Title: Relativity - Lecture 221028

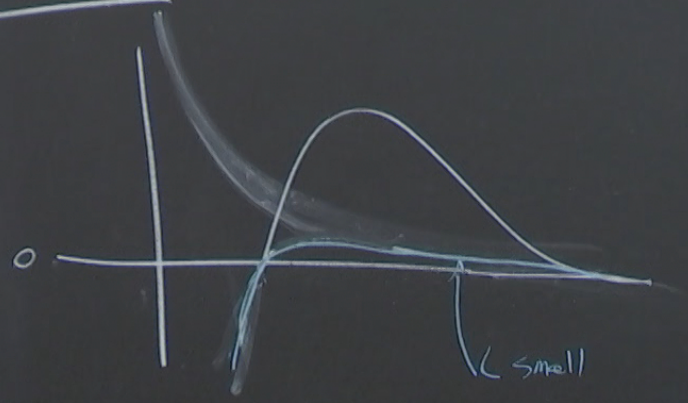
Speakers:

Collection: Relativity (2022/2023)

Date: October 28, 2022 - 9:00 AM

URL: https://pirsa.org/22100086

Mercury

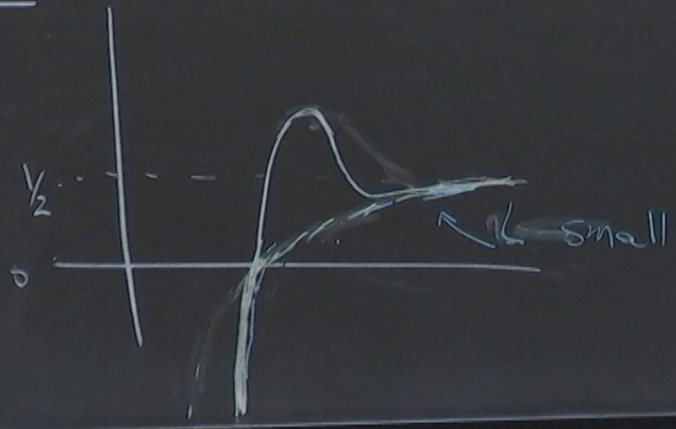


$$V = \frac{k}{2} - \frac{M}{r}k + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

$$= k\left(\frac{1}{2} - \frac{M}{r}\right) + \frac{L^2}{r^2}\left(\frac{1}{2} - \frac{M}{r}\right)$$

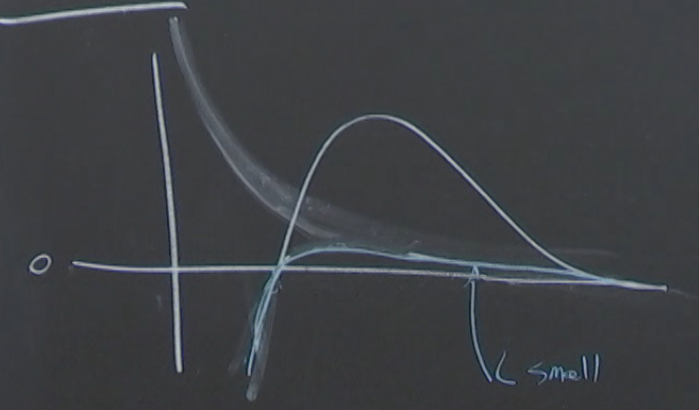
$$V(r) = \left(\frac{1}{2} - \frac{M}{r}\right)\cdot\left(k + \frac{L^2}{r^2}\right)$$

Mars



$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2$$

Massless

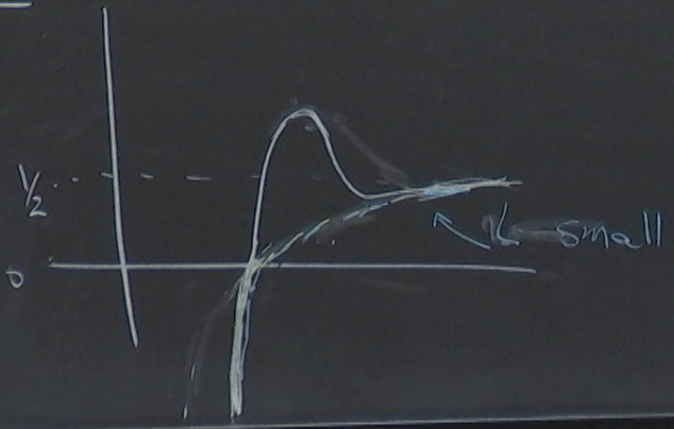


$$V = \frac{K}{2} - \frac{M}{r}K + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$$

$$= K\left(\frac{1}{2} - \frac{M}{r}\right) + \frac{L^2}{r^2}\left(\frac{1}{2} - \frac{M}{r}\right)$$

$$V(r) = \left(\frac{1}{2} - \frac{M}{r}\right)\left(K + \frac{L^2}{r^2}\right)$$

Massive



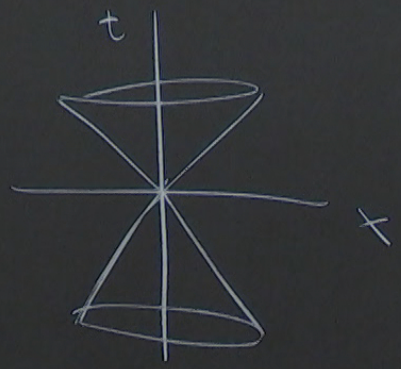
$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2$$

$t \to a t'$

$dt = a\, dt'$

$ds^2 = -a(dt')^2 + dx^2$

$ds^2 = -dt^2 + dx^2 = 0 \qquad \frac{dt}{dx} = \pm 1$



$t \to a t'$

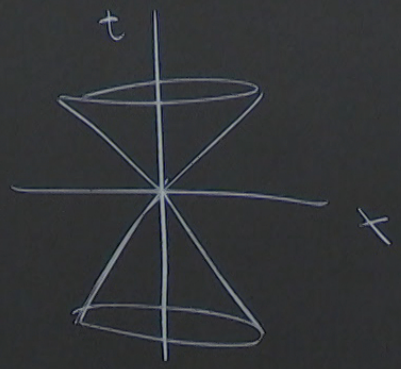
$dt = a\, dt'$

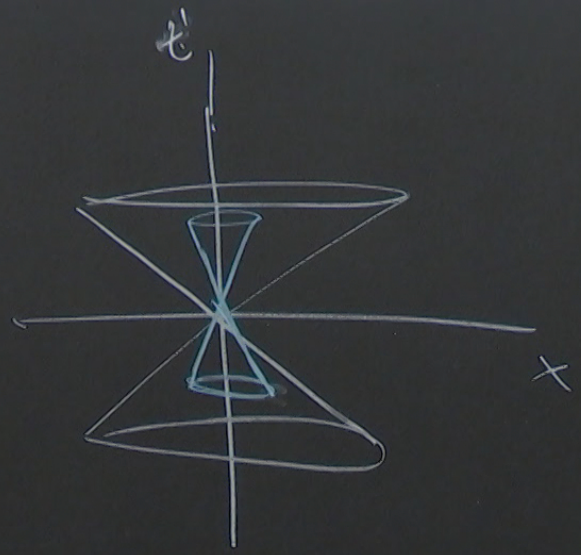
$ds^2 = -a^2 (dt')^2 + dx^2$

$ds^2 = 0 \Rightarrow a^2 dt'^2 = dx^2$

$dt' = \pm \frac{1}{a} dx$

$ds^2 = -dt^2 + dx^2 = 0 \qquad \frac{dt}{dx} = \pm 1$





$t \to a t'$

$dt = a\, dt'$

$ds^2 = -a^2 (dt')^2 + dx^2$

$ds^2 = 0 \Rightarrow a^2 dt'^2 = dx^2$

$dt' = \pm \frac{1}{a} dx$

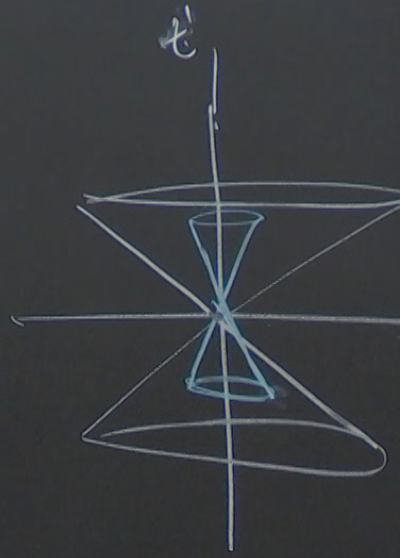
$ds^2 = -d$

t

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}dr^2 + r^2 d\Omega^2$$

$$ds^2 = 0 \qquad \left(1-\frac{2M}{r}\right)dt^2 = \frac{1}{\left(1-\frac{2M}{r}\right)}dr^2$$

