Title: Relativity - Lecture 221026

Speakers:

Collection: Relativity (2022/2023)

Date: October 26, 2022 - 9:00 AM

URL: https://pirsa.org/22100085

$$p^a \nabla_a p_b = m \frac{dx^a}{d\tau} \nabla_a p_b - \frac{1}{2} (\partial_b g_{cd}) p^c p^d = 0$$

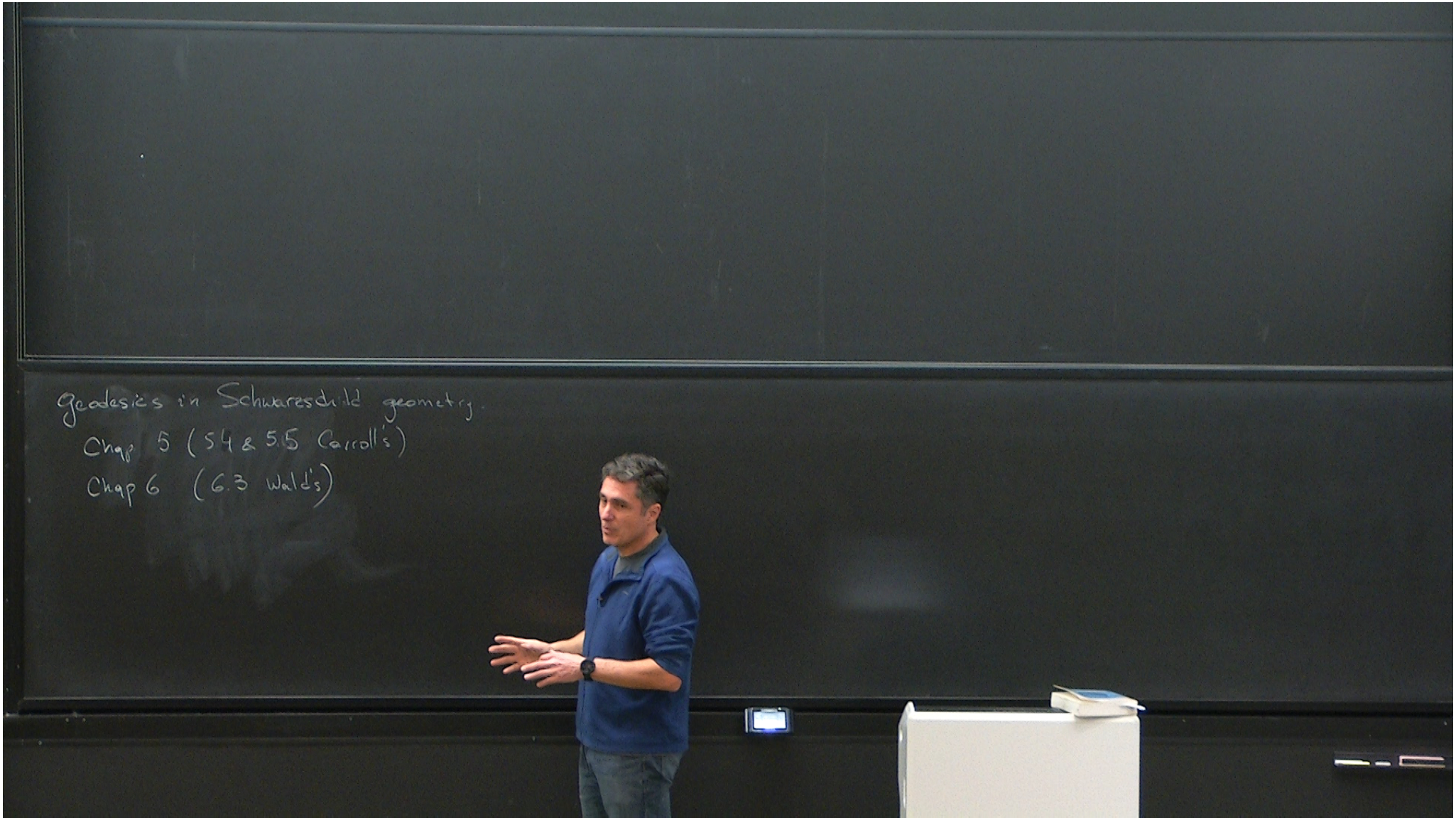
$$m \frac{dp_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) p^c p^d = 0$$

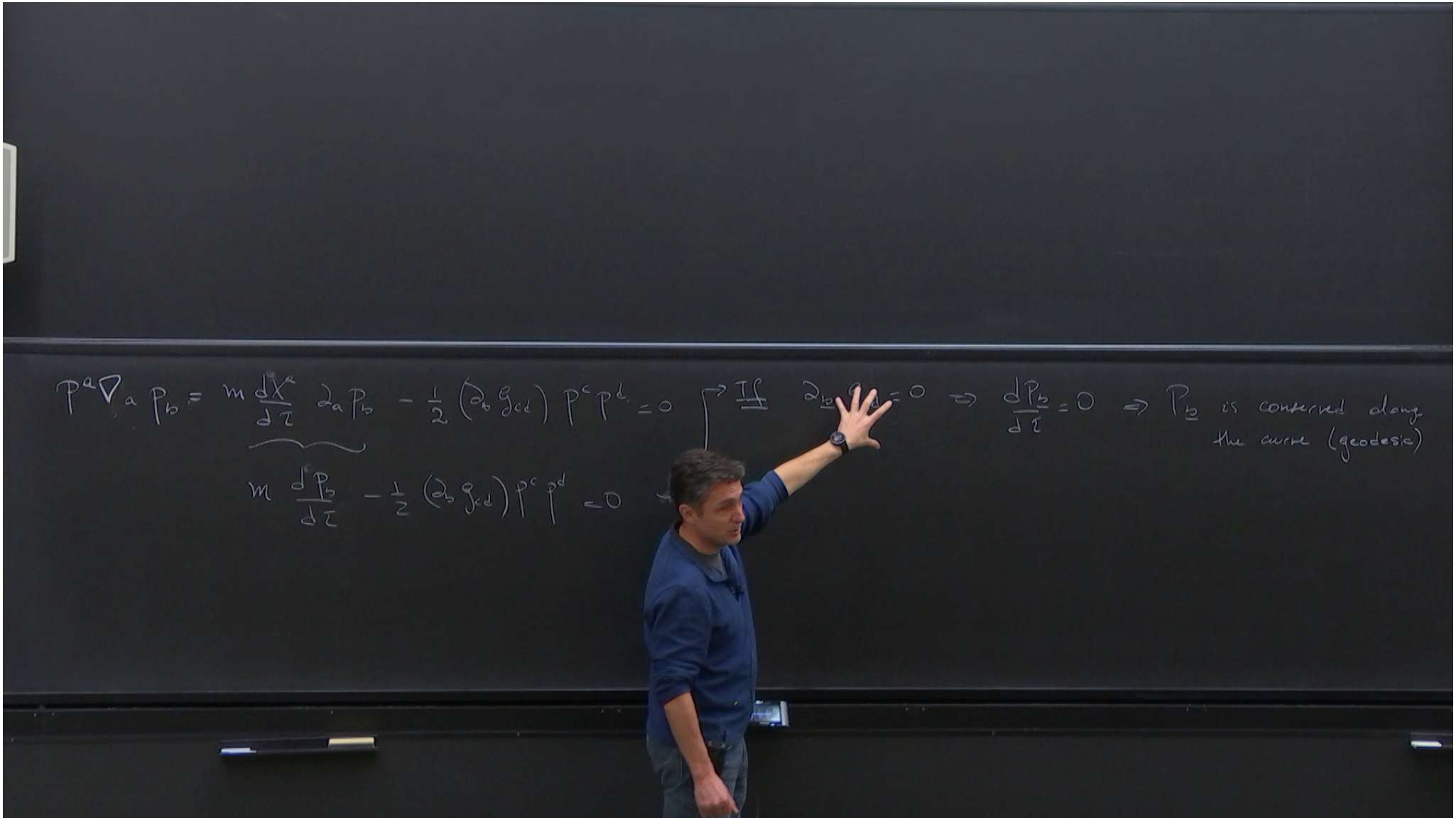
If $\partial_b g_{cd} = 0 \Rightarrow \frac{dp_b}{d\tau} = 0 \Rightarrow p_b$ is conserved along the curve (geodesic)

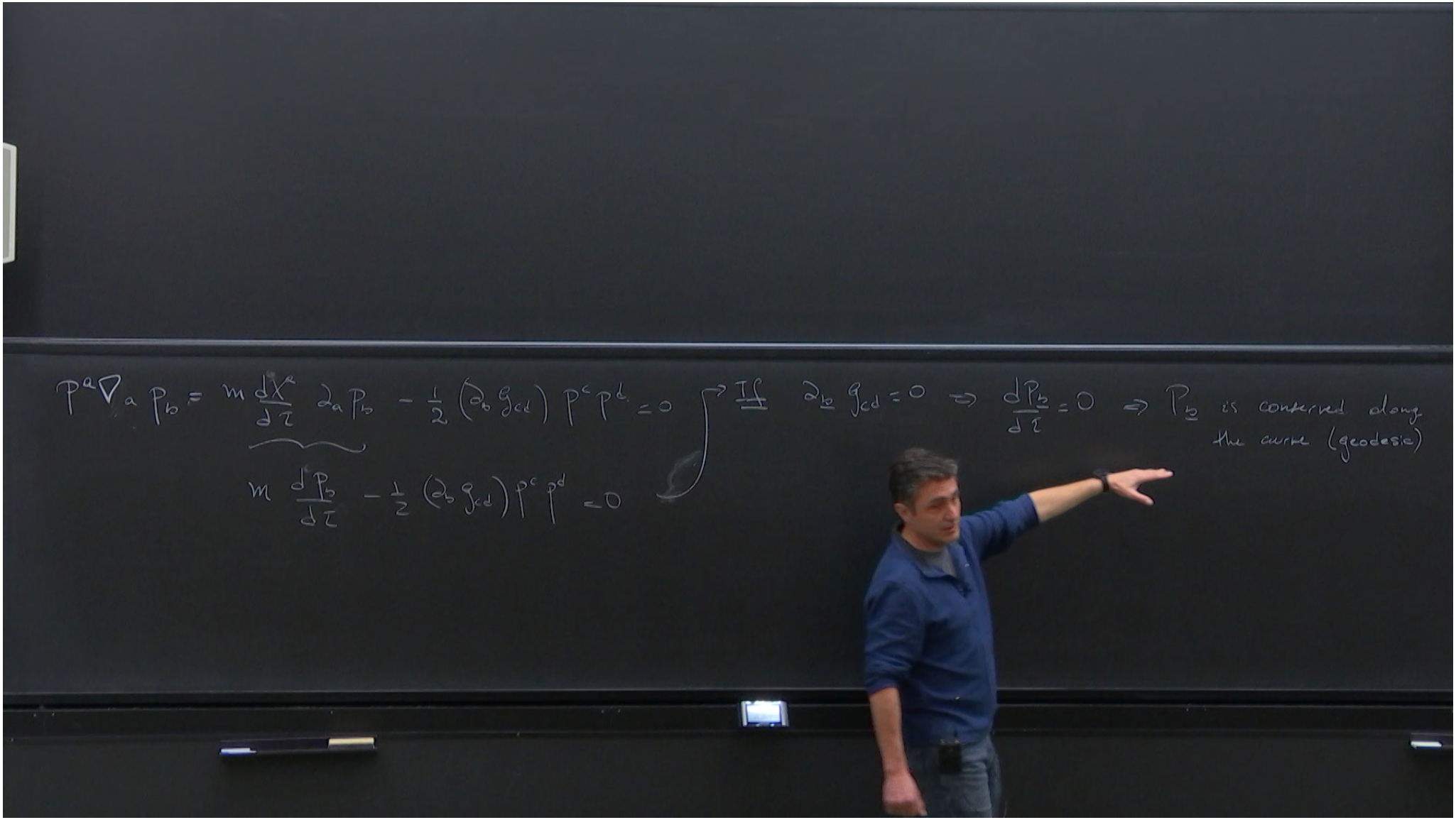
Geodesics in Schwarzschild geometry.

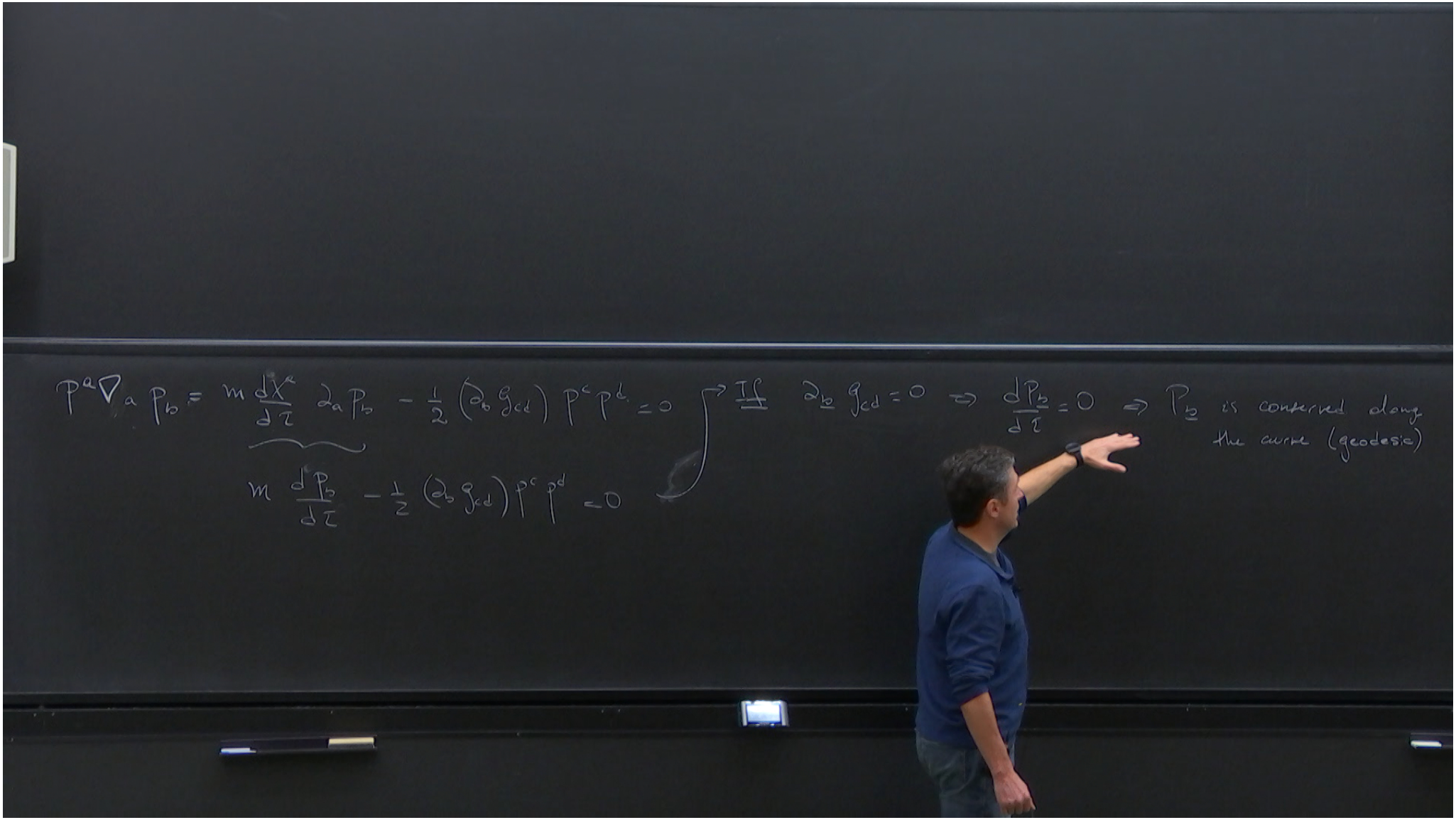
Chap. 5 (5.4 & 5.4 Carroll's)

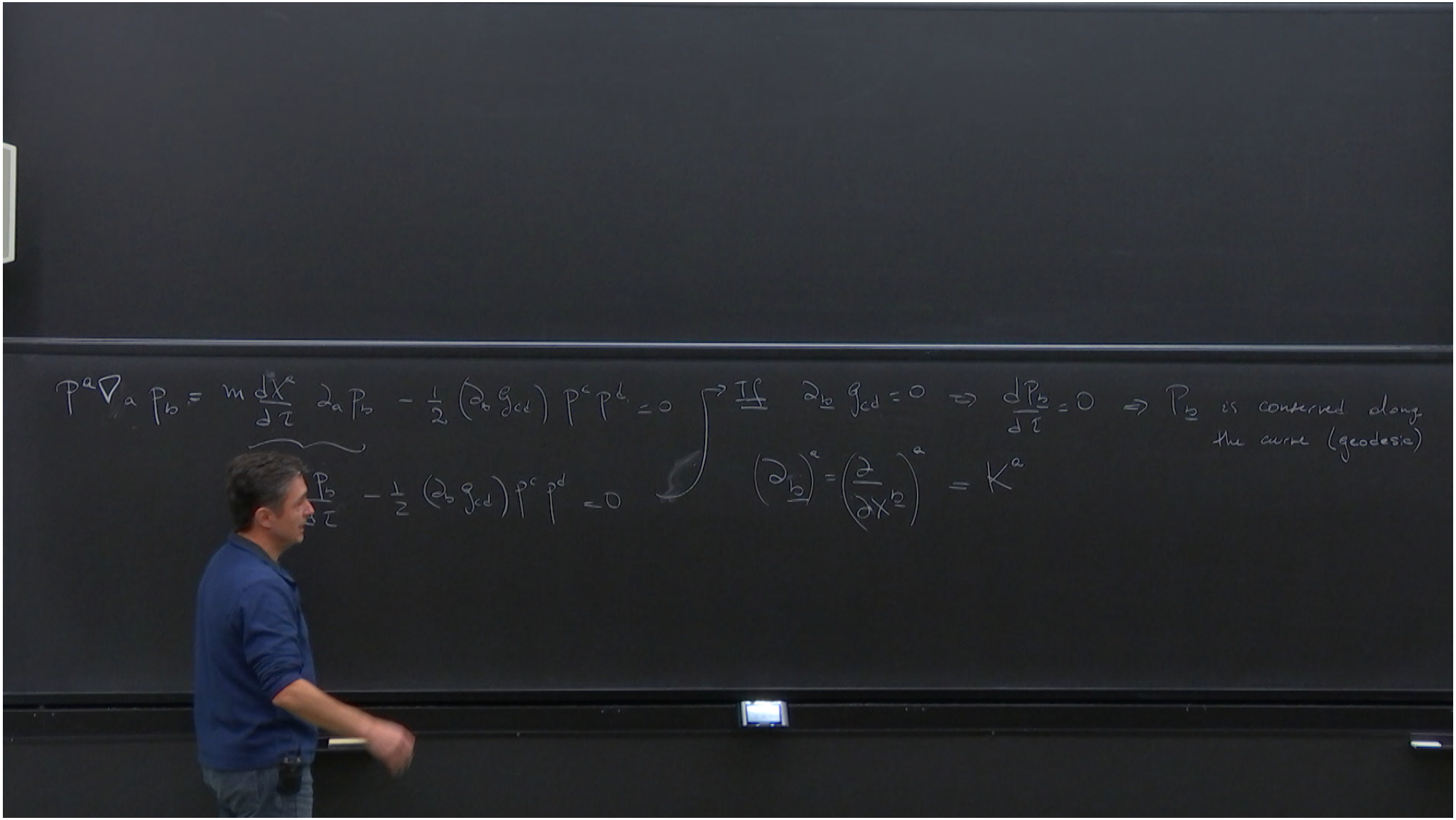
Chap 6 (6.3 Wald's)

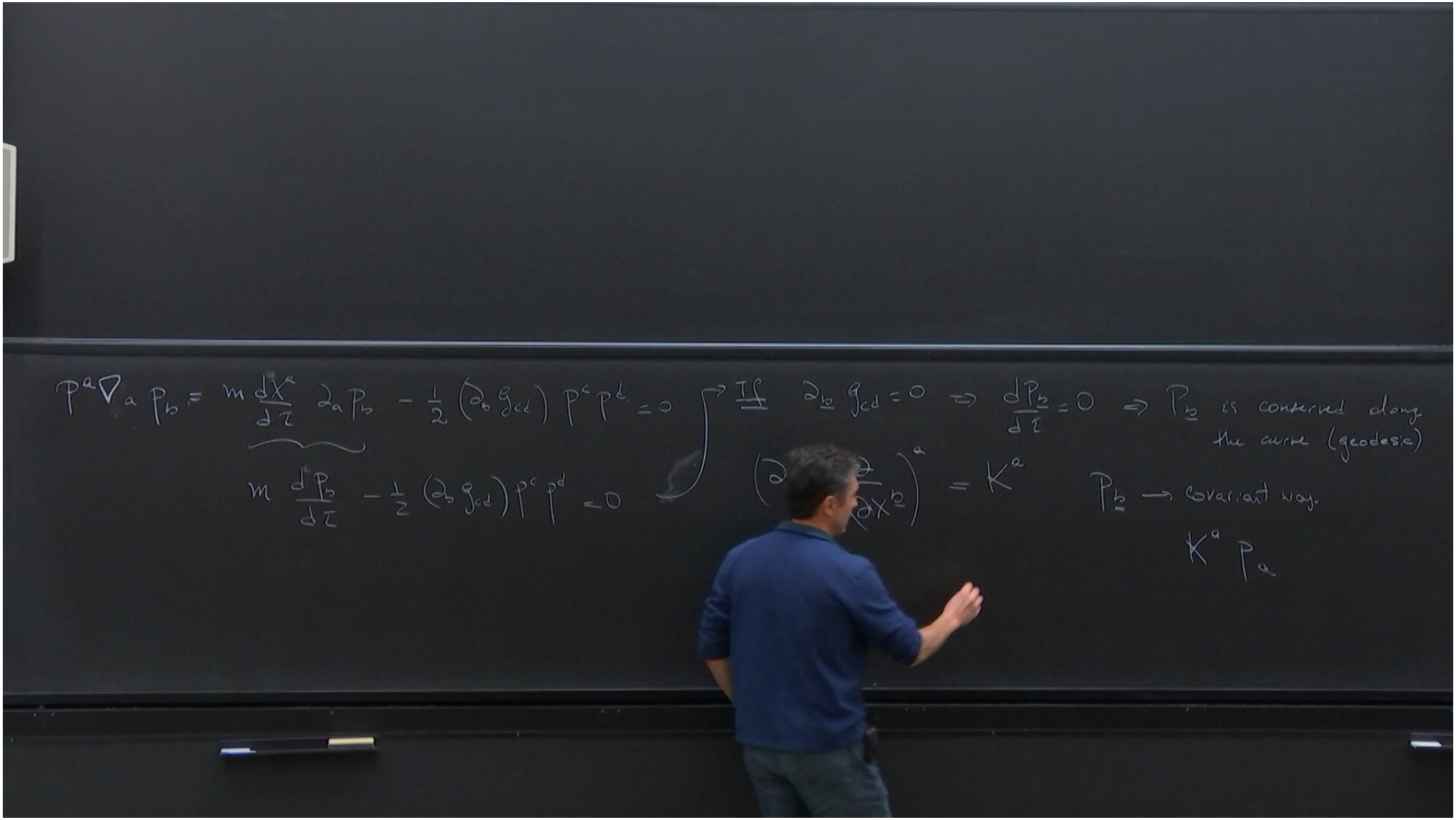

Geodesics in Schwarzschild geometry.
Chap 5 (5.4 & 5.5 Carroll's)
Chap 6 (6.3 Wald's)

