Title: Modular commutators in conformal field theory, topological order, and holography

Speakers: Yijian Zou

Series: Quantum Matter

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Abstract: The modular commutator is a recently discovered multipartite entanglement measure that quantifies the chirality of the underlying many-body quantum state. In this Letter, we derive a universal expression for the modular commutator in conformal field theories in 1+1 dimensions and discuss its salient features. We show that the modular commutator depends only on the chiral central charge and the conformal cross ratio. We test this formula for a gapped (2+1)-dimensional system with a chiral edge, i.e., the quantum Hall state, and observe excellent agreement with numerical simulations. Furthermore, we propose a geometric dual for the modular commutator in certain preferred states of the AdS/CFT correspondence. For these states, we argue that the modular commutator can be obtained from a set of crossing angles between intersecting Ryu-Takayanagi surfaces.

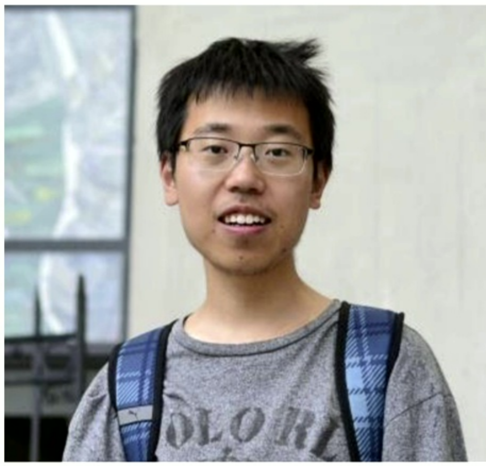
Zoom link: https://pitp.zoom.us/j/94069836709?pwd=RlA2ZUsxdXlPTlh2TStObHFDNUY0Zz09

# Modular Commutators in conformal field theory, topological phases and holography

Yijian Zou

Stanford University

# Collaborators



Bowen Shi
UC San Diego

Jonathan Sorce
Stanford -> MIT

Ian Lim
UC Davis

Issac Kim
UC Davis

# Motivation

Characterizing universality is one of the main themes of condensed matter physics

Motto: It from Qubit

| UV theory (lattice model) | RG flow → | IR theory (field theory) |
| --- | --- | --- |
| Lattice observables, such as | | Universal data, such as |
| *Wavefunctions* | | *Conformal data (CFT)* |
| *Correlation functions* | | *Modular tensor category (TQFT)* |
| *Entanglement quantities* | | *Emergent geometry (AdS/CFT)* |

# Motivation

- 2+1D topological phases (with gapped bulk)
  - Gappable topological phases (e.g. string-net states)
  - Ungappable topological phases
    - Non-chiral edge states (Levin 2013)
    - **Chiral edge states**

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2+1D topological phases
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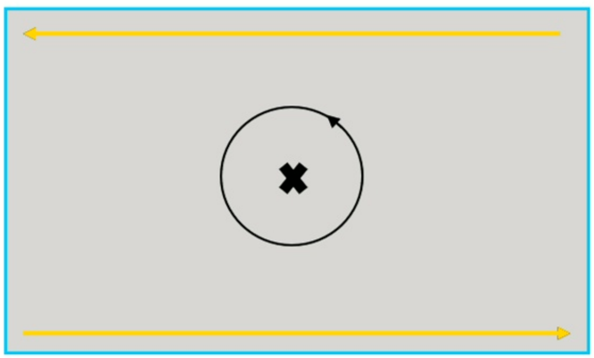
→ **Chiral edge states**



Quantum Hall states
(electron gas in magnetic field)

# Motivation

Long range entanglement is fundamental to topological phases

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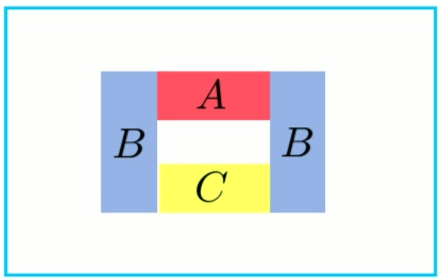


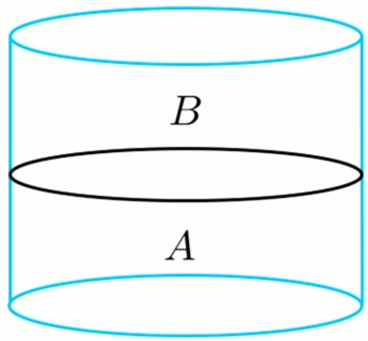


$$I(A:C|B) = 2\gamma$$

Topological entanglement entropy
(Levin, Wen; Kitaev, Preskill 2005)

$$K_A \sim H_L^{CFT}$$

Entanglement spectrum
(Li, Haldane 2008)

$$\langle T_{A,x}^L \rangle_a = e^{\frac{2\pi i p_a}{L_x} - \alpha L_x}$$

Momentum polarization
(Tu, Qi 2013)