$$\frac{dt}{dr} = \pm$$

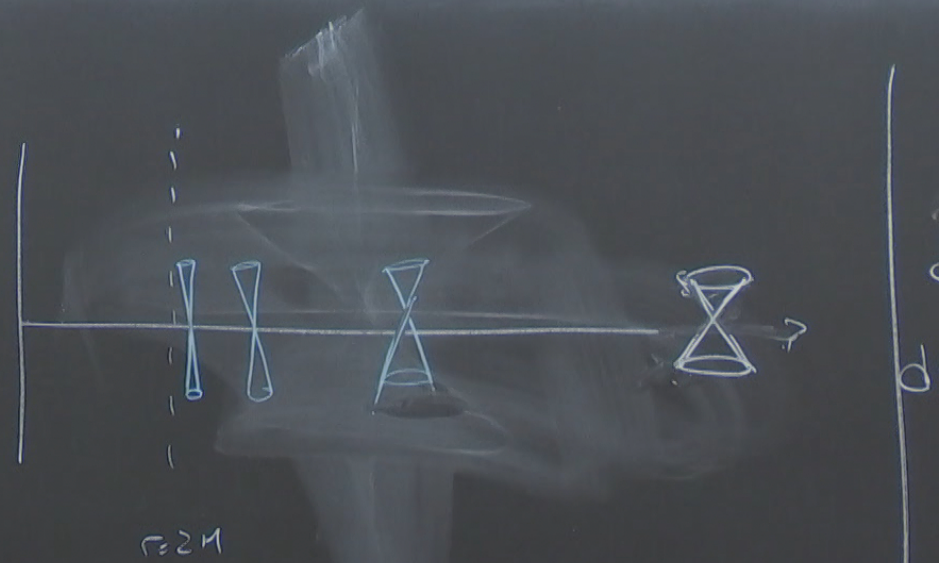


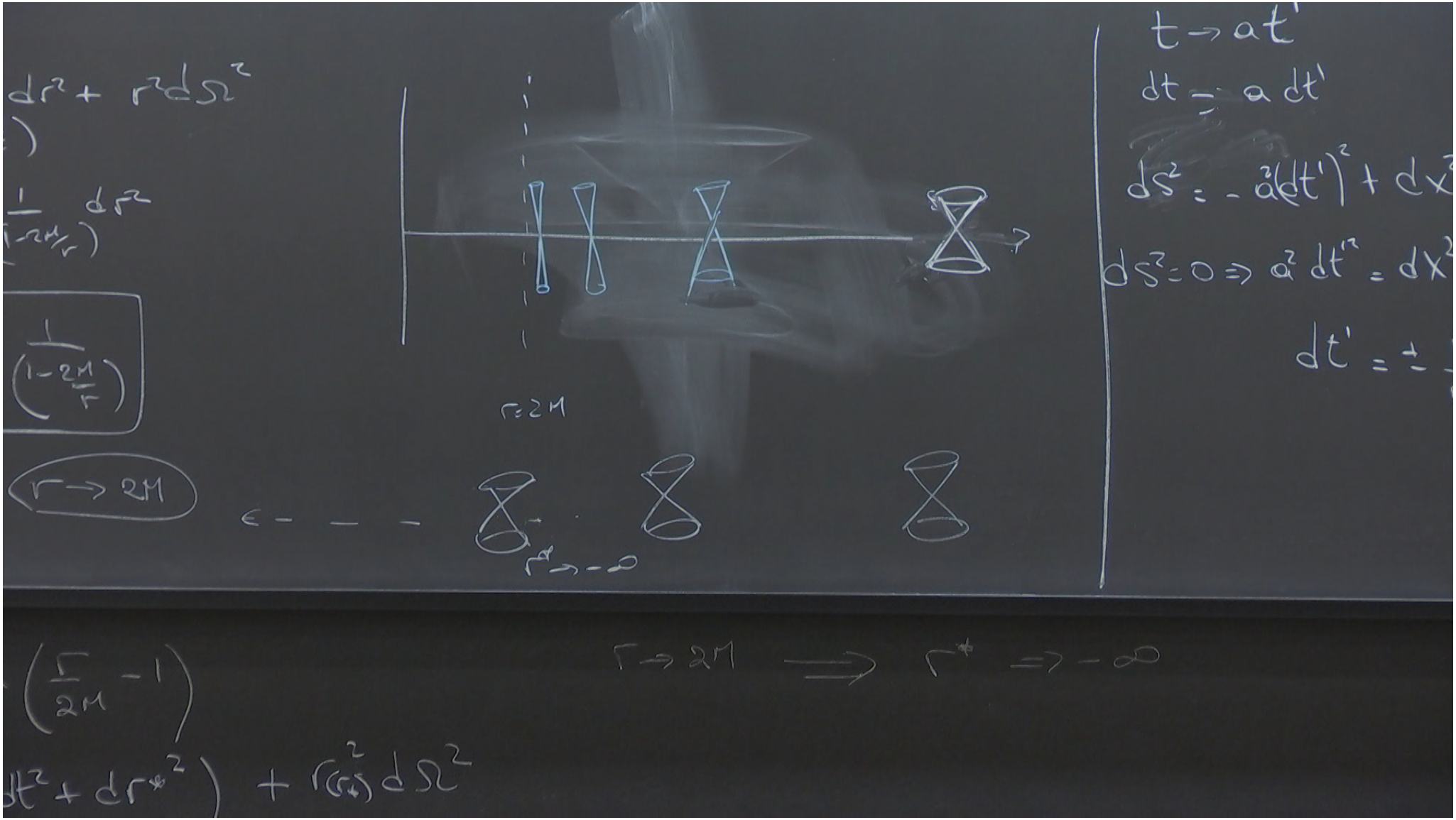
$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \frac{1}{\left(1-\frac{2M}{r}\right)}dr^2 + r^2 d\Omega^2$$

$$ds^2 = 0 \qquad \left(1-\frac{2M}{r}\right)dt^2 = \frac{1}{\left(1-2M/r\right)}dr^2$$

$$\boxed{\frac{dt}{dr} = \pm \frac{1}{\left(1-\frac{2M}{r}\right)}}$$

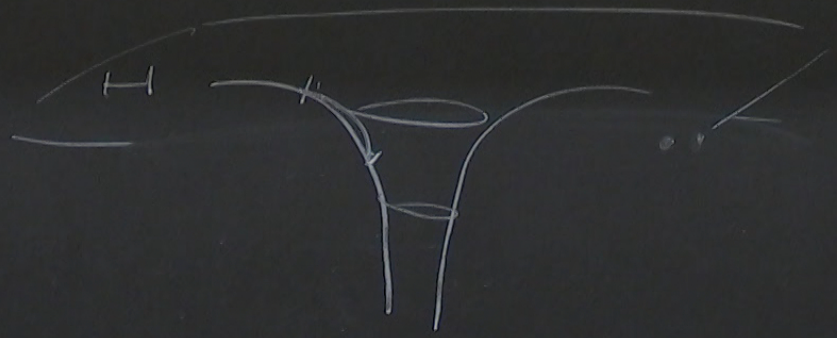
$r \to 2M$




$dr^2 + r^2 d\Omega^2$
$\frac{1}{(1-\frac{2M}{r})} dr^2$
$\left(\frac{1}{1-\frac{2M}{r}}\right)$
$r \to 2M$
$r = 2M$
$r^* \to -\infty$
$t \to at'$
$dt = a\,dt'$
$ds^2 = -a^2(dt')^2 + dx^2$
$ds^2 = 0 \Rightarrow a^2 dt'^2 = dx^2$
$dt' = \pm$
$\left(\frac{r}{2M} - 1\right)$
$r \to 2M \Longrightarrow r^* \to -\infty$
$dt^2 + dr^{*2}) + r(r^*)^2 d\Omega^2$

$r \to -\infty$

$r \to 2M \Longrightarrow r^* \to -\infty$



$t = \text{const}$

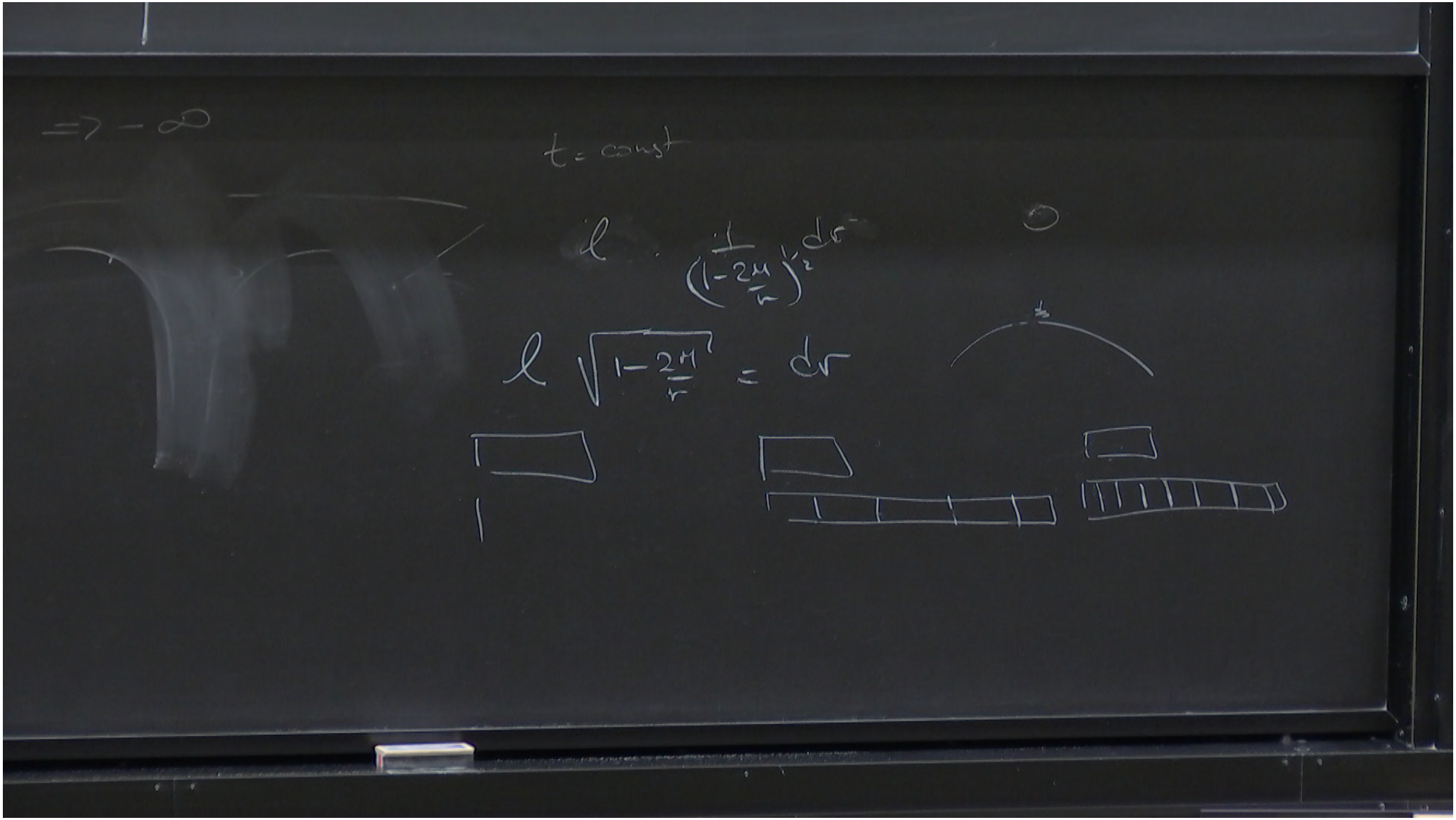
$$l = \int \frac{1}{\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}} dr$$

