Title: Relativity - Lecture 221031

Speakers:

Collection: Relativity (2022/2023)

Date: October 31, 2022 - 9:00 AM

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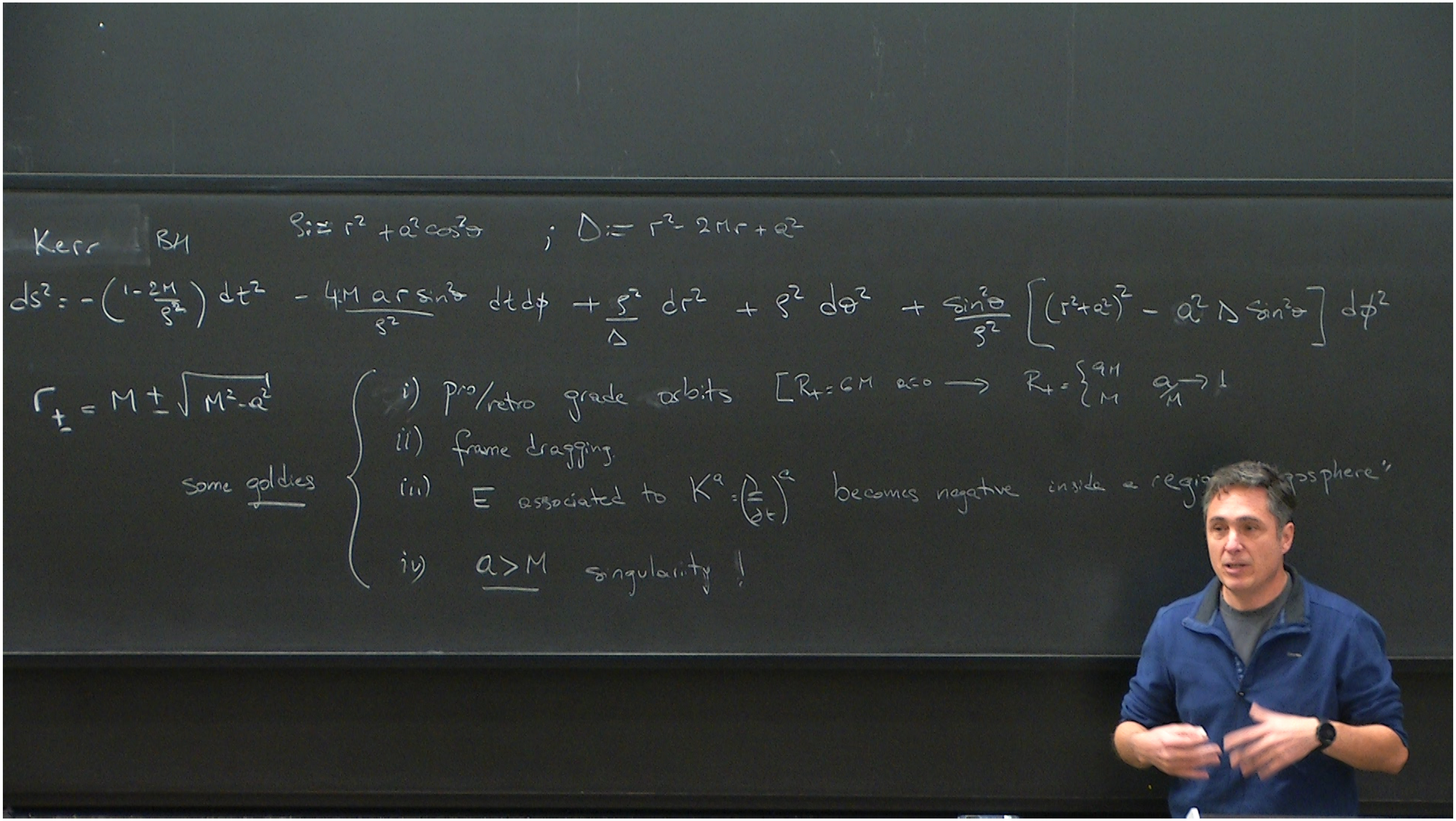


Kerr BH
$\rho := r^2 + a^2\cos^2\theta$; $\Delta := r^2 - 2Mr + a^2$
$ds^2 = -\left(1 - \frac{2M}{\rho^2}\right) dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\, dt\, d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right] d\phi^2$
$r_\pm = M \pm \sqrt{M^2 - a^2}$
some goldies
i) pro/retro grade orbits [$R_+ = 6M$ $a=0 \longrightarrow$ $R_+ = \begin{cases} 9M \\ M \end{cases}$ $\frac{a}{M} \to 1$
ii) frame dragging
iii) E associated to $K^a = \left(\frac{\partial}{\partial t}\right)^a$ becomes negative inside a regi osphere"
iv) $\underline{a > M}$ singularity !