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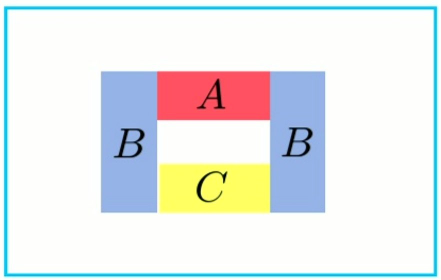


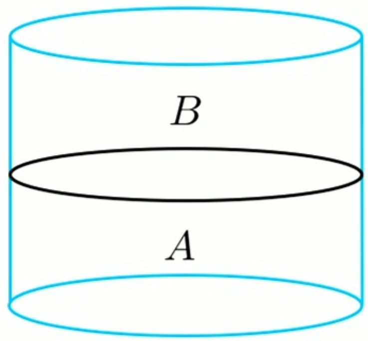


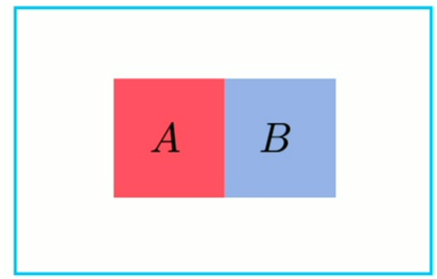


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Momentum polarization
(Tu, Qi 2013)

$$h(A:B) = \frac{c}{3}\log 2$$

Markov gap
(Siva, Zou, Soejima, Mong, Zaletel 2021)

# Result overview

Modular commutator: an entanglement measure to detect chirality

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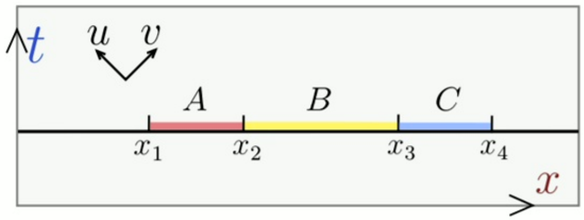
Modular commutator: an entanglement measure to detect chirality

(1) 1+1D CFT

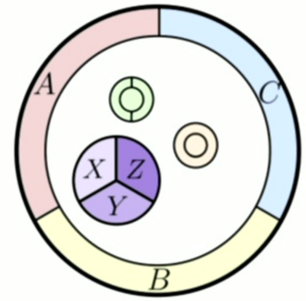



$$J(A,B,C)_{|\Omega\rangle} = \frac{\pi c_-}{6}(2\eta - 1)$$

$$(c_- \equiv c_L - c_R)$$

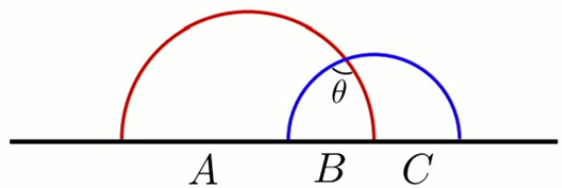
Bulk-edge correspondence

Geometric dual




(2) 2+1D Topological phase

$$J(A,B,C) = -J(X,Y,Z) = \frac{\pi c}{3}$$




(3) Holography $J(A,B,C) = \frac{\pi c_-}{6} \sum_i \cos\theta_i$

# Modular commutator

- Usual entanglement measures are *invariant* under *time reversal (TR).*

$$S(\rho^*) = S(\rho)$$

- In order to detect chirality, we need an entanglement measure that is *odd* under TR

$$J(\rho^*) = -J(\rho)$$

- Modular commutator: $J(A, B, C) = i\mathrm{Tr}([\log \rho_{AB}, \log \rho_{BC}]\rho_{ABC})$

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$$\begin{aligned} J(\rho^*_{ABC}) &= i\mathrm{Tr}([\log \rho^*_{AB}, \log \rho^*_{BC}]\rho^*_{ABC}) \\ &= i\mathrm{Tr}([\log \rho^T_{AB}, \log \rho^T_{BC}]\rho^T_{ABC}) \\ &= -i\mathrm{Tr}([\log \rho_{AB}, \log \rho_{BC}]^T \rho^T_{ABC}) \\ &= -i\mathrm{Tr}([\log \rho_{AB}, \log \rho_{BC}]\rho_{ABC}) = -J(\rho_{ABC}) \end{aligned}$$