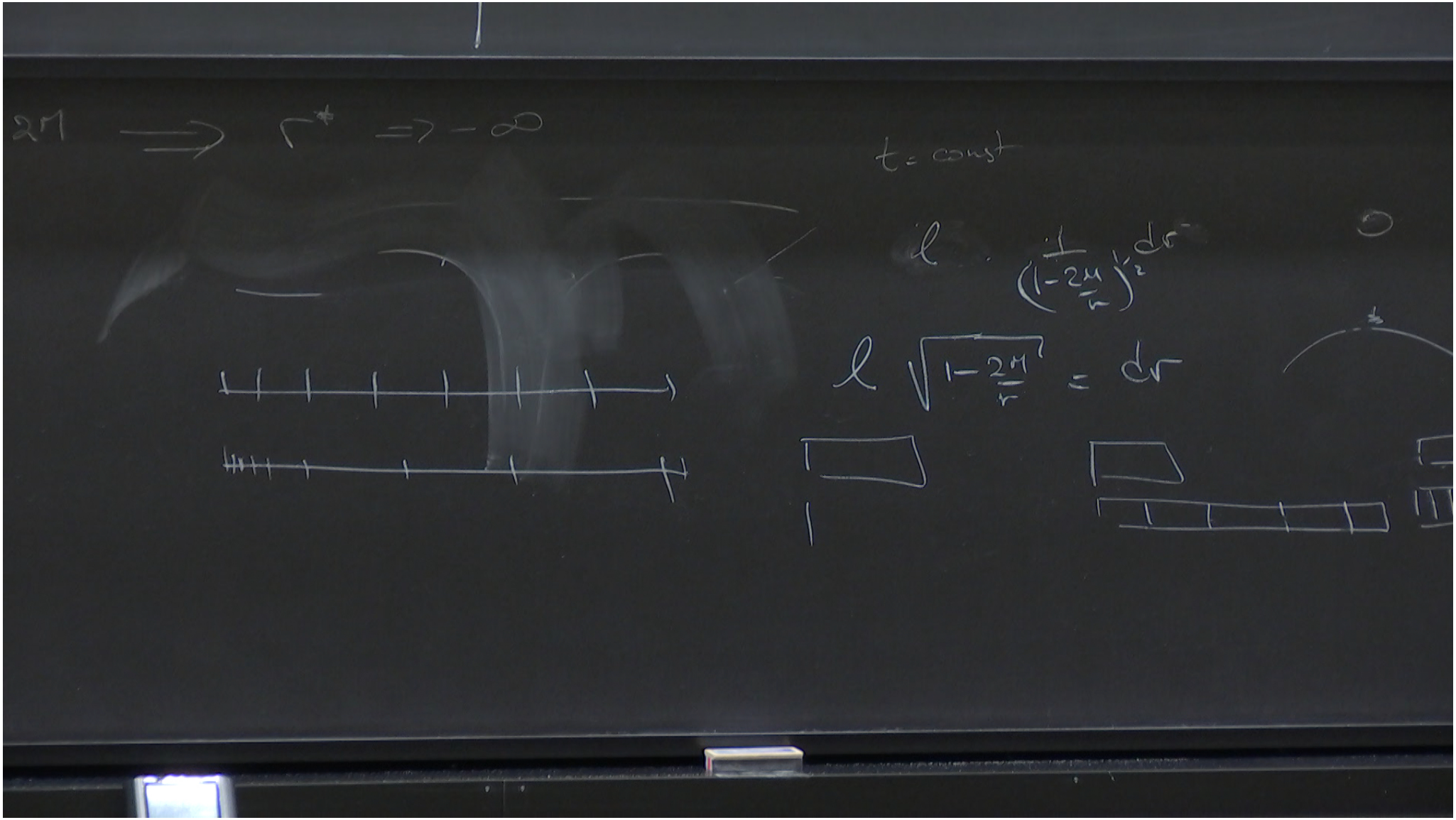
$$l\sqrt{1-\frac{2M}{r}} = dr$$

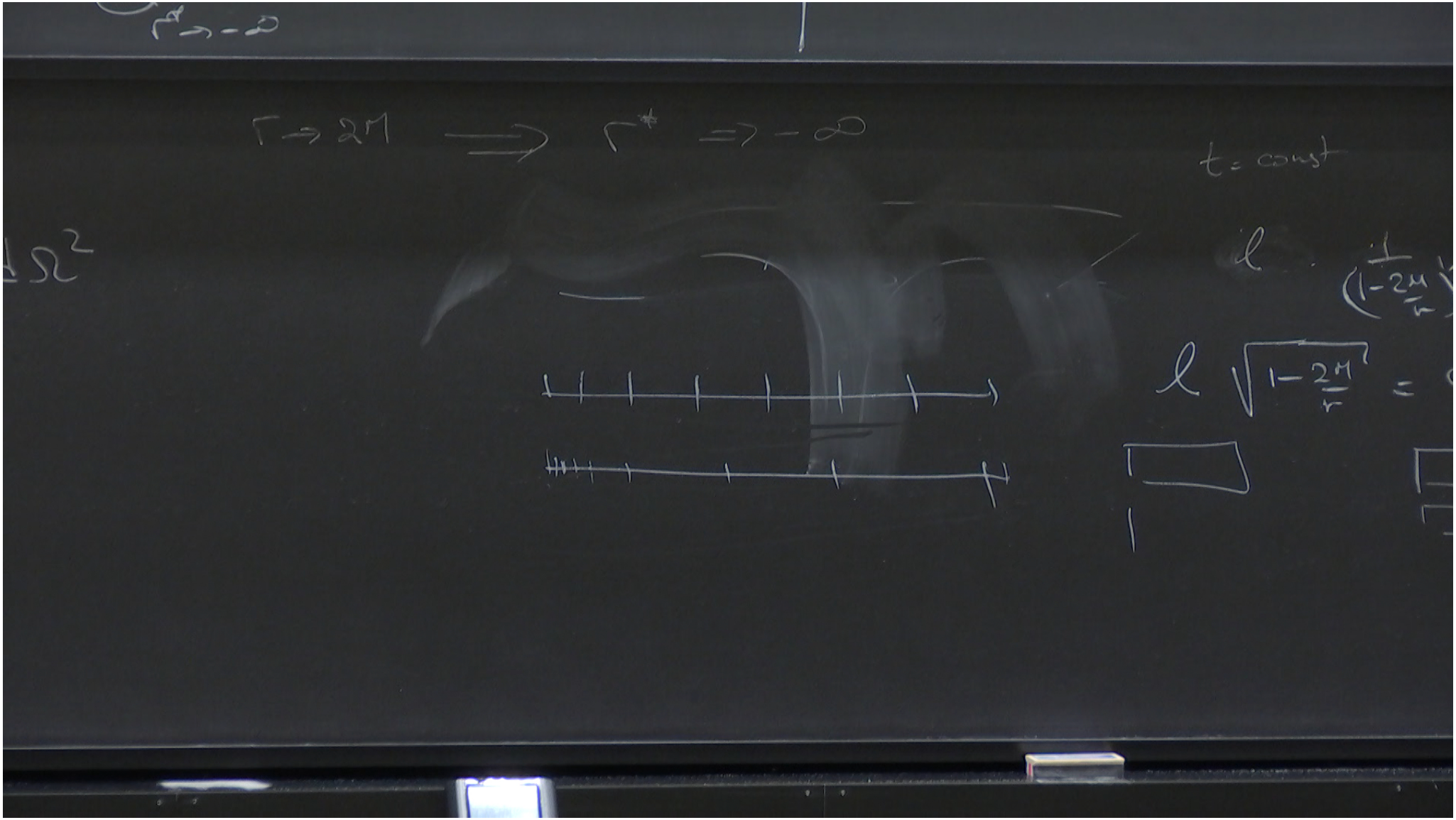
define $r^* = r + 2M \ln\left(\frac{r}{2M} - 1\right)$

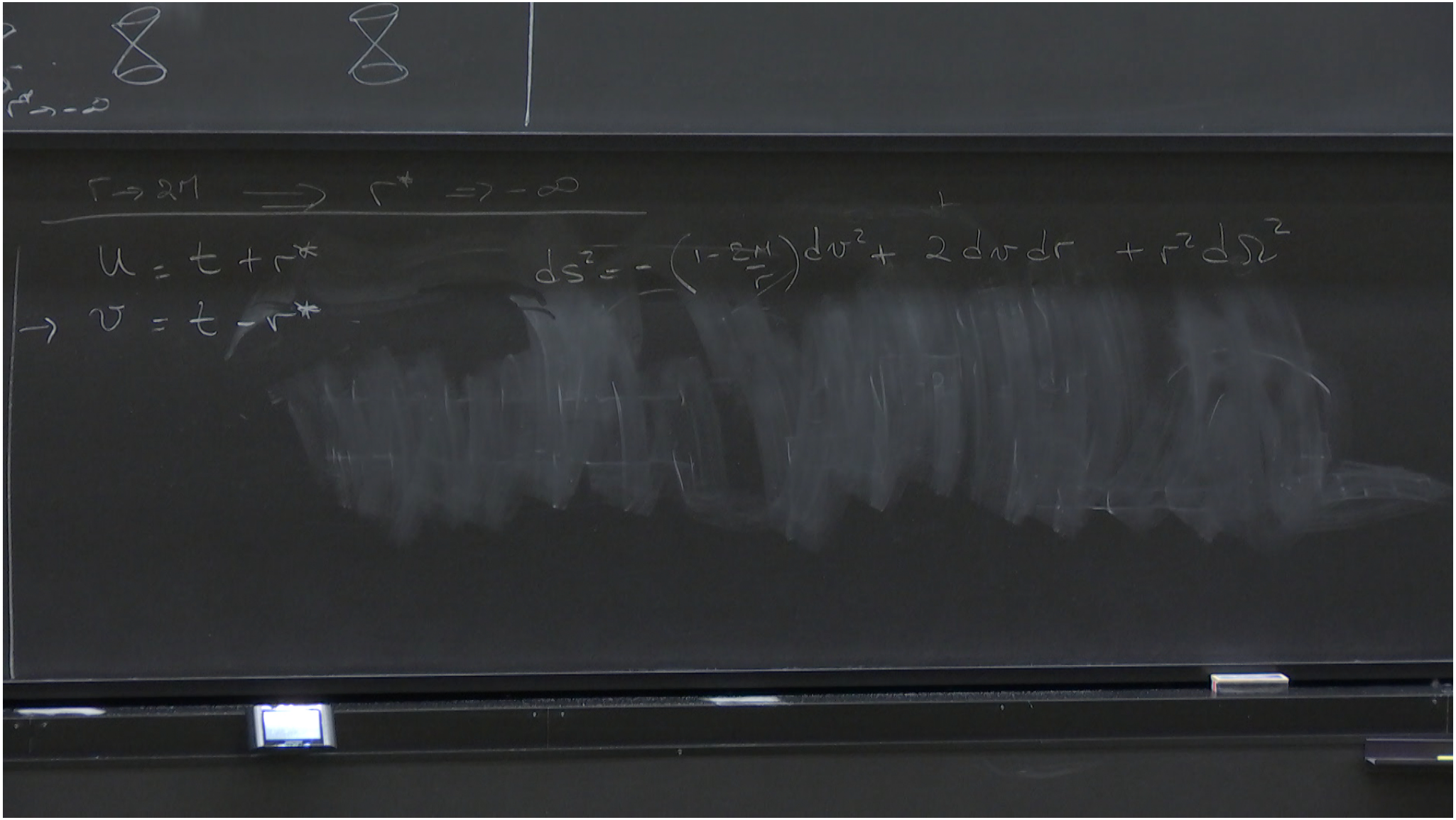
$$\hookrightarrow ds^2 = \left(1 - \frac{2M}{r(r_*)}\right)\left(-dt^2 + dr_*^2\right) + r(r_*)^2 d\Omega^2$$

