

Title: Modular commutators in conformal field theory, topological order, and holography

Speakers: Yijian Zou

Series: Quantum Matter

Date: October 11, 2022 - 3:30 PM

URL: <https://pirsa.org/22100092>

Abstract: The modular commutator is a recently discovered multipartite entanglement measure that quantifies the chirality of the underlying many-body quantum state. In this Letter, we derive a universal expression for the modular commutator in conformal field theories in 1+1 dimensions and discuss its salient features. We show that the modular commutator depends only on the chiral central charge and the conformal cross ratio. We test this formula for a gapped (2+1)-dimensional system with a chiral edge, i.e., the quantum Hall state, and observe excellent agreement with numerical simulations. Furthermore, we propose a geometric dual for the modular commutator in certain preferred states of the AdS/CFT correspondence. For these states, we argue that the modular commutator can be obtained from a set of crossing angles between intersecting Ryu-Takayanagi surfaces.

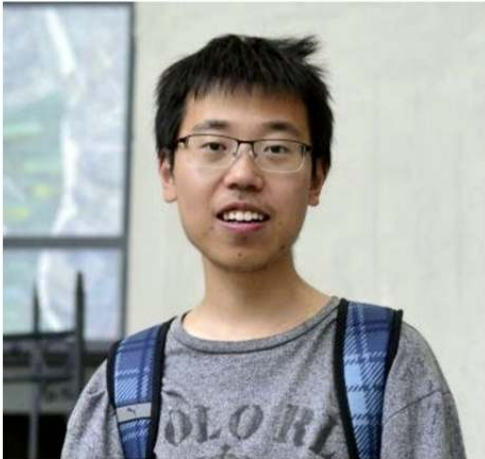
Zoom link: <https://pitp.zoom.us/j/94069836709?pwd=R1A2ZUsxdXlPTlh2TSStObHFDNUY0Zz09>

Modular Commutators in conformal field theory, topological phases and holography

Yijian Zou

Stanford University

Collaborators



Bowen Shi
UC San Diego



Jonathan Sorce
Stanford -> MIT



Ian Lim
UC Davis

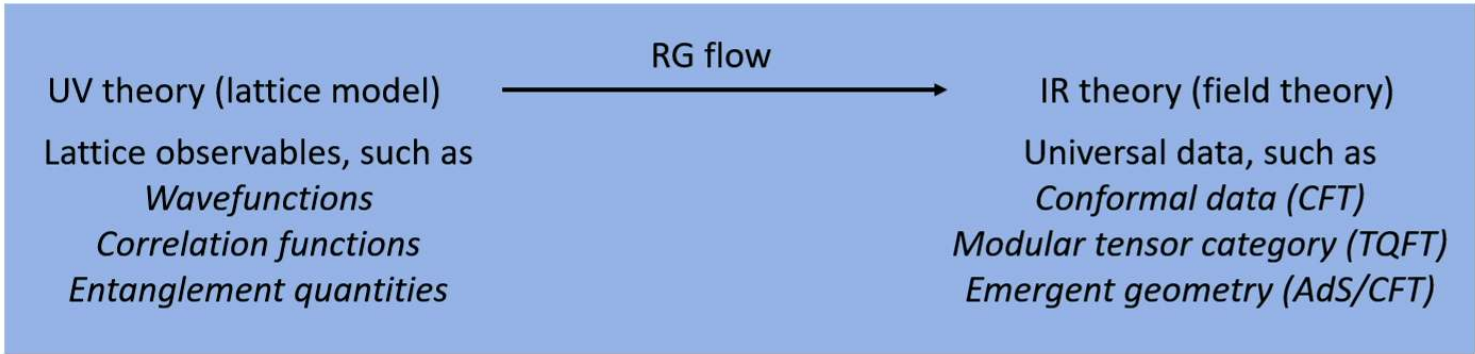


Issac Kim
UC Davis

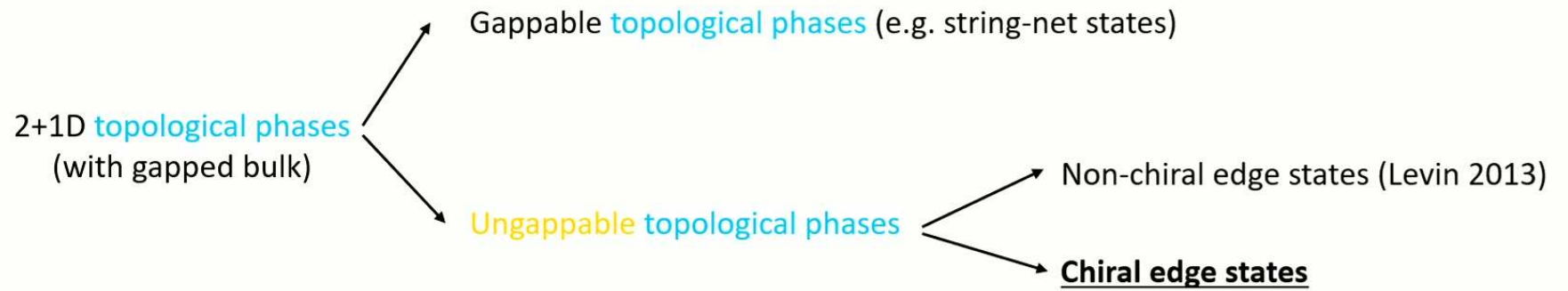
Motivation

Characterizing universality is one of the main themes of condensed matter physics