# Modular commutator

- Vanishes for classical states $\rho_{ABC} = \sum_i p_i \Pi_i^A \otimes \Pi_i^B \otimes \Pi_i^C \to J(A,B,C) = 0$
- Vanishes for pure states $\rho_{ABC} = |\psi\rangle\langle\psi| \to J(A,B,C) = 0$
- Vanishes for Markov states

$$\begin{aligned} I(A:C|B) &\equiv S(AB) + S(BC) - S(ABC) - S(B) \geq 0 \\ I(A:C|B) &= 0 \;\to\; J(A,B,C) = 0 \end{aligned}$$

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$$J(A,B,C) = i\langle[K_{AB}, K_{BC}]\rangle = i\langle[K_C, K_A]\rangle = 0 \quad (K_X := -\log \rho_X)$$

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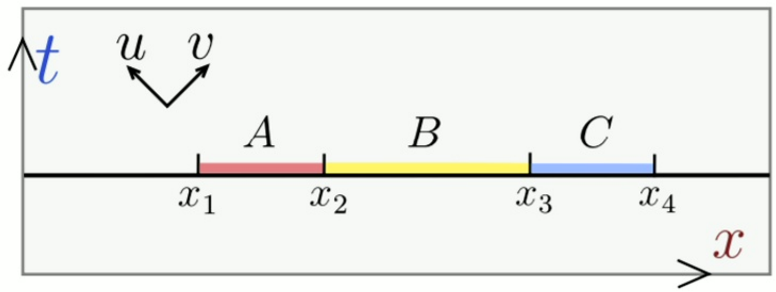
$$I(A:C|B) = 0 \rightarrow K_{ABC} = K_{AB} + K_{BC} - K_B \quad \text{(Petz 2002)}$$

$$\begin{aligned} J(A,B,C) &= i\mathrm{Tr}([K_{AB}, K_{BC}]\rho_{ABC}) \\ &= i\mathrm{Tr}([K_{ABC}, K_{BC}]\rho_{ABC}) + i\mathrm{Tr}([K_B, K_{BC}]\rho_{AB}) \\ &= -i\mathrm{Tr}([K_{ABC}, \rho_{ABC}]K_{BC}) + i\mathrm{Tr}([K_{BC}, \rho_{AB}]K_B) \\ &= 0 \end{aligned}$$

# 1. Modular commutators in CFT




$$J(A,B,C)_{|\Omega\rangle} = \frac{\pi c_-}{6}(2\eta - 1) \qquad \eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

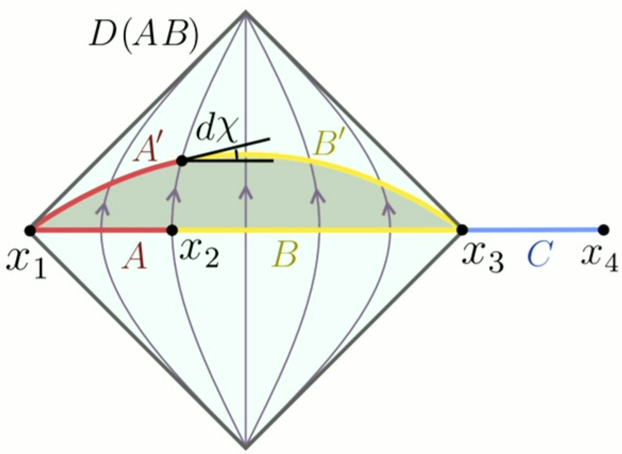
$$(c_- \equiv c_L - c_R)$$

# Derivation sketch

Modular commutator from *modular flow*

*Chen, Dong, Lewkowycz, Qi '18*
*Faulkner, Li, Wang '19*
*Kim, Shi, Kato, Albert '21*

$$\rho_{BC}(s) = \mathrm{Tr}_A(\rho_{AB}^{is}\rho_{ABC}\rho_{AB}^{-is}) \qquad \left.\frac{dS(\rho_{BC}(s))}{ds}\right|_{s=0} = -J(A,B,C)_\rho$$




Within the causal diamond:

$$\begin{pmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{pmatrix} = 2\pi \begin{pmatrix} \frac{(u-u_1)(u_3-u)}{u_1-u_3} \\ \frac{(v-v_1)(v_3-v)}{v_3-v_1} \end{pmatrix}$$