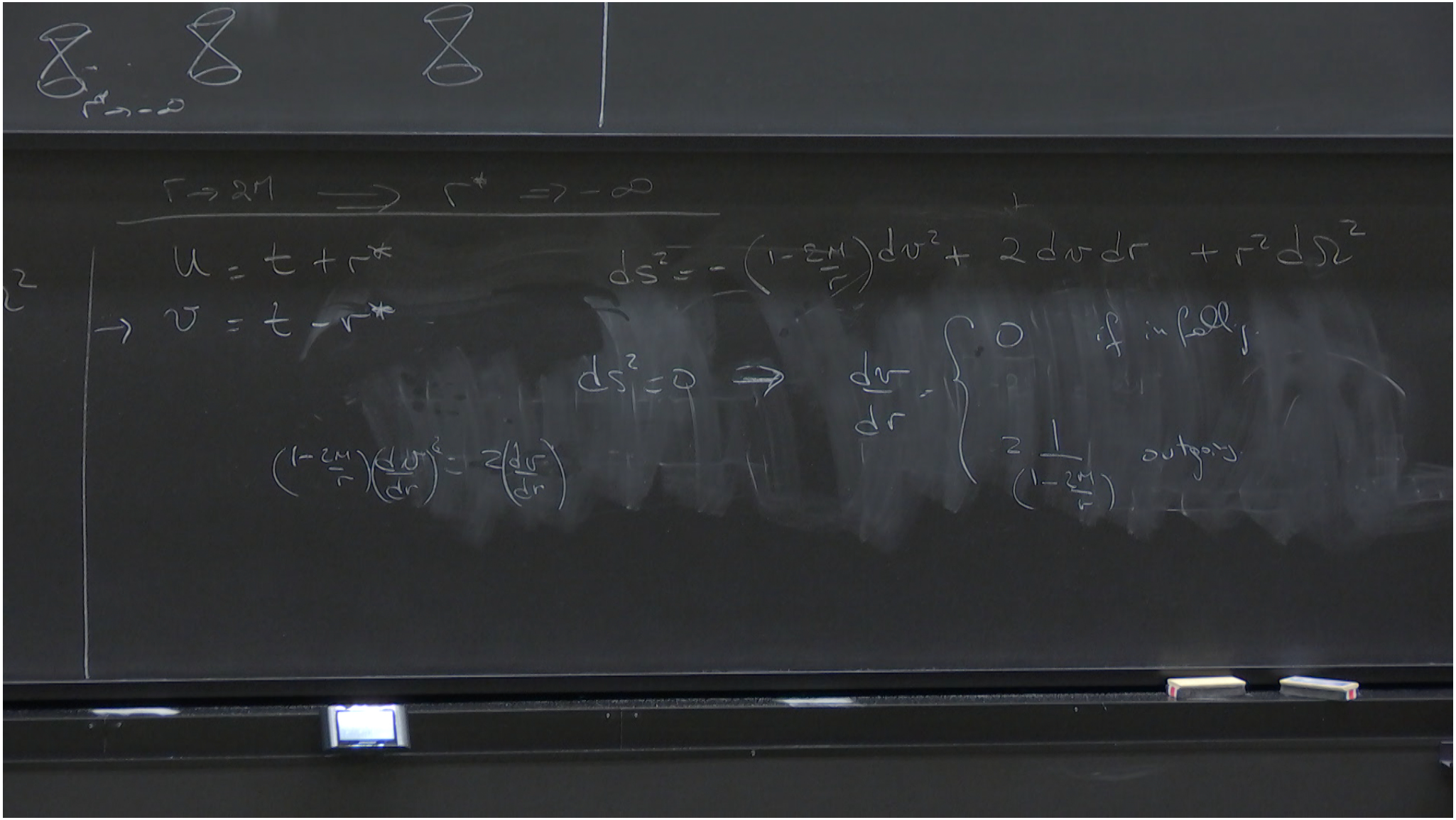
$$\rightarrow \left(\frac{dt}{dr_*}\right) = \pm 1$$

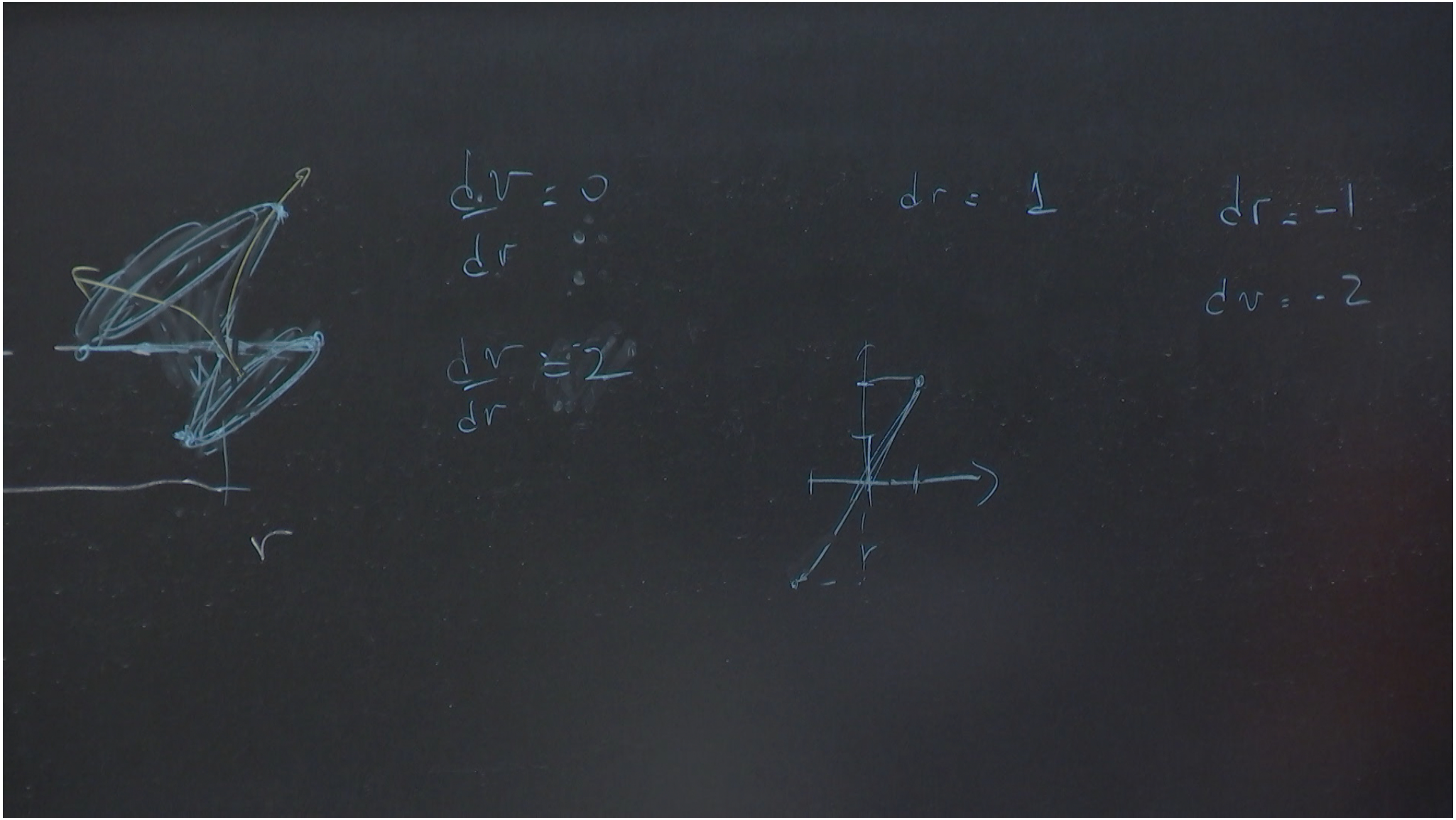

$\Rightarrow -\infty$
$t = const$
$l \cdot \frac{1}{\left(1-\frac{2M}{r}\right)^{\frac{1}{2}}} dr$
$l\sqrt{1-\frac{2M}{r}} = dr$