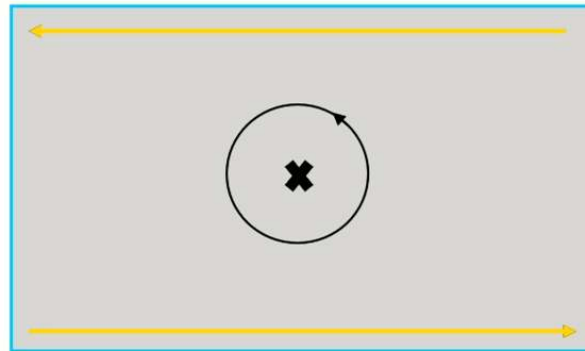
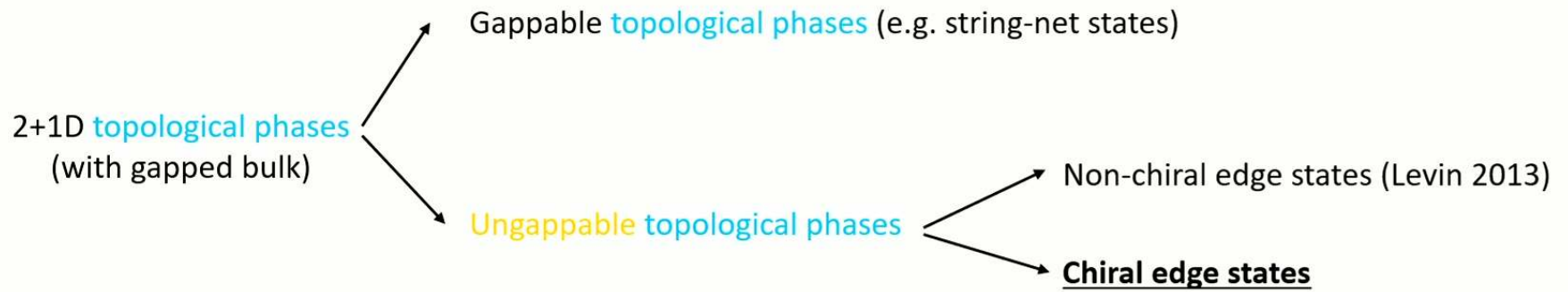
Motto: It from Qubit



Motivation



Motivation



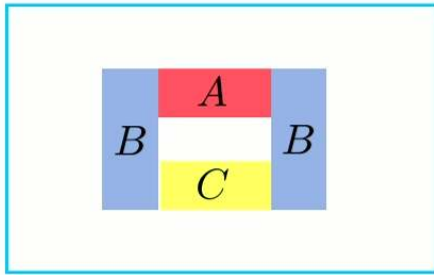
Quantum Hall states
(electron gas in magnetic field)

Motivation

Long range entanglement is fundamental to topological phases

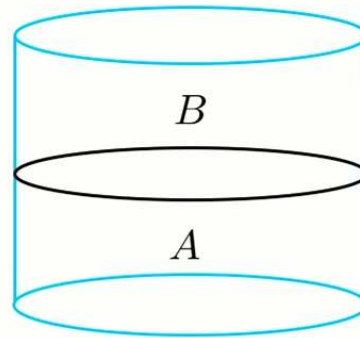
Motivation

Long range entanglement is fundamental to topological phases



$$I(A : C|B) = 2\gamma$$

Topological entanglement entropy
(Levin, Wen; Kitaev, Preskill 2005)



$$K_A \sim H_L^{CFT}$$

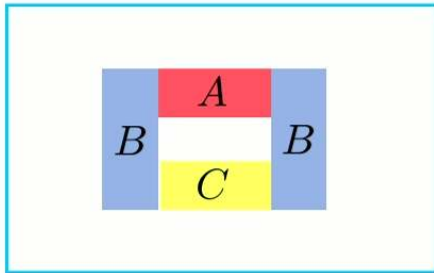
Entanglement spectrum
(Li, Haldane 2008)

$$\langle T_{A,x}^L \rangle_a = e^{\frac{2\pi i p a}{Lx} - \alpha Lx}$$

Momentum polarization
(Tu, Qi 2013)

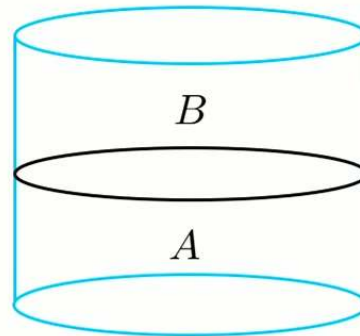
Motivation

Long range entanglement is fundamental to topological phases



$$I(A : C|B) = 2\gamma$$

Topological entanglement entropy
(Levin, Wen; Kitaev, Preskill 2005)

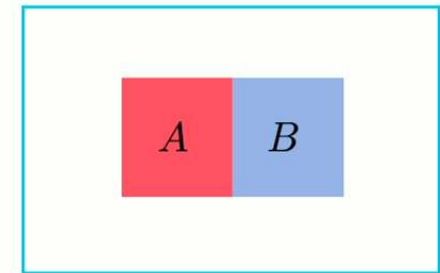


$$K_A \sim H_L^{CFT}$$

Entanglement spectrum
(Li, Haldane 2008)

$$\langle T_{A,x}^L \rangle_a = e^{\frac{2\pi i p_a}{Lx} - \alpha Lx}$$

Momentum polarization
(Tu, Qi 2013)



$$h(A : B) = \frac{c}{3} \log 2$$

Markov gap
(Siva, Zou, Soejima, Mong, Zaletel 2021)