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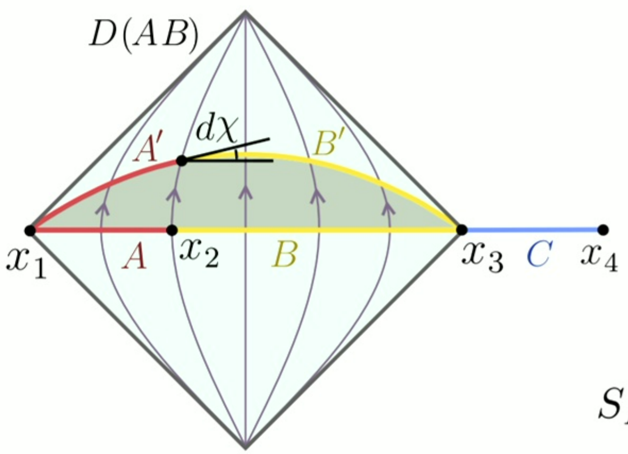


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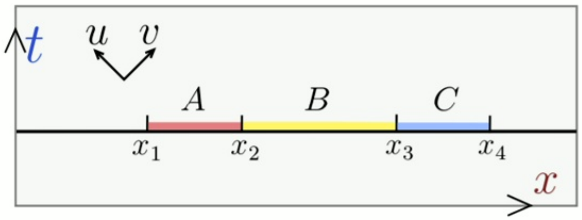
$$S_{BC} = \frac{c_L}{12}\ln\frac{(v_4-v_2)^2}{\epsilon_{v2}\epsilon_{v4}} + \frac{c_R}{12}\ln\frac{(u_4-u_2)^2}{\epsilon_{u2}\epsilon_{u4}}$$ *[Iqbal, Wall '16]*

$$d\ln\epsilon_{v2} = -d\ln\epsilon_{u2} = d\chi$$

# Properties




$$J(A,B,C)_{|\Omega\rangle} = \frac{\pi c_-}{6}(2\eta - 1) \qquad \eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

$$(c_- \equiv c_L - c_R)$$

- Changes sign under "crossing symmetry" $\eta \to 1 - \eta$
- $J = \dfrac{\pi c_-}{6}$ as $x_1 \to -\infty, x_4 \to \infty$ or equivalently $x_2 \to x_3$. This value is *topological.*
- Generalization to finite size or finite temperature $\eta_{\text{eff}}^{(\beta;L)} = \begin{cases} \dfrac{\sin(\pi x_{12}/L)\sin(\pi x_{34}/L)}{\sin(\pi x_{13}/L)\sin(\pi x_{24}/L)}, & \beta/L \to \infty, \\ \dfrac{\sinh(\pi x_{12}/\beta)\sinh(\pi x_{34}/\beta)}{\sinh(\pi x_{13}/\beta)\sinh(\pi x_{24}/\beta)}, & L/\beta \to \infty. \end{cases}$

# Gravitational anomaly

- If $c_L \neq c_R$ then $J(A,B,C) \neq 0$ for pure state $|\psi\rangle_{ABC}$. Contradiction!
- This means that a chiral CFT cannot admit a lattice regularization.
- Entanglement is still well defined, but *frame dependent*.

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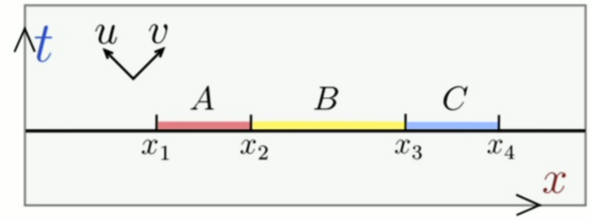

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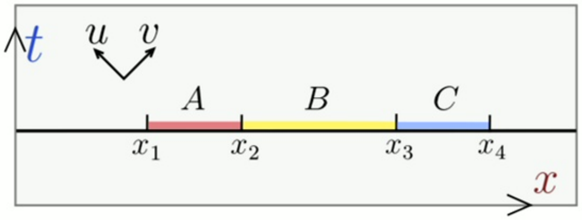


- Entanglement is still well defined, but *frame dependent*.
- This is the effect of *gravitational anomaly* (stress tensor not conserved in curved spacetimes)
- Alternative argument: no conformal boundary condition due to net energy flow

*[Hellerman, Orlando, Watanabe '21]*

# Derivation sketch

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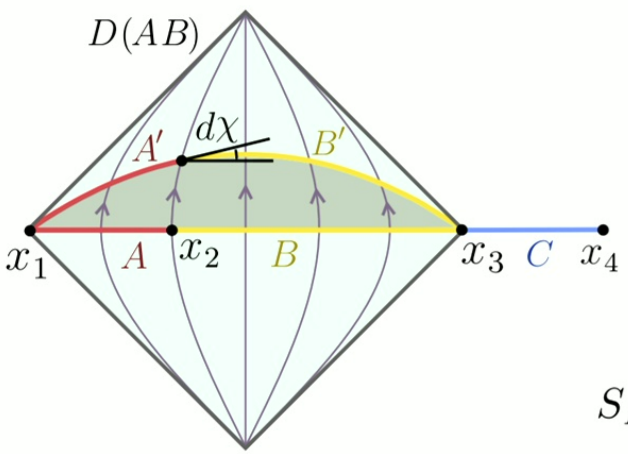