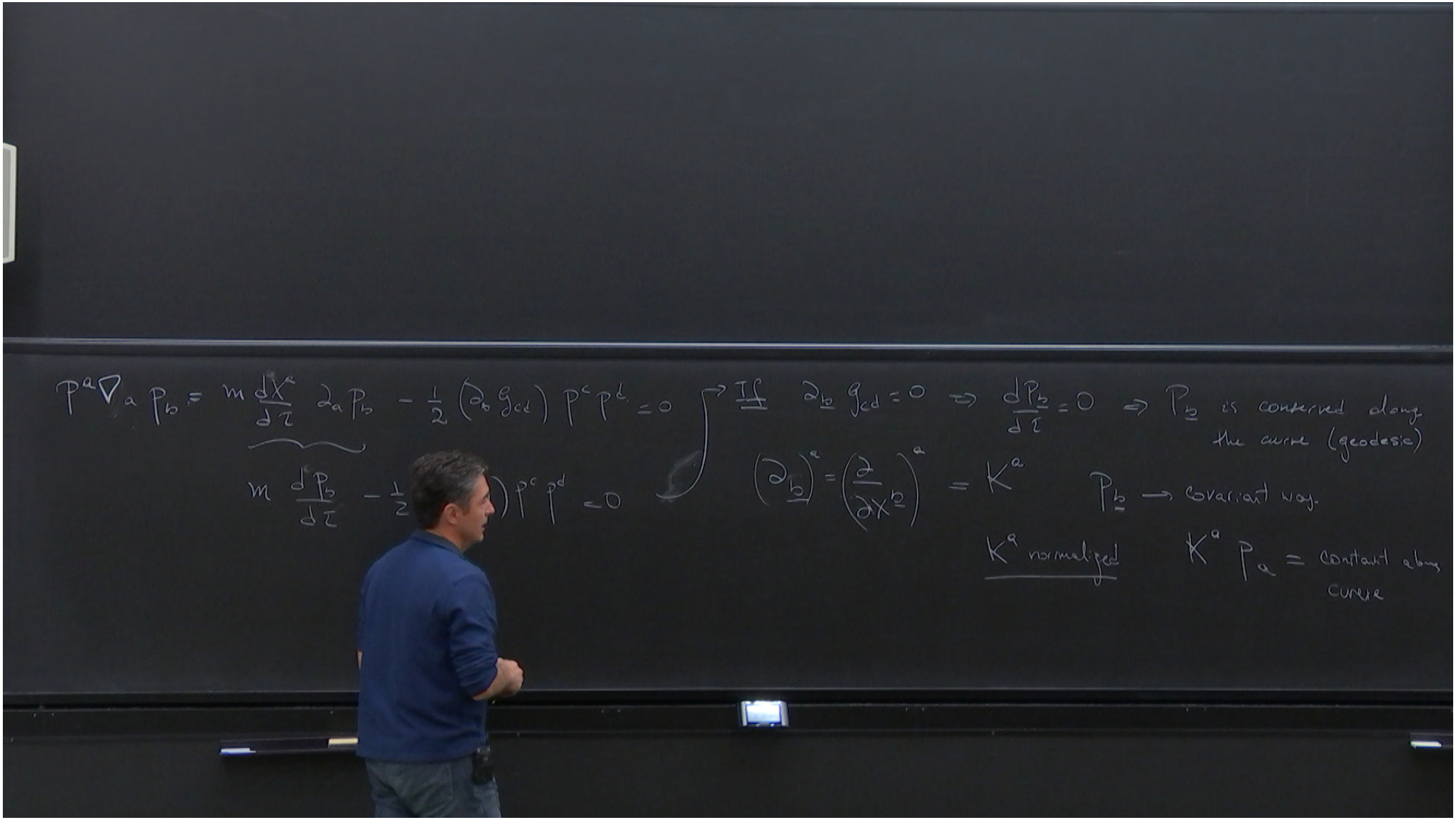

$P^a \nabla_a P_b = m \frac{dx^a}{d\tau} \partial_a P_b - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
If $\partial_b g_{cd} = 0 \Rightarrow \frac{dP_b}{d\tau} = 0 \Rightarrow P_b$ is conserved along the curve (geodesic)
$m \frac{dP_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$

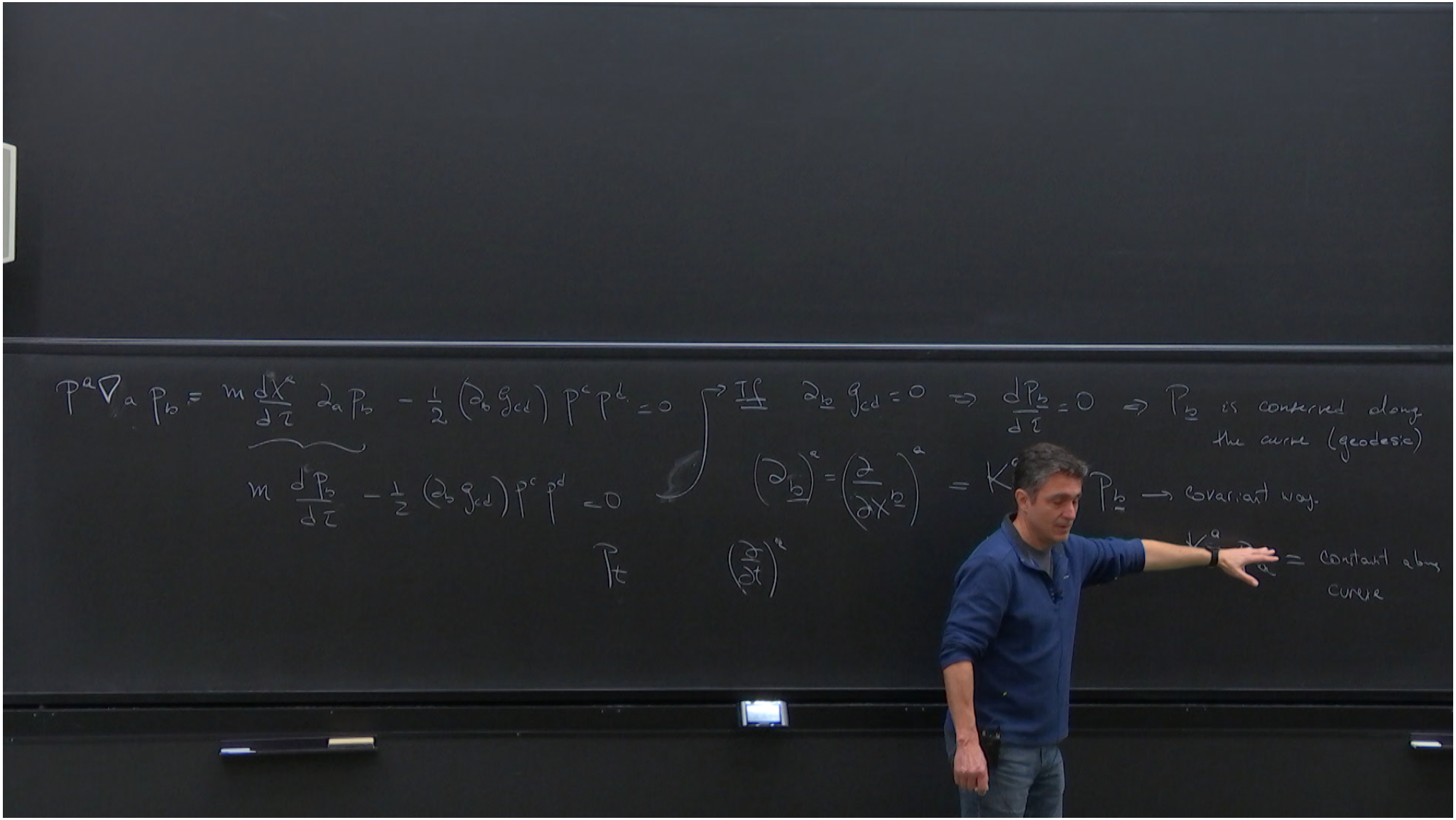

$P^a \nabla_a P_b = m \frac{dx^a}{d\tau} \partial_a P_b - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
$m \frac{dP_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
If $\partial_b g_{cd} = 0 \Rightarrow \frac{dP_b}{d\tau} = 0 \Rightarrow P_b$ is conserved along the curve (geodesic)


$P^a \nabla_a P_b = m \frac{dx^c}{d\tau} \partial_a P_b - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
$m \frac{dP_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
If $\partial_b g_{cd} = 0 \Rightarrow \frac{dP_b}{d\tau} = 0 \Rightarrow P_b$ is conserved along the curve (geodesic)


$P^a \nabla_a P_b = m \frac{dx^a}{d\tau} \partial_a P_b - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
$\frac{dP_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
If $\partial_{\underline{b}} g_{cd} = 0 \Rightarrow \frac{dP_{\underline{b}}}{d\tau} = 0 \Rightarrow P_{\underline{b}}$ is conserved along the curve (geodesic)
$(\partial_{\underline{b}})^a = \left(\frac{\partial}{\partial x^{\underline{b}}}\right)^a = K^a$


$P^a \nabla_a P_b = m \frac{dx^a}{d\tau} \partial_a P_b - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
If $\partial_b g_{cd} = 0 \Rightarrow \frac{dP_b}{d\tau} = 0 \Rightarrow P_b$ is conserved along the curve (geodesic)
$m \frac{dP_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
$\left(\partial \frac{\partial}{\partial x^b} \right)^a = K^a$
$P_b \rightarrow$ covariant way.
$K^a P_a$


$P^a \nabla_a P_b = m \frac{dx^a}{d\tau} \partial_a P_b - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
If $\partial_b g_{cd} = 0 \Rightarrow \frac{dP_b}{d\tau} = 0 \Rightarrow P_b$ is conserved along the curve (geodesic)
$m \frac{dP_b}{d\tau} - \frac{1}{2}) P^c P^d = 0$
$(\partial_b)^a = \left(\frac{\partial}{\partial x^b}\right)^a = K^a$
$P_b \rightarrow$ covariant way.
K^a normalized
$K^a P_a =$ Constant along curve


$P^a \nabla_a P_b = m \frac{dx^c}{d\tau} \partial_a P_b - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
$m \frac{dP_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) P^c P^d = 0$
If $\partial_b g_{cd} = 0 \Rightarrow \frac{dP_b}{d\tau} = 0 \Rightarrow P_b$ is conserved along the curve (geodesic)
$(\partial_b)^a = \left(\frac{\partial}{\partial x^b}\right)^a = K^a$
P_b → covariant way.
P_t
$\left(\frac{\partial}{\partial t}\right)^a$
$K^a P_a$ = Constant along curve

$$p^a \nabla_a p_b = m \frac{dx^a}{d\tau} \partial_a p_b - \frac{1}{2} (\partial_b g_{cd}) p^c p^d = 0$$

$$m \frac{dp_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) p^c p^d = 0$$

If $\partial_b g_{cd} = 0 \Rightarrow \frac{dp_b}{d\tau} = 0 \Rightarrow p_b$ is conserved along the curve (geodesic)

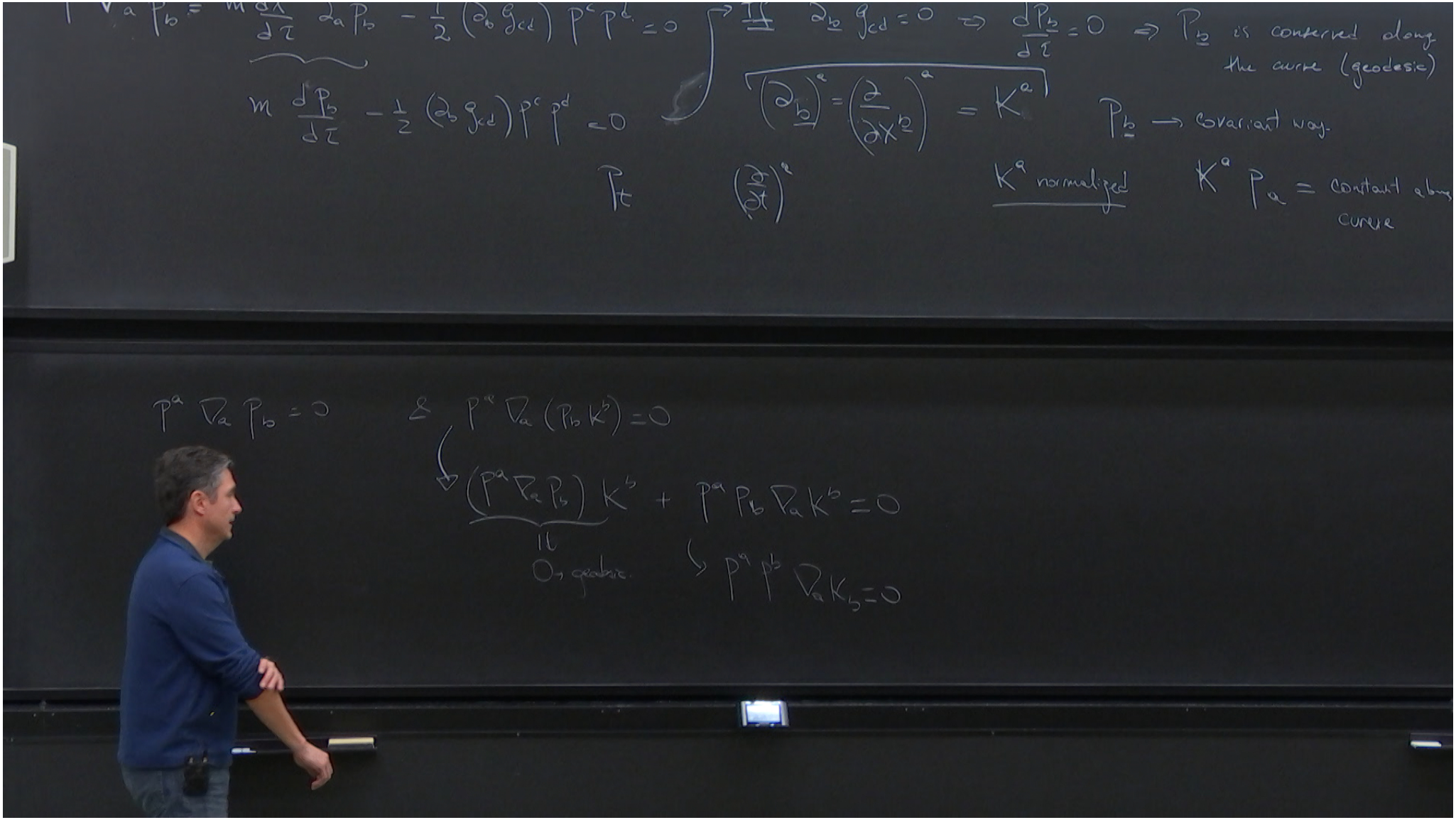
$\left(\partial_b\right)^a = \left(\frac{\partial}{\partial x^b}\right)^a = K^a$

$p_b \rightarrow$ covariant way.

$p_t \quad \left(\frac{\partial}{\partial t}\right)^a$

K^a normalized $\quad K^a p_a =$ constant along curve

$$p^a \nabla_a p_b = 0 \quad \& \quad p^a \nabla_a (p_b k^b) = 0$$


$P^a \nabla_a P_b = m \frac{dx}{d\tau} \partial_a P_b - \frac{1}{2}(\partial_b g_{cd}) P^c P^d = 0$
$\partial_b g_{cd} = 0 \Rightarrow \frac{dP_b}{d\tau} = 0 \Rightarrow P_b$ is conserved along the curve (geodesic)
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$(\partial_b)^a = \left(\frac{\partial}{\partial x^b}\right)^a = K^a$
$P_b \rightarrow$ covariant way.
$P_t \quad \left(\frac{\partial}{\partial t}\right)^a$
K^a normalized $K^a P_a =$ constant along curve
$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
0, geodesic. $\hookrightarrow P^a P^b \nabla_a K_b = 0$

$$p^a \nabla_a p_b = m \frac{dx^a}{d\tau} \partial_a p_b - \frac{1}{2} (\partial_b g_{cd}) p^c p^d = 0$$

$$m \frac{dp_b}{d\tau} - \frac{1}{2} (\partial_b g_{cd}) p^c p^d = 0$$

p_t $\left(\frac{\partial}{\partial t}\right)^a$

$\partial_b g_{cd} = 0 \Rightarrow \frac{dp_b}{d\tau} = 0 \Rightarrow p_b$ is conserved along the curve (geodesic)

$(\partial_b)^a = \left(\frac{\partial}{\partial x^b}\right)^a = K^a$

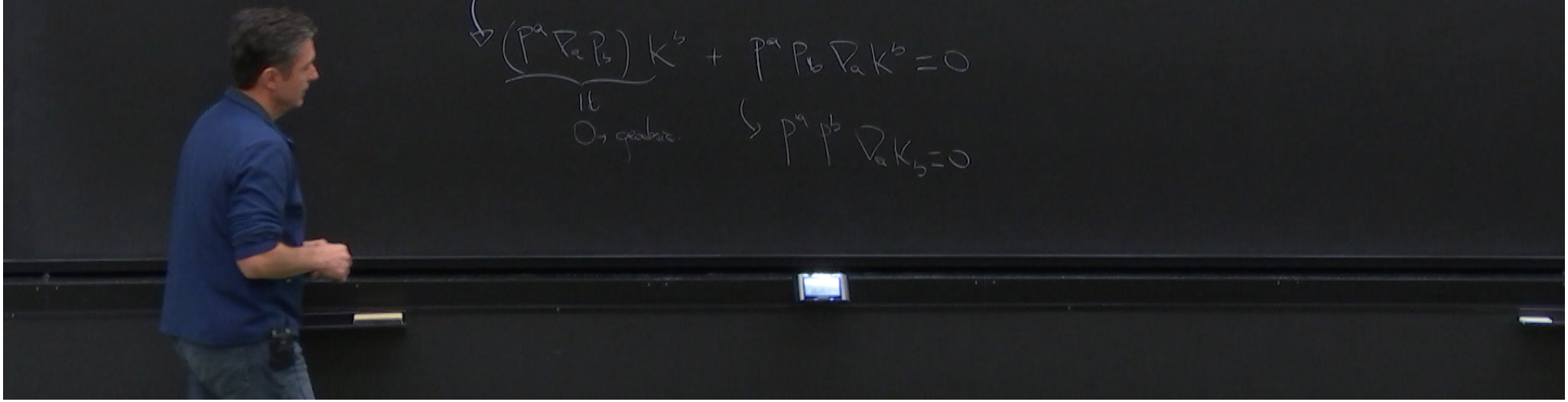
$p_b \rightarrow$ covariant way.

K^a normalized $K^a p_a =$ constant along curve

$p^a \nabla_a p_b = 0$ & $p^a \nabla_a (p_b K^b) = 0$

$(p^a \nabla_a p_b) K^b + p^a p_b \nabla_a K^b = 0$

$0 \rightarrow$ geodesic $\hookrightarrow p^a p^b \nabla_a K_b = 0$



$$m\frac{dp_b}{d\tau} - \frac{1}{2}(\partial_b g_{cd})p^c p^d = 0$$

$$(\partial_b)^a = \left(\frac{\partial}{\partial x^b}\right)^a = K^a$$

$p_b \rightarrow$ covariant va

$\tau_t \quad \left(\frac{\partial}{\partial t}\right)^a$

K^a normalized $\quad K^a p_a =$

$$p^a \nabla_a p_b = 0 \quad \& \quad p^a \nabla_a (p_b K^b) = 0$$

$$\underbrace{(p^a \nabla_a p_b)}_{0 \text{ geodesic.}} K^b + p^a p_b \nabla_a K^b = 0$$

$$\hookrightarrow p^a p^b \nabla_a K_b = 0 \Rightarrow \boxed{\nabla_{(a} K_{b)} = 0}$$



$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$

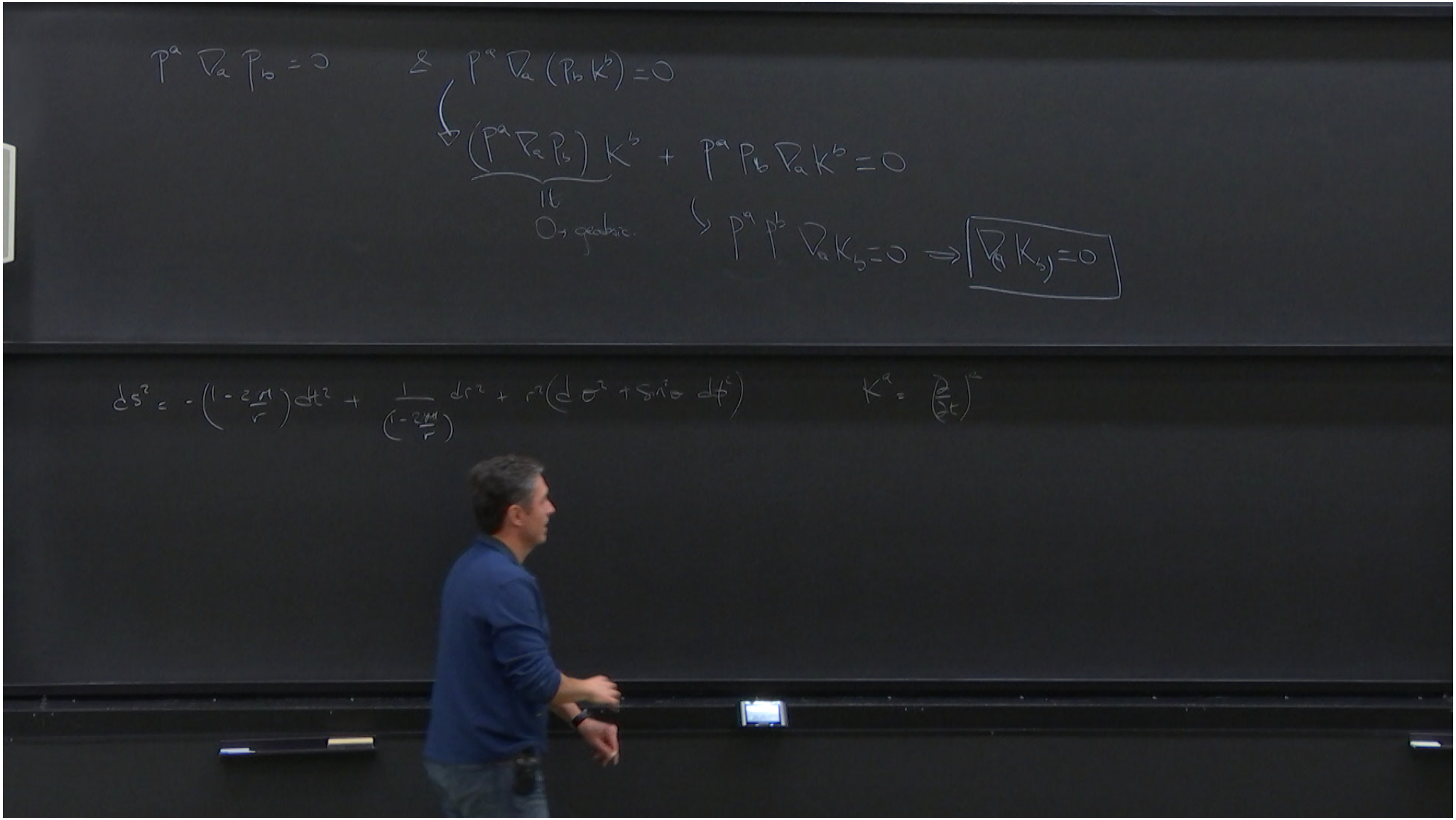
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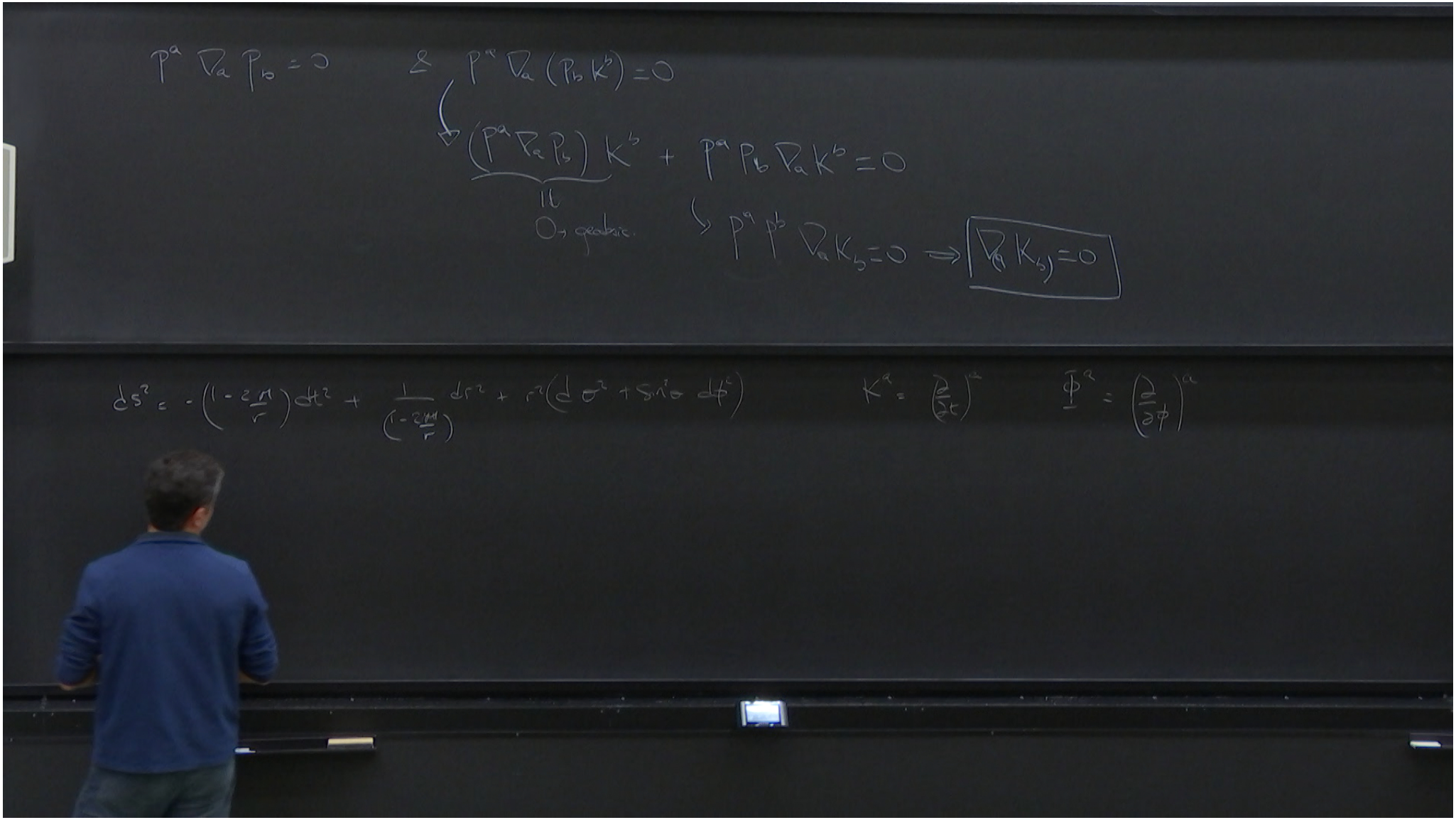
$\underbrace{(P^a \nabla_a P_b)}_{\text{0 geodesic.}} K^b + P^a P_b \nabla_a K^b = 0$