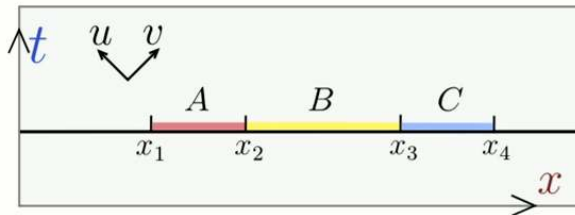
Result overview

Modular commutator: an entanglement measure to detect chirality

Result overview

Modular commutator: an entanglement measure to detect chirality

(1) 1+1D CFT

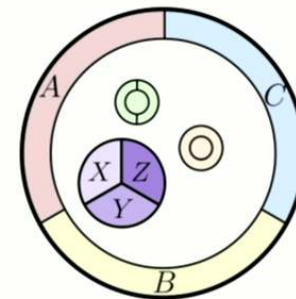


$$J(A, B, C)_{|\Omega\rangle} = \frac{\pi c_-}{6} (2\eta - 1)$$

$$(c_- \equiv c_L - c_R)$$

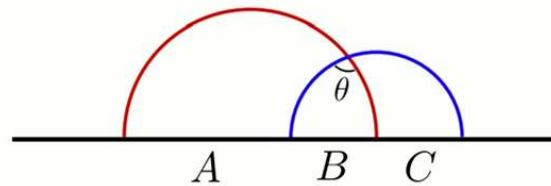
Bulk-edge correspondence

Geometric dual



(2) 2+1D
Topological phase

$$J(A, B, C) = -J(X, Y, Z) = \frac{\pi c}{3}$$



(3) Holography

$$J(A, B, C) = \frac{\pi c_-}{6} \sum_i \cos \theta_i$$

Modular commutator

- Usual entanglement measures are *invariant* under *time reversal (TR)*.

$$S(\rho^*) = S(\rho)$$

- In order to **detect chirality**, we need an entanglement measure that is *odd* under TR

$$J(\rho^*) = -J(\rho)$$

- Modular commutator: $J(A, B, C) = i\text{Tr}([\log \rho_{AB}, \log \rho_{BC}]\rho_{ABC})$

Modular commutator

- Usual entanglement measures are *invariant* under *time reversal (TR)*.

$$S(\rho^*) = S(\rho)$$

- In order to **detect chirality**, we need an entanglement measure that is *odd* under TR

$$J(\rho^*) = -J(\rho)$$

- Modular commutator: $J(A, B, C) = i\text{Tr}([\log \rho_{AB}, \log \rho_{BC}]\rho_{ABC})$

$$\begin{aligned} J(\rho_{ABC}^*) &= i\text{Tr}([\log \rho_{AB}^*, \log \rho_{BC}^*]\rho_{ABC}^*) \\ &= i\text{Tr}([\log \rho_{AB}^T, \log \rho_{BC}^T]\rho_{ABC}^T) \\ &= -i\text{Tr}([\log \rho_{AB}, \log \rho_{BC}]^T \rho_{ABC}^T) \\ &= -i\text{Tr}([\log \rho_{AB}, \log \rho_{BC}]\rho_{ABC}) = -J(\rho_{ABC}) \end{aligned}$$

Modular commutator

- Vanishes for classical states $\rho_{ABC} = \sum_i p_i \Pi_i^A \otimes \Pi_i^B \otimes \Pi_i^C \rightarrow J(A, B, C) = 0$
- Vanishes for pure states $\rho_{ABC} = |\psi\rangle\langle\psi| \rightarrow J(A, B, C) = 0$
- Vanishes for Markov states $I(A : C|B) \equiv S(AB) + S(BC) - S(ABC) - S(B) \geq 0$
 $I(A : C|B) = 0 \rightarrow J(A, B, C) = 0$

Modular commutator

- Vanishes for classical states $\rho_{ABC} = \sum_i p_i \Pi_i^A \otimes \Pi_i^B \otimes \Pi_i^C \rightarrow J(A, B, C) = 0$

- Vanishes for pure states $\rho_{ABC} = |\psi\rangle\langle\psi| \rightarrow J(A, B, C) = 0$

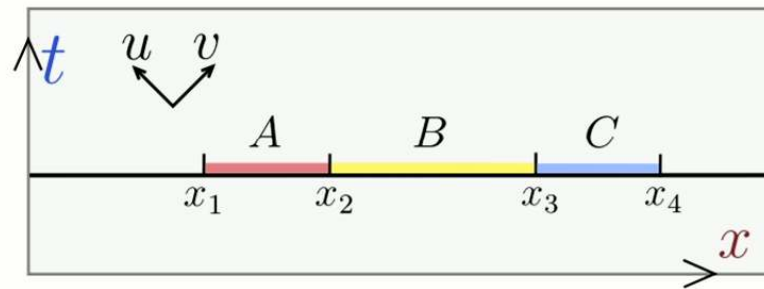
$$J(A, B, C) = i\langle [K_{AB}, K_{BC}] \rangle = i\langle [K_C, K_A] \rangle = 0 \quad (K_X := -\log \rho_X)$$

- Vanishes for Markov states $I(A : C|B) \equiv S(AB) + S(BC) - S(ABC) - S(B) \geq 0$
 $I(A : C|B) = 0 \rightarrow J(A, B, C) = 0$

$$I(A : C|B) = 0 \rightarrow K_{ABC} = K_{AB} + K_{BC} - K_B \quad (\text{Petz 2002})$$

$$\begin{aligned} J(A, B, C) &= i\text{Tr}([K_{AB}, K_{BC}]\rho_{ABC}) \\ &= i\text{Tr}([K_{ABC}, K_{BC}]\rho_{ABC}) + i\text{Tr}([K_B, K_{BC}]\rho_{AB}) \\ &= -i\text{Tr}([K_{ABC}, \rho_{ABC}]K_{BC}) + i\text{Tr}([K_{BC}, \rho_{AB}]K_B) \\ &= 0 \end{aligned}$$