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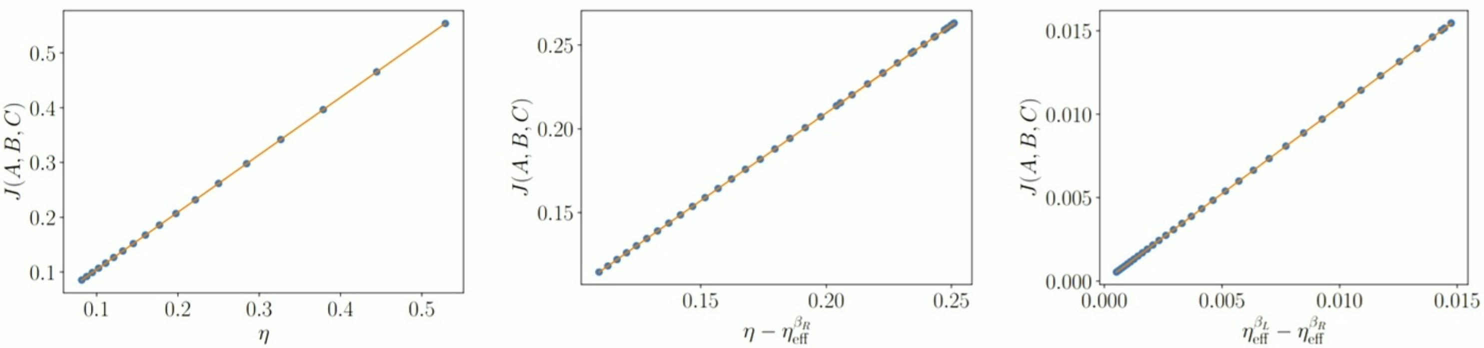
# Chiral thermal states

- Non-chiral CFT with different left/right temperatures (*grand canonical ensemble*)

$$\rho^{(\beta_L,\beta_R)} = \frac{1}{Z} e^{-\beta_L H_L} e^{-\beta_R H_R}$$

- Modular commutator $J(A,B,C)_{\rho^{(\beta_L,\beta_R;L)}} = \frac{\pi}{3} c(\eta_{\text{eff}}^{(\beta_L;L)} - \eta_{\text{eff}}^{(\beta_R;L)})$



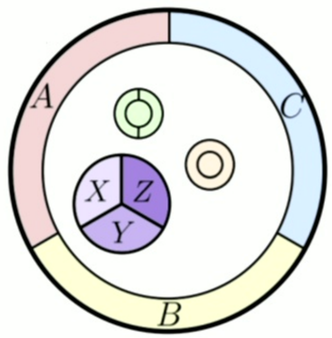

(a) $(\beta_L,\beta_R) = (\infty, 80), L_A = L_C = 320, 120 \le L_B \le 800$

(b) $(\beta_L,\beta_R) = (\infty, 80), L_A = L_C = 40, 10 \le L_B \le 80$

(c) $(\beta_L,\beta_R) = (82, 78), L_A = L_C = 40, 10 \le L_B \le 80$

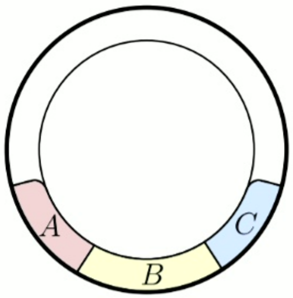
Free fermion CFT
Numerical results

## 2. Modular commutators in topological order




$$J(A, B, C) = -J(X, Y, Z) = \frac{\pi c}{3}$$

# Edge of quantum Hall states




$$\rho_{\text{edge}} = \frac{1}{Z} e^{-\beta_L H_L} e^{-\beta_R H_R} \qquad \beta_L = \infty, \beta_R \to 0$$

*[Tu, Qi '13]*

$$J(A,B,C)_{\rho^{(\beta_L,\beta_R;L)}} = \frac{\pi}{3} c(\eta_{\text{eff}}^{(\beta_L;L)} - \eta_{\text{eff}}^{(\beta_R;L)})$$