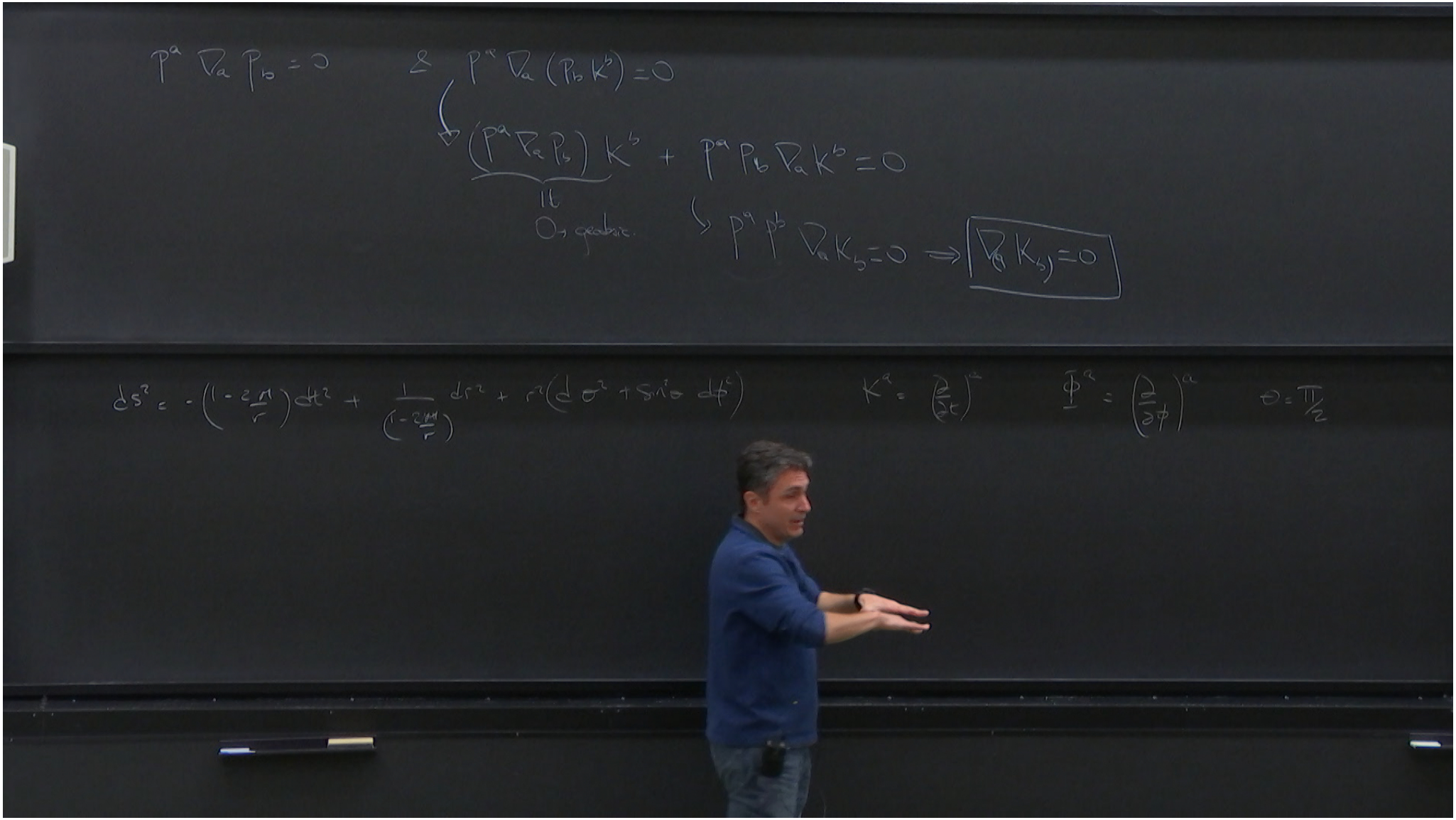
$\hookrightarrow P^a P^b \nabla_a K_b = 0 \Rightarrow \boxed{\nabla_{(a} K_{b)} = 0}$

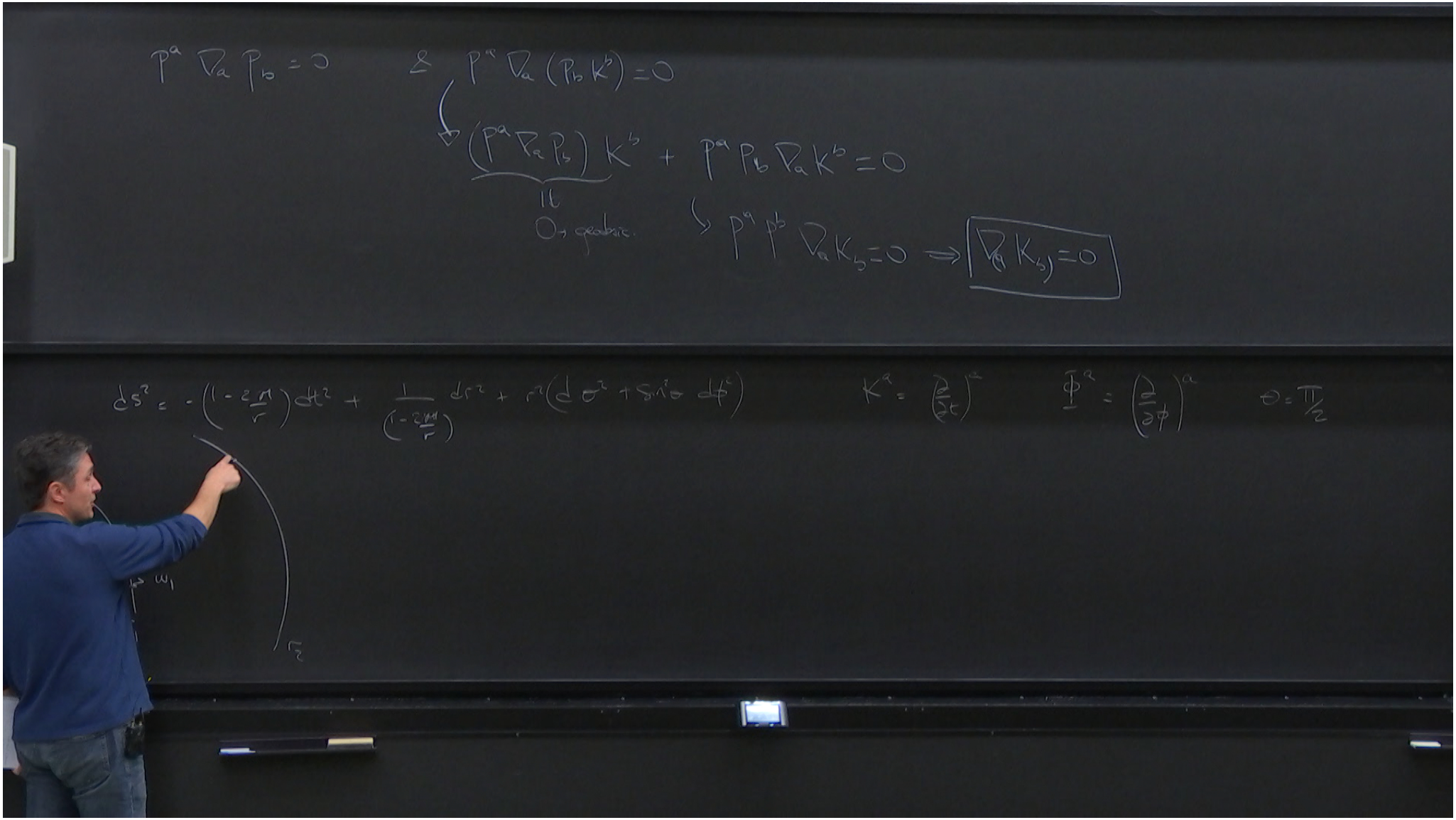
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$




$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
0 by geodesic.
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\sigma^2 + \sin^2\sigma \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$


$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
$\to$ geodesic.
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$


$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
0 → geodesic
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$


$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
0 → geodesic.
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
w_1
r_2

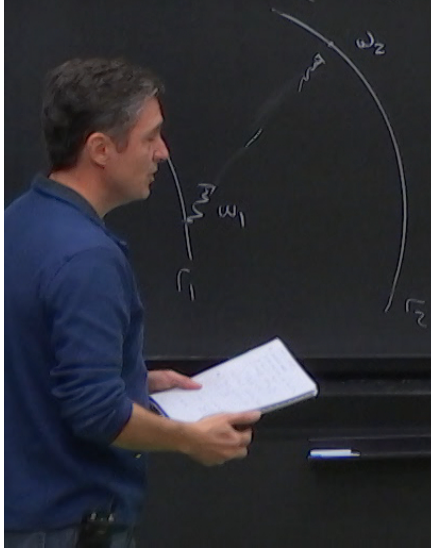
$P^a \nabla_a P_b = 0 \quad \& \quad P^a \nabla_a (P_b K^b) = 0$

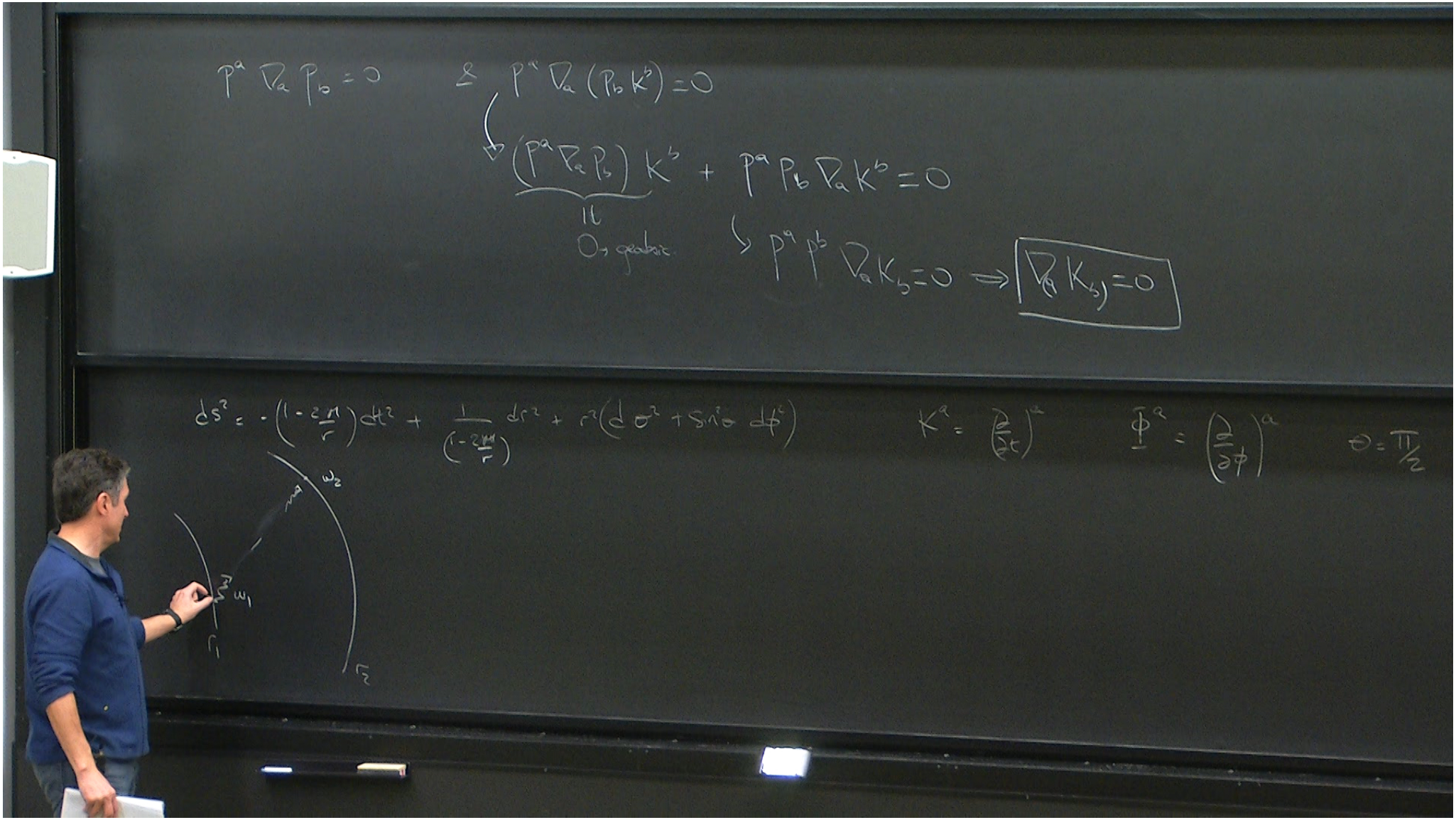
$\Rightarrow \underbrace{(P^a \nabla_a P_b)}_{\text{0 geodesic.}} K^b + P^a P_b \nabla_a K^b = 0$

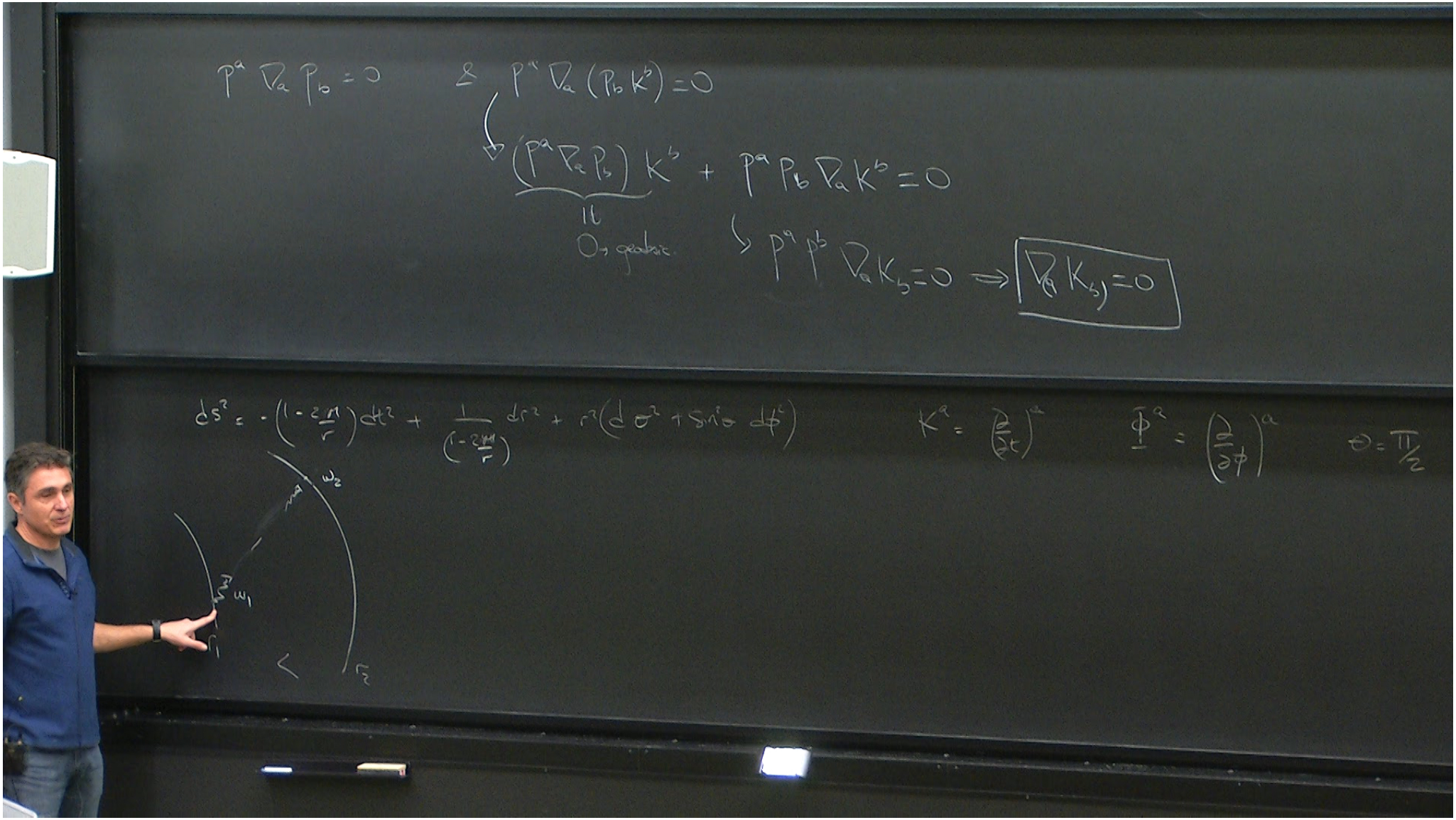
$\hookrightarrow P^a P^b \nabla_a K_b = 0 \Rightarrow \boxed{\nabla_{(a} K_{b)} = 0}$

$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\sigma^2 + \sin^2\theta \, d\phi^2)$

$K^a = \left(\frac{\partial}{\partial t}\right)^a \qquad \Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \qquad \theta = \pi/2$




$P^a \nabla_a P_b = 0$ &
$P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
$= 0$, geodésica
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
ω_2
ω_1
r_1
r_2


$P^a \nabla_a P_b = 0 \quad \& \quad P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
$0 \Rightarrow$ geodesic
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
ω_2
ω_1
r_1
r_2

$$P^a \nabla_a P_b = 0 \quad \& \quad P^a \nabla_a (P_b K^b) = 0$$

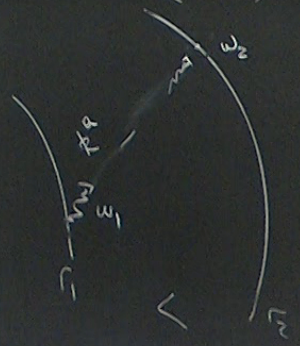
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$$\underbrace{(P^a \nabla_a P_b)}_{\text{0, geodesic.}} K^b + P^a P_b \nabla_a K^b = 0$$

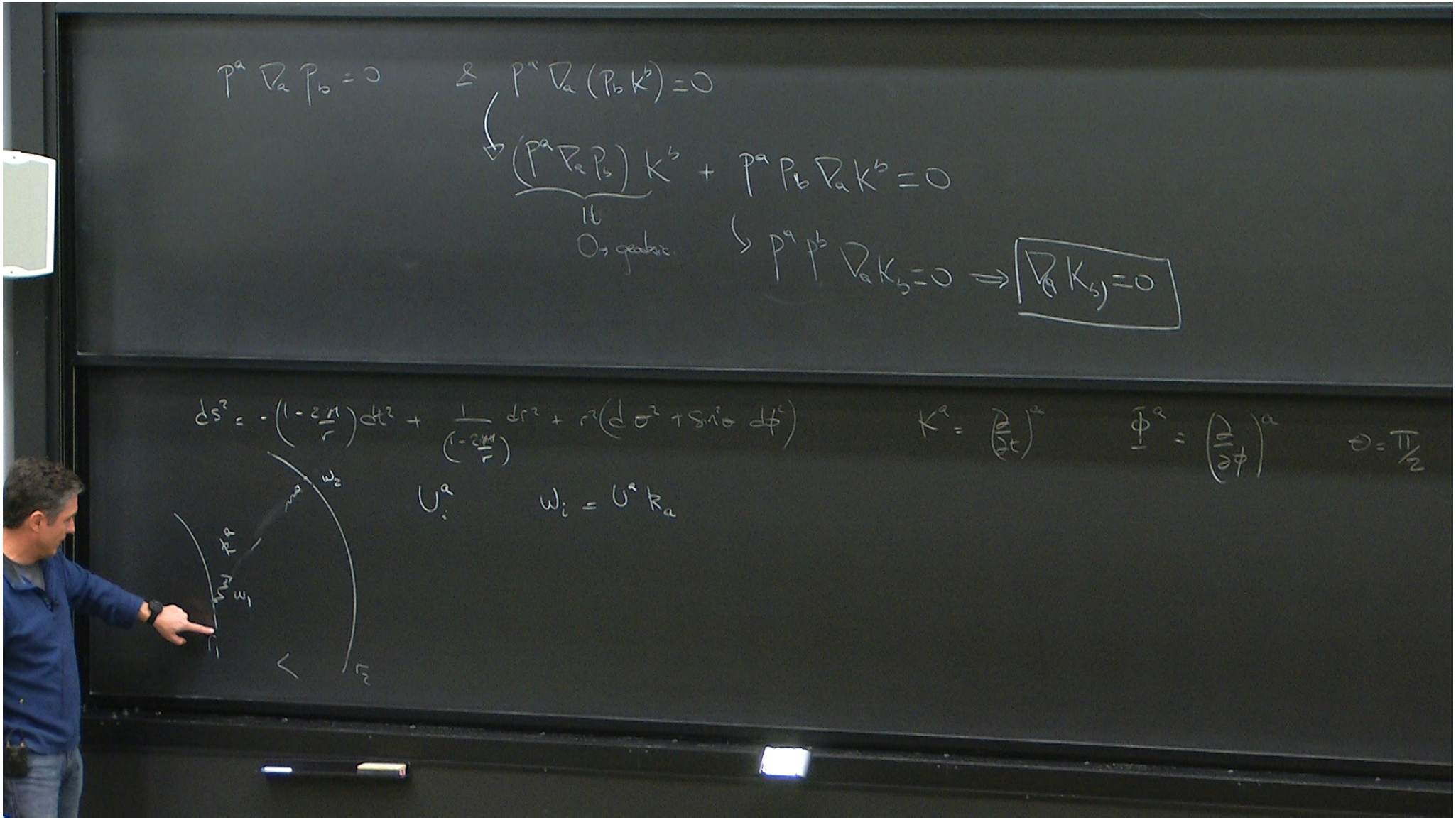
$$\hookrightarrow P^a P^b \nabla_a K_b = 0 \Rightarrow \boxed{\nabla_{(a} K_{b)} = 0}$$