1. Modular commutators in CFT



$$J(A, B, C)|_{\Omega} = \frac{\pi c_-}{6} (2\eta - 1) \quad \eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$
$$(c_- \equiv c_L - c_R)$$

Derivation sketch

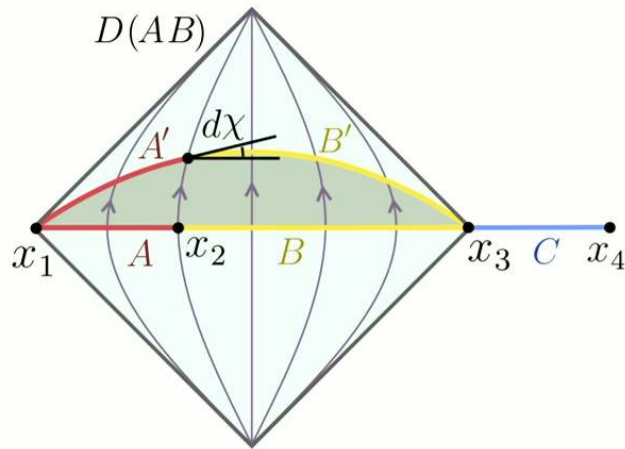
Modular commutator from *modular flow*

Chen, Dong, Lewkowycz, Qi '18

Faulkner, Li, Wang '19

Kim, Shi, Kato, Albert '21

$$\rho_{BC}(s) = \text{Tr}_A(\rho_{AB}^{is} \rho_{ABC} \rho_{AB}^{-is}) \quad \left. \frac{dS(\rho_{BC}(s))}{ds} \right|_{s=0} = -J(A, B, C)_\rho$$



Within the causal diamond:

$$\begin{pmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{pmatrix} = 2\pi \begin{pmatrix} \frac{(u-u_1)(u_3-u)}{u_1-u_3} \\ \frac{(v-v_1)(v_3-v)}{v_3-v_1} \end{pmatrix}$$

Derivation sketch

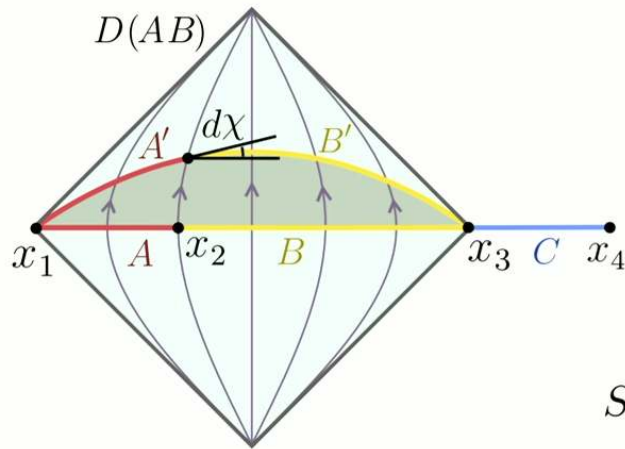
Modular commutator from *modular flow*

Chen, Dong, Lewkowycz, Qi '18

Faulkner, Li, Wang '19

Kim, Shi, Kato, Albert '21

$$\rho_{BC}(s) = \text{Tr}_A(\rho_{AB}^{is} \rho_{ABC} \rho_{AB}^{-is}) \quad \left. \frac{dS(\rho_{BC}(s))}{ds} \right|_{s=0} = -J(A, B, C)_\rho$$



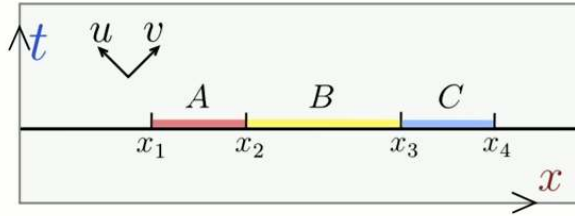
Within the causal diamond:

$$\begin{pmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{pmatrix} = 2\pi \begin{pmatrix} \frac{(u-u_1)(u_3-u)}{u_1-u_3} \\ \frac{(v-v_1)(v_3-v)}{v_3-v_1} \end{pmatrix}$$

$$S_{BC} = \frac{c_L}{12} \ln \frac{(v_4 - v_2)^2}{\epsilon_{v2} \epsilon_{v4}} + \frac{c_R}{12} \ln \frac{(u_4 - u_2)^2}{\epsilon_{u2} \epsilon_{u4}} \quad [Iqbal, Wall '16]$$

$$d \ln \epsilon_{v2} = -d \ln \epsilon_{u2} = d\chi$$

Properties



$$J(A, B, C)_{|\Omega\rangle} = \frac{\pi c_-}{6} (2\eta - 1) \quad \eta = \frac{x_{12}x_{34}}{x_{13}x_{24}}$$

$$(c_- \equiv c_L - c_R)$$

- Changes sign under “crossing symmetry” $\eta \rightarrow 1 - \eta$

- $J = \frac{\pi c_-}{6}$ as $x_1 \rightarrow -\infty, x_4 \rightarrow \infty$ or equivalently $x_2 \rightarrow x_3$. This value is *topological*.

- Generalization to finite size or finite temperature $\eta_{\text{eff}}^{(\beta; L)} = \begin{cases} \frac{\sin(\pi x_{12}/L) \sin(\pi x_{34}/L)}{\sin(\pi x_{13}/L) \sin(\pi x_{24}/L)}, & \beta/L \rightarrow \infty, \\ \frac{\sinh(\pi x_{12}/\beta) \sinh(\pi x_{34}/\beta)}{\sinh(\pi x_{13}/\beta) \sinh(\pi x_{24}/\beta)}, & L/\beta \rightarrow \infty. \end{cases}$

Gravitational anomaly

- If $c_L \neq c_R$ then $J(A, B, C) \neq 0$ for pure state $|\psi\rangle_{ABC}$. Contradiction!
- This means that a **chiral CFT** cannot admit a lattice regularization.
- Entanglement is still well defined, but *frame dependent*.