Kerr BH

$\rho := r^2 + a^2\cos^2\theta \quad ; \quad \Delta := r^2 - 2Mr + a^2$

$$ds^2 = -\left(1 - \frac{2M}{\rho^2}\right)dt^2 - \frac{4Mar\sin^2\theta}{\rho^2}\,dt\,d\phi + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)^2 - a^2\Delta\sin^2\theta\right]d\phi^2$$

$$r_\pm = M \pm \sqrt{M^2 - a^2}$$

some goldies

i) pro/retro grade orbits $\left[R_+ = 6M \quad a=0 \longrightarrow R_+ = \begin{cases} 9M \\ M \end{cases} \quad \frac{a}{M} \to 1\right.$

ii) frame dragging

iii) E associated to $K^a = \left(\frac{\partial}{\partial t}\right)^a$ becomes negative inside a region "ergosphere"

iv) $\underline{a > M}$ singularity !

$$g_{ab} = \eta_{ab} + h_{ab} \qquad |h_{ab}| \ll 1$$

$$\hookrightarrow g^{ab} = \eta^{ab} - h^{ab}$$

$$G_{ab}(g) \longrightarrow G_{ab}(h)$$

$$\Gamma = g\,\partial g \;;\; \longrightarrow (\eta + h)\,\partial(\eta + h)$$

$$\Gamma^2 \sim h^2$$

$$\int$$

$$G_{\mu\nu} = \frac{1}{2}\left[\partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial \ldots\right.$$

$| \ll 1$

$$G_{\mu\nu} = \frac{1}{2}\left[\partial_\sigma \partial_\nu h^\sigma_{\ \mu} + \partial_\sigma \partial_\mu h^\sigma_{\ \nu} - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \Box h\right]$$

with $h = \eta^{ab} h_{ab}$

$\Rightarrow \Gamma \sim h$

$$\left[\partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu} \partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu} \Box h\right]$$

with $h = \eta^{ab} h_{ab}$

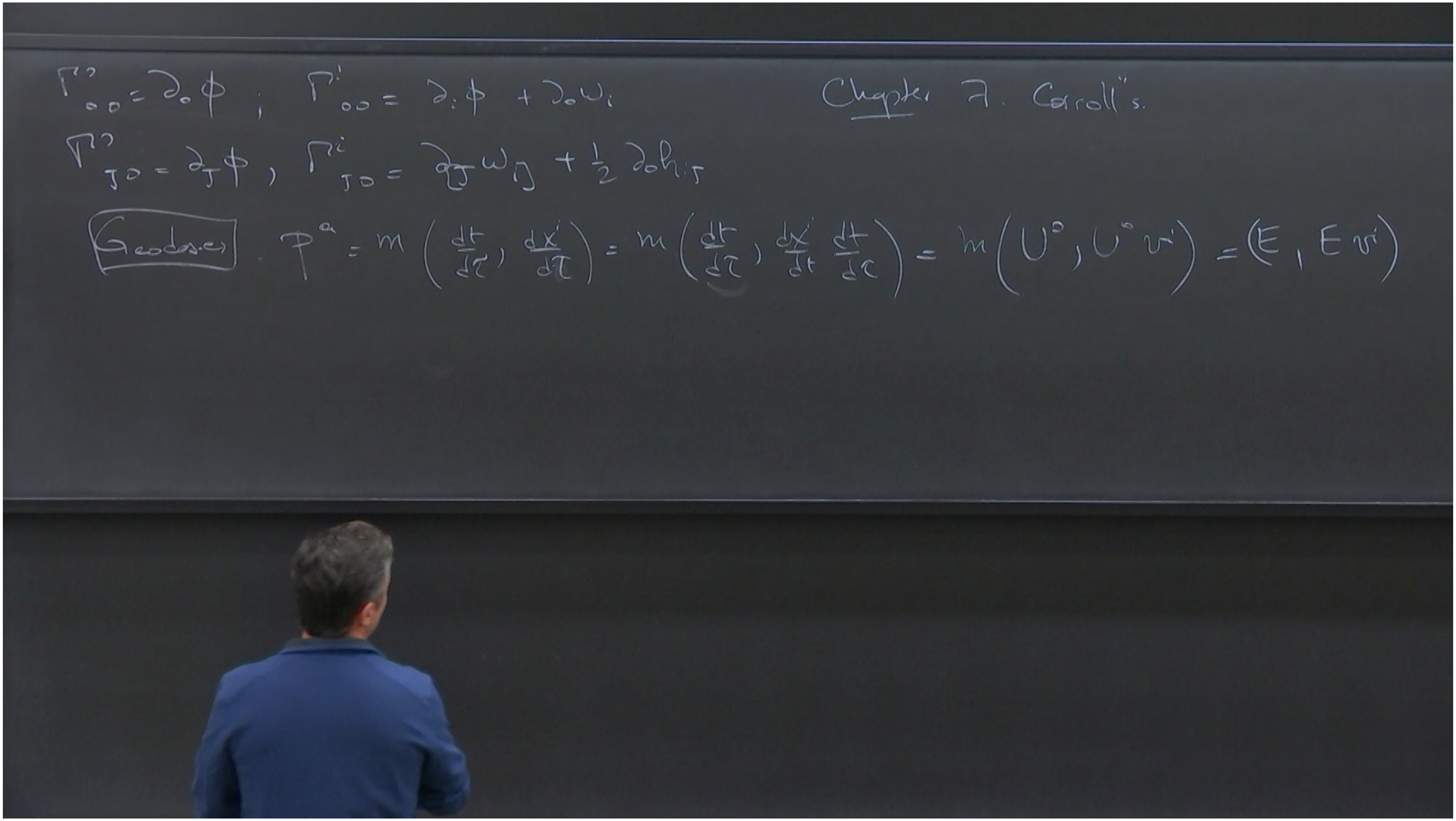
Introduce some convenient notation

$$h_{00} = -2\Phi \; ; \; h_{0i} = w_i \; ;$$

$$h_{ij} = 2 s_{ij} - \psi \delta_{ij}$$

$$\psi = -\frac{1}{6} \delta^{ij} h_{ij}$$

$$s_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3} \delta^{kl} h_{kl} \delta_{ij}\right)$$