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right)$$

$$K^a = \left(\frac{\partial}{\partial t}\right)^a \qquad \Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \qquad \theta = \pi/2$$



U^a


$P^a \nabla_a P_b = 0 \quad \& \quad P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
0, geodésica
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
U_i^a
$w_i = U^a k_a$
w_2
w_1
r_1
r_2

$$P^a \nabla_a P_b = 0 \quad \& \quad P^a \nabla_a (P_b K^b) = 0$$

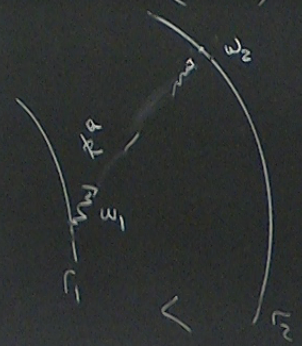
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$$\underbrace{(P^a \nabla_a P_b)}_{\text{0, gradient.}} K^b + P^a P_b \nabla_a K^b = 0$$

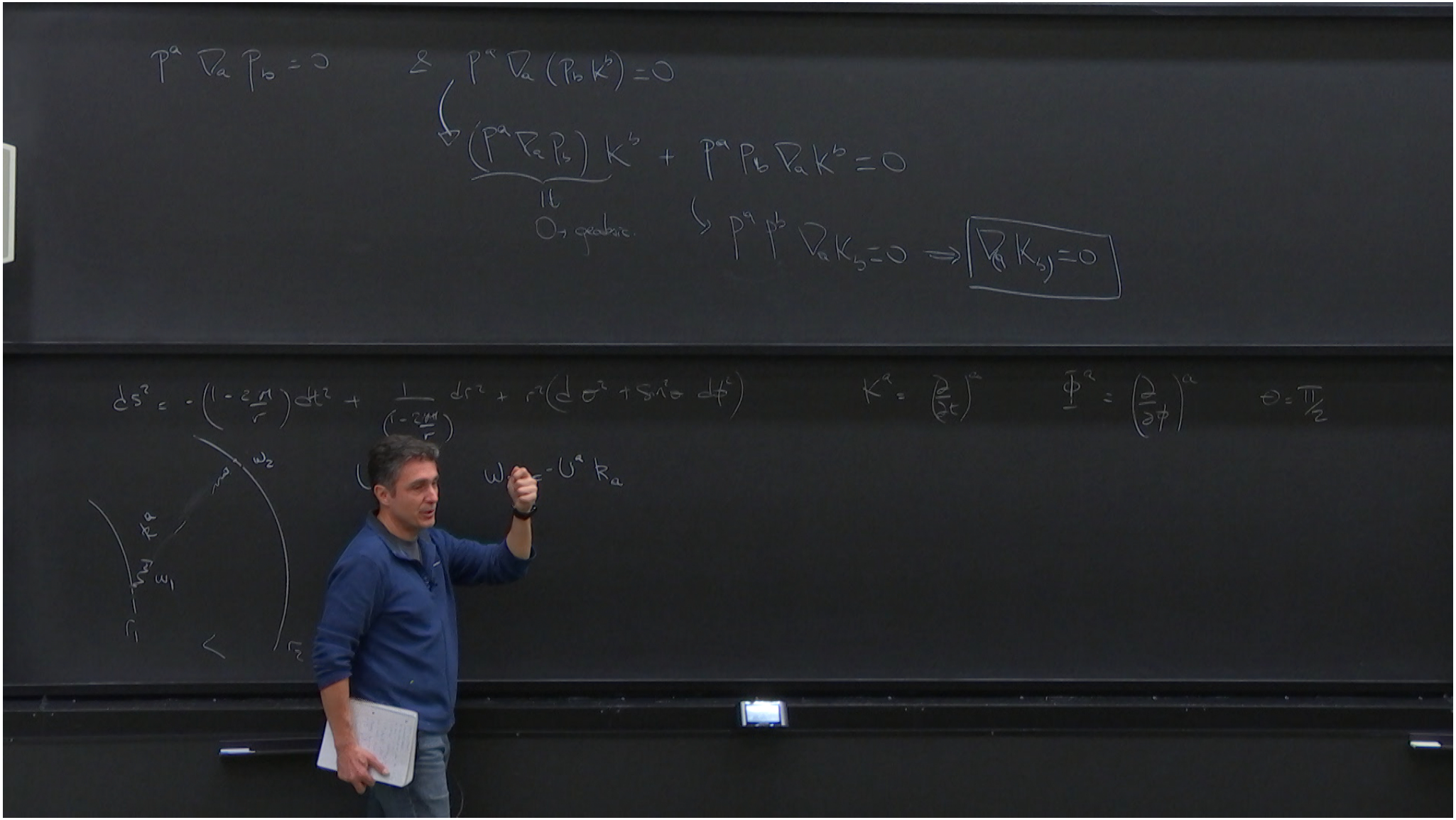
$$\hookrightarrow P^a P^b \nabla_a K_b = 0 \Rightarrow \boxed{\nabla_{(a} K_{b)} = 0}$$

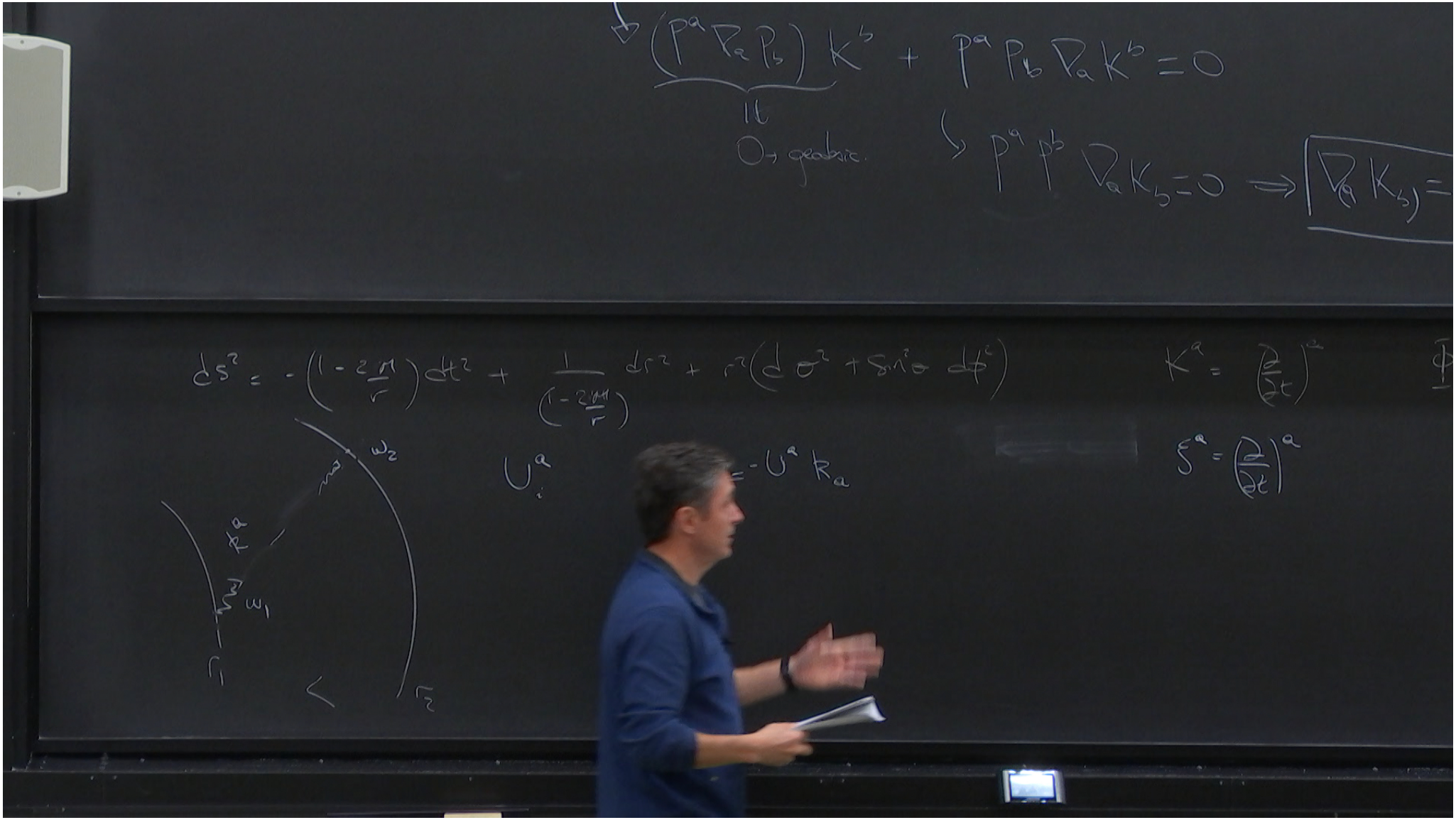
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$$

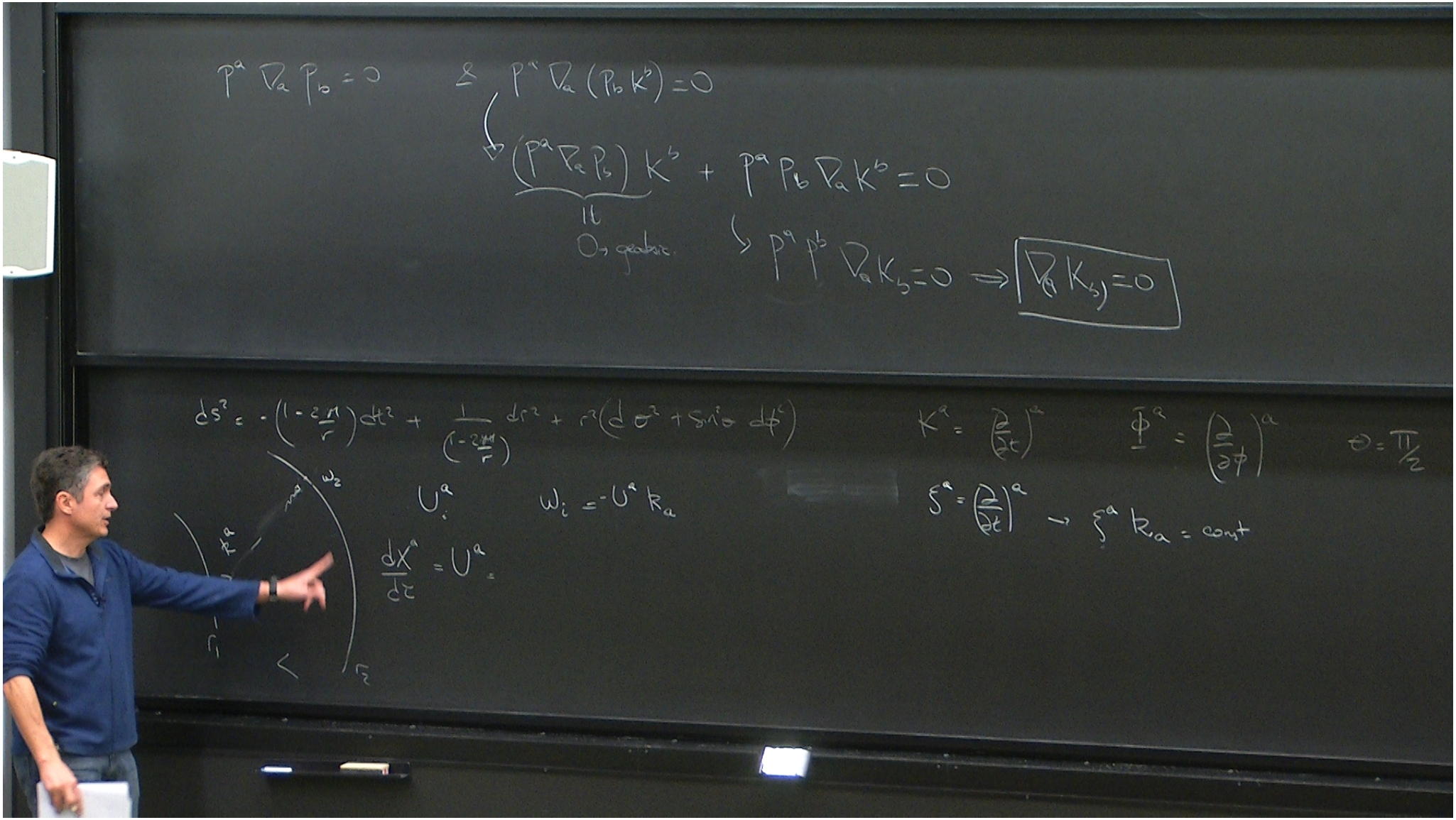
$$K^a = \left(\frac{\partial}{\partial t}\right)^a \qquad \Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \qquad \theta = \pi/2$$

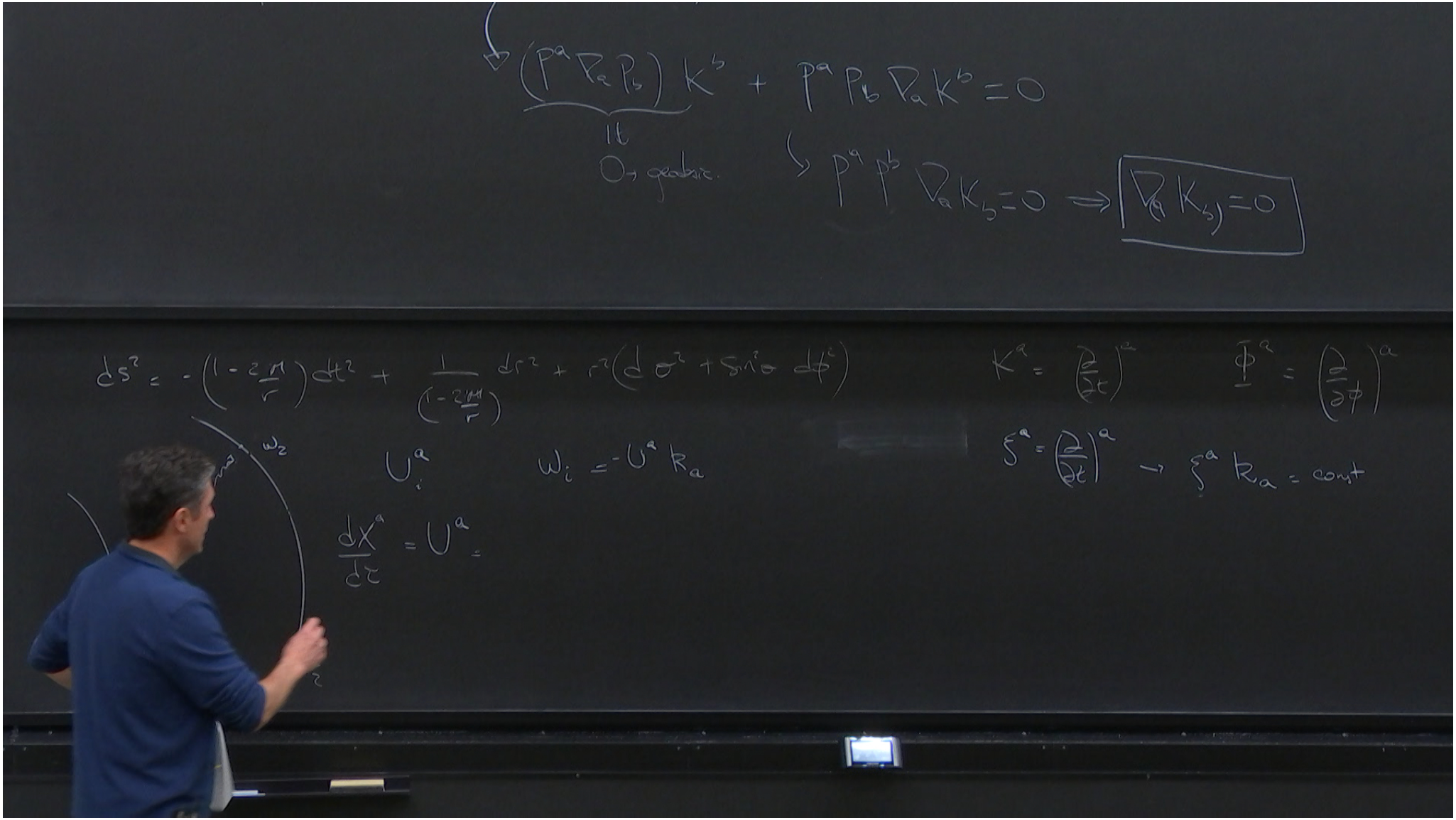


$$U_i^a \qquad \omega_i = U^a k_a$$


$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
0 → geodesic
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\sigma^2 + \sin^2\sigma \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\sigma = \pi/2$
$W = -U^a k_a$
ω_2
R^a
ω_1
r_1
r_2


$(P^a P_a P_b) K^b + P^a P_b P_a K^b = 0$
$\Rightarrow P^a P^b P_a K_b = 0 \Rightarrow \boxed{P_a K_b =}$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a$
U^a
$= -U^a R_a$
ω_2
R^a
ω_1


$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
$= 0$ geodésic.
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
U^a
$W_i = -U^a k_a$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a k_a = \text{cont}$
$\frac{dx^a}{d\tau} = U^a$
k^a
W_2
r_1
r_2


$(P^a P_a P_b) K^b + P^a P_b P_a K^b = 0$
$\Rightarrow$ 0 $\rightarrow$ geodesic.
$P^a P^b P_a K_b = 0 \Rightarrow \boxed{P_a K_b = 0}$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\sigma^2 + \sin^2\sigma\, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
U^a
$W_i = -U^a R_a$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a R_a = \text{const}$
$\frac{dx^a}{d\tau} = U^a$

$$(P^a P_a P_b) K^b + P^a P_b P_a K^b = 0$$

$= 0$ geodesic. $\hookrightarrow P^a P^b P_a K_b = 0 \Rightarrow \boxed{P_a K_b = 0}$

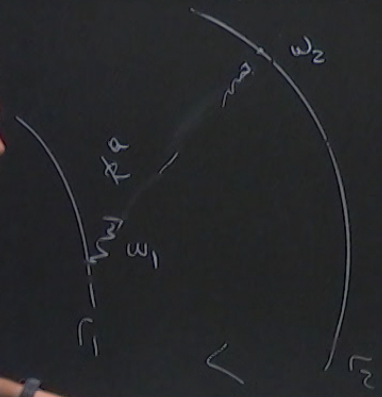
$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\sigma^2 + \sin^2\sigma \, d\phi^2)$$

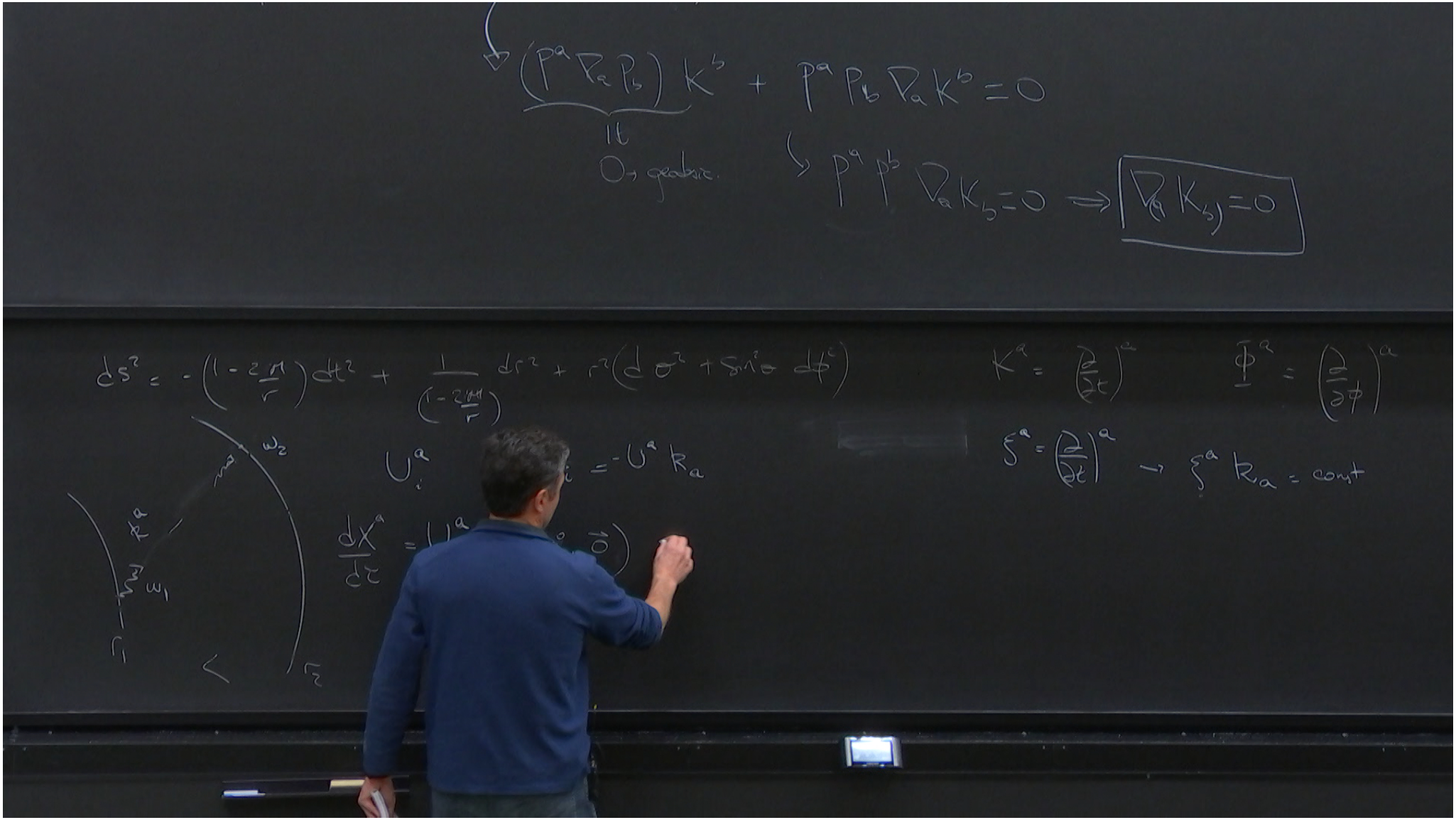
$$K^a = \left(\frac{\partial}{\partial t}\right)^a \qquad \Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$$