Modular commutator

- Vanishes for classical states $\rho_{ABC} = \sum_i p_i \Pi_i^A \otimes \Pi_i^B \otimes \Pi_i^C \rightarrow J(A, B, C) = 0$

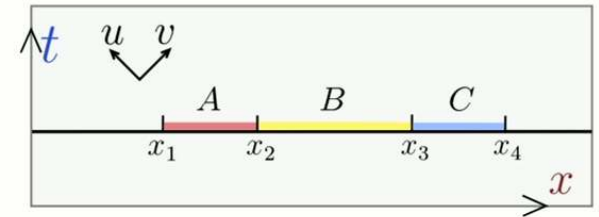
- Vanishes for pure states $\rho_{ABC} = |\psi\rangle\langle\psi| \rightarrow J(A, B, C) = 0$

$$J(A, B, C) = i\langle [K_{AB}, K_{BC}] \rangle = i\langle [K_C, K_A] \rangle = 0 \quad (K_X := -\log \rho_X)$$

- Vanishes for Markov states $I(A : C|B) \equiv S(AB) + S(BC) - S(ABC) - S(B) \geq 0$
 $I(A : C|B) = 0 \rightarrow J(A, B, C) = 0$

Gravitational anomaly

- If $c_L \neq c_R$ then $J(A, B, C) \neq 0$ for pure state $|\psi\rangle_{ABC}$. Contradiction!
- This means that a **chiral CFT** cannot admit a lattice regularization.
- Entanglement is still well defined, but *frame dependent*.

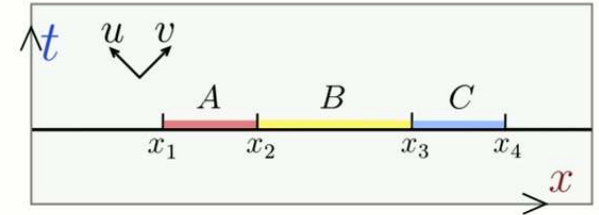


Gravitational anomaly

- If $c_L \neq c_R$ then $J(A, B, C) \neq 0$ for pure state $|\psi\rangle_{ABC}$. Contradiction!

- This means that a **chiral CFT** cannot admit a lattice regularization.

- Entanglement is still well defined, but *frame dependent*.



- This is the effect of **gravitational anomaly** (stress tensor not conserved in curved spacetimes)

- Alternative argument: no conformal boundary condition due to net energy flow

[Hellerman, Orlando, Watanabe '21]

Derivation sketch

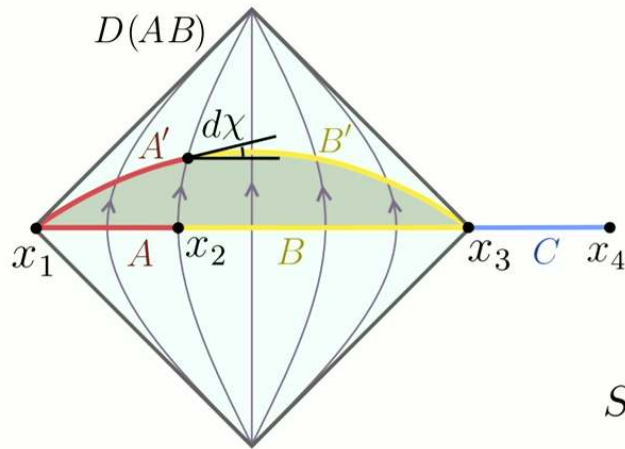
Modular commutator from *modular flow*

Chen, Dong, Lewkowycz, Qi '18

Faulkner, Li, Wang '19

Kim, Shi, Kato, Albert '21

$$\rho_{BC}(s) = \text{Tr}_A(\rho_{AB}^{is} \rho_{ABC} \rho_{AB}^{-is}) \quad \left. \frac{dS(\rho_{BC}(s))}{ds} \right|_{s=0} = -J(A, B, C)_\rho$$



Within the causal diamond:

$$\begin{pmatrix} \frac{du}{ds} \\ \frac{dv}{ds} \end{pmatrix} = 2\pi \begin{pmatrix} \frac{(u-u_1)(u_3-u)}{u_1-u_3} \\ \frac{(v-v_1)(v_3-v)}{v_3-v_1} \end{pmatrix}$$

$$S_{BC} = \frac{c_L}{12} \ln \frac{(v_4 - v_2)^2}{\epsilon_{v2} \epsilon_{v4}} + \frac{c_R}{12} \ln \frac{(u_4 - u_2)^2}{\epsilon_{u2} \epsilon_{u4}} \quad [Iqbal, Wall '16]$$

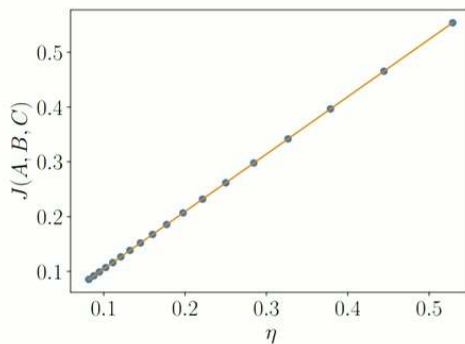
$$d \ln \epsilon_{v2} = -d \ln \epsilon_{u2} = d\chi$$