$\Gamma^0_{00} = \partial_0 \phi \; ; \; \Gamma^i_{00} = \partial_i \phi + \partial_0 w_i$
Chapter 7. Carroll's.
$\Gamma^0_{j0} = \partial_j \phi \; , \; \Gamma^i_{j0} = \partial_{[j} w_{i]} + \frac{1}{2} \partial_0 h_{ij}$
$\boxed{\text{Geodesics}}$. $p^a = m \left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau} \right) = m \left(\frac{dt}{d\tau}, \frac{dx^i}{dt} \frac{dt}{d\tau} \right) = m \left(U^0 ; U^0 v^i \right) = (E, E v^i)$

$\Gamma^0_{00} = \partial_0 \phi \;;\; \Gamma^i_{00} = \partial_i \phi + \partial_0 \omega_i$

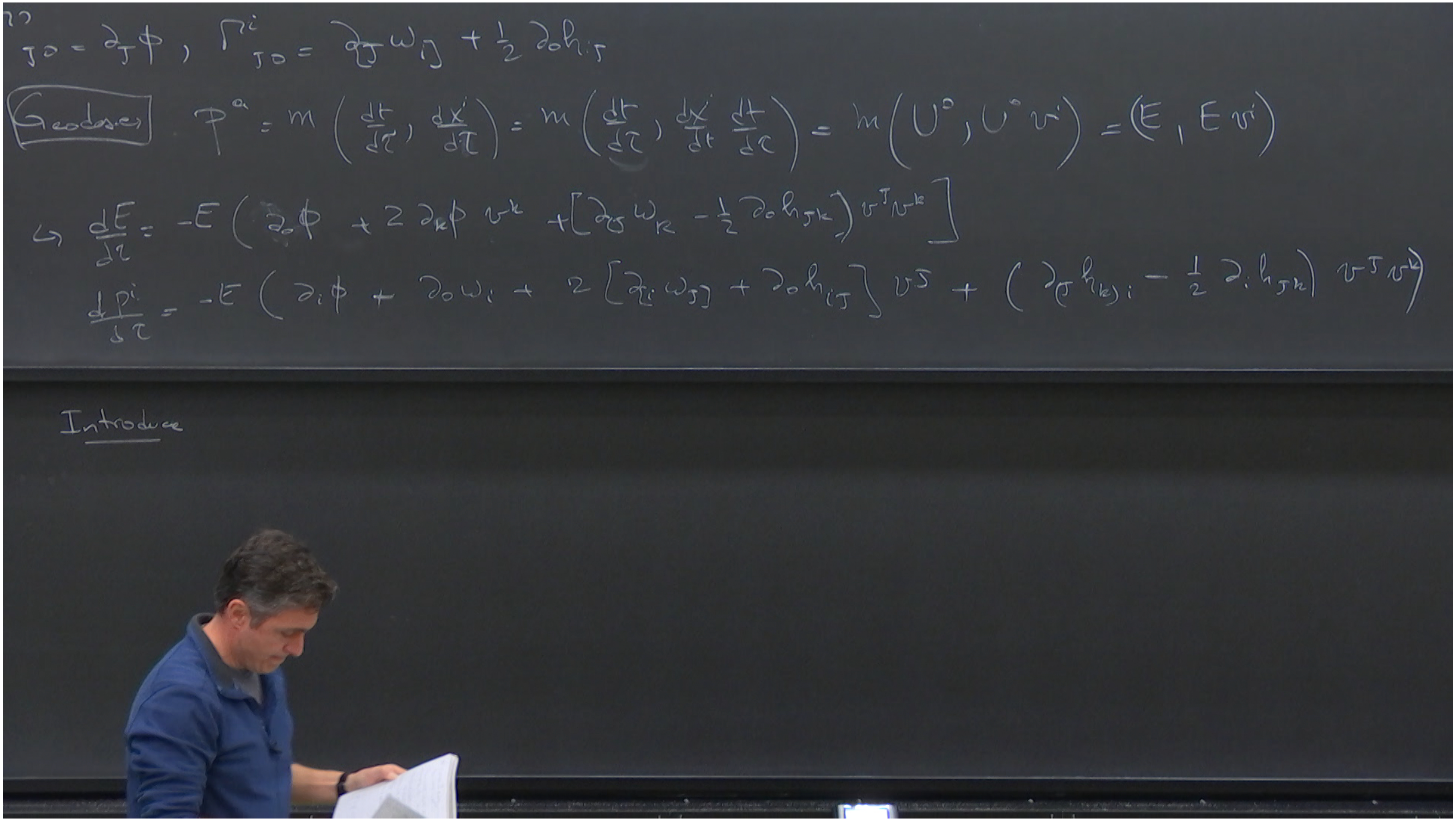
Chapter 7. Carroll's.