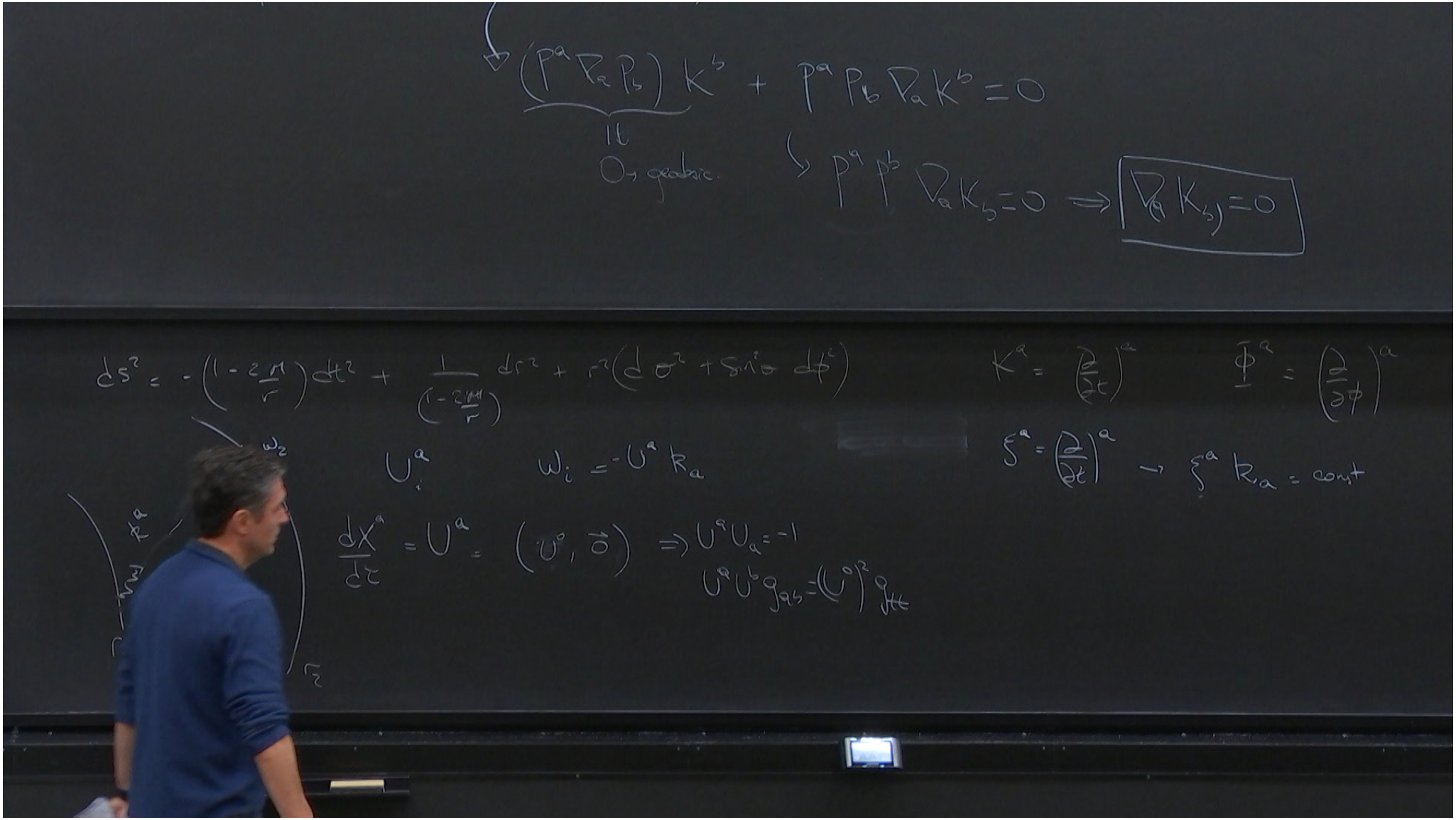
U_i^a $\qquad w_i = -U^a R_a$

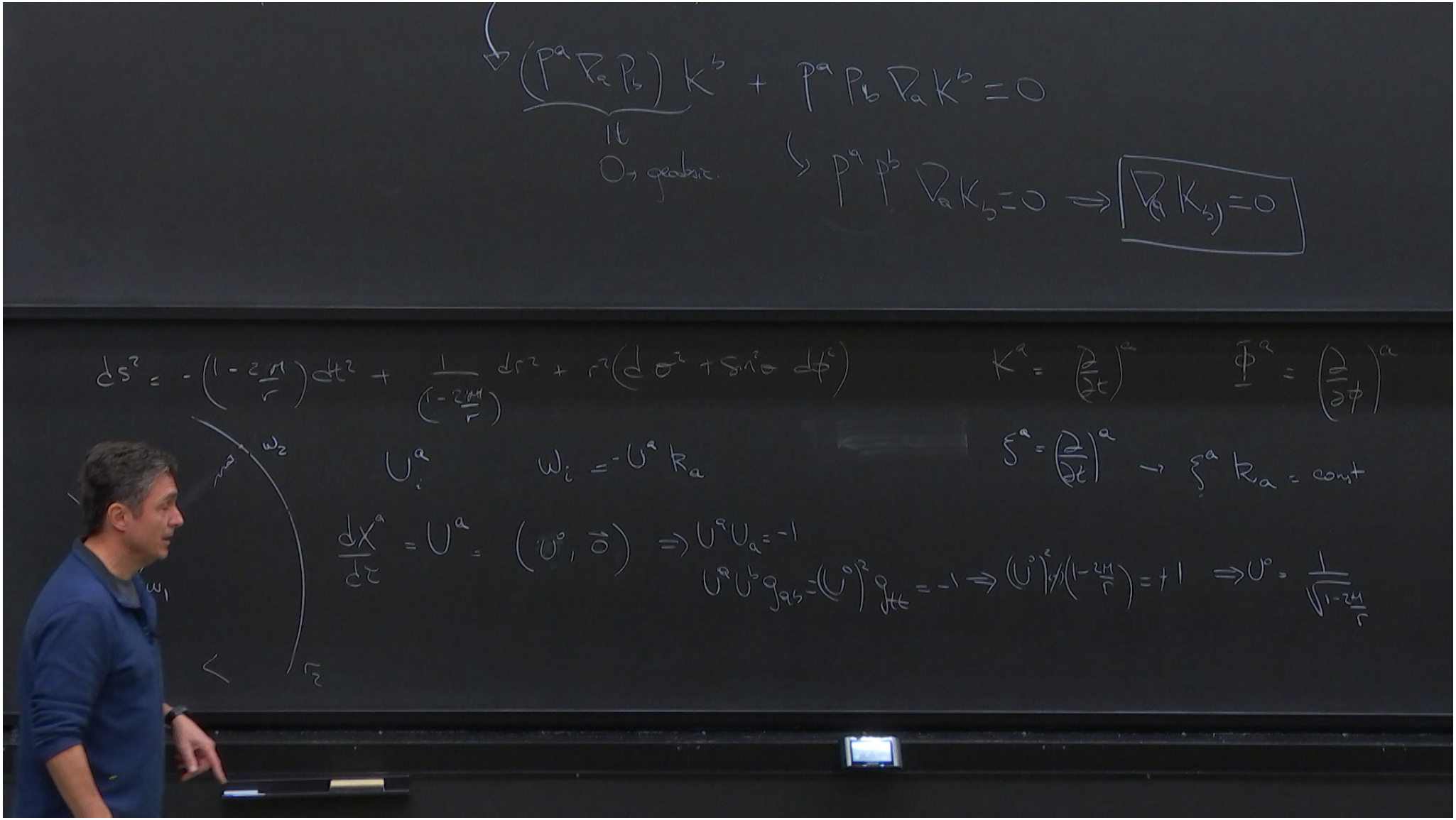
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a R_a = \text{const}$

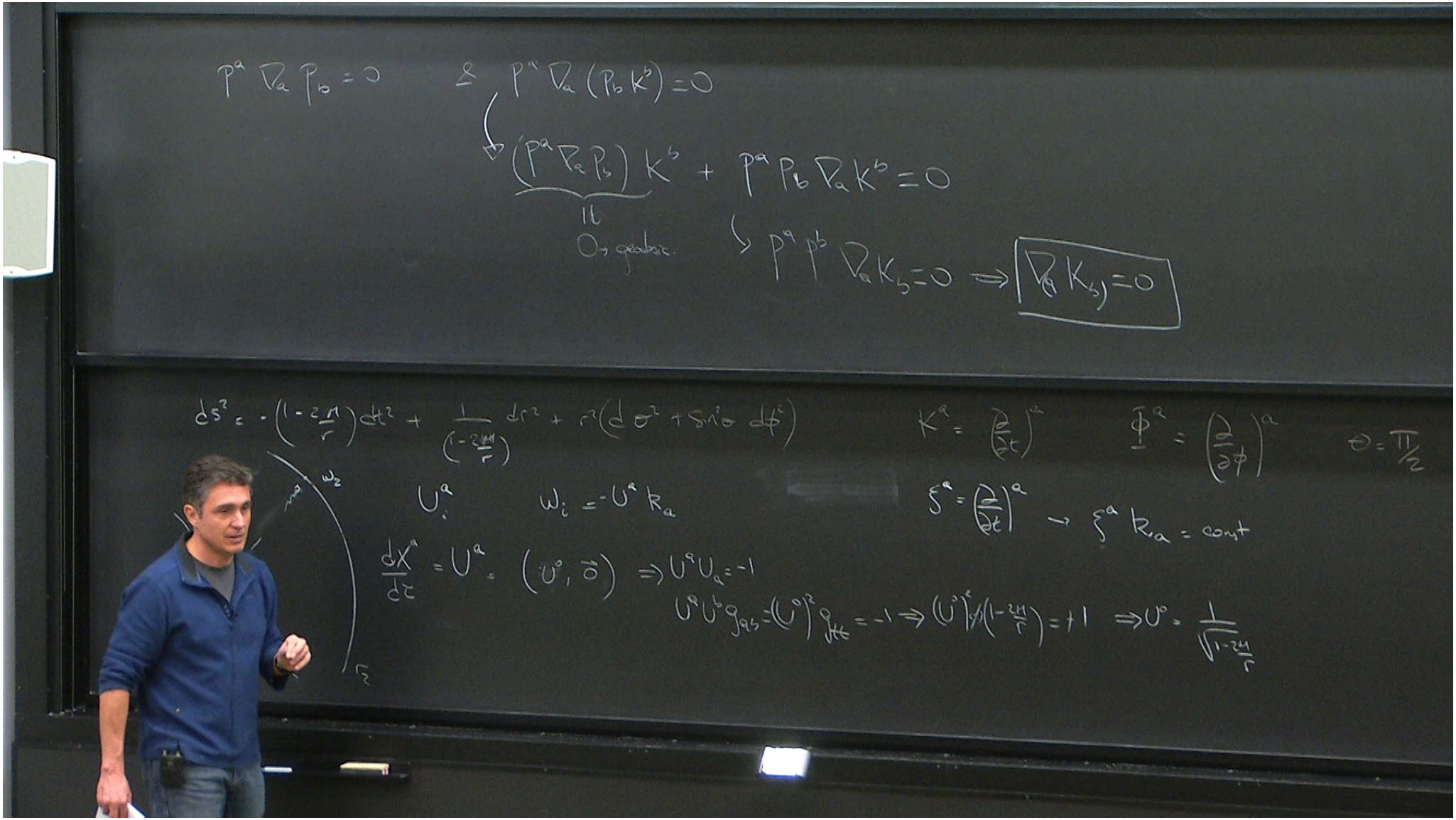
$\frac{dx^a}{d\tau} = U^a = (\#, \vec{0})$

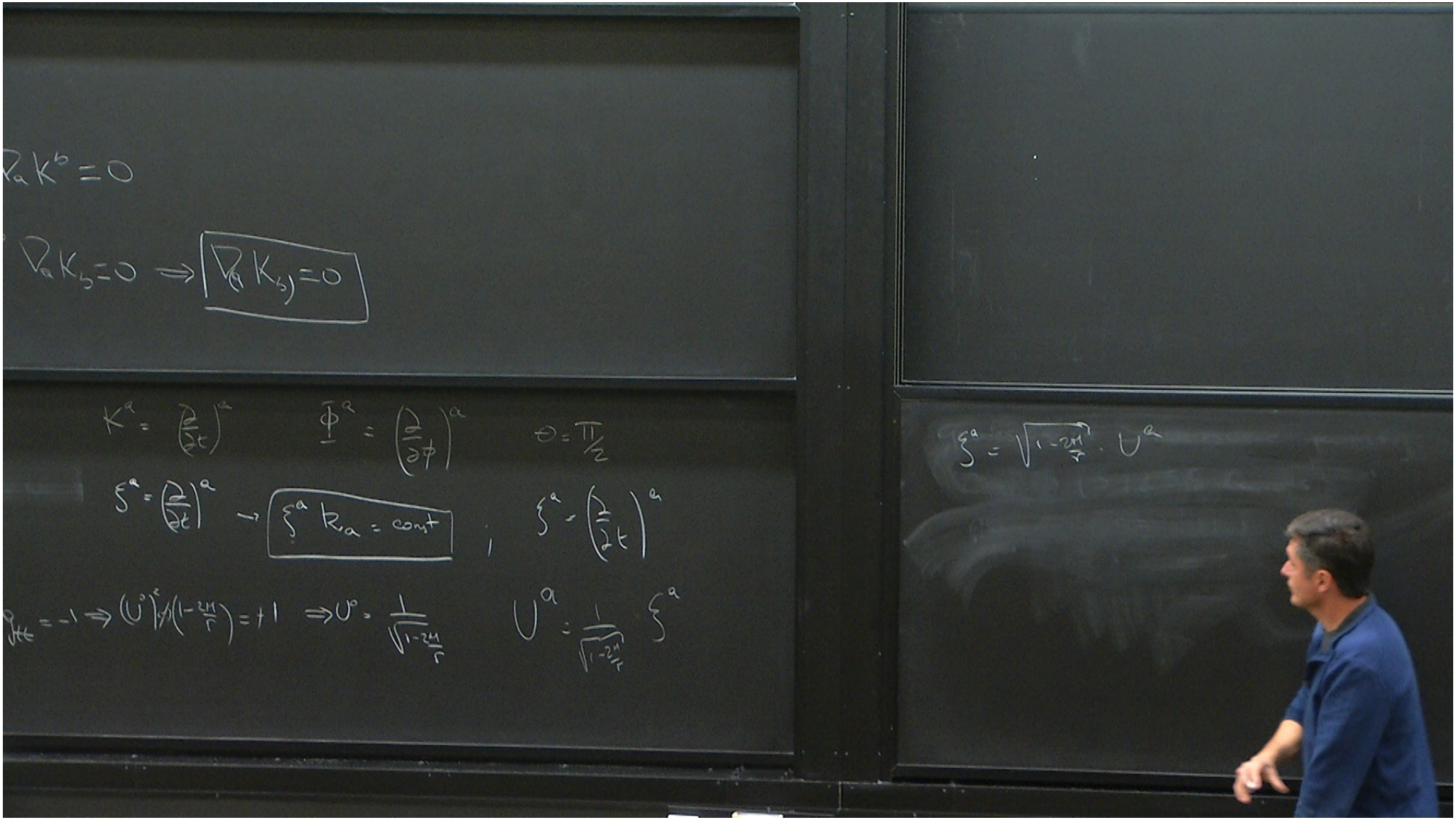


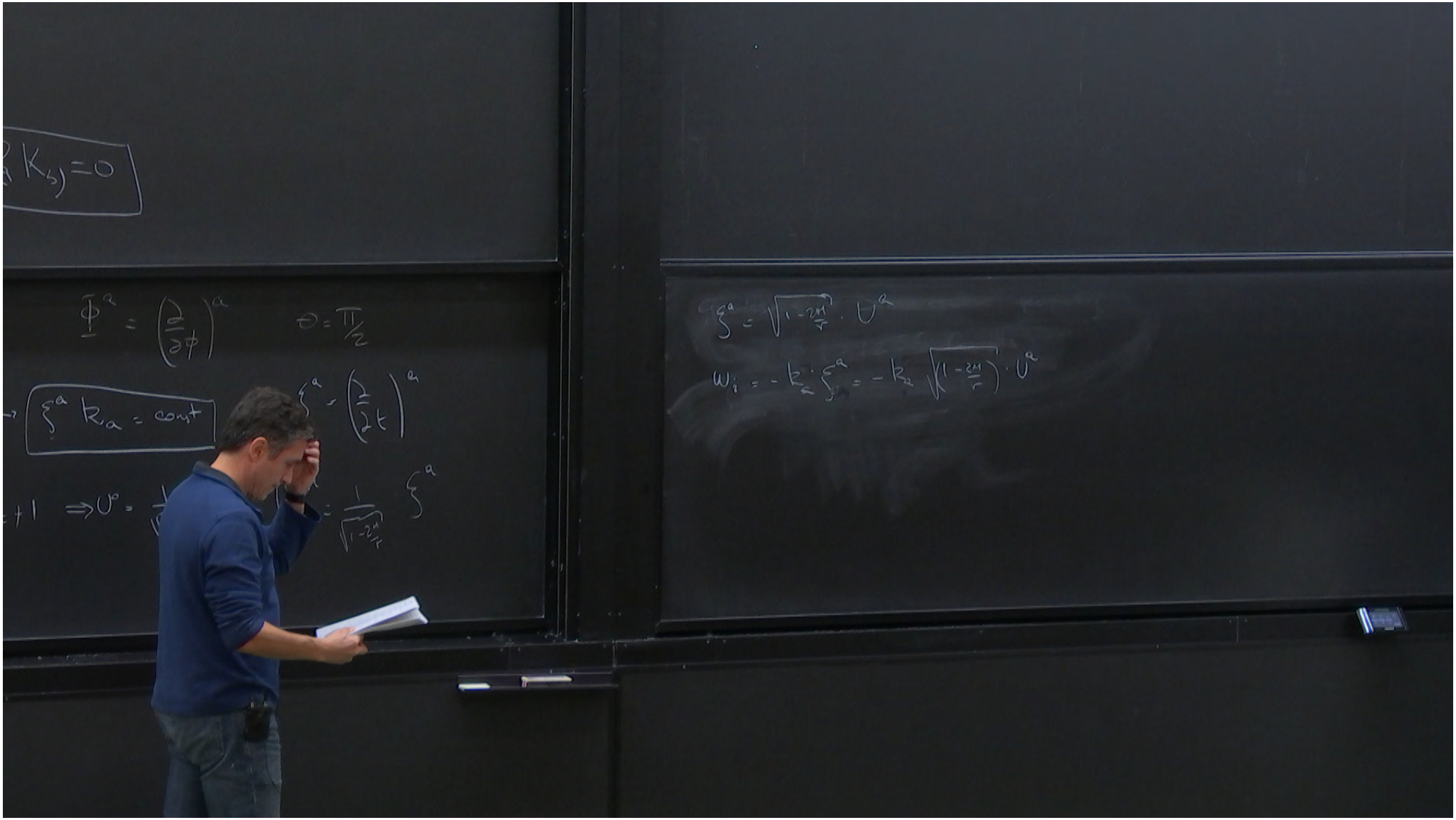

$(P^a P_a P_b) K^b + P^a P_b P_a K^b = 0$
$\Downarrow$
0 → geodesic
$\Rightarrow P^a P^b \nabla_a K_b = 0 \Rightarrow \boxed{P^{(a} \nabla^{b)} K_b = 0}$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a K_a = \text{const}$
U^a
$= -U^a K_a$
$\frac{dx^a}{d\tau} = (U^a$
$\vec{0})$
ω_2
ω_1
r_1
r_2

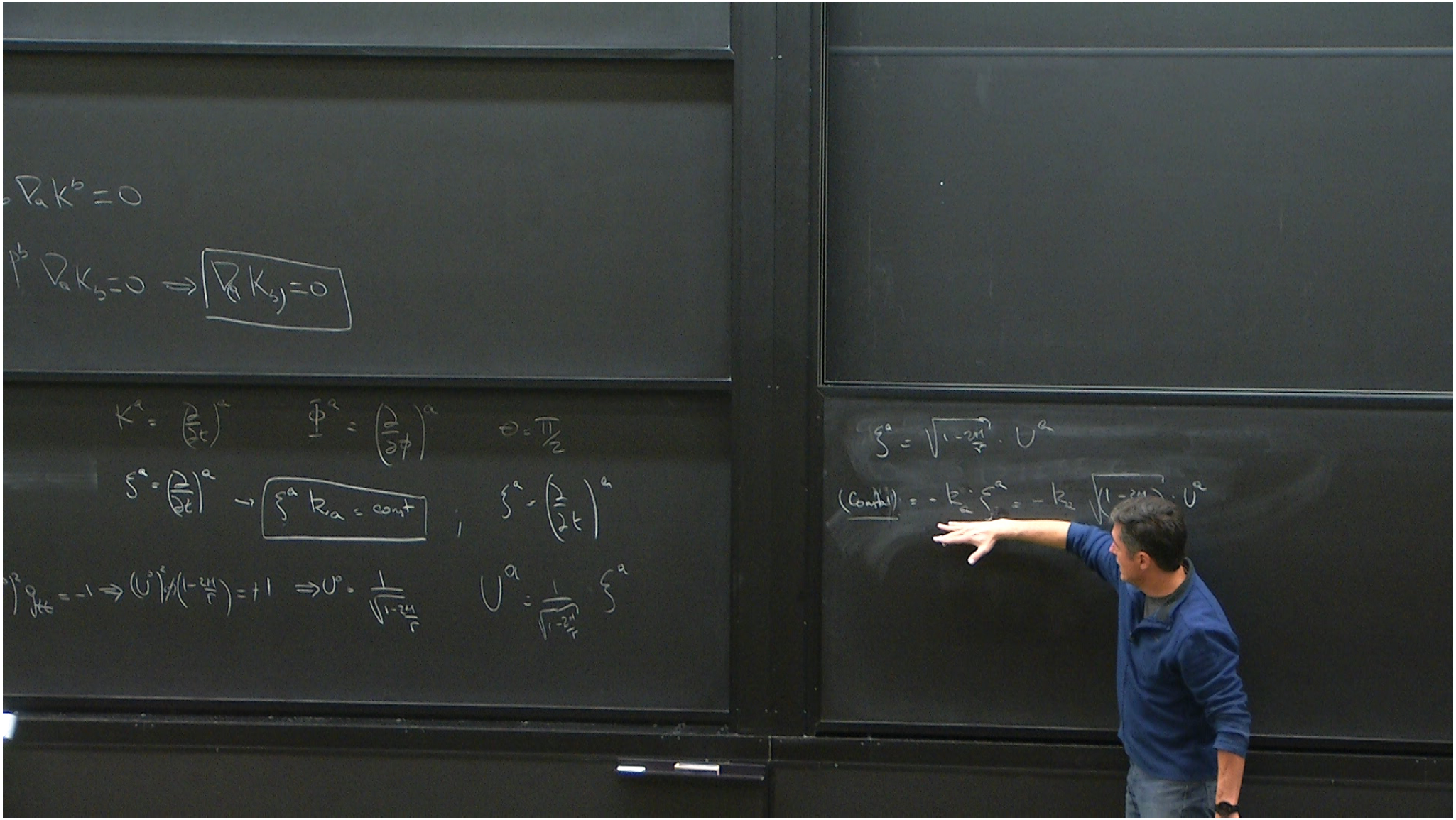

$(P^a P_a P_b) K^b + P^a P_b P_a K^b = 0$
$\Downarrow$
0 → geodesic.
$\hookrightarrow P^a P^b P_a K_b = 0 \Rightarrow \boxed{P_a K_b = 0}$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
U^a
$W_i = -U^a R_a$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a R_a = \text{const}$
$\frac{dx^a}{d\tau} = U^a = (U^0, \vec{0}) \Rightarrow U^a U_a = -1$
$U^a U^b g_{ab} = (U^0)^2 g_{tt}$