Chiral thermal states

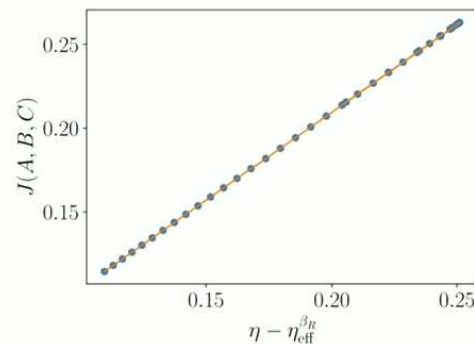
- Non-chiral CFT with different left/right temperatures (*grand canonical ensemble*)

$$\rho^{(\beta_L, \beta_R)} = \frac{1}{Z} e^{-\beta_L H_L} e^{-\beta_R H_R}$$

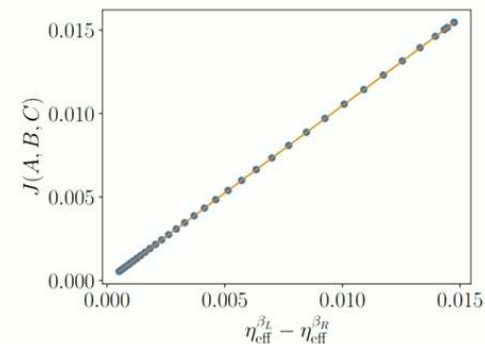
- Modular commutator $J(A, B, C)_{\rho^{(\beta_L, \beta_R; L)}} = \frac{\pi}{3} c(\eta_{\text{eff}}^{(\beta_L; L)} - \eta_{\text{eff}}^{(\beta_R; L)})$



(a) $(\beta_L, \beta_R) = (\infty, 80), L_A = L_C = 320, 120 \leq L_B \leq 800$



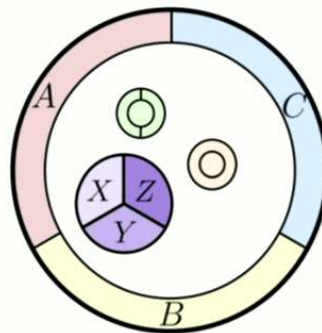
(b) $(\beta_L, \beta_R) = (\infty, 80), L_A = L_C = 40, 10 \leq L_B \leq 80$



(c) $(\beta_L, \beta_R) = (82, 78), L_A = L_C = 40, 10 \leq L_B \leq 80$

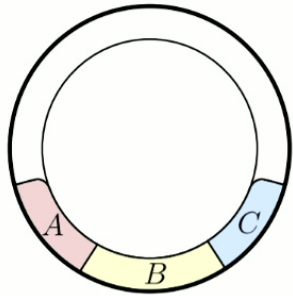
Free fermion CFT
Numerical results

2. Modular commutators in topological order



$$J(A, B, C) = -J(X, Y, Z) = \frac{\pi C}{3}$$

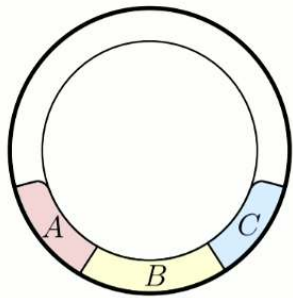
Edge of quantum Hall states



$$\rho_{\text{edge}} = \frac{1}{Z} e^{-\beta_L H_L} e^{-\beta_R H_R} \quad \beta_L = \infty, \beta_R \rightarrow 0 \quad [\text{Tu, Qi '13}]$$

$$J(A, B, C)_{\rho^{(\beta_L, \beta_R; L)}} = \frac{\pi}{3} c(\eta_{\text{eff}}^{(\beta_L; L)} - \eta_{\text{eff}}^{(\beta_R; L)})$$

Edge of quantum Hall states



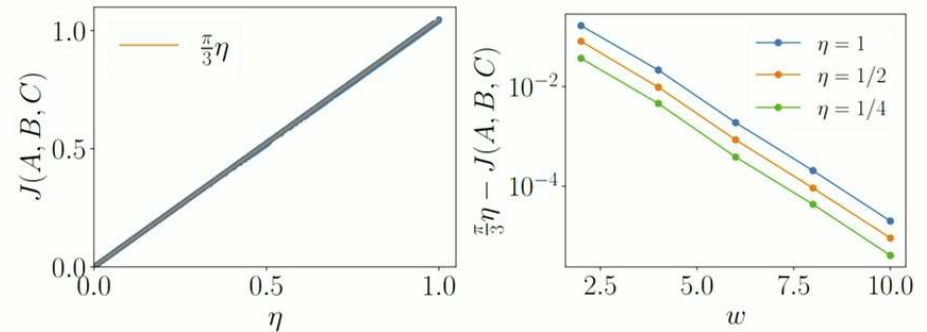
$$\rho_{\text{edge}} = \frac{1}{Z} e^{-\beta_L H_L} e^{-\beta_R H_R} \quad \beta_L = \infty, \beta_R \rightarrow 0 \quad [\text{Tu, Qi '13}]$$

$$J(A, B, C)_{\rho^{(\beta_L, \beta_R; L)}} = \frac{\pi}{3} c(\eta_{\text{eff}}^{(\beta_L; L)} - \eta_{\text{eff}}^{(\beta_R; L)})$$

$$\eta_{\text{eff}}^{(\beta_L, L)} = \frac{\sin(\pi x_{12}/L) \sin(\pi x_{34}/L)}{\sin(\pi x_{13}/L) \sin(\pi x_{24}/L)} \quad \eta_{\text{eff}}^{(\beta_R, L)} = \frac{\sinh(\pi x_{12}/\beta_R) \sinh(\pi x_{34}/\beta_R)}{\sinh(\pi x_{13}/\beta_R) \sinh(\pi x_{24}/\beta_R)}$$