$\Gamma^0_{j0} = \partial_j \phi \,,\; \Gamma^i_{j0} = \partial_{[j} \omega_{i]} + \frac{1}{2} \partial_0 h_{ij}$

$\boxed{\text{Geodesics}}$ $p^a = m\left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) = m\left(\frac{dt}{d\tau}, \frac{dx^i}{dt}\frac{dt}{d\tau}\right) = m\left(U^0, U^0 v^i\right) = \left(E, E v^i\right)$

$\hookrightarrow \frac{dE}{d\tau} = -E\left(\partial_0 \phi + 2\partial_k \phi\, v^k + \left[\partial_{(j} \omega_{k)} - \frac{1}{2}\partial_0 h_{jk}\right) v^j v^k\right]$

$\frac{dp^i}{d\tau} = -E\left(\partial_i \phi + \partial_0 \omega_i + 2\left[\partial_{[i} \omega_{j]} + \partial_0 h_{ij}\right] v^j + \left(\partial_{(j} h_{k)i} - \frac{1}{2}\partial_i h_{jk}\right) v^j v^k\right)$


$\Gamma^0_{j0} = \partial_j \phi$, $\Gamma^i_{j0} = \partial_{[j}\omega_{i]} + \frac{1}{2}\partial_0 h_{ij}$
$\boxed{\text{Geodesics}}$ $P^a = m\left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) = m\left(\frac{dt}{d\tau}, \frac{dx^i}{dt}\frac{dt}{d\tau}\right) = m\left(U^0, U^0 v^i\right) = \left(E, E v^i\right)$
$\hookrightarrow \frac{dE}{d\tau} = -E\left(\partial_0\phi + 2\partial_k\phi\, v^k + [\partial_{(j}\omega_{k)} - \frac{1}{2}\partial_0 h_{jk}) v^j v^k\right]$
$\frac{dp^i}{d\tau} = -E\left(\partial_i\phi + \partial_0\omega_i + 2\left[\partial_{[i}\omega_{j]} + \partial_0 h_{ij}\right] v^j + \left(\partial_{(j}h_{k)i} - \frac{1}{2}\partial_i h_{jk}\right) v^j v^k\right)$
Introduce

Geodesics $\quad P^a = m\left(\frac{dt}{d\tau}, \frac{dx^i}{d\tau}\right) = m\left(\frac{dt}{d\tau}, \frac{dx^i}{dt}\frac{dt}{d\tau}\right) = m\left(U^0, U^0 v^i\right) = \left(E, E v^i\right)$

$$\hookrightarrow \frac{dE}{d\tau} = -E\left(\partial_0\phi + 2\partial_k\phi\, v^k + \left[\partial_j \omega_k - \frac{1}{2}\partial_0 h_{jk}\right) v^j v^k\right]$$

$$\frac{dp^i}{d\tau} = -E\left(\partial_i\phi + \partial_0\omega_i + 2\left[\partial_{[i}\omega_{j]} + \partial_0 h_{ij}\right] v^j + \left(\partial_{(j} h_{k)i} - \frac{1}{2}\partial_i h_{jk}\right) v\right.$$

Introduce $\quad G^i = -\partial_i\phi - \partial_0\omega_i \quad ; \quad H^i = (\nabla\wedge\omega)^i \equiv \varepsilon^{ijk}\partial_j\omega_k$

$$\frac{dp^i}{d\tau} = -E\left[G^i + (v\wedge H)^i - 2(\partial_0 h_{ij})v^j - \left(\partial_j h_{kj;} - \frac{1}{2}\partial_i h_{jk}\right)v^j v^k\right]$$

$\partial_j \omega_k$

$$p^a \nabla_a p^b = 0$$

$\Big) v^j v^k \Big]$

$\varepsilon^{ijk}\partial_j\omega_k$

$\partial_i h_{jk})v^j v^k]$

$p^a \nabla_a p^b = 0$

$G \sim$ "gravito-electric part"

H " " magnetic "