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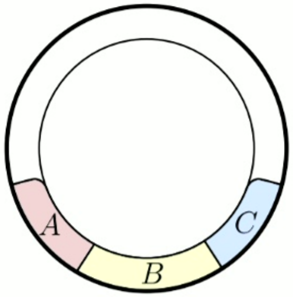
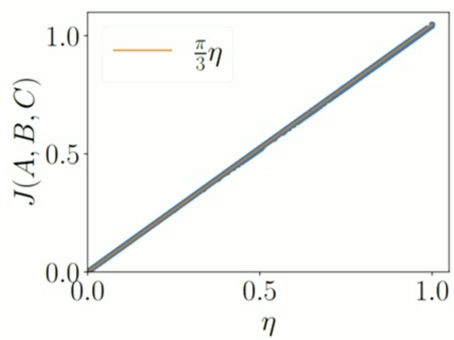


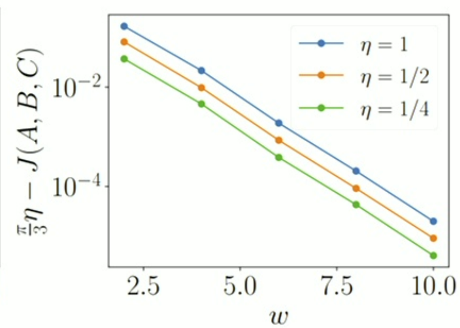

$$\rho_{\text{edge}} = \frac{1}{Z} e^{-\beta_L H_L} e^{-\beta_R H_R} \quad \beta_L = \infty, \beta_R \to 0$$ *[Tu, Qi '13]*

$$J(A,B,C)_{\rho^{(\beta_L,\beta_R;L)}} = \frac{\pi}{3} c(\eta_{\text{eff}}^{(\beta_L;L)} - \eta_{\text{eff}}^{(\beta_R;L)})$$

$$\eta_{\text{eff}}^{(\beta_L,L)} = \frac{\sin(\pi x_{12}/L)\sin(\pi x_{34}/L)}{\sin(\pi x_{13}/L)\sin(\pi x_{24}/L)} \equiv \eta \qquad \eta_{\text{eff}}^{(\beta_R,L)} = \frac{\sinh(\pi x_{12}/\beta_R)\sinh(\pi x_{34}/\beta_R)}{\sinh(\pi x_{13}/\beta_R)\sinh(\pi x_{24}/\beta_R)} = 0$$

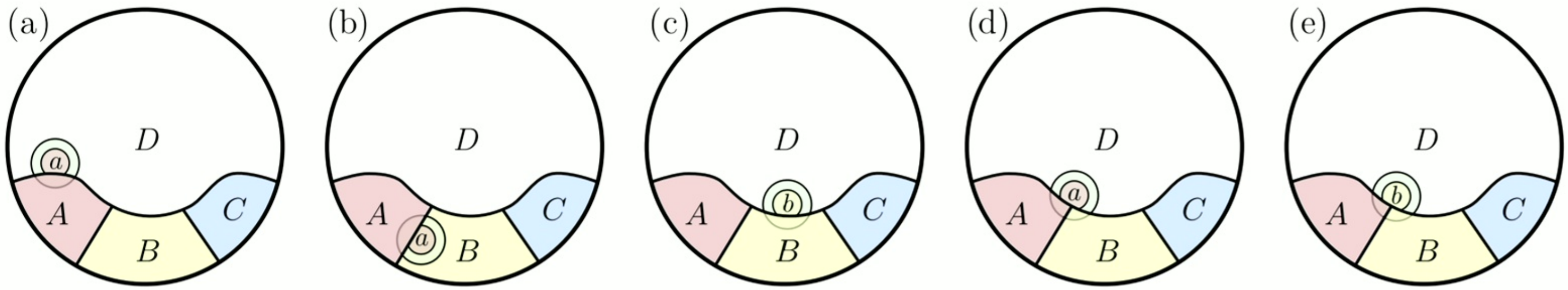
$$\boxed{J(A,B,C) = \frac{\pi c \eta}{3}}$$