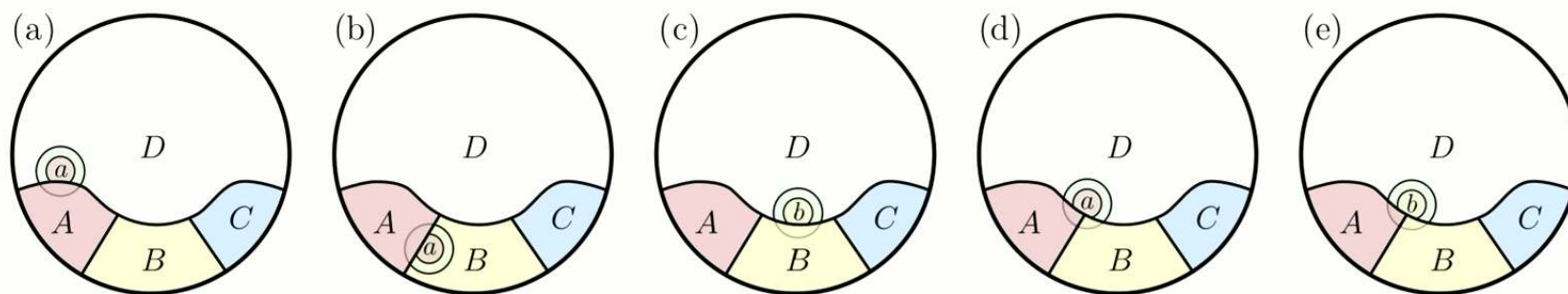
$$\equiv \eta \quad \quad \quad = 0$$

$$J(A, B, C) = \frac{\pi c \eta}{3}$$

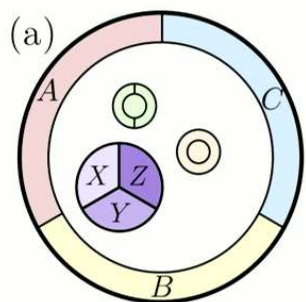


Topological robustness

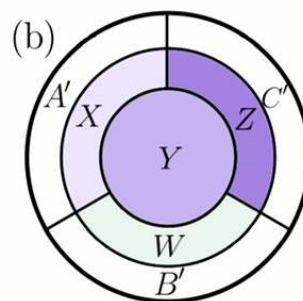
- Invariant under local deformations



- If A, B, C fills the whole annulus, then we have a stronger result



$$J(A, B, C) = -J(X, Y, Z)$$



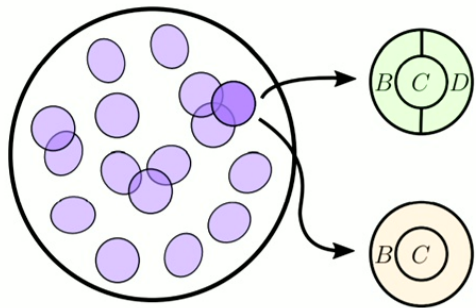
$$J(A'X, B'W, C'Z) = \frac{\pi C}{3}$$

for all $|\tilde{\psi}\rangle = U_{A'B'C'}|\psi\rangle$

Entanglement bootstrap

- Entanglement bootstrap: a systematic quantum information technique to understand entanglement in 2+1D topological order [Shi, Kim 2021; Shi, Kato, Kim 2020; Shi 2020]

- Axioms:

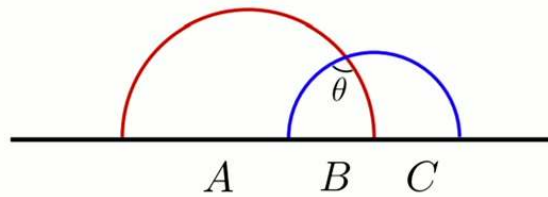


$$\mathbf{A1:} (S_{BC} + S_{CD} - S_B - S_D)_{|\psi_{2D}\rangle} = 0.$$

$$\mathbf{A0:} (S_{BC} + S_C - S_B)_{|\psi_{2D}\rangle} = 0.$$

- These axioms are very powerful. They generate a series of quantum Markov chains in the bulk. They have been used to derive fusion rules of anyons, Verlinde formula, gapped boundaries, etc.
- No extra assumption for topological phases with a boundary!

3. Modular commutators in holography



$$J(A, B, C) = \frac{\pi c_-}{6} \sum_i \cos \theta_i$$

Holographic duality

1+1 dimensional large-N CFT

CFT state

CFT ground state

CFT (chiral) thermal state

Global conformal symmetry

Central charge c

Entanglement entropy

Modular Hamiltonian

2+1 dimensional semiclassical gravity
in asymptotic AdS spacetimes

Semiclassical geometry

Vacuum AdS

(Rotating) BTZ black hole

Isometry of AdS

AdS curvature l [Brown, Henneaux '86]

Area of RT surfaces [Ryu, Takayanagi '06]

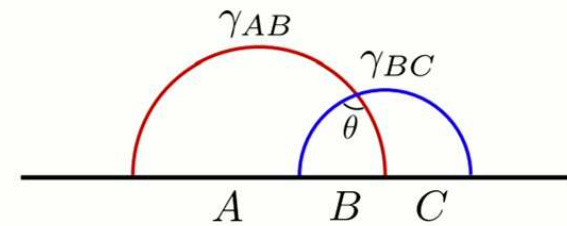
Area operator [JLMS '14]