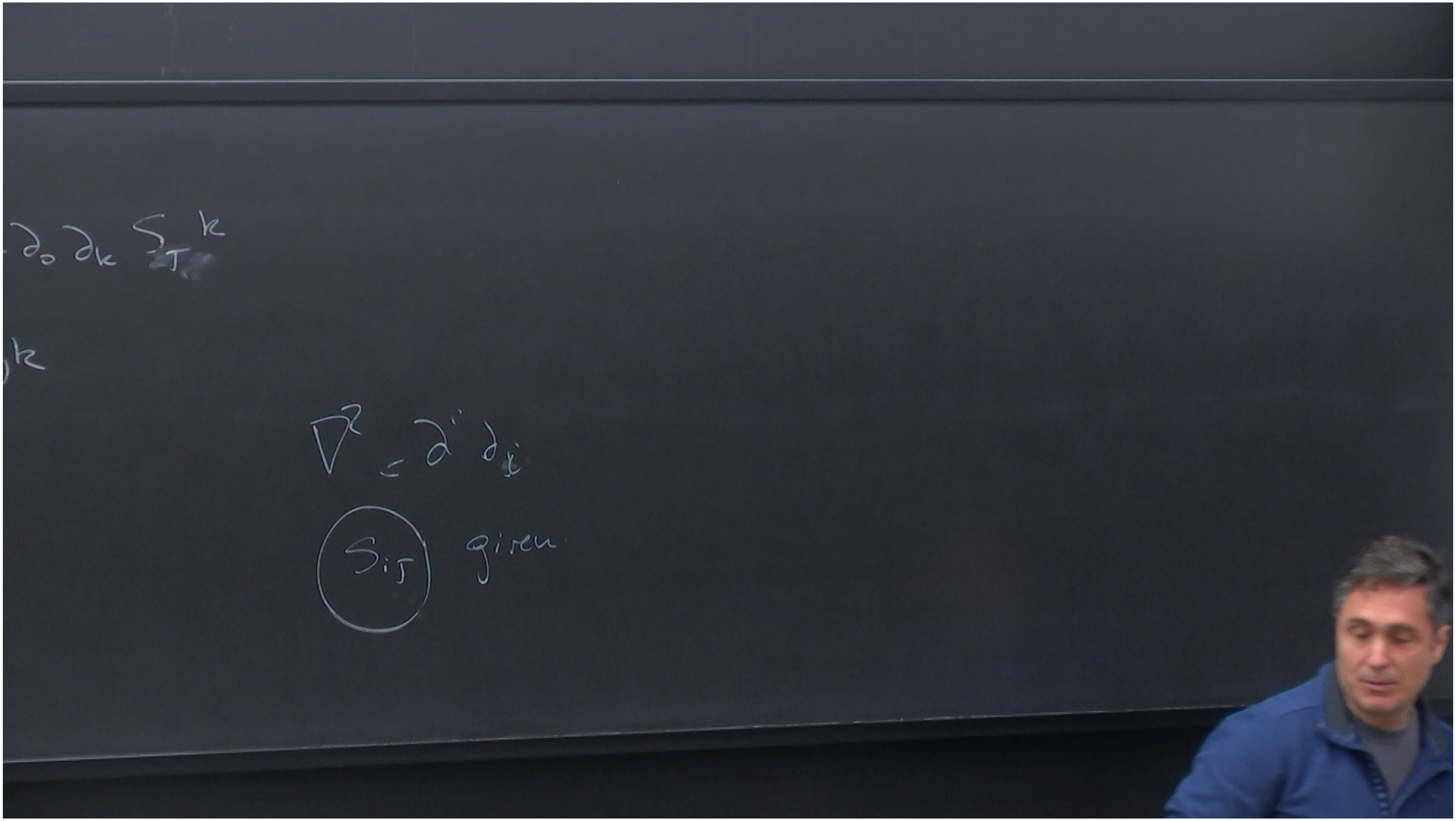
$$G_{00} = 2\nabla^2\psi + \partial_k\partial_\ell S^{k\ell}$$

$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\psi + \partial_0\partial_k S_j{}^k$$

$$G_{ij} = \left(\delta_j{}^k\nabla^2 - \partial_i\partial_j\right)(\phi-\psi) + \delta_{ij}\partial_0\partial_k w^k$$

$$-\partial_0\partial_{(i}w_{j)} + 2\delta_{ij}\partial_0^2\psi - \Box S_{ij}$$

$$+2\partial_k\partial_{(i}S_{j)}{}^k - \delta_{ij}\partial_k\partial_\ell S^{k\ell}$$


$\partial_0 \partial_k S_J^{\ k}$
^{k}
$\nabla^2 = \partial^i \partial_i$
S_{ij} given