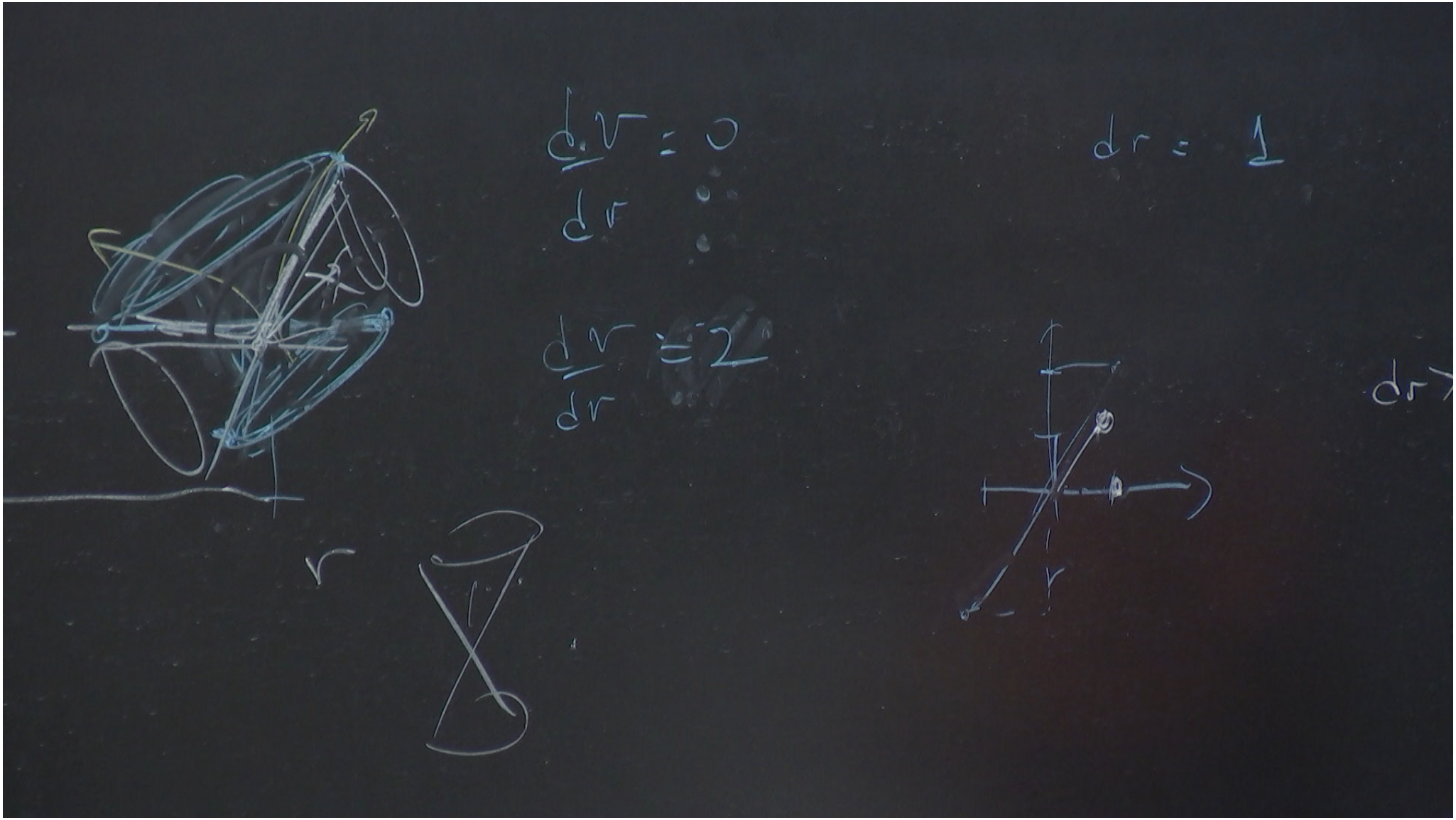

2M ⟹ r* ⇒ −∞
t = const
l · 1/(1−2M/r)^½ dr
l √(1−2M/r) = dr


$r^* \to -\infty$
$r \to 2M \Rightarrow r^* \to -\infty$
$t = \text{const}$
$d\Omega^2$
$l \cdot \frac{1}{\sqrt{1-\frac{2M}{r}}}$
$l\sqrt{1-\frac{2M}{r}}$


$r \to -\infty$
$r \to 2M \Rightarrow r^* \to -\infty$
$u = t + r^*$
$\to v = t - r^*$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2$


$r^* \to -\infty$
$r \to 2M \Rightarrow r^* \to -\infty$
$u = t + r^*$
$\to v = t - r^*$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dv^2 + 2\,dv\,dr + r^2 d\Omega^2$
$ds^2 = 0 \Rightarrow \frac{dv}{dr} = \begin{cases} 0 & \text{if in fall.} \\ 2\frac{1}{\left(1 - \frac{2M}{r}\right)} & \text{outgoing} \end{cases}$
$\left(1 - \frac{2M}{r}\right)\left(\frac{dv}{dr}\right)^2 = 2\left(\frac{dv}{dr}\right)$


$\frac{dv}{dr} = 0$
$dr = 1$
$dr = -1$
$dv = -2$
$\frac{dv}{dr} = 2$
r
r


dV = 0
dr
dr = 1
dV = 2
dr
r
r

$\frac{dv}{dr} = 0$

$\frac{dv}{dr} = 2$

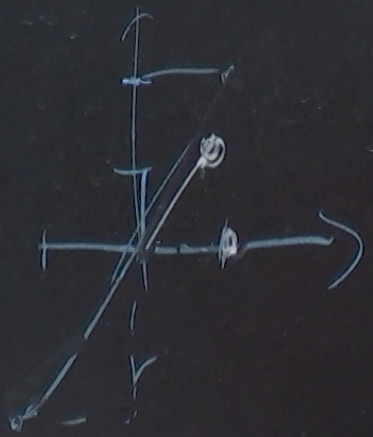
$dr = 1$

$dr = -1$

$dv = -2$

$dr > 0 \rightarrow dv = 0$

$dv = 2$



$\cdot d\Omega^2$ $\quad G_{ab} = 8\pi T_{ab}$.

$T_{ab} = (\rho + p) u_a u_b + p g_{ab}$

$= \rho u_a u_b + p (g_{ab} + u_a u_b)$

$u^a u_a = -1$

$u^a = (\vec{e}^{\,\alpha}, \vec{0})$

$u_a = (e^{\alpha}, \vec{0})$

$G_{ab} = 8\pi T_{ab}$.

$T_{ab} = (\rho + p) u_a u_b + p\, g_{ab}$

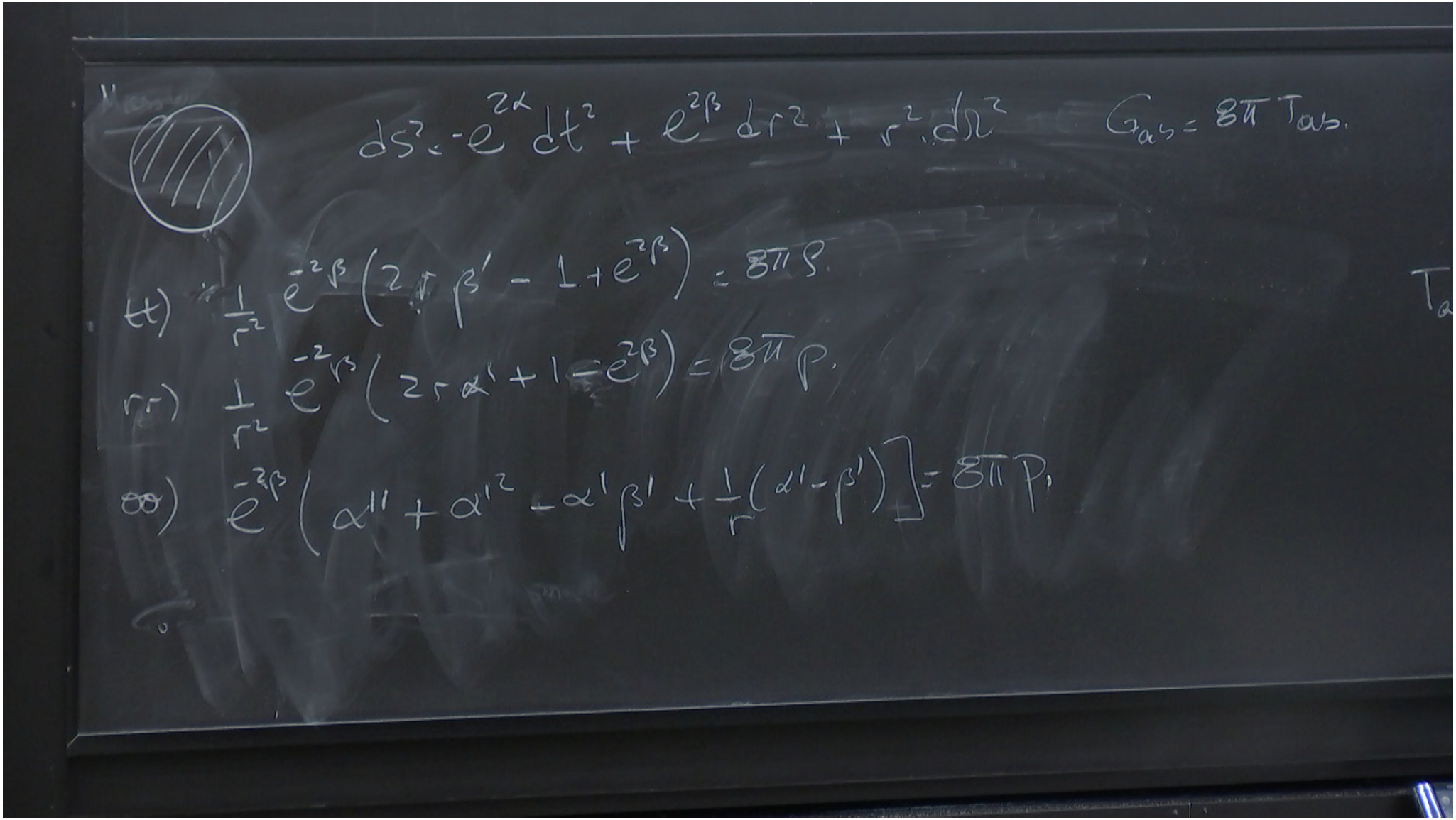
$= \rho\, u_a u_b + p(g_{ab} + u_a u_b)$

$u^a u_a = -1$

$u^a = (\vec{e}^{\,\alpha}, \vec{0})$

$u_a = (e^{\alpha}, \vec{0})$

$$T_{ab} = \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\alpha}p & & \\ & & r^2 p & \\ & & & r^2 p \sin^2\theta \end{pmatrix}$$


$ds^2 = -e^{2\alpha} dt^2 + e^{2\beta} dr^2 + r^2 d\Omega^2$
$G_{ab} = 8\pi T_{ab}$.
tt) $\frac{1}{r^2} e^{-2\beta}\left(2r\beta' - 1 + e^{2\beta}\right) = 8\pi\rho$.
rr) $\frac{1}{r^2} e^{-2\beta}\left(2r\alpha' + 1 - e^{2\beta}\right) = 8\pi p$.
$\theta\theta$) $e^{-2\beta}\left(\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{1}{r}(\alpha' - \beta')\right] = 8\pi p$.