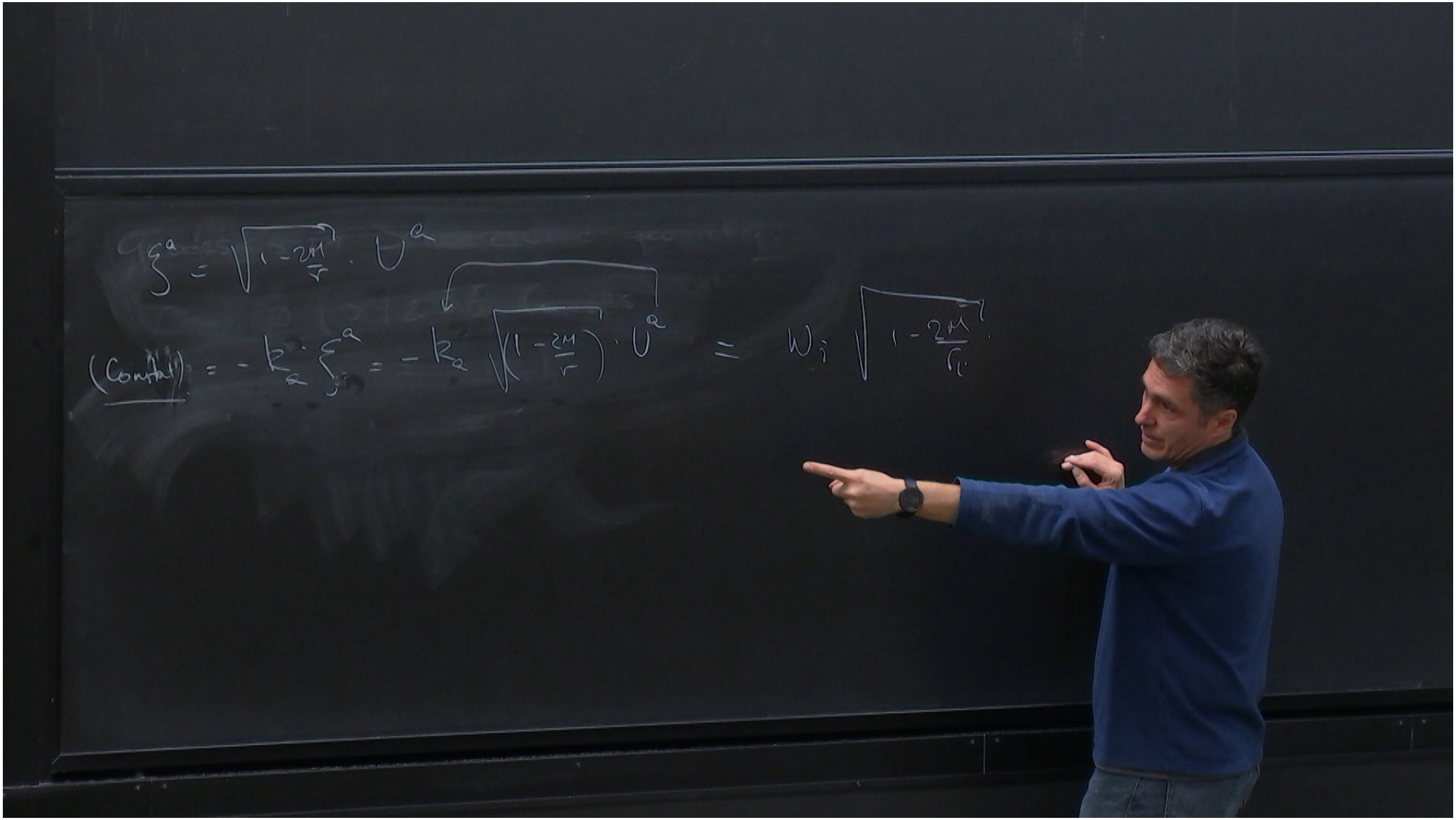

$(P^a P_a P_b) K^b + P^a P_b P_a K^b = 0$
0 → geodesic
$P^a P^b P_a K_b = 0 \Rightarrow P_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
U^a
$w_i = -U^a R_a$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a R_a = \text{const}$
$\frac{dx^a}{d\tau} = U^a = (U^0, \vec{0}) \Rightarrow U^a U_a = -1$
$U^a U^b g_{ab} = (U^0)^2 g_{tt} = -1 \Rightarrow (U^0)^2 \left(1 - \frac{2M}{r}\right) = +1 \Rightarrow U^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$
w_2
w_1
r_2

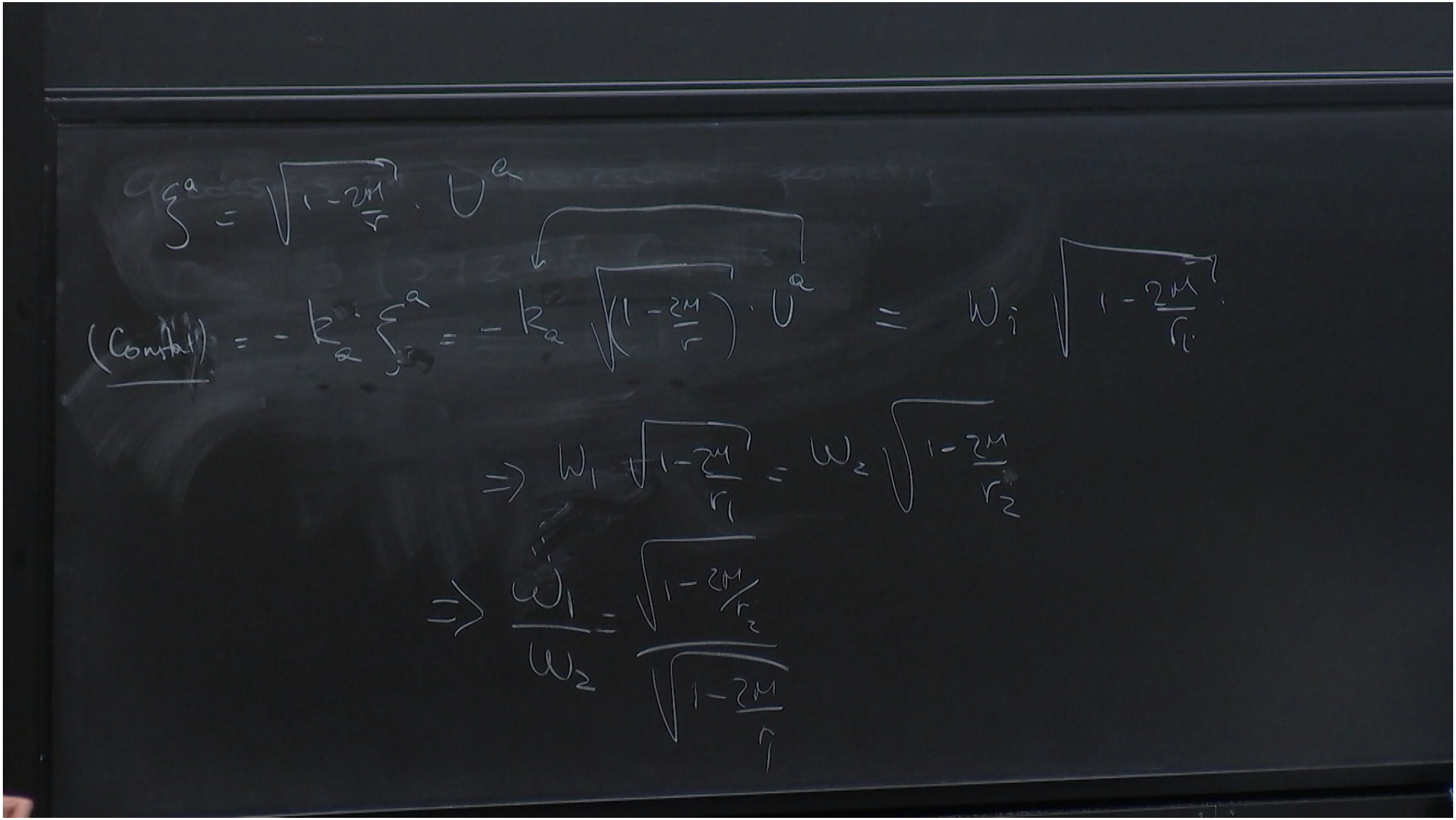

$P^a P^b \nabla_a P_b = 0 \iff P^a \nabla_a (P_b K^b) = 0$
$(P^a P^b \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
0 → geodesic
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\Theta = \pi/2$
U^a
$w_i = -U^a k_a$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a k_a = \text{const}$
$\frac{dx^a}{d\tau} = U^a = (U^0, \vec{0}) \Rightarrow U^a U_a = -1$
$U^a U^b g_{ab} = (U^0)^2 g_{tt} = -1 \Rightarrow (U^0)^2 \left(1 - \frac{2M}{r}\right) = +1 \Rightarrow U^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$
w_2
r_2

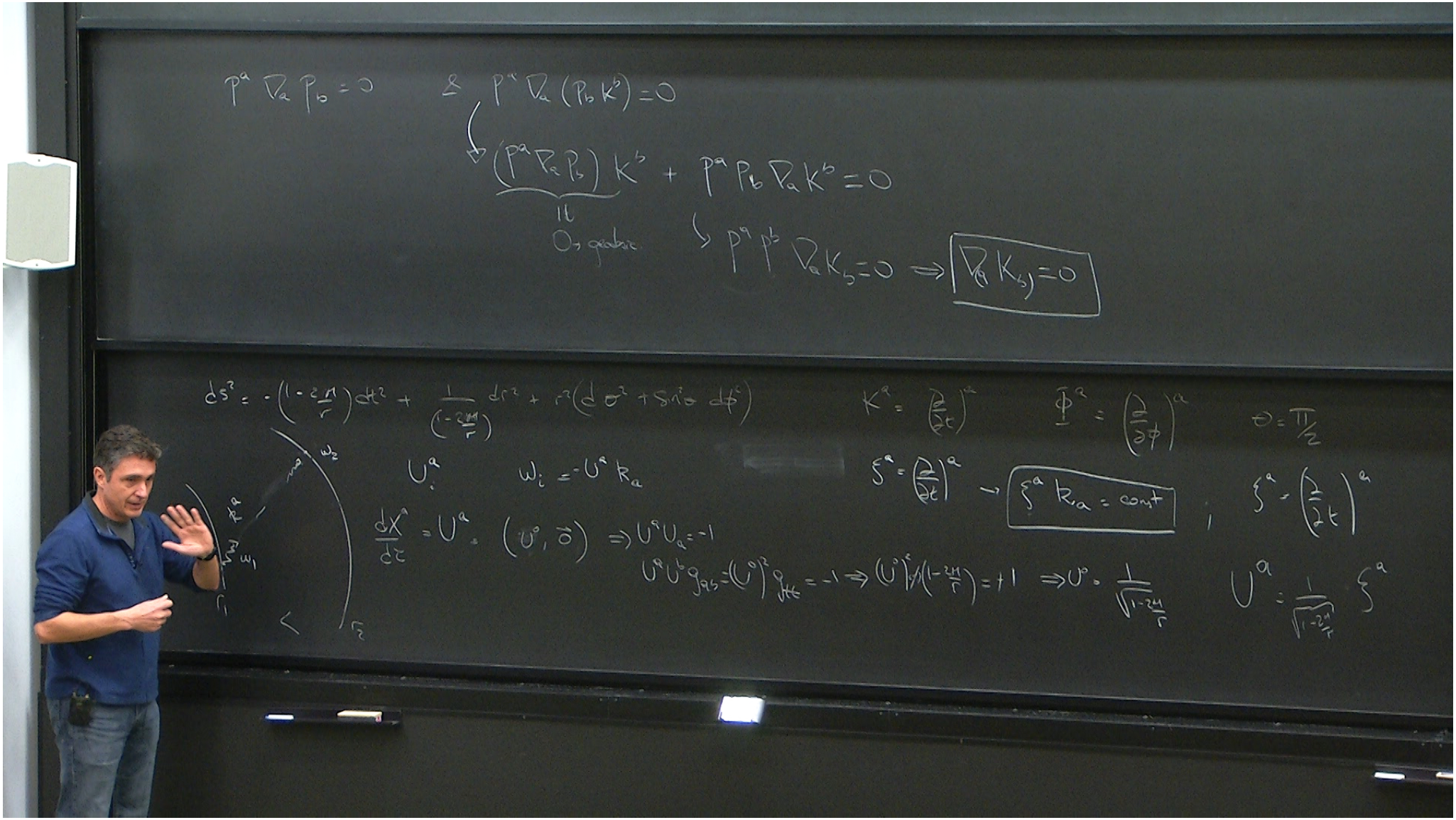

$\nabla_a K^b = 0$
$\nabla_a K_b = 0 \Rightarrow \boxed{\nabla_{(a} K_{b)} = 0}$
$K^a = \left(\frac{\partial}{\partial t}\right)^a \qquad \Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \qquad \Theta = \frac{\pi}{2}$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \boxed{\xi^a K_a = \text{cont}}$; $\xi^a = \left(\frac{\partial}{\partial t}\right)^a$
$g_{tt} = -1 \Rightarrow (U^0)^2 (1 - \frac{2M}{r}) = +1 \Rightarrow U^0 = \frac{1}{\sqrt{1-\frac{2M}{r}}}$
$U^a = \frac{1}{\sqrt{1-\frac{2M}{r}}} \xi^a$
$\xi^a = \sqrt{1-\frac{2M}{r}} \cdot U^a$

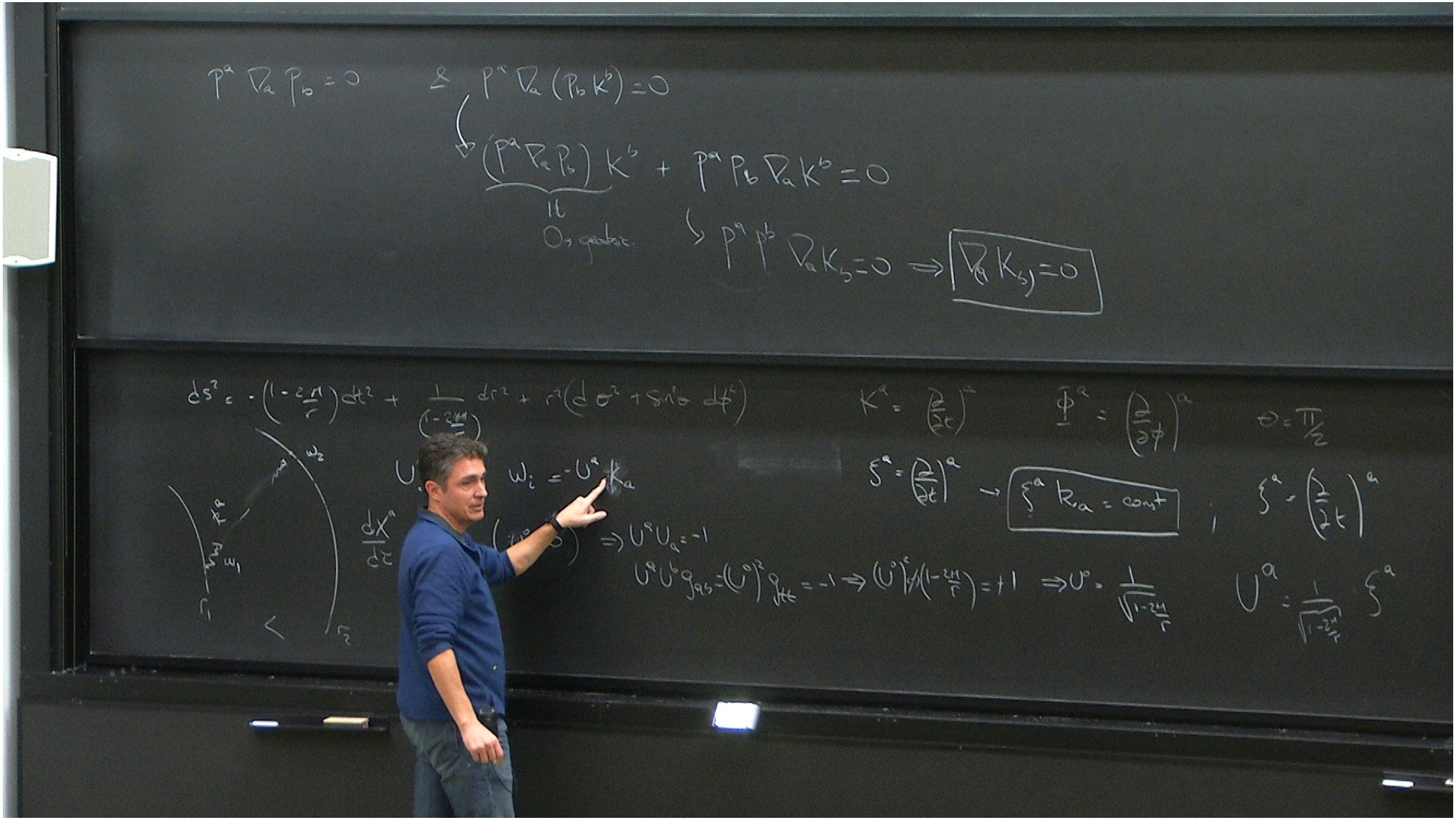

$\xi^a = \left(\frac{\partial}{\partial t}\right)^a$
$\theta = \frac{\pi}{2}$
$\xi^a k_a = \text{cost}$
$\xi^a = \sqrt{1-\frac{2M}{r}} \cdot U^a$
$w_i = -k_a \xi^a = -k_a \sqrt{1-\frac{2M}{r}} \cdot U^a$

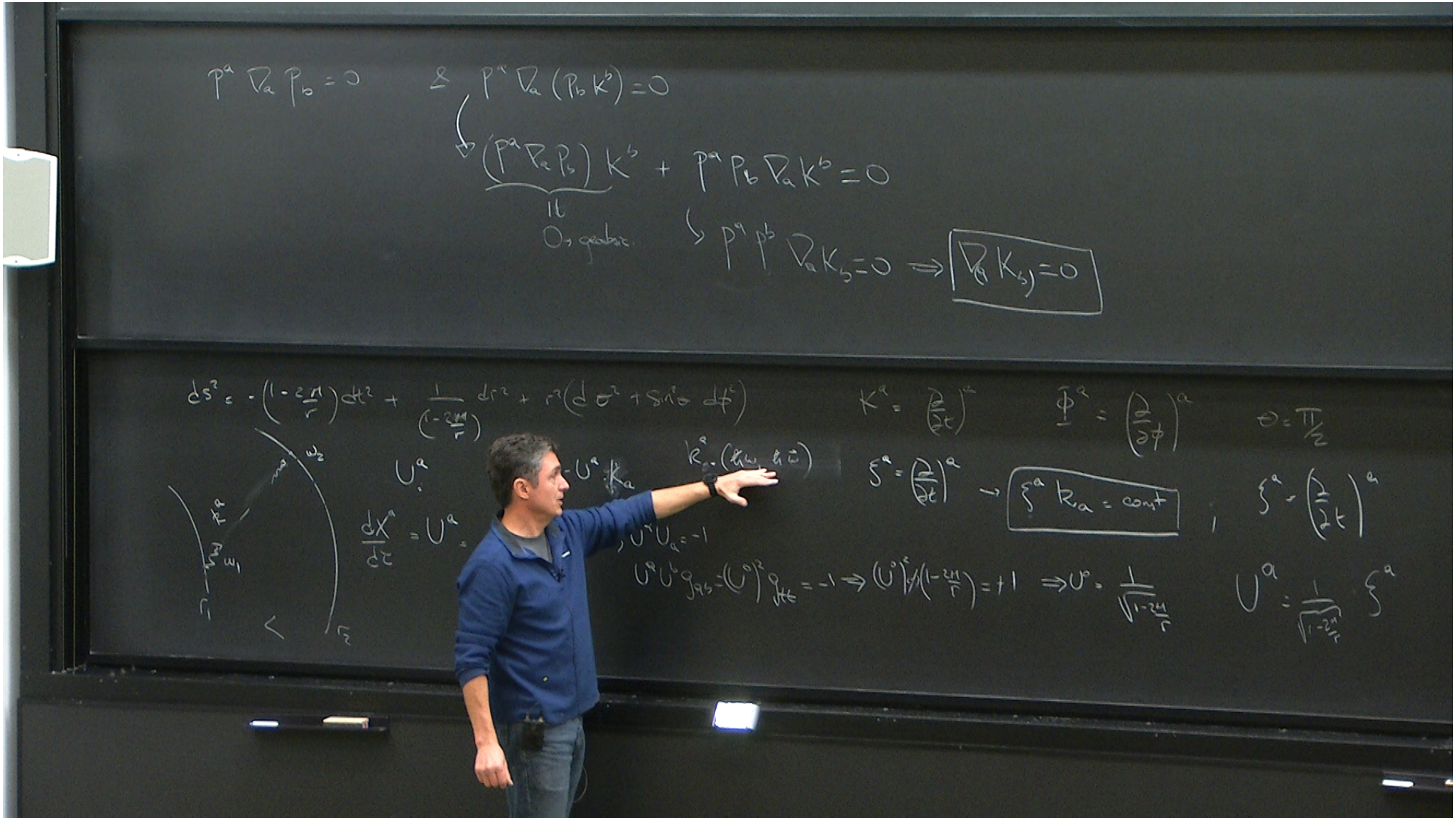

$\nabla_a K^b = 0$
$\Rightarrow \nabla_{(a} K_{b)} = 0$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\Theta = \pi/2$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a k_{ra} = \text{const}$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Rightarrow U^0 = \frac{1}{\sqrt{1-\frac{2M}{r}}}$
$U^a = \frac{1}{\sqrt{1-\frac{2M}{r}}} \xi^a$
$\xi^a = \sqrt{1-\frac{2M}{r}} \cdot U^a$
(Constant) $= -k_a \xi^a = -k_a \sqrt{1-\frac{2M}{r}} \cdot U^a$