$$G_{00} = 2\nabla^2\psi + \partial_k\partial_l S^{kl}$$

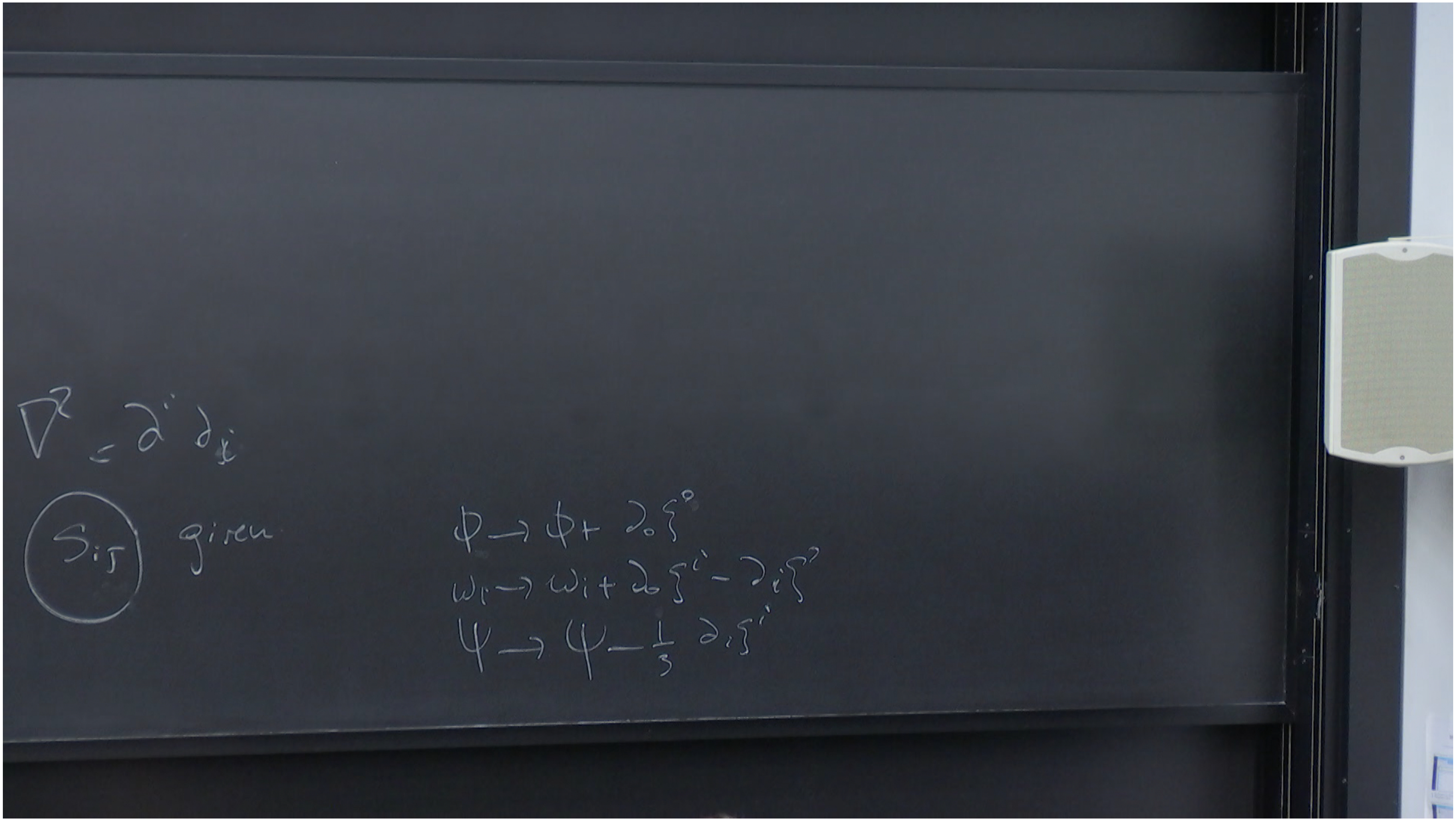
$$G_{0j} = -\frac{1}{2}\nabla^2 w_j + \frac{1}{2}\partial_j\partial_k w^k + 2\partial_0\partial_j\psi + \partial_0\partial_k S_j{}^k$$

$$G_{ij} = (\delta_j{}^k\nabla^2 - \partial_i\partial_j)(\phi - \psi) + \delta_{ij}\partial_0\partial_k w^k$$
$$- \partial_0\partial_{(i} w_{j)} + 2\delta_{ij}\partial_0^2\psi - \Box S_{ij}$$
$$+ 2\partial_k\partial_{(i} S_{j)}{}^k - \delta_{ij}\partial_k\partial_l S^{kl}$$

$$h_{ab} \to h_{ab} + \partial_{(a}\xi_{b)}$$

$$\nabla^2 = \partial^i\partial_i$$

S_{ij} given


$\nabla^2 = \partial^i \partial_i$
S_{ij} given
$\phi \to \phi + \partial_0 \xi^0$
$\omega_i \to \omega_i + \partial_0 \xi^i - \partial_i \xi^0$
$\psi \to \psi - \frac{1}{3} \partial_i \xi^i$

$$\nabla^2 = \partial^i \partial_i$$

$$S_{ij} \to S_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}(\partial_k \xi^k)\,\delta_{ij}$$

$$\phi \to \phi + \partial_0 \xi^0$$

$$\omega_i \to \omega_i + \partial_0 \xi^i - \partial_i \xi^0$$

$$\psi \to \psi - \frac{1}{3}\,\partial_i \xi^i$$

$$-\partial_0\partial_{(i}\omega_{j)} + 2\partial_{(i}\partial_{j)}\partial_0\psi - \Box S_{ij}$$

$$+2\partial_k\partial_{(i}S_{j)}{}^k - \delta_{ij}\partial_k\partial_l S^{kl}$$

$$h_{ab} \to h_{ab} + \partial_{(a}\xi_{b)}$$

$$S_{ij} \to S_{ij} + \partial_{(i}\xi_{j)} - \frac{1}{3}(\partial_k\xi^k)\delta_{ij}$$

$\Phi \to$

$\omega_i \to$

$\psi \to$

Transverse gauge

$$\partial_i S^{ij} = 0 \Rightarrow \nabla^2\xi^j + \frac{1}{3}\partial_j\partial_i\xi^i = -2\partial_i S^{ij}$$

$$\partial_i w^i = 0 \Rightarrow \nabla^2\xi^0 = \partial_i w^i + \partial_0\partial_i\xi^i$$

$$- \partial_0 \partial_{(i} \omega_{j)} + 2 \delta_{ij} \partial_0^2 \psi - \Box S_{ij}$$

$$+ 2 \partial_k \partial_{(i} S_{j)}^{k} - \delta_{ij} \partial_k \partial_l S^{kl}$$

$$h_{ab} \to h_{ab} + \partial_{(a} \xi_{b)}$$

$$S_{ij} \to S_{ij} + \partial_{(i} \xi_{j)} - \frac{1}{3} (\partial_k \xi^k) \delta_{ij}$$