$+ r^2 d\Omega^2$ $\quad G_{ab} = 8\pi T_{ab}$ $\quad T_{ab} = (\rho + p) u_a u_b + p g_{ab}$ $\quad u^a u_a = -1$

$= \rho u_a u_b + p(g_{ab} + u_a u_b)$

$8\pi\rho$ ✓

$8\pi p$ ✓

$\nabla_a T^{ab} = 0$

$T_{ab} = \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\beta}p & & \\ & & r^2 p & \\ & & & r^2 p \sin^2\theta \end{pmatrix}$

$u^a = (e^{\alpha}, \vec{0})$

$u_a = (e^{\alpha}, \vec{0})$

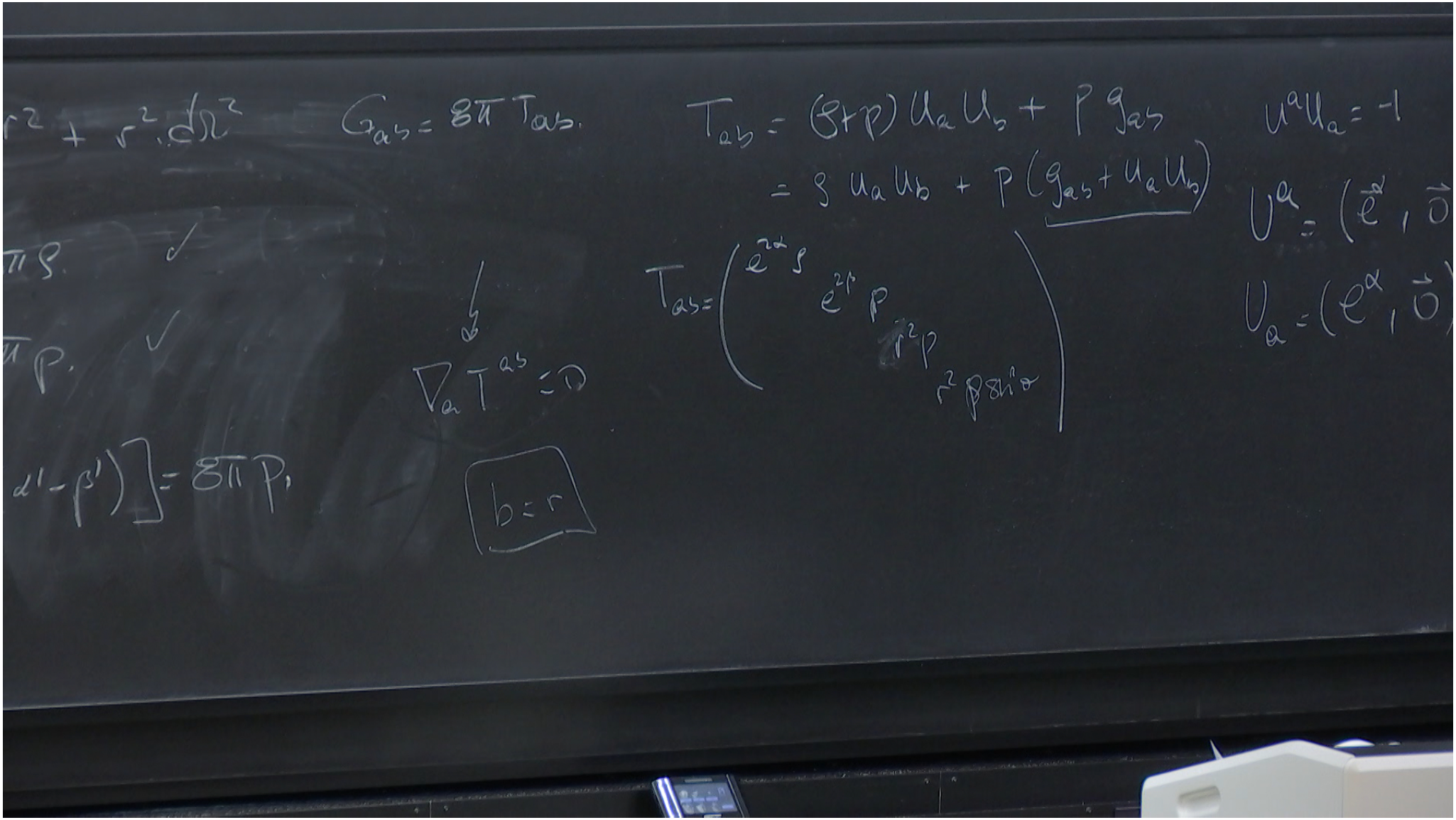
$(d' - p')] = 8\pi p$

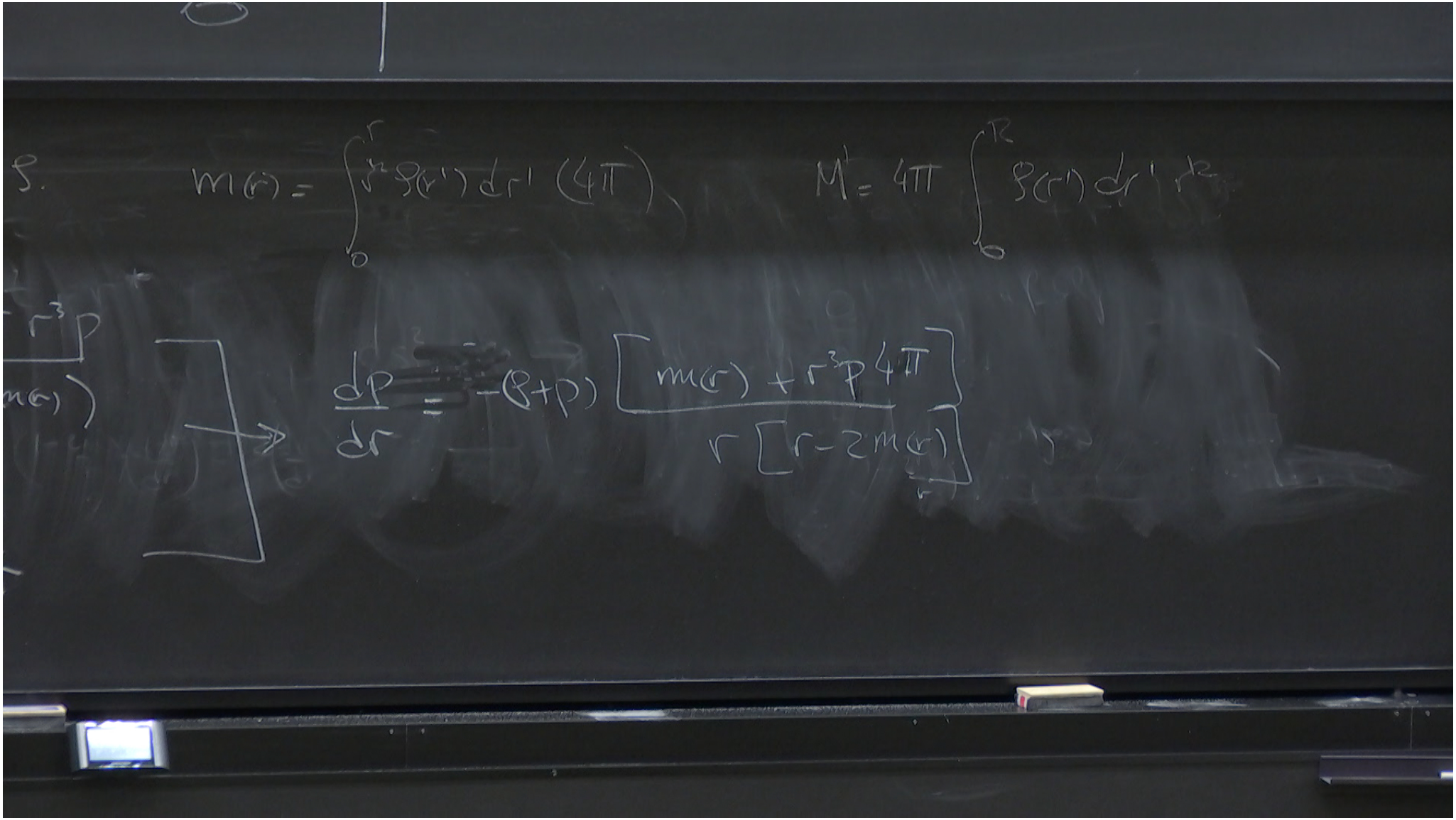
$r^2 \to -\infty$

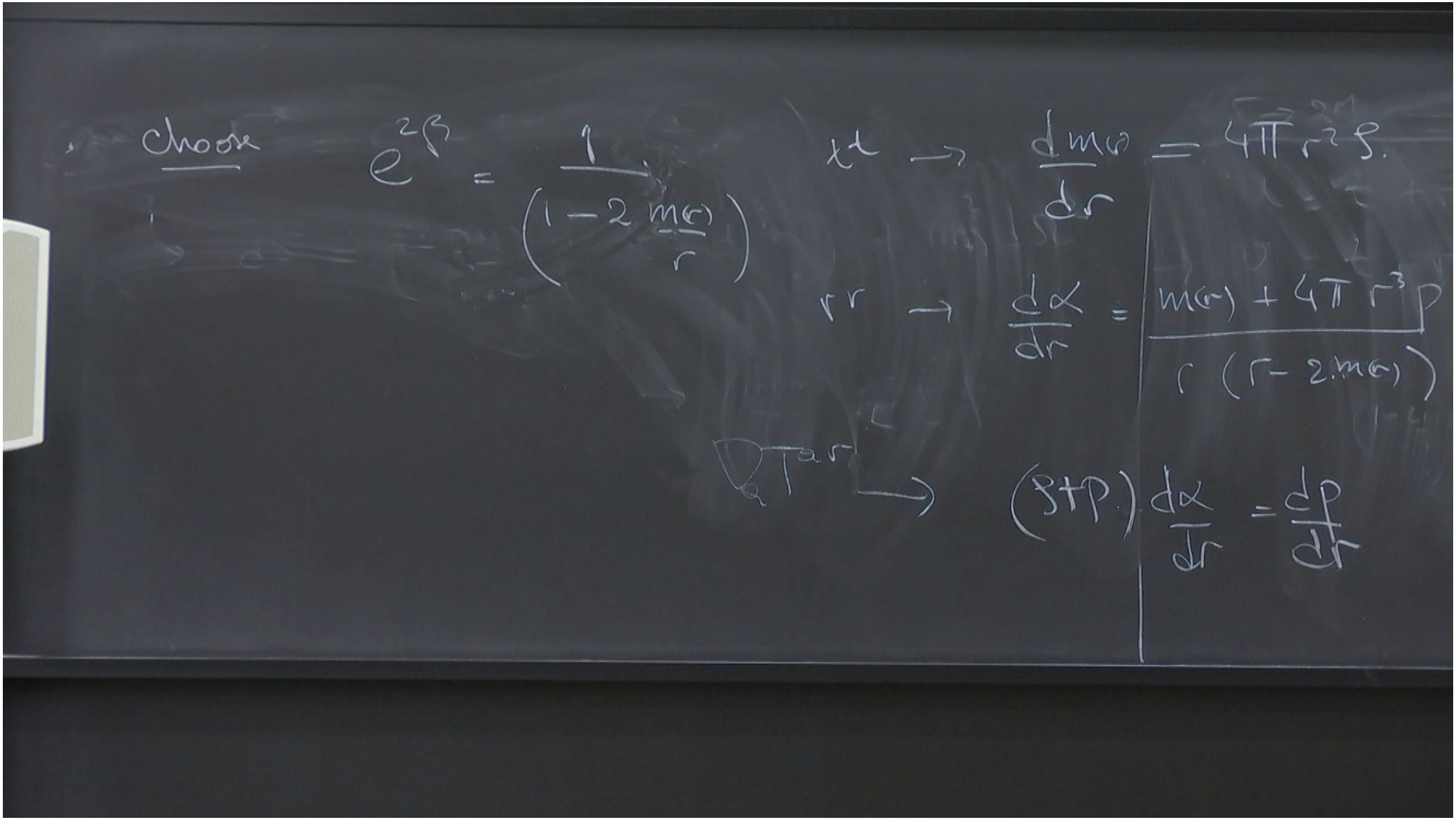
$$tt \to \frac{dm(r)}{dr} = 4\pi r^2 \rho. \qquad m(r) = \int_0^r r'^2 \rho(r')\, dr'$$

