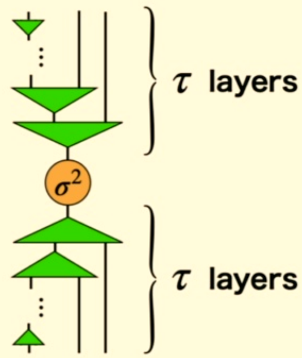


Entanglement distillation in MPS

[**TM**-Manabe-Matsueda]

The reduced transition matrix and the trace distance are

$\rho_{\gamma(\tau)} =$ 

$$D_{\gamma(\tau)} = \sqrt{1 - \frac{e^{S_{1/2}}}{\chi^{\tau+1}}}, \quad \tau = 0,1,2$$

$$D_{\gamma(3)} = D_{\gamma(2)}$$

The distillation by pushing the foliation equals removing redundant tensors.

The entanglement spectrum σ remains unchanged.

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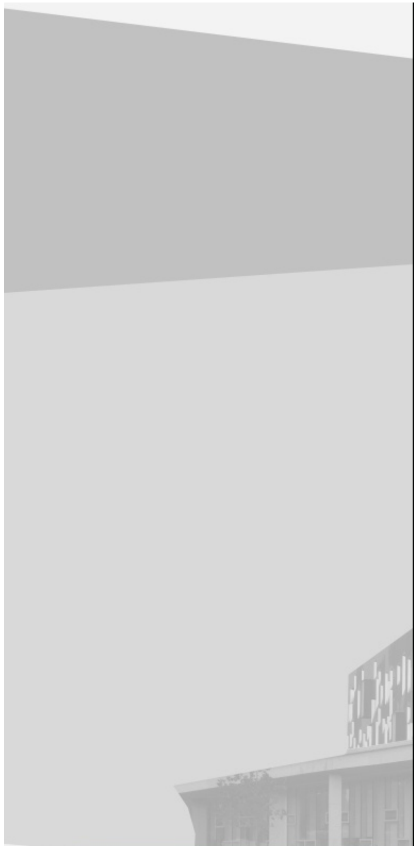
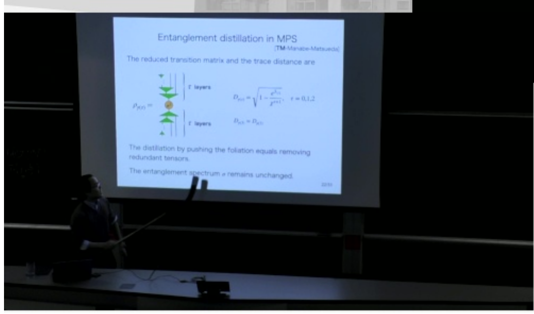


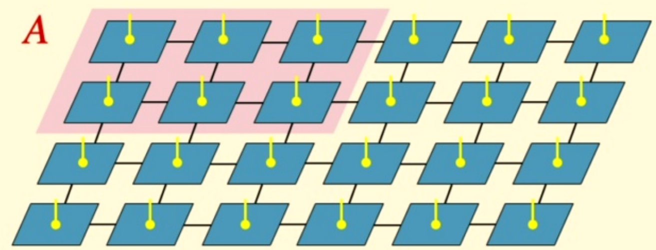


Entanglement distillation in higher dimensional TNs

[work in progress]

In fact, one can take a similar canonical form for left-right bipartitioning of any higher-dimensional open boundary systems!