Transverse gauge

$$\partial_i S^{ij} = 0 \Rightarrow \nabla^2 \xi^j + \frac{1}{3} \partial_j \partial_i \xi^i = -2 \partial_i S^{ij}$$

$$\partial_i \omega^i = 0 \Rightarrow \nabla^2 \xi^0 = \partial_i \omega^i + \partial_0 \partial_i \xi^i$$

$\boxed{\text{EoM}}$

$$\nabla^2 \psi = 0$$

$$\nabla^2 \omega_{ij} = 0$$

$$\nabla^2 \phi = 0$$

$$\Box S_{ij} = 0$$

$$\nabla^2 = \partial^i \partial_i$$

$$-\partial_0 \partial_{(i}\omega_{j)} + 2\delta_{ij}\partial_0^2\psi - \Box S_{ij}$$

$$+ 2\partial_k \partial_{(i} S_{j)}^{\ k} - \delta_{ij}\partial_k\partial_l S^{kl}$$

$$h_{ab} \to h_{ab} + \partial_{(a}\xi_{b)}$$

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Transverse gauge

$$\partial_i S^{ij} = 0 \Rightarrow \nabla^2 \xi^j + \tfrac{1}{3}\partial_j \partial_i \xi^i = -2\partial_i S^{ij}$$

$$\partial_i w^i = 0 \Rightarrow \nabla^2 \xi^0 = \partial_i w^i + \partial_0 \partial_i \xi^i$$

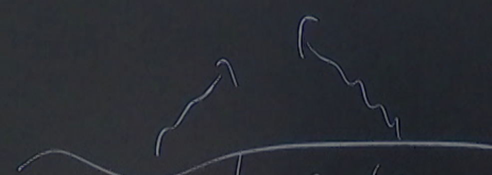
EOM

$$\nabla^2 \psi = 0 \to \psi = 0$$

$$\nabla^2 w_i = 0 \to w_i = 0$$

$$\nabla^2 \phi = 0 \to \Phi = 0$$

$$\boxed{\Box S_{ij} = 0}$$



$\partial_i S_{ij} \neq 0$ $\quad S_{ij} \neq 0$

$$-\partial_0 \partial_{(i} \omega_{j)} + 2\delta_{ij} \partial_0^2 \psi - \Box S_{ij}$$

$$+ 2 \partial_k \partial_{(i} S_{j)}^k - \delta_{ij} \partial_k \partial_\ell S^{k\ell}$$

$$h_{ab} \to h_{ab} + \partial_{(a} \xi_{b)}$$

$$S_{ij} \to S_{ij} + \partial_{(i} \xi_{j)} - \frac{1}{3}(\partial_k \xi^k)\delta_{ij}$$

Transverse gauge

$$\partial_i S^{ij} = 0 \Rightarrow \nabla^2 \xi^j + \frac{1}{3} \partial_j \partial_i \xi^i = -2 \partial_i S^{ij}$$

$$\partial_i w^i = 0 \Rightarrow \nabla^2 \xi^0 = \partial_i w^i + \partial_0 \partial_i \xi^i$$

$\boxed{\text{EoM}}$

$$\nabla^2 \psi = 0 \to \psi = 0$$

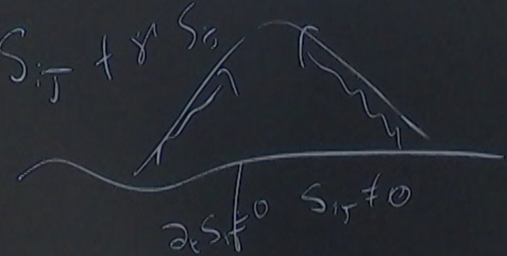
$$\nabla^2 w_i = 0 \to w_i = 0$$

$$\nabla^2 \phi = 0 \to \Phi = 0$$

$$\boxed{\Box S_{ij} = 0}$$

$$\Box S_{ij} = k \partial_0 S_{ij} + \gamma S_{ij}$$

$$\partial_t^2 S = k \partial_t S$$



$$\tfrac{1}{2}\left[\partial_\sigma \partial_\nu h^\sigma_\mu + \partial_\sigma \partial_\mu h^\sigma_\nu - \partial_\mu \partial_\nu h - \Box h_{\mu\nu} - \eta_{\mu\nu}\partial_\rho \partial_\lambda h^{\rho\lambda} + \eta_{\mu\nu}\Box h\right]$$

with $h = \eta^{ab} h_{ab}$

Introduce some convenient notation

$$h_{00} = -2\Phi \;;\; h_{0i} = w_i \;;$$

$$\psi \equiv -\frac{1}{6}\delta^{ij} h_{ij}$$

$$h_{ij} = 2S_{ij} - \psi\,\delta_{ij}\,2$$

$$S_{ij} = \frac{1}{2}\left(h_{ij} - \frac{1}{3}\delta^{kl} h_{kl}\,\delta_{ij}\right)$$

$$h^i_i = -\psi \cdot 6$$

$$m\left(\frac{dt}{d\tau}, \frac{dx^i}{dt}\frac{dt}{d\tau}\right) = m\left(U^0, U^0 v^i\right) = \left(E, E v^i\right)$$

$$+\left[\partial_j w_k - \tfrac{1}{2}\partial_0 h_{jk}\right) v^j v^k\Big]$$

$$2\left[\partial_{[i} w_{j]} + \partial_0 h_{ij}\right] v^j + \left(\partial_{(j} h_{k)i} - \tfrac{1}{2}\partial_i h_{jk}\right) v^j v^k\Big)$$