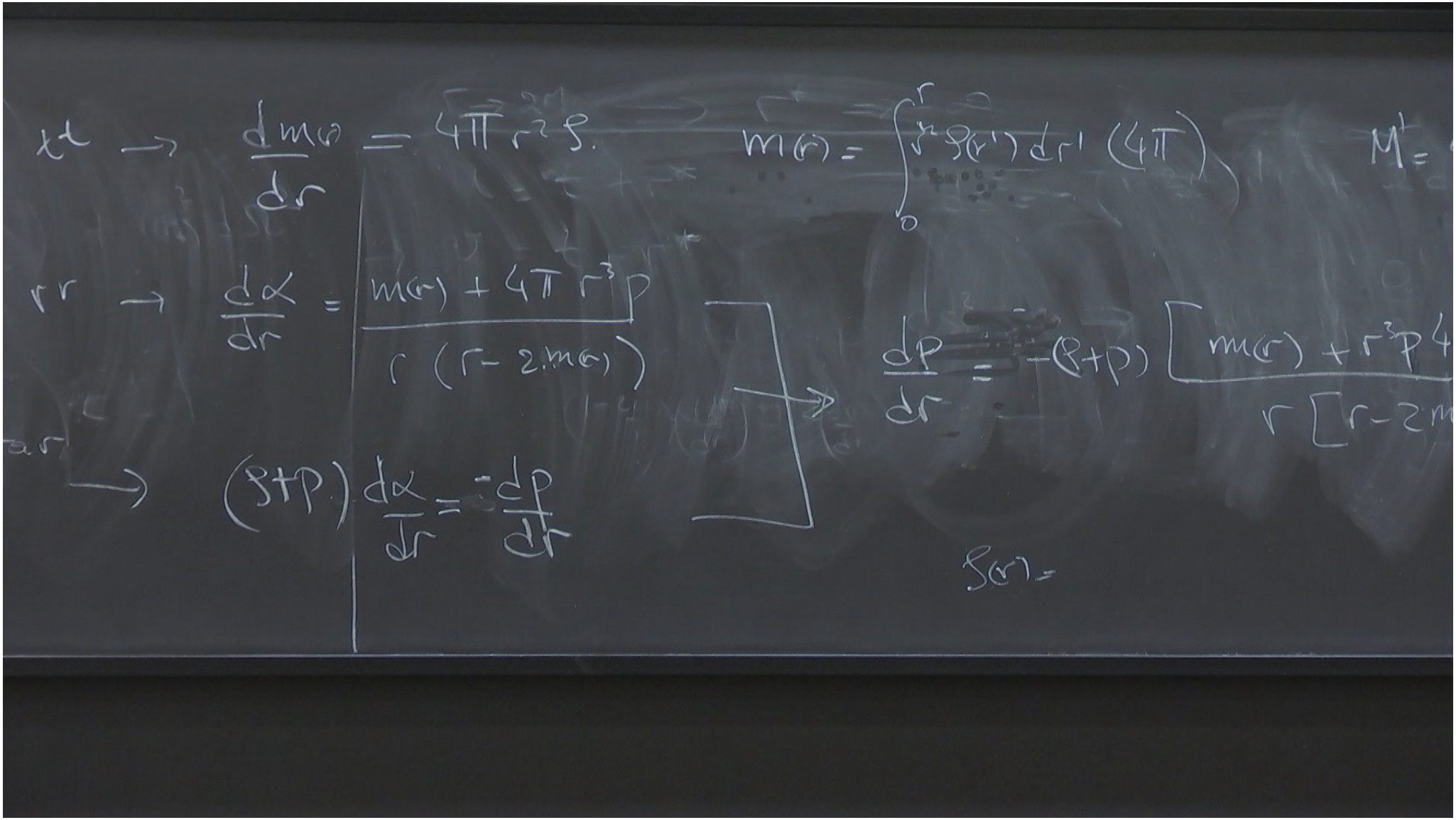
$$\left(1 - \frac{2m(r)}{r}\right)$$

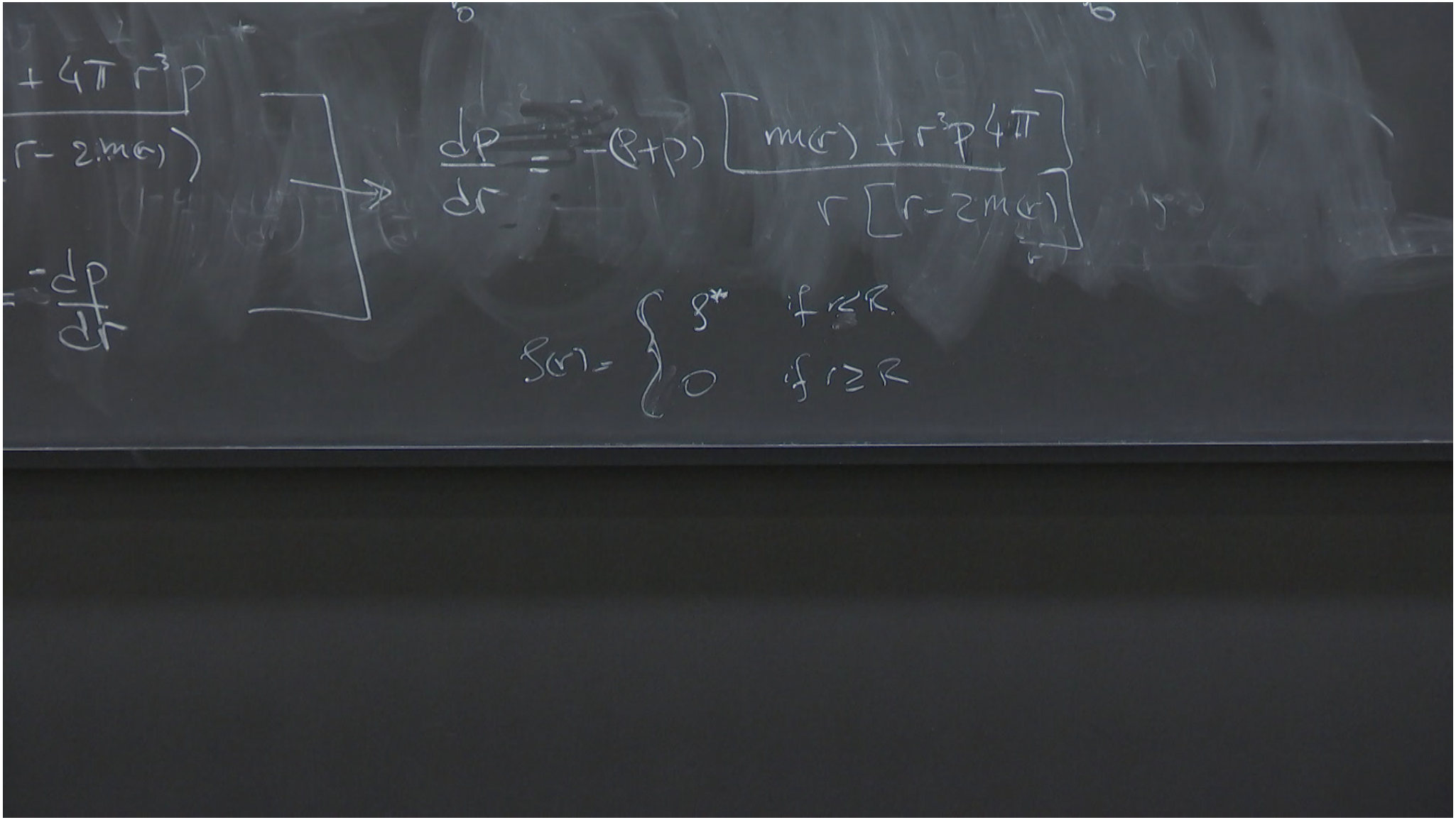
$$rr \to \frac{d\alpha}{dr} = \frac{m(r) + 4\pi r^3 p}{r\left(r - 2m(r)\right)}$$


$r^2 + r^2 d\Omega^2$
$G_{ab} = 8\pi T_{ab}$
$T_{ab} = (\rho + p) u_a u_b + p g_{ab}$
$= \rho u_a u_b + p(g_{ab} + u_a u_b)$
$u^a u_a = -1$
$u^a = (e^{\alpha}, \vec{0})$
$u_a = (e^{\alpha}, \vec{0})$
$T_{ab} = \begin{pmatrix} e^{2\alpha}\rho & & & \\ & e^{2\alpha}p & & \\ & & r^2 p & \\ & & & r^2 p \sin^2\theta \end{pmatrix}$
$\nabla_a T^{ab} = 0$
$b = r$
$\pi\rho$
πp
$\alpha' - \beta')] = 8\pi p$


$m(r) = \int_0^r r'^2 \rho(r')\, dr'\, (4\pi)$
$M = 4\pi \int_0^R \rho(r)\, dr\, r^2$
$\frac{dP}{dr} = -(\rho + P)\left[\frac{m(r) + r^3 P 4\pi}{r[r - 2m(r)]}\right]$


choose
$e^{2\beta} = \frac{1}{\left(1 - 2\frac{m(r)}{r}\right)}$
$tt \rightarrow \frac{dm(r)}{dr} = 4\pi r^2 \rho$.
$rr \rightarrow \frac{d\alpha}{dr} = \frac{m(r) + 4\pi r^3 p}{r\,(r - 2m(r))}$
$\nabla_a T^{ar} \rightarrow (\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr}$