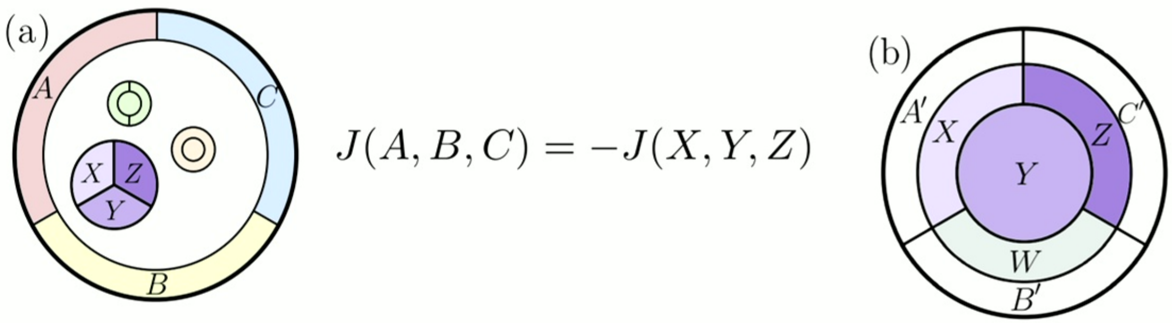



# Topological robustness

- Invariant under local deformations




- If $A, B, C$ fills the whole annulus, then we have a stronger result




$$J(A,B,C) = -J(X,Y,Z)$$

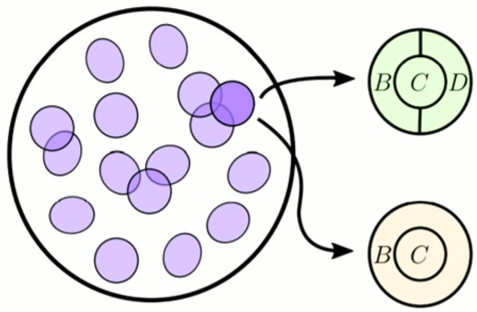
$$J(A'X, B'W, C'Z) = \frac{\pi c}{3}$$

for all $|\tilde{\psi}\rangle = U_{A'B'C'}|\psi\rangle$

# Entanglement bootstrap

- Entanglement bootstrap: a systematic quantum information technique to understand entanglement in 2+1D topological order *[Shi, Kim 2021; Shi, Kato, Kim 2020; Shi 2020]*

- Axioms:



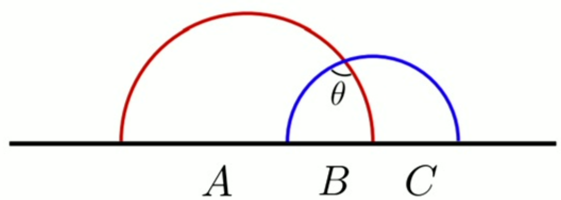

$$\textbf{A1:}\ (S_{BC} + S_{CD} - S_B - S_D)_{|\psi_{2D}\rangle} = 0.$$

$$\textbf{A0:}\ (S_{BC} + S_C - S_B)_{|\psi_{2D}\rangle} = 0.$$

- These axioms are very powerful. They generate a series of quantum Markov chains in the bulk. They have been used to derive fusion rules of anyons, Verlinde formula, gapped boundaries, etc.

- No extra assumption for topological phases with a boundary!

# 3. Modular commutators in holography




$$J(A,B,C) = \frac{\pi c_-}{6} \sum_i \cos\theta_i$$

# Holographic duality

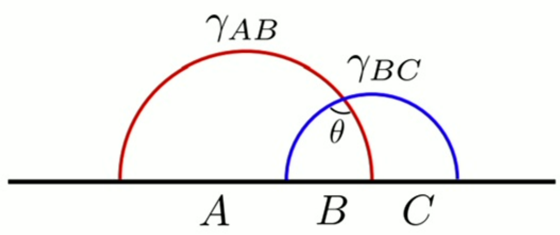
| 1+1 dimensional large-N CFT | 2+1 dimensional semiclassical gravity in asymptotic AdS spacetimes | |
|---|---|---|
| CFT state | Semiclassical geometry | |
| CFT ground state | Vacuum AdS | |
| CFT (chiral) thermal state | (Rotating) BTZ black hole | |
| Global conformal symmetry | Isometry of AdS | |
| Central charge $c$ | AdS curvature $l$ | *[Brown, Henneaux '86]* |
| **Entanglement entropy** | **Area of RT surfaces** | *[Ryu, Takayanagi '06]* |
| **Modular Hamiltonian** | **Area operator** | *[JLMS '14]* |

# RT formula & JLMS formula

Hyperbolic plane
(a time slice of vacuum AdS) $ds^2 = \frac{l^2}{z^2}(dx^2 + dz^2)$

$$\gamma_X = \text{argmin}\{\mathcal{A}_\gamma, \gamma \text{ homologous to } X\}$$

$$S_X = \frac{\mathcal{A}_{\gamma_X}}{4G_N} + O(1) \qquad \mathcal{A}_\gamma = \int_\gamma ds \sqrt{g_{\mu\nu} v^\mu v^\nu}$$