$h_{ab} \to h_{0b} = 0 \to$ purely spatial

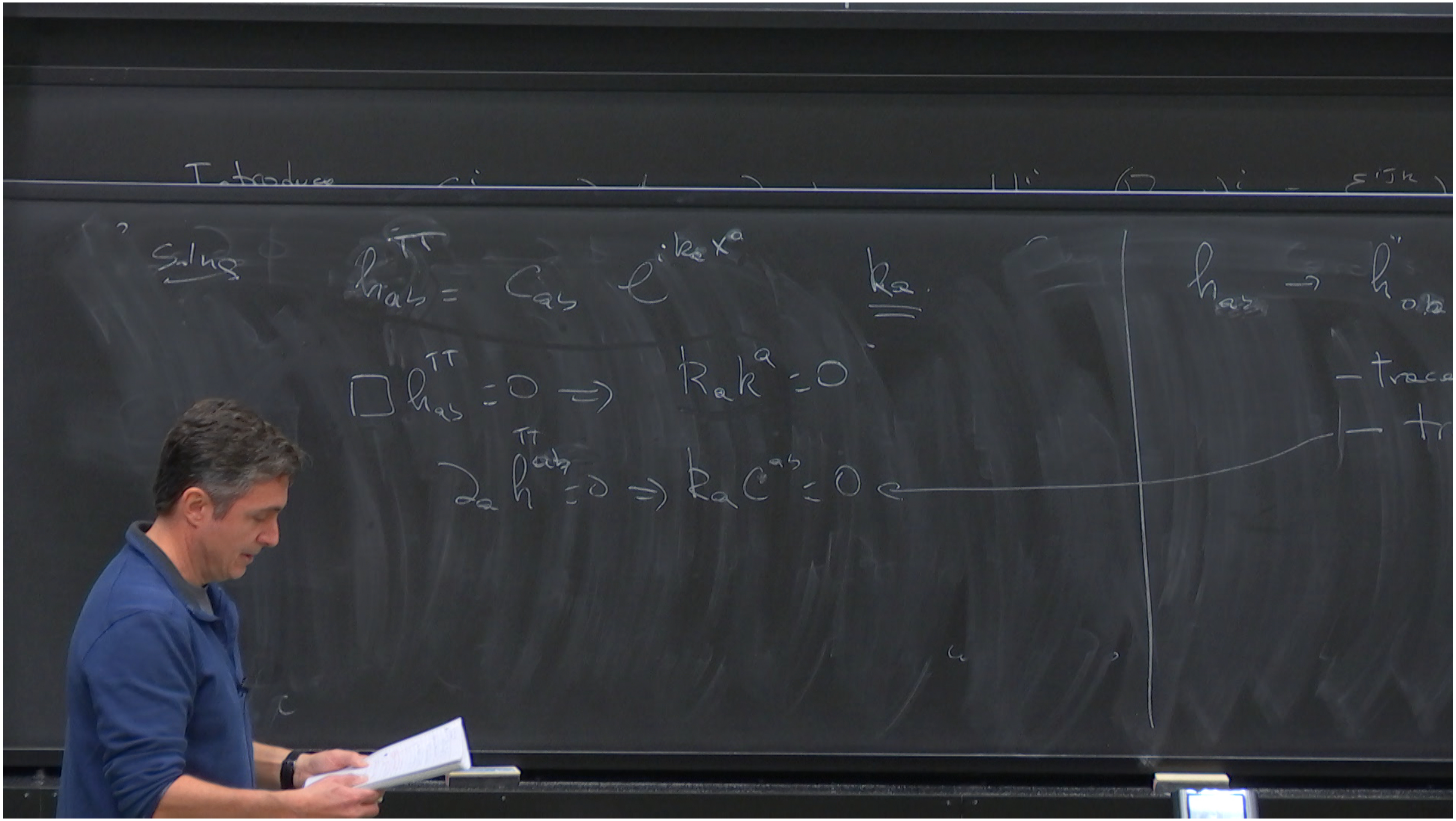
- traceless
- transverse

$$h^{TT}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2S_{ij} & \\ 0 & & & \end{pmatrix}$$

Solns

$$h^{TT}_{ab} = C_{ab}\, e^{ik_a x^a}$$

$h_{ab} \to$


Solns
h^TT_ab = C_ab e^{ik_a x^a}
k_a
□ h^TT_ab = 0 ⇒ k_a k^a = 0
∂_a h^{TT ab} = 0 ⇒ k_a C^{ab} = 0
h_ab → h'_ab
− trace

$h_{ab} \to h_{0b} = 0 \to$ purely spatial

- traceless
- transverse

Suppose $\to$ wave moves along z

$$k^a = (\omega, 0, 0, \omega)$$

$$h^{TT}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2S_{ij} & \\ 0 & & & \end{pmatrix}$$

$$C_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & c_{11} & c_{12} & 0 \\ 0 & c_{12} & -c_{11} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$h_{ab} \to h_{0a} = 0 \to$ purely spatial

- traceless
- transverse

Suppose $\to$ wave moves along z

$$k^a = (\omega, 0, 0, \omega)$$

$$h^{TT}_{ab} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & 2S_{ij} & \\ 0 & & & \end{pmatrix}$$

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