For example, a projected entangled-pair state (PEPS)



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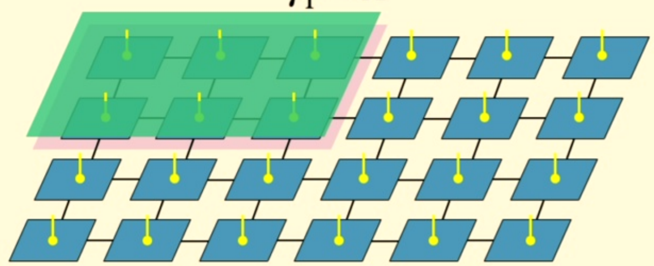
$\gamma_1 = A$



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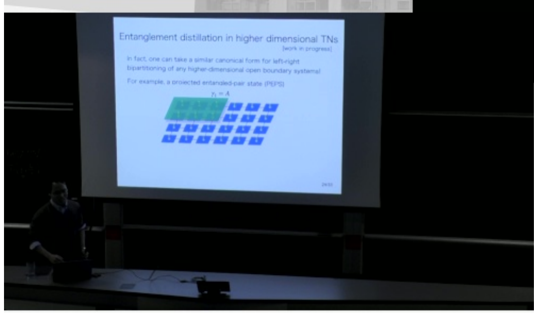

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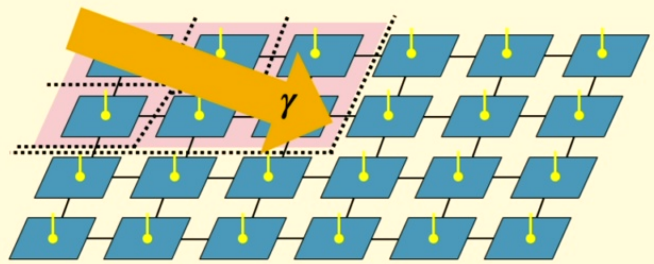

In the normal direction (foliating direction), the geometric distillation is a 1-dimensional array.

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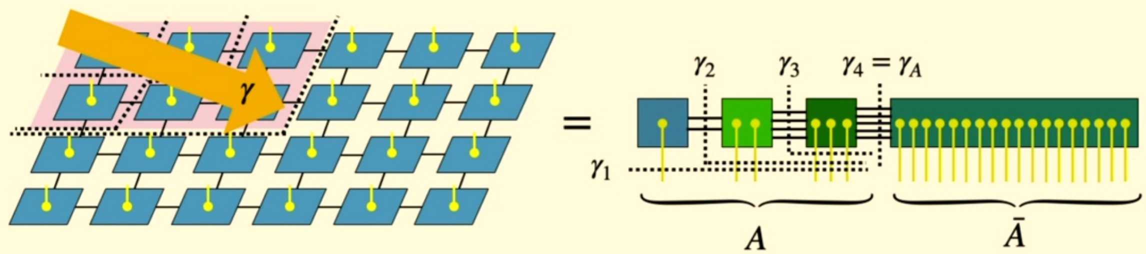

In the normal direction (foliating direction), the geometric distillation is a 1-dimensional array. One can always make it MPS and canonicalize it.