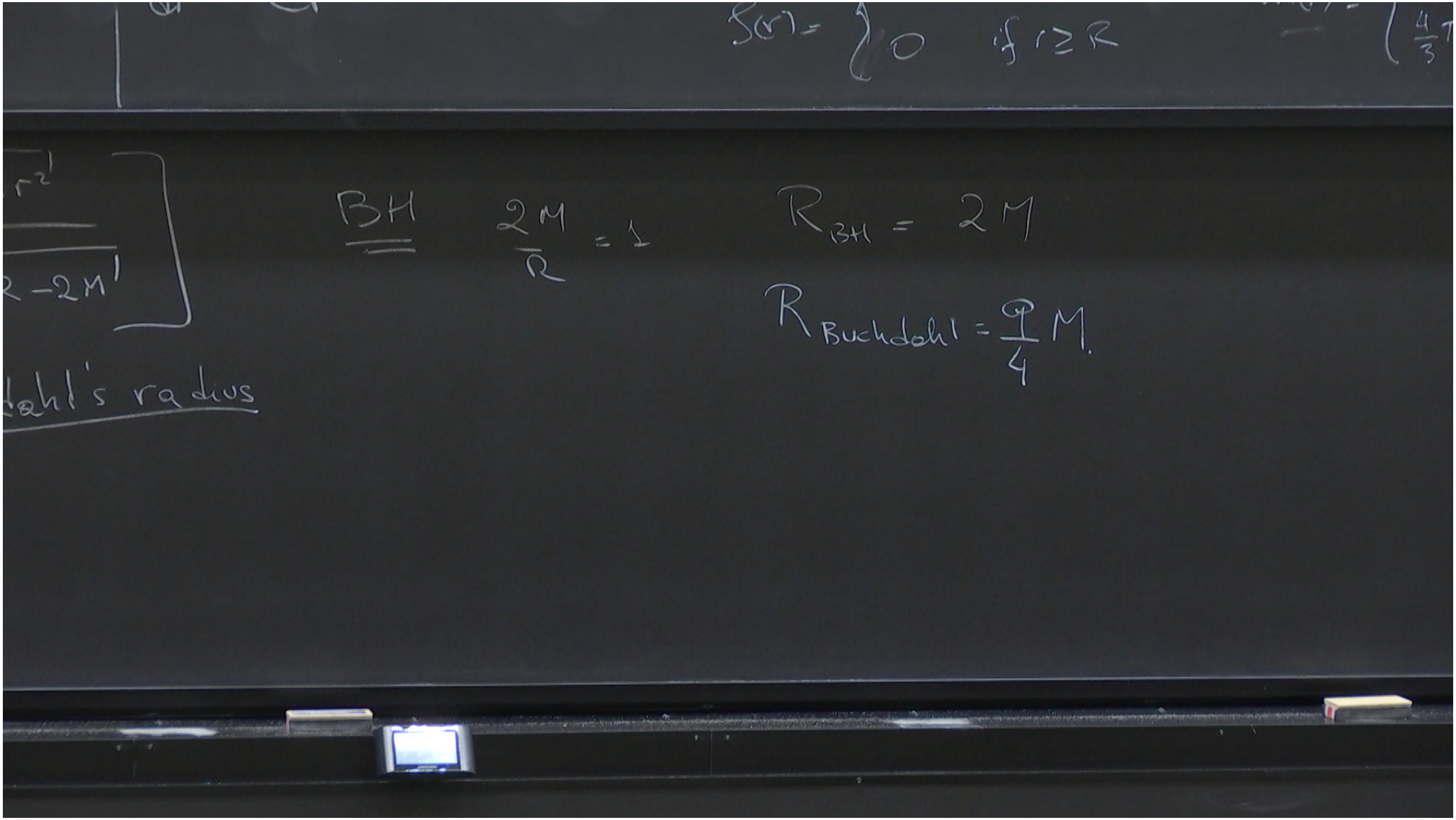

$tt \rightarrow \frac{dm(r)}{dr} = 4\pi r^2 \rho$.
$m(r) = \int_0^r r^2 \rho(r') dr' (4\pi)$
$M =$
$rr \rightarrow \frac{d\alpha}{dr} = \frac{m(r) + 4\pi r^3 p}{r((r - 2m(r))}$
$\frac{dp}{dr} = -(\rho + p)\left[\frac{m(r) + r^3 p 4}{r[r - 2m}\right.$
$ar \rightarrow (\rho + p)\frac{d\alpha}{dr} = -\frac{dp}{dr}$
$\rho(r) =$


$+4\pi r^3 p$
$r - 2m(r)$
$\frac{dp}{dr} = -(\rho+p)\left[\frac{m(r) + r^3 p 4\pi}{r[r-2m(r)]}\right]$
$= -\frac{dp}{dr}$
$\rho(r) = \begin{cases} \rho^* & \text{if } r \leq R \\ 0 & \text{if } r \geq R \end{cases}$

$$p(r) = \rho^* \left[\frac{R\sqrt{R-2M'} - \sqrt{R^3-2Mr^2}}{\sqrt{R^3-2Mr^2} - 3R\sqrt{R-2M'}} \right]$$

Keep M const

if $\frac{M}{R} = \frac{4}{9}$ Buchdahl's radius


$\rho(r) = \begin{cases} \rho_0 & \text{if } r \geq R \end{cases}$
$\frac{4}{3}\pi$
r^2
$R - 2M'$
BH
$\frac{2M}{R} = 1$
$R_{BH} = 2M$
$R_{Buchdahl} = \frac{9}{4}M.$
dahl's radius

$$\rho(r) = \begin{cases} \rho^* & \text{if } r < R \\ 0 & \text{if } r \geq R \end{cases} \qquad m(r) = \begin{cases} \frac{4}{3}\pi r^3 \rho^* & r < R \\ \frac{4}{3}\pi R^3 \rho^* \equiv M & r \geq R \end{cases}$$

$$\frac{2M}{R} = 1 \qquad R_{BH} = 2M$$

$$R_{Buchdahl} = \frac{9}{4} M$$

$$M_{NS} \simeq 2\text{-}3\, M_\odot$$

$$R_{NS} \simeq 10\, M_\odot$$

$\rho(r) = \begin{cases} \\ 0 \quad \text{if } r \geq R \end{cases}$

$-2Mr^2$

$R\sqrt{R-2M}$

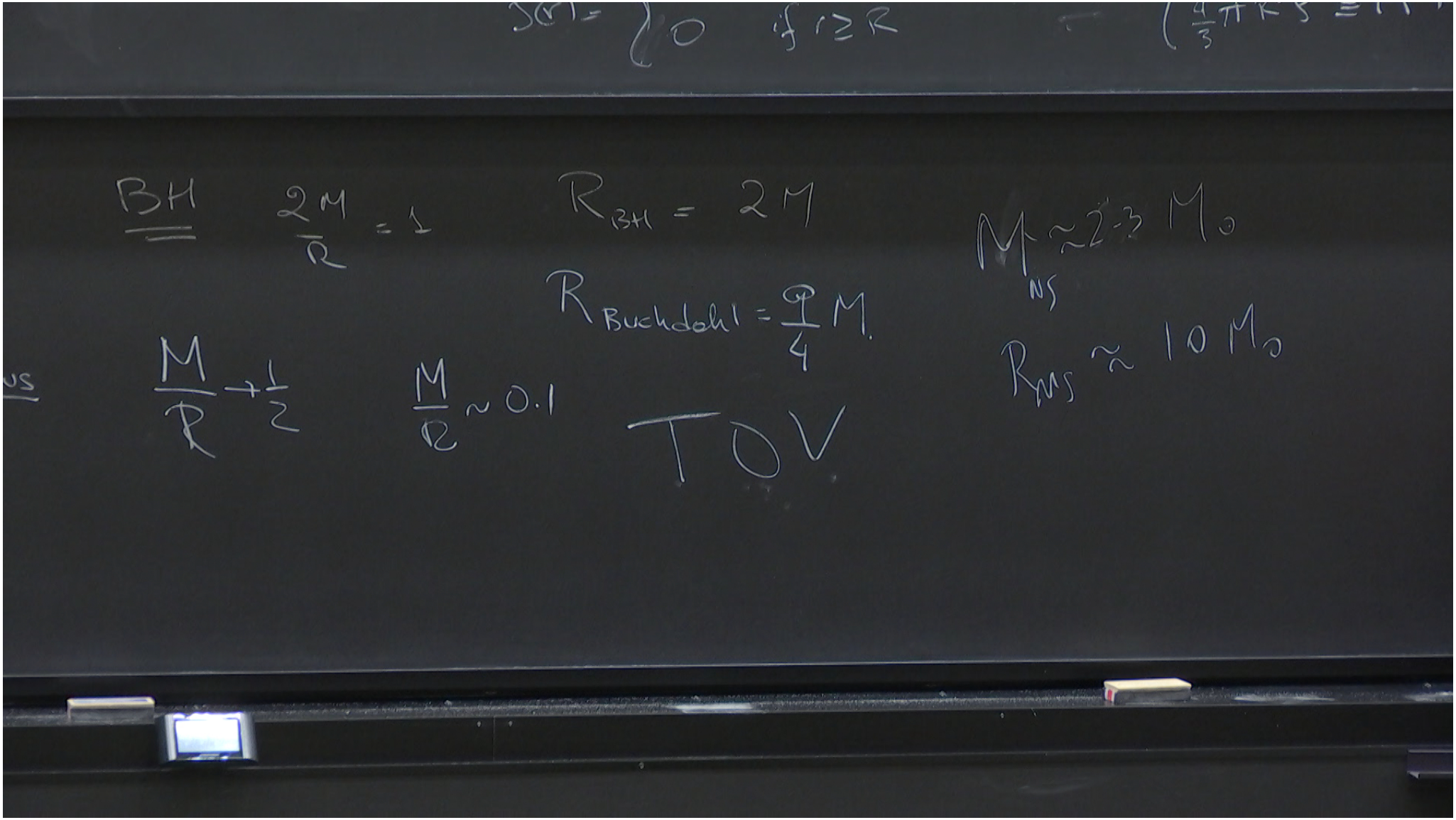
BH $\quad \frac{2M}{R} = 1 \qquad R_{BH} = 2M$

$R_{Buchdahl} = \frac{9}{4}M$

$M_{NS} \sim 2$

Buchdahl's radius $\quad \frac{M}{R} \to \frac{1}{2} \qquad \frac{M}{R} \sim 0.1$

$R_{NS} \sim$


BH
$\frac{2M}{R} = 1$
$R_{BH} = 2M$
$M_{NS} \simeq 2\text{-}3\, M_\odot$
$R_{Buchdahl} = \frac{9}{4} M$
$R_{NS} \simeq 10\, M_\odot$
$\frac{M}{R} \to \frac{1}{2}$
$\frac{M}{R} \sim 0.1$
TOV