RT formula & JLMS formula

Hyperbolic plane
(a time slice of vacuum AdS) $ds^2 = \frac{l^2}{z^2}(dx^2 + dz^2)$

$$\gamma_X = \operatorname{argmin}\{\mathcal{A}_\gamma, \gamma \text{ homologous to } X\}$$

$$S_X = \frac{\mathcal{A}_{\gamma_X}}{4G_N} + O(1) \quad \mathcal{A}_\gamma = \int_\gamma ds \sqrt{g_{\mu\nu} v^\mu v^\nu}$$



RT formula & JLMS formula

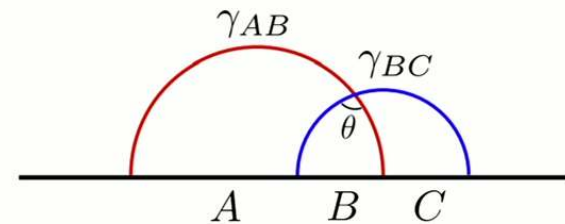
Hyperbolic plane
(a time slice of vacuum AdS) $ds^2 = \frac{l^2}{z^2}(dx^2 + dz^2)$

$$\gamma_X = \operatorname{argmin}\{\mathcal{A}_\gamma, \gamma \text{ homologous to } X\}$$

$$S_X = \frac{\mathcal{A}_{\gamma_X}}{4G_N} + O(1) \quad \mathcal{A}_\gamma = \int_\gamma ds \sqrt{g_{\mu\nu} v^\mu v^\nu}$$

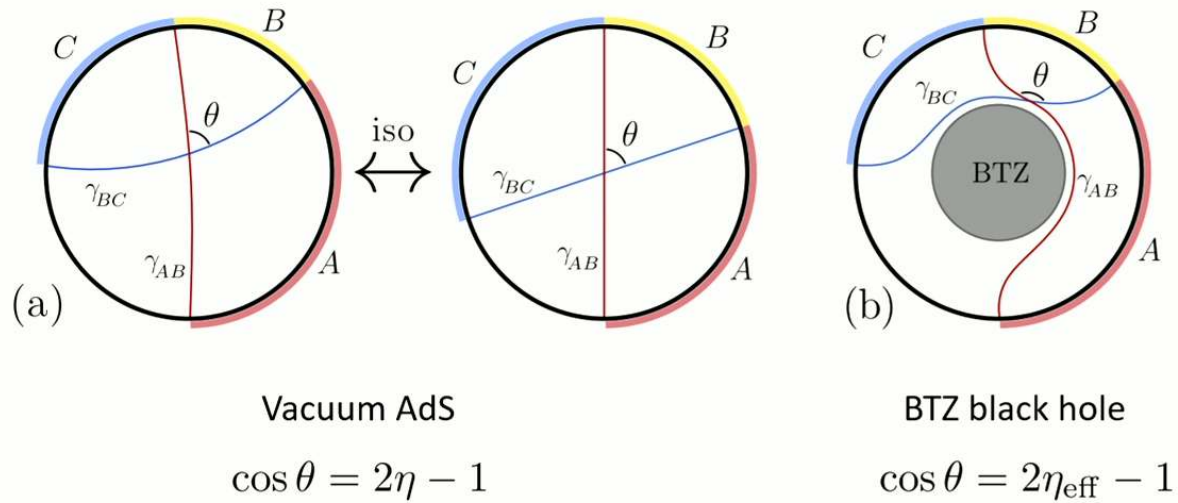
Regarding area as an operator

$$S_X = \langle K_X \rangle \quad K_X = \frac{\mathcal{A}_{\gamma_X}}{4G_N} + K_{\text{bulk}}$$



Modular commutator in CFT = commutator of area operator in AdS

Modular commutators in chiral AdS/CFT



$$J(A, B, C) = \frac{\pi c_-}{6} \cos \theta$$

Uncertainty relation of geometric operators

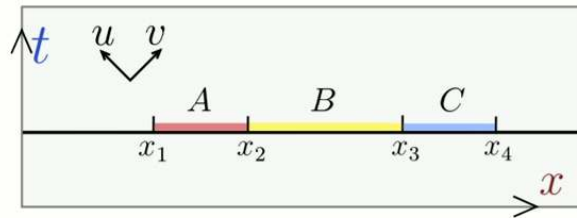
- In chiral AdS/CFT, modular Hamiltonian K_X is dual to some geometric operator F_X

$$J(A, B, C) = \frac{\pi c_-}{6} \cos \theta \quad \longrightarrow \quad \Delta F_{AB} \Delta F_{BC} \geq \frac{\pi c_-}{12} |\cos \theta|$$
$$J(A, B, C) = i \langle [K_{AB}, K_{BC}] \rangle$$

Conclusion

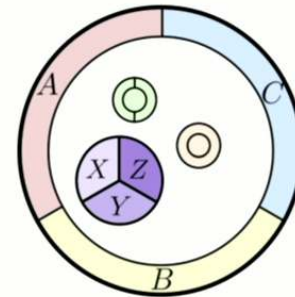
Modular commutator: an entanglement measure to detect chirality

(1) 1+1D CFT



$$J(A, B, C)_{|\Omega\rangle} = \frac{\pi c_-}{6} (2\eta - 1)$$

$$(c_- \equiv c_L - c_R)$$



(2) 2+1D
Topological phase

$$J(A, B, C) = -J(X, Y, Z) = \frac{\pi c}{3}$$

Open questions

- Beyond conformal family of identity states
- Modular commutators under scrambling or integrable dynamics
- Higher dimensions

- Nontrivial anyon flux in fractional quantum Hall states
- Making use of global symmetries, detecting SPT phases
- Topological quantum field theory derivation for the bulk formula

- Beyond contiguous intervals
- Warped black holes in topologically massive gravity
- Non-stationary spacetimes, such as black hole formation

Thank you for listening!