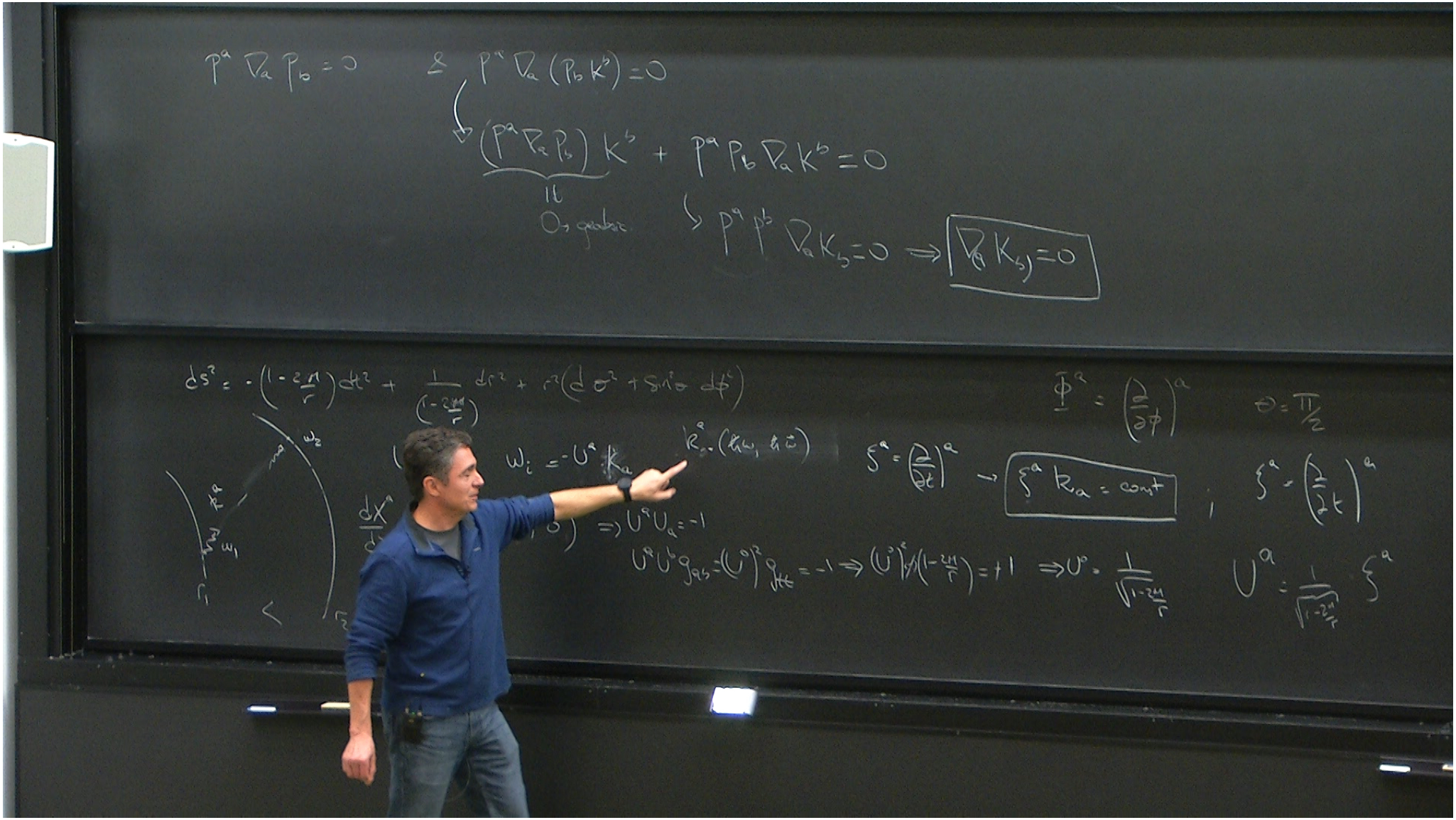

$\xi^a = \sqrt{1-\frac{2M}{r}}\cdot U^a$
(Constat) $= -k_a \xi^a = -k_a \sqrt{\left(1-\frac{2M}{r}\right)}\cdot U^a = W_i \sqrt{1-\frac{2M}{r_i}}$

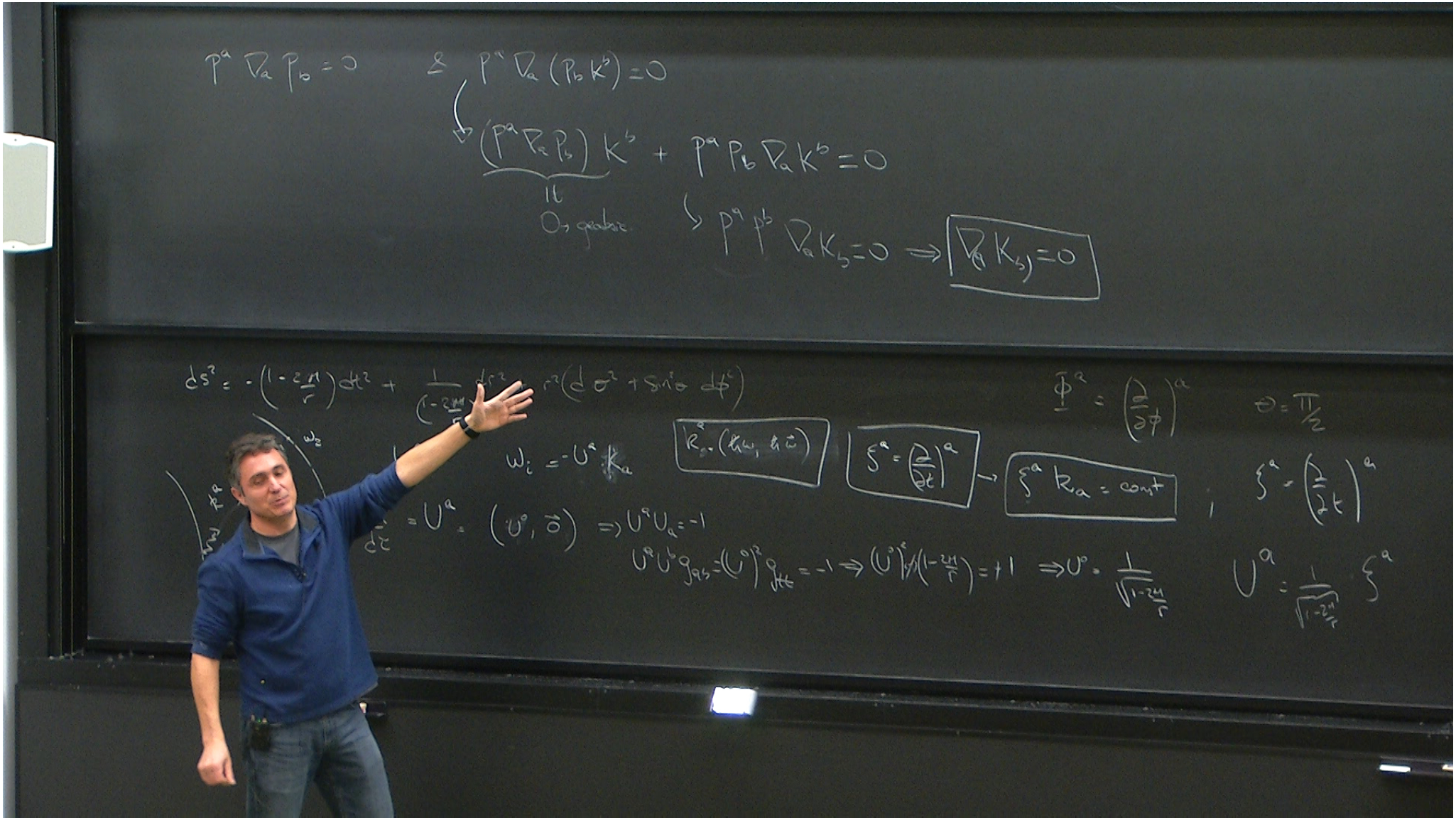

$\xi^a = \sqrt{1-\frac{2M}{r}}\cdot U^a$
(Constant) $= -k_a \xi^a = -k_a \sqrt{\left(1-\frac{2M}{r}\right)}\cdot U^a = \omega_i \sqrt{1-\frac{2M}{r_i}}$
$\Rightarrow \omega_1 \sqrt{1-\frac{2M}{r_1}} = \omega_2 \sqrt{1-\frac{2M}{r_2}}$
$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{\sqrt{1-2M/r_2}}{\sqrt{1-\frac{2M}{r_1}}}$

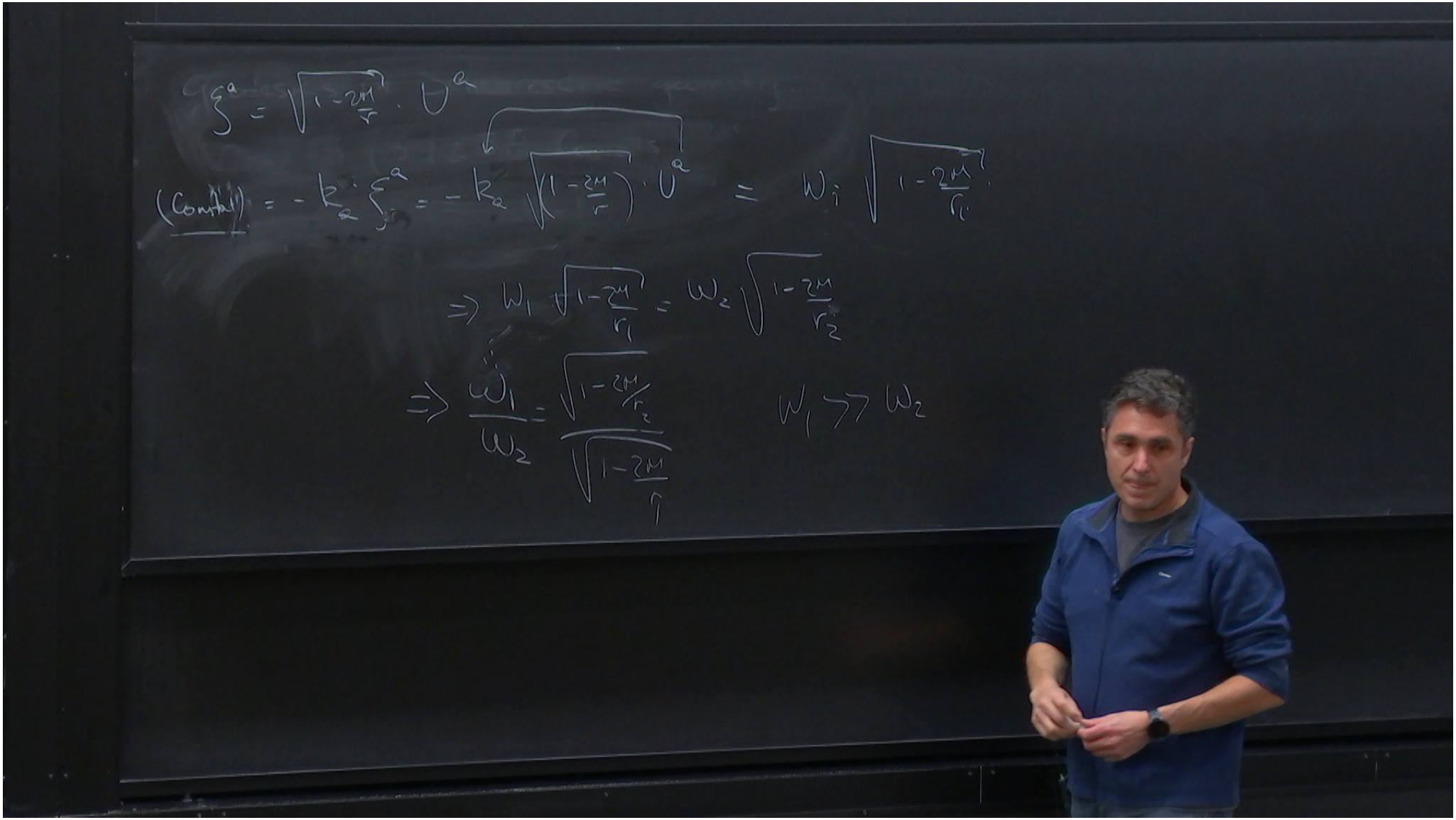

$p^a \nabla_a p_b = 0$ & $p^a \nabla_a (p_b k^b) = 0$
$(p^a \nabla_a p_b) k^b + p^a p_b \nabla_a k^b = 0$
0, geodesic
$p^a p^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
U^a
$w_i = -U^a k_a$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \xi^a k_a = \text{cont}$
$\frac{dx^a}{d\tau} = U^a = (U^0, \vec{0}) \Rightarrow U^a U_a = -1$
$U^a U^b g_{ab} = (U^0)^2 g_{tt} = -1 \Rightarrow (U^0)^2 \left(1 - \frac{2M}{r}\right) = +1 \Rightarrow U^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$
$U^a = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \xi^a$

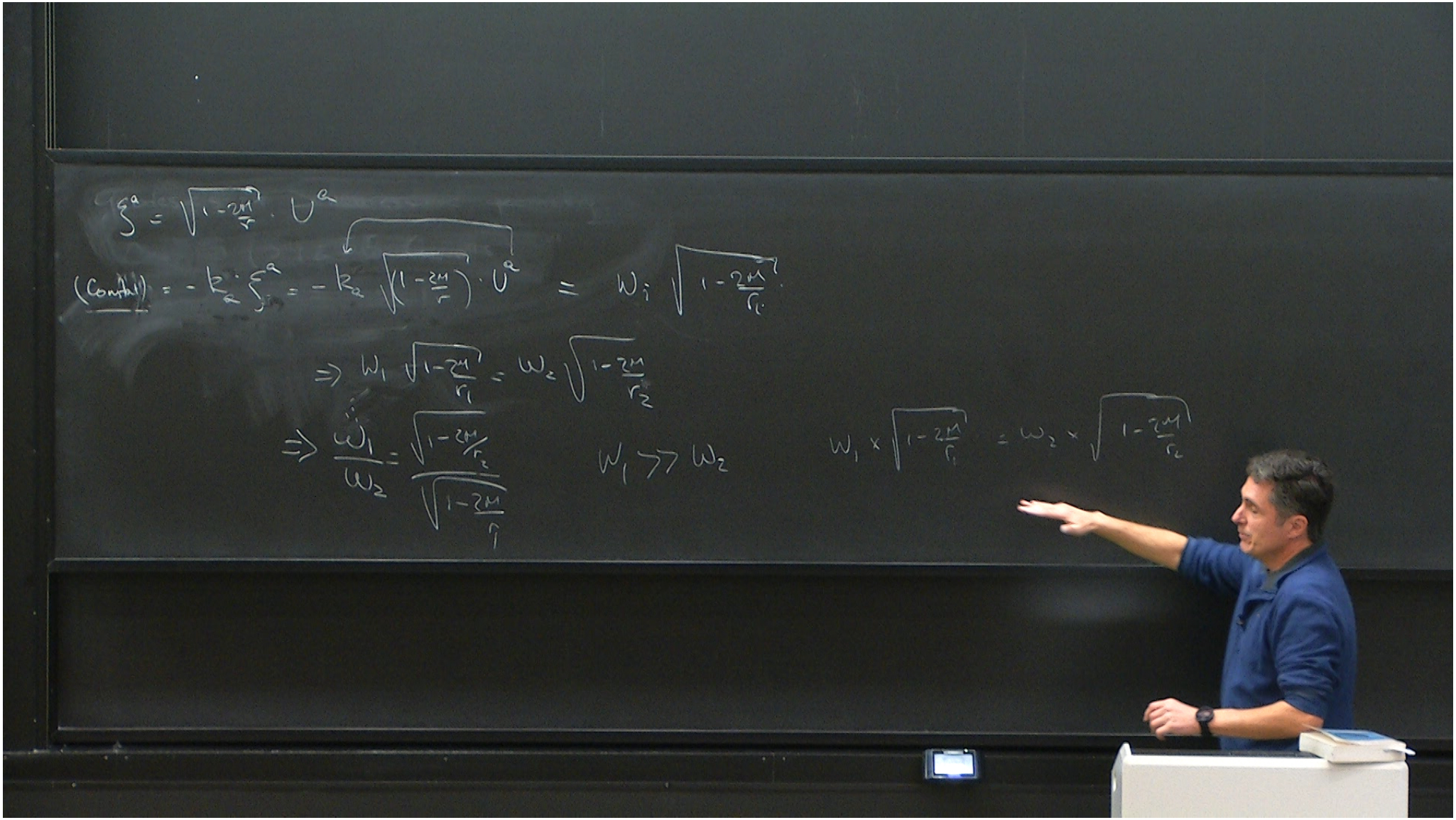

$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b k^b) = 0$
$(P^a \nabla_a P_b) k^b + P^a P_b \nabla_a k^b = 0$
0, geodesic
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \boxed{\xi^a k_a = \text{const}}$
$W_i = -U^a k_a$
$\frac{dx^a}{d\tau}$
$\Rightarrow U^a U_a = -1$
$U^a U^b g_{ab} = (U^0)^2 g_{tt} = -1 \Rightarrow (U^0)^2 \left(1 - \frac{2M}{r}\right) = +1 \Rightarrow U^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$
$U^a = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \xi^a$
ω_2
ω_1
r_1
r_2


$p^a \nabla_a p_b = 0$ & $p^a \nabla_a (p_b k^b) = 0$
$(p^a \nabla_a p_b) k^b + p^a p_b \nabla_a k^b = 0$
0, geodat.
$p^a p^b \nabla_a k_b = 0 \Rightarrow \nabla_{(a} k_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$K^a = \left(\frac{\partial}{\partial t}\right)^a$
$\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \boxed{\xi^a k_a = \text{cont}}$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a$
U^a
$\frac{dx^a}{d\tau} = U^a$
$U^a U_a = -1$
$U^a U^b g_{ab} = (U^0)^2 g_{tt} = -1 \Rightarrow (U^0)^2 \left(1 - \frac{2M}{r}\right) = +1 \Rightarrow U^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$
$U^a = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \xi^a$
ω_2
ω_1
r_1
r_2


$P^a \nabla_a P_b = 0$ & $P^a \nabla_a (P_b K^b) = 0$
$(P^a \nabla_a P_b) K^b + P^a P_b \nabla_a K^b = 0$
0, geodesic
$P^a P^b \nabla_a K_b = 0 \Rightarrow \nabla_{(a} K_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow \boxed{\xi^a k_a = \text{const}}$
$\xi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
$U^a U_a = -1$
$U^a U^b g_{ab} = (U^t)^2 g_{tt} = -1 \Rightarrow (U^t)^2 \left(1 - \frac{2M}{r}\right) = +1 \Rightarrow U^t = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$
$U^a = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \xi^a$


$p^a \nabla_a p_b = 0$
$\Rightarrow p^a \nabla_a (p_b k^b) = 0$
$(p^a \nabla_a p_b) k^b + p^a p_b \nabla_a k^b = 0$
$= 0$ geodesic
$\Rightarrow p^a p^b \nabla_a k_b = 0 \Rightarrow \nabla_{(a} k_{b)} = 0$
$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2M}{r}\right)} dr^2 + r^2 (d\theta^2 + \sin^2\theta \, d\phi^2)$
$\vec{\Phi}^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$\theta = \pi/2$
$w_i = -u^a k_a$
$k^a = (\omega, \vec{0})$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a$
$\xi^a k_a = \text{const}$
$\xi^a = \left(\frac{\partial}{\partial t}\right)^a$
$u^a = (u^0, \vec{0}) \Rightarrow u^a u_a = -1$
$u^a u^b g_{ab} = (u^0)^2 g_{tt} = -1 \Rightarrow (u^0)^2 \left(1 - \frac{2M}{r}\right) = +1 \Rightarrow u^0 = \frac{1}{\sqrt{1 - \frac{2M}{r}}}$
$u^a = \frac{1}{\sqrt{1 - \frac{2M}{r}}} \xi^a$

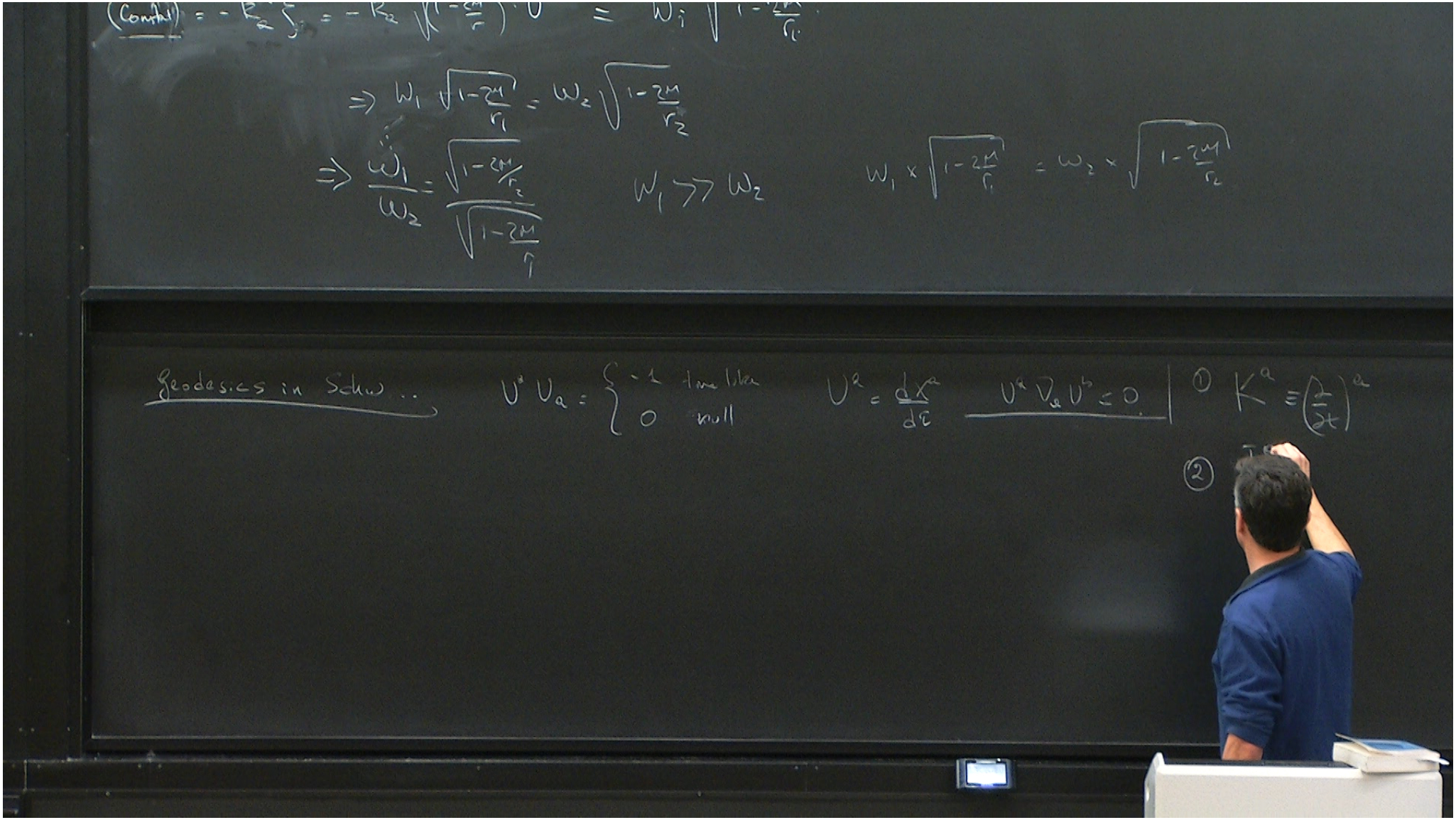

$\xi^a = \sqrt{1-\frac{2M}{r}}\cdot U^a$
$(\text{Constant}) = -k_a \xi^a = -k_a \sqrt{1-\frac{2M}{r}}\cdot U^a = \omega_i \sqrt{1-\frac{2M}{r_i}}$
$\Rightarrow \omega_1 \sqrt{1-\frac{2M}{r_1}} = \omega_2 \sqrt{1-\frac{2M}{r_2}}$
$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{\sqrt{1-\frac{2M}{r_2}}}{\sqrt{1-\frac{2M}{r_1}}}$
$\omega_1 \gg \omega_2$