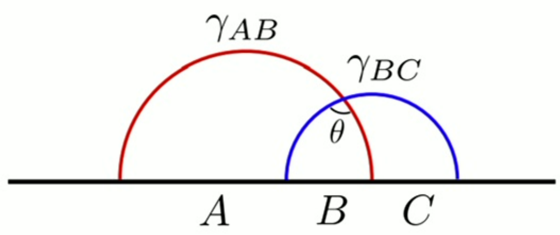


# RT formula & JLMS formula

Hyperbolic plane
(a time slice of vacuum AdS) $ds^2 = \frac{l^2}{z^2}(dx^2 + dz^2)$

$$\gamma_X = \text{argmin}\{\mathcal{A}_\gamma, \gamma \text{ homologous to } X\}$$

$$S_X = \frac{\mathcal{A}_{\gamma_X}}{4G_N} + O(1) \qquad \mathcal{A}_\gamma = \int_\gamma ds\, \sqrt{g_{\mu\nu} v^\mu v^\nu}$$

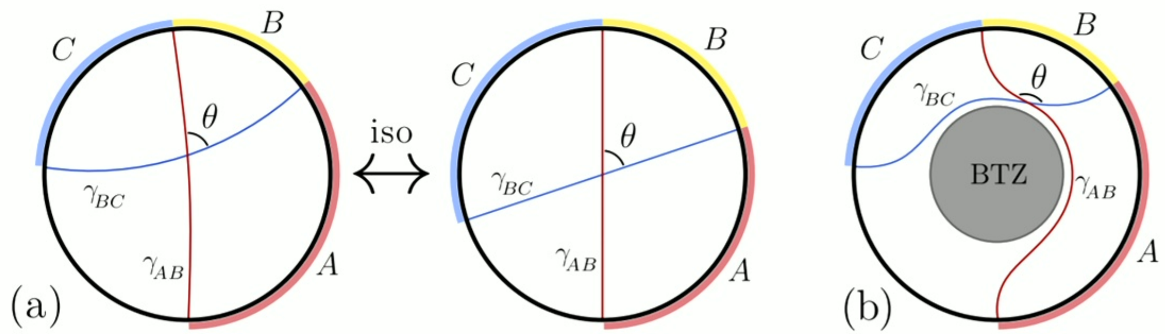



Regarding area as an operator

$$S_X = \langle K_X \rangle \qquad K_X = \frac{\mathcal{A}_{\gamma_X}}{4G_N} + K_{\text{bulk}}$$

**Modular commutator in CFT = commutator of area operator in AdS**

# Modular commutators in chiral AdS/CFT




Vacuum AdS

$$\cos\theta = 2\eta - 1$$

BTZ black hole

$$\cos\theta = 2\eta_{\text{eff}} - 1$$

$$J(A,B,C) = \frac{\pi c_-}{6}\cos\theta$$

# Uncertainty relation of geometric operators

- In chiral AdS/CFT, modular Hamiltonian $K_X$ is dual to some geometric operator $F_X$

$$J(A,B,C) = \frac{\pi c_-}{6}\cos\theta$$

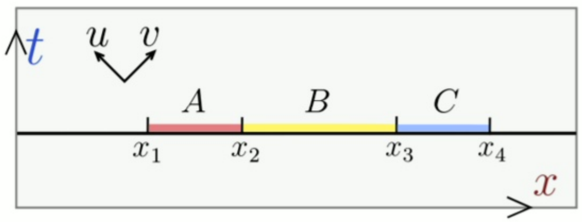
$$J(A,B,C) = i\langle[K_{AB}, K_{BC}]\rangle$$

$$\Longrightarrow \quad \Delta F_{AB}\Delta F_{BC} \geq \frac{\pi c_-}{12}|\cos\theta|$$

# Conclusion

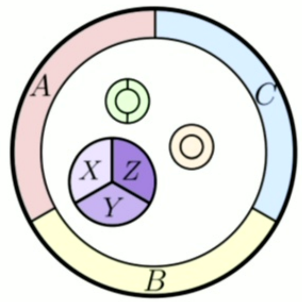
Modular commutator: an entanglement measure to detect chirality

(1) 1+1D CFT




$$J(A,B,C)_{|\Omega\rangle} = \frac{\pi c_-}{6}(2\eta - 1)$$

$$(c_- \equiv c_L - c_R)$$




(2) 2+1D Topological phase

$$J(A,B,C) = -J(X,Y,Z) = \frac{\pi c}{3}$$

# Open questions

- Beyond conformal family of identity states
- Modular commutators under scrambling or integrable dynamics
- Higher dimensions

- Nontrivial anyon flux in fractional quantum Hall states
- Making use of global symmetries, detecting SPT phases
- Topological quantum field theory derivation for the bulk formula

- Beyond contiguous intervals
- Warped black holes in topologically massive gravity
- Non-stationary spacetimes, such as black hole formation

Thank you for listening!