➡ Geometric entanglement distillation indeed removes isometric dofs

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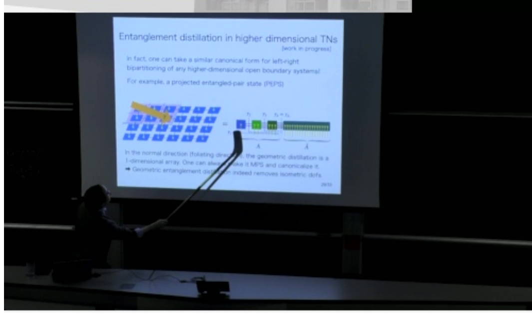
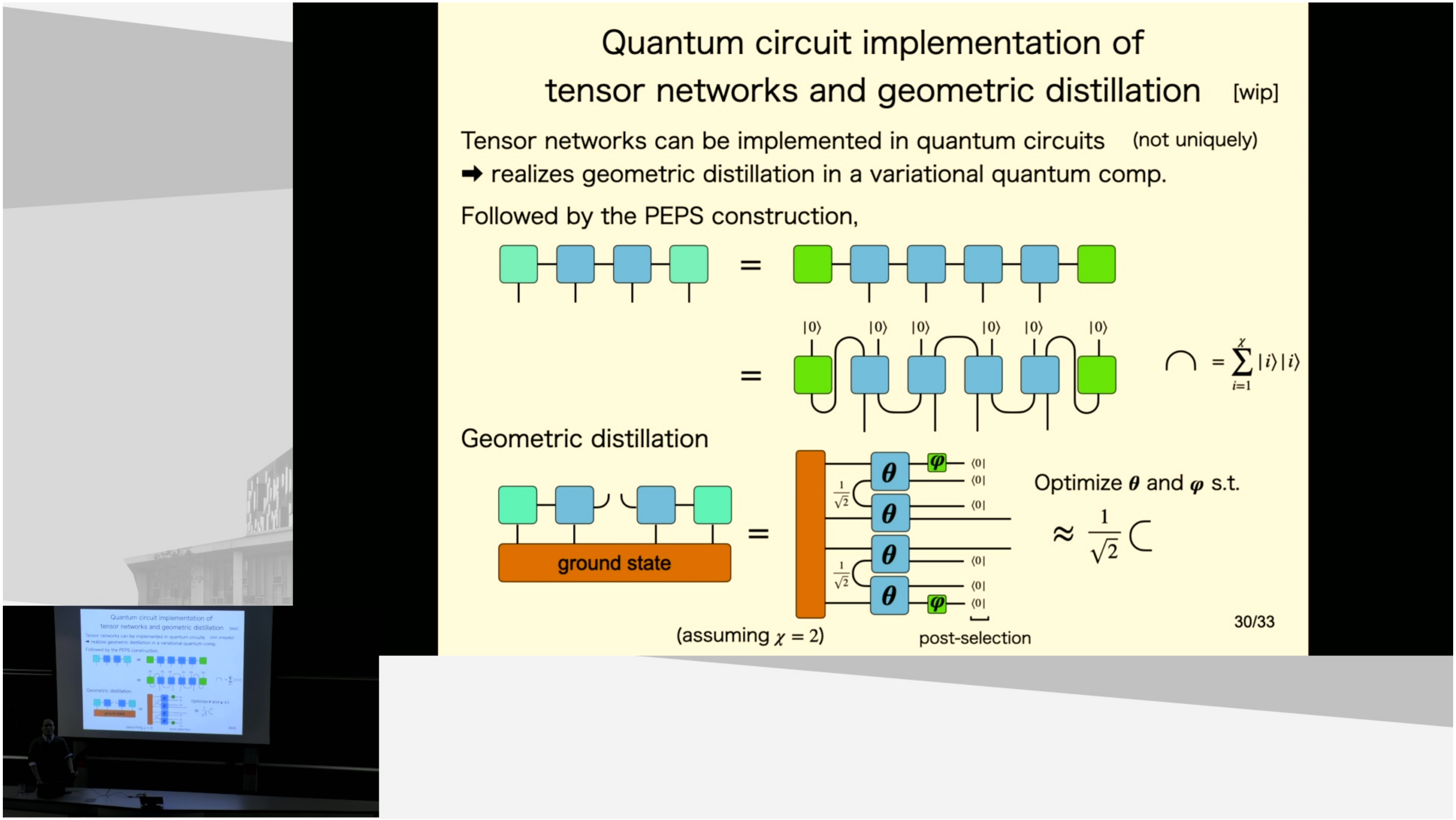
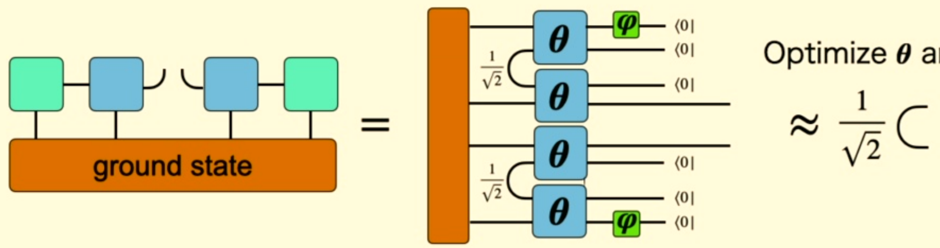



Quantum circuit implementation of
tensor networks and geometric distillation [wip]
Tensor networks can be implemented in quantum circuits (not uniquely)
➡ realizes geometric distillation in a variational quantum comp.
Followed by the PEPS construction,
=
$|0\rangle$ $|0\rangle$ $|0\rangle$ $|0\rangle$ $|0\rangle$ $|0\rangle$
=
$= \sum_{i=1}^{\chi} |i\rangle|i\rangle$
Geometric distillation
ground state
=
θ
φ
$\langle 0|$
$\frac{1}{\sqrt{2}}$
Optimize θ and φ s.t.
$\approx \frac{1}{\sqrt{2}}$
(assuming $\chi = 2$)
post-selection
30/33

Quantum circuit implementation of tensor networks and geometric distillation [wip]



Since we already consumes EPR pairs as a resource, it is not so meaningful as a process of distilling an EPR. ("entanglement-assisted distillation")

Rather, this is an ***entanglement teleportation***! The above ansatz answers to the question

> Assume we can prepare EPR pairs inside each subregion.
> Using EPR pairs in each subregion as a resource, how much EPR pairs can be harvested (teleported) between A and $\bar{A}$ from a non-maximal entangled many-body ground state by LOCC? How?

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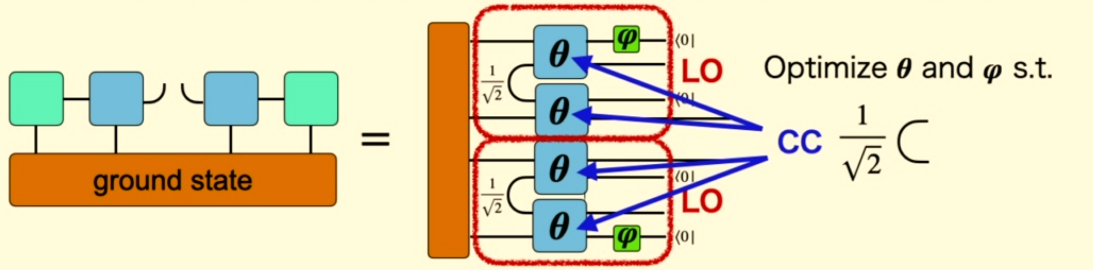
Quantum circuit implementation of tensor networks and geometric distillation [wip]



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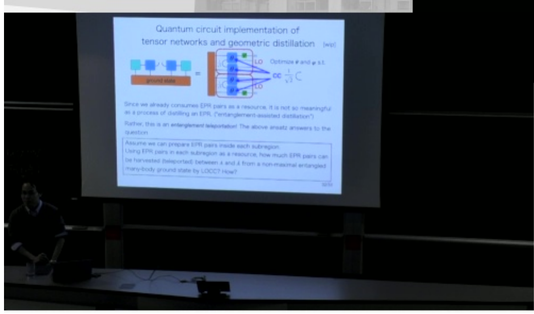
32/33



Summary

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arXiv:2205.06633

- By pushing the bond cut surface γ to the minimal bond cut surface, strongly entangled pairs are geometrically distilled in tensor networks while retaining the entanglement spectrum
- It is essential to consider reduced transition matrices rather than a reduced density matrix
- Our method works for various TNs including higher dimensional TNs. This suggests geometry of TN is intimately related to distillation

  Emergent geometry from distillation
- Quantum circuit implementation implies another interpretation as entanglement teleportation through an entangled many-body ground state

Future directions

- Operational interpretation of geometric distillation [Milsted-Vidal]
  (Geometric distillation is nontrivial in local, non-canonical form!)
- Universal TN-based VQE to distill EPRs?
- CFT realization? Quant. adiabatic comp. (~annealing) with $T\bar{T}$
  [McGough-Mezei-Verlinde]

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