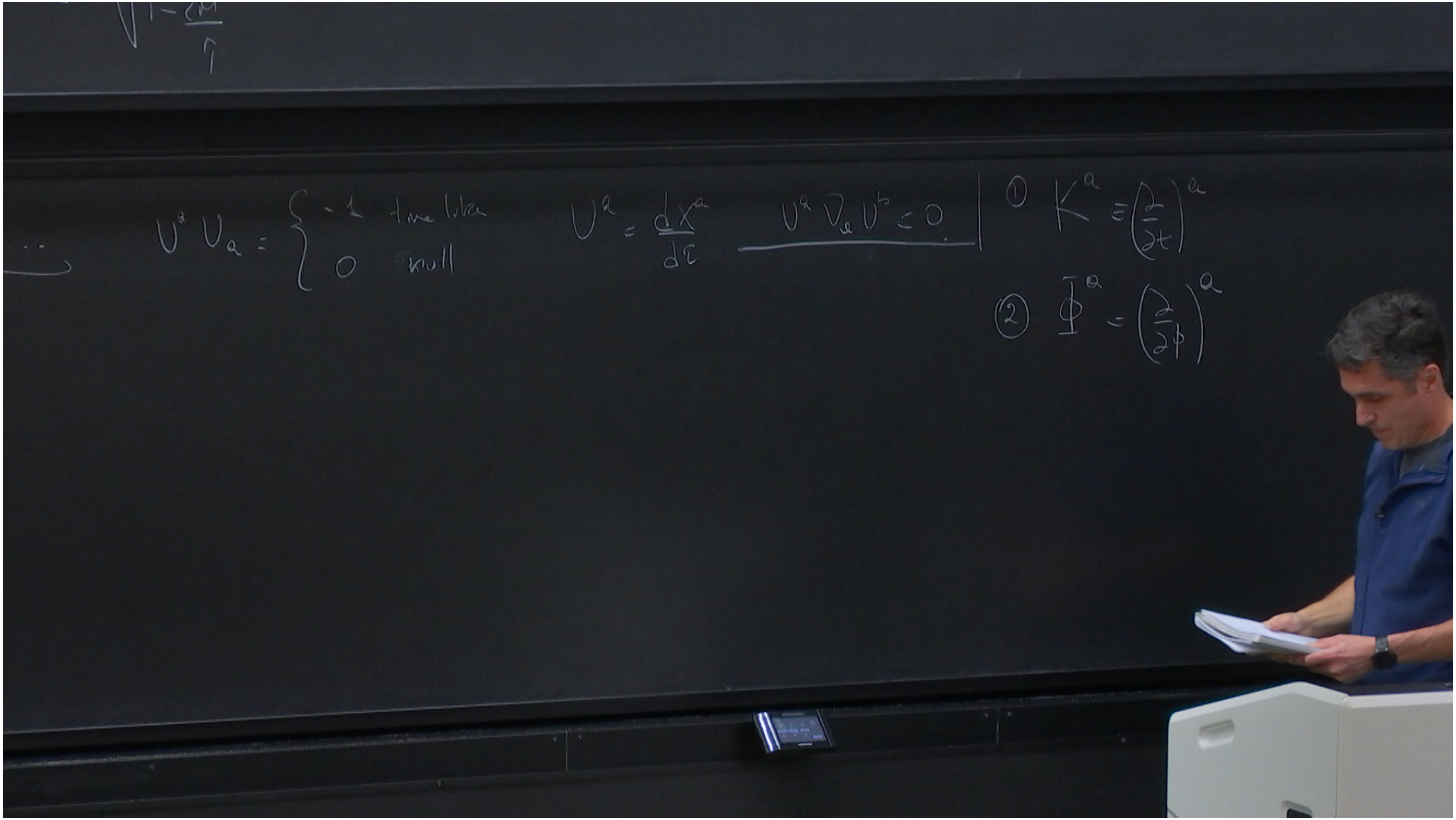

$\xi^a = \sqrt{1-\frac{2M}{r}}\cdot U^a$
$(\text{Const.}) = -k_a \cdot \xi^a = -k_a \sqrt{1-\frac{2M}{r}}\cdot U^a = w_1\sqrt{1-\frac{2M}{r_1}}$
$\Rightarrow w_1\sqrt{1-\frac{2M}{r_1}} = w_2\sqrt{1-\frac{2M}{r_2}}$
$\Rightarrow \frac{w_1}{w_2} = \frac{\sqrt{1-\frac{2M}{r_2}}}{\sqrt{1-\frac{2M}{r_1}}}$
$w_1 \gg w_2$
$w_1 \times \sqrt{1-\frac{2M}{r_1}} = w_2 \times \sqrt{1-\frac{2M}{r_2}}$

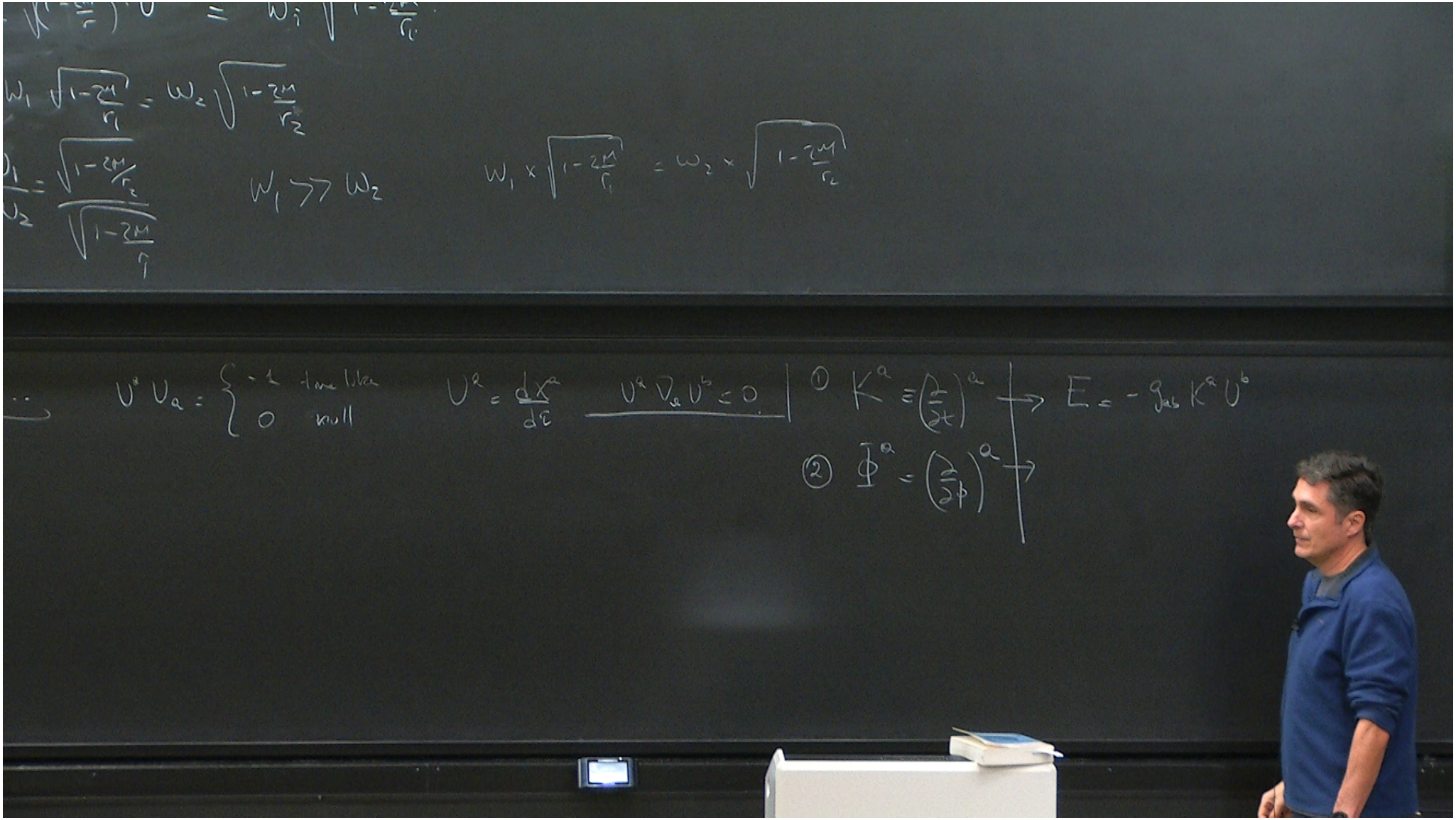
Geodesics in Schw...

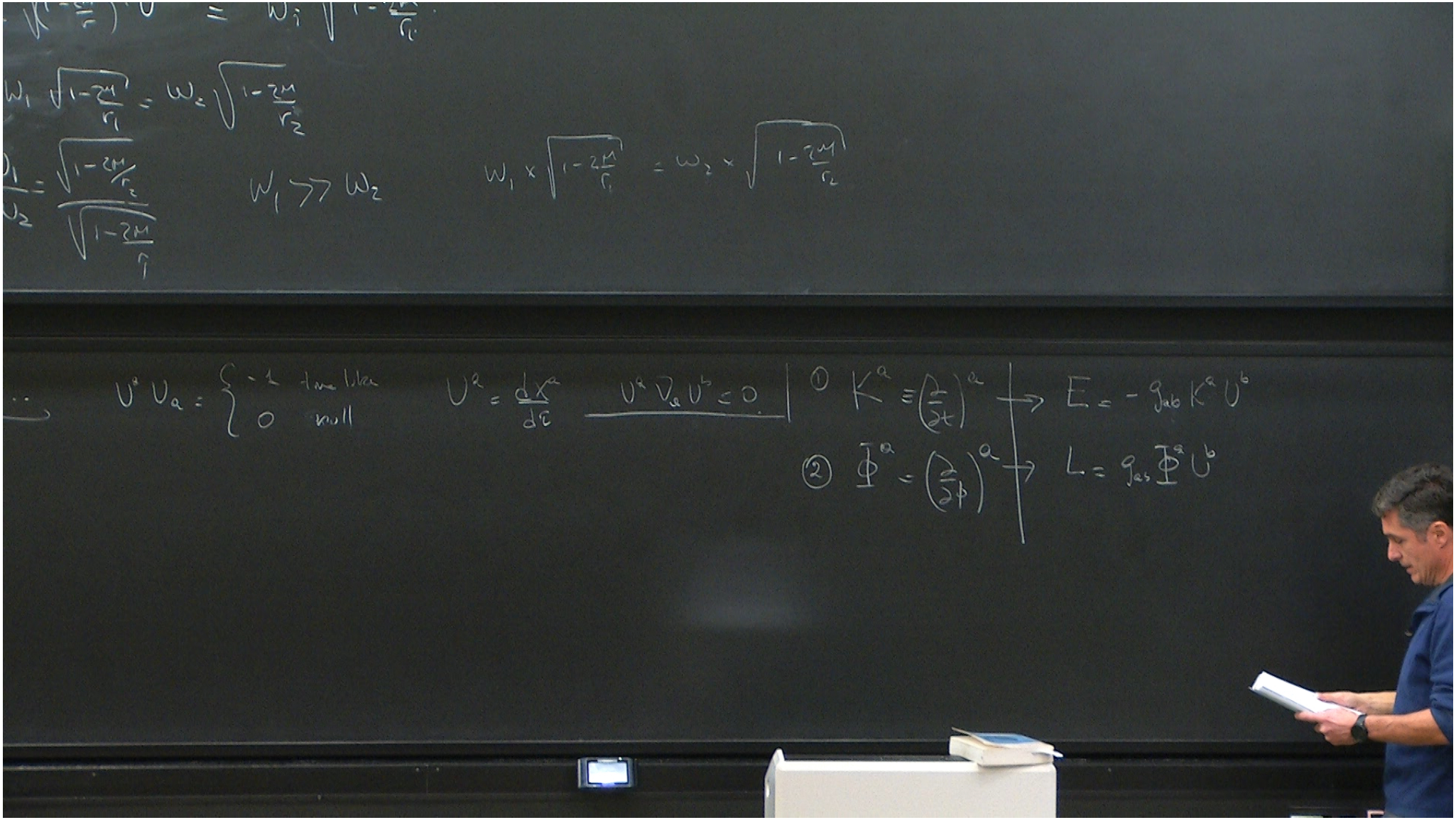
$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$

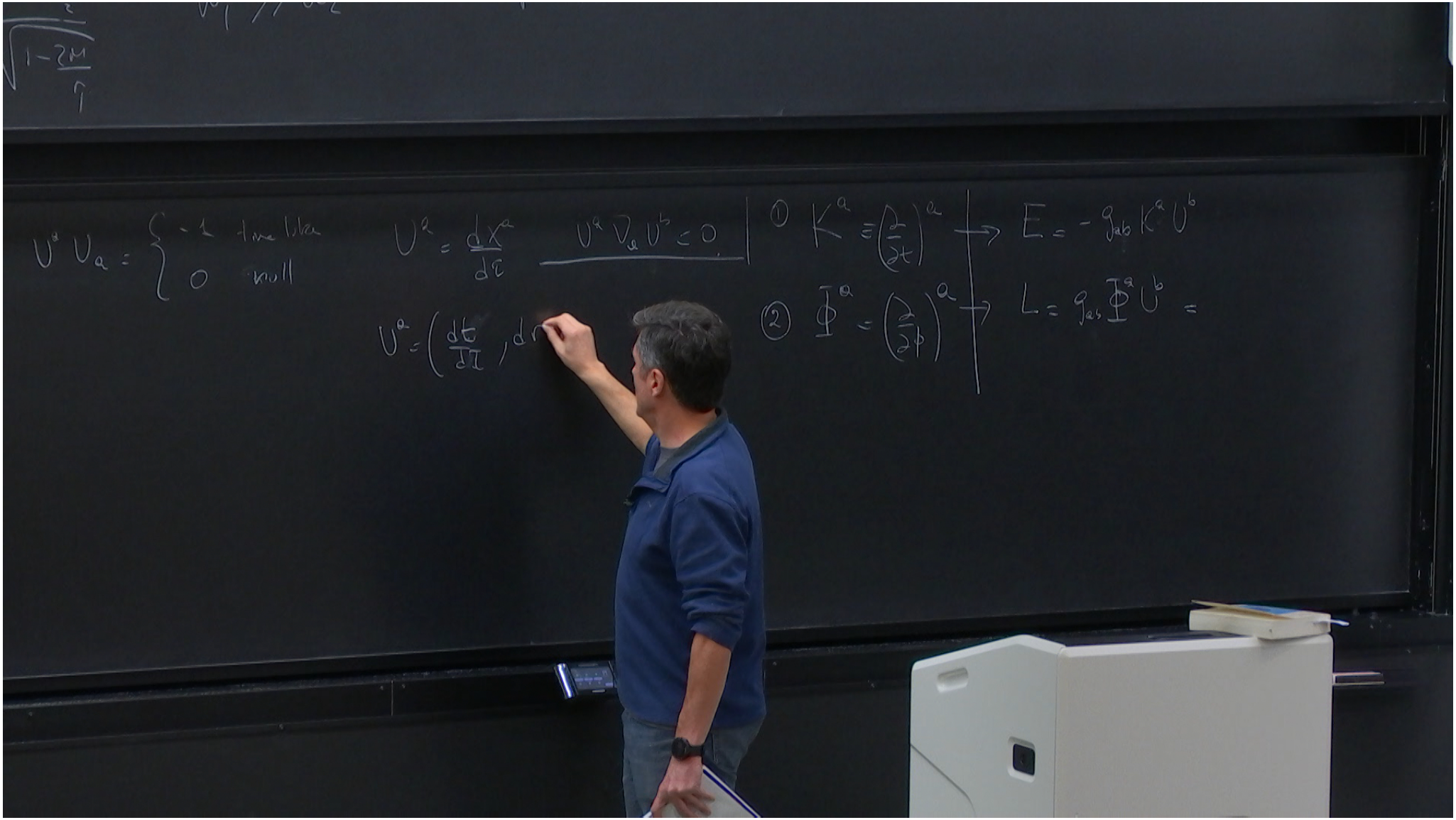


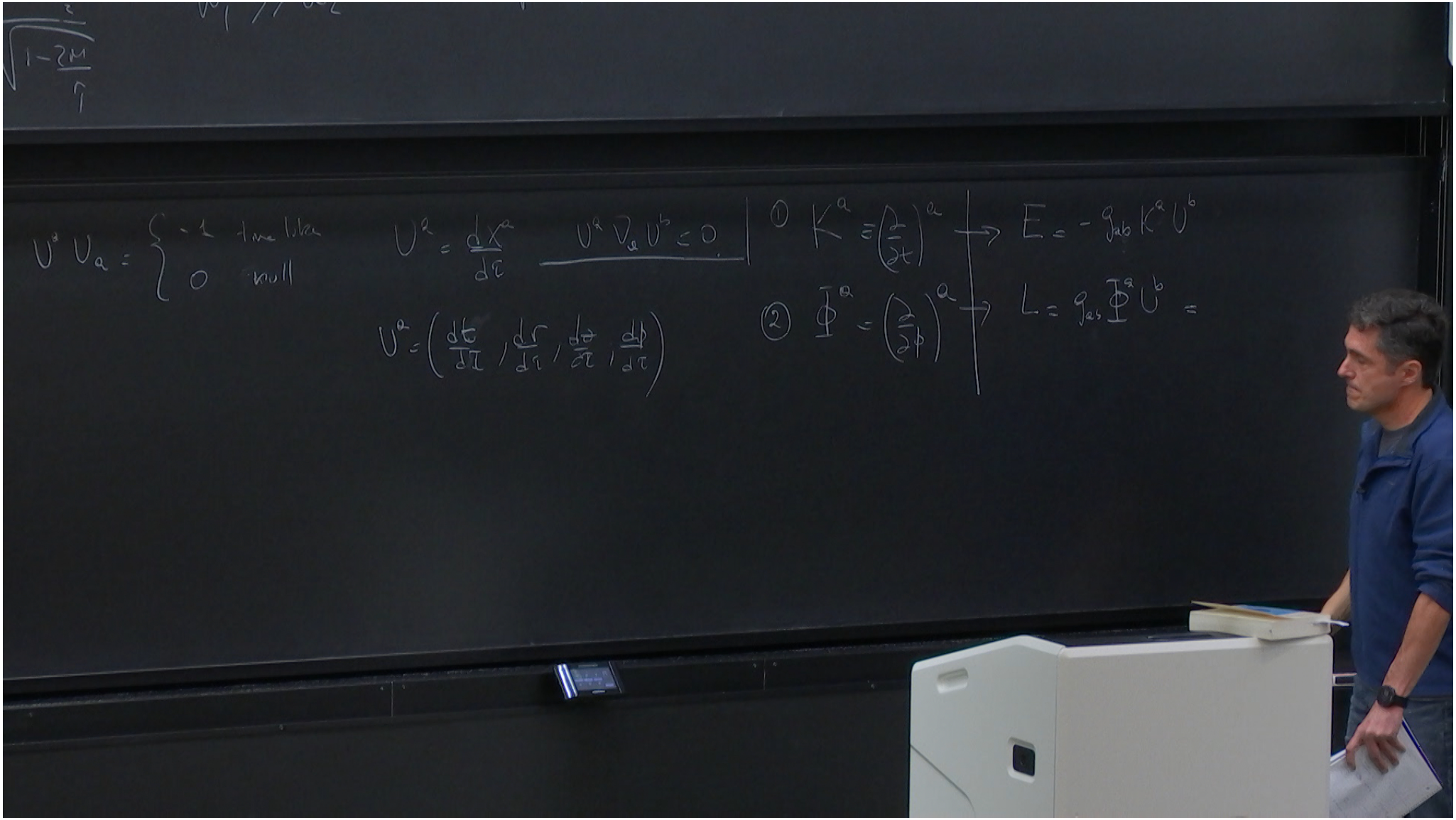

(Contd)
$\Rightarrow w_1\sqrt{1-\frac{2M}{r_1}} = w_2\sqrt{1-\frac{2M}{r_2}}$
$\Rightarrow \frac{\omega_1}{\omega_2} = \frac{\sqrt{1-\frac{2M}{r_2}}}{\sqrt{1-\frac{2M}{r_1}}}$
$w_1 \gg w_2$
$w_1 \times \sqrt{1-\frac{2M}{r_1}} = w_2 \times \sqrt{1-\frac{2M}{r_2}}$
Geodesics in Schw..
$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$
$U^a = \frac{dx^a}{d\tau}$
$U^a \nabla_a U^b = 0$
① $K^a \equiv \left(\frac{\partial}{\partial t}\right)^a$
②

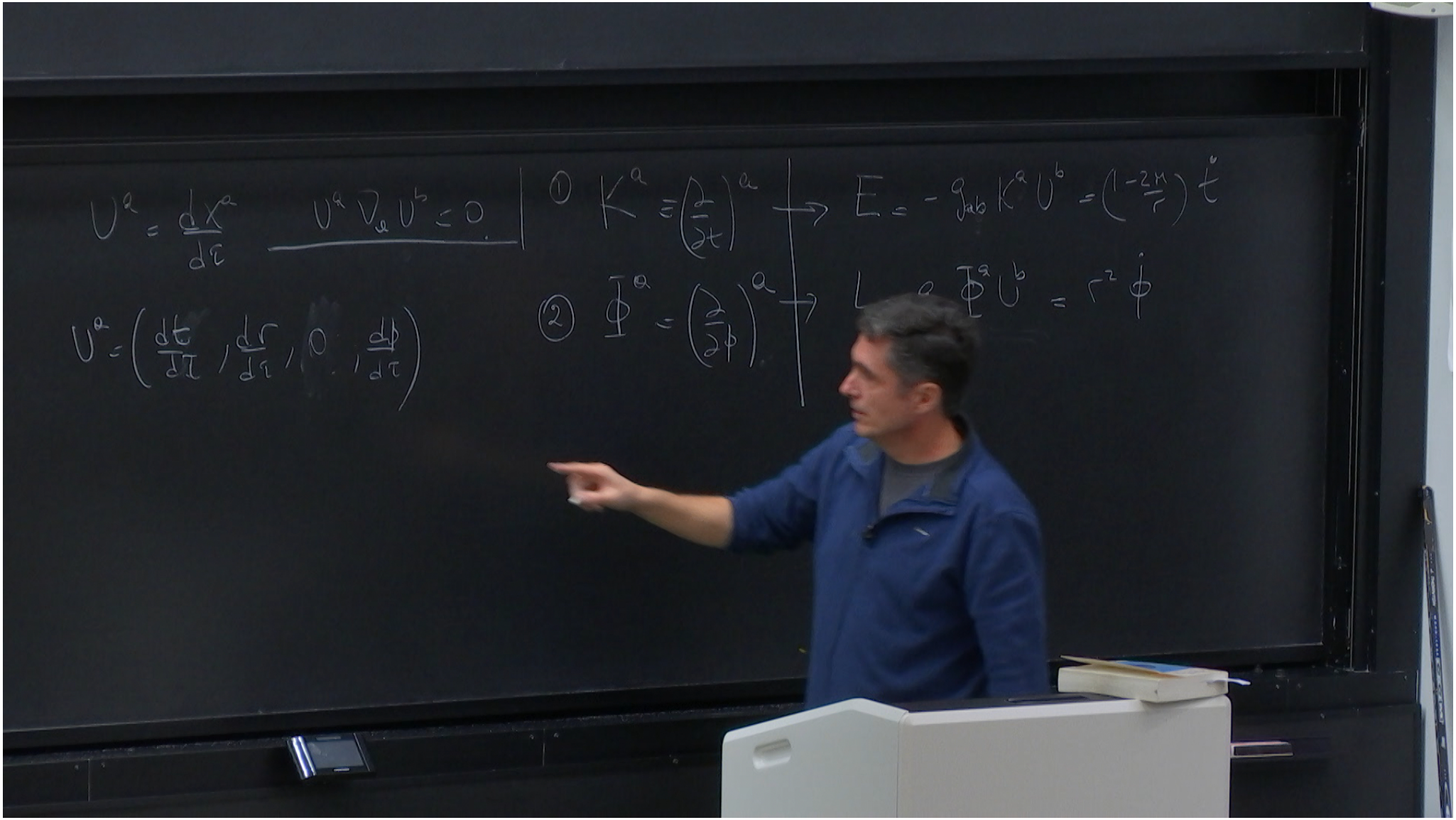

$\sqrt{1-\frac{2M}{r}}$
$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$
$U^a = \frac{dx^a}{d\tau}$
$U^a \nabla_a U^b = 0$
① $K^a \equiv \left(\frac{\partial}{\partial t}\right)^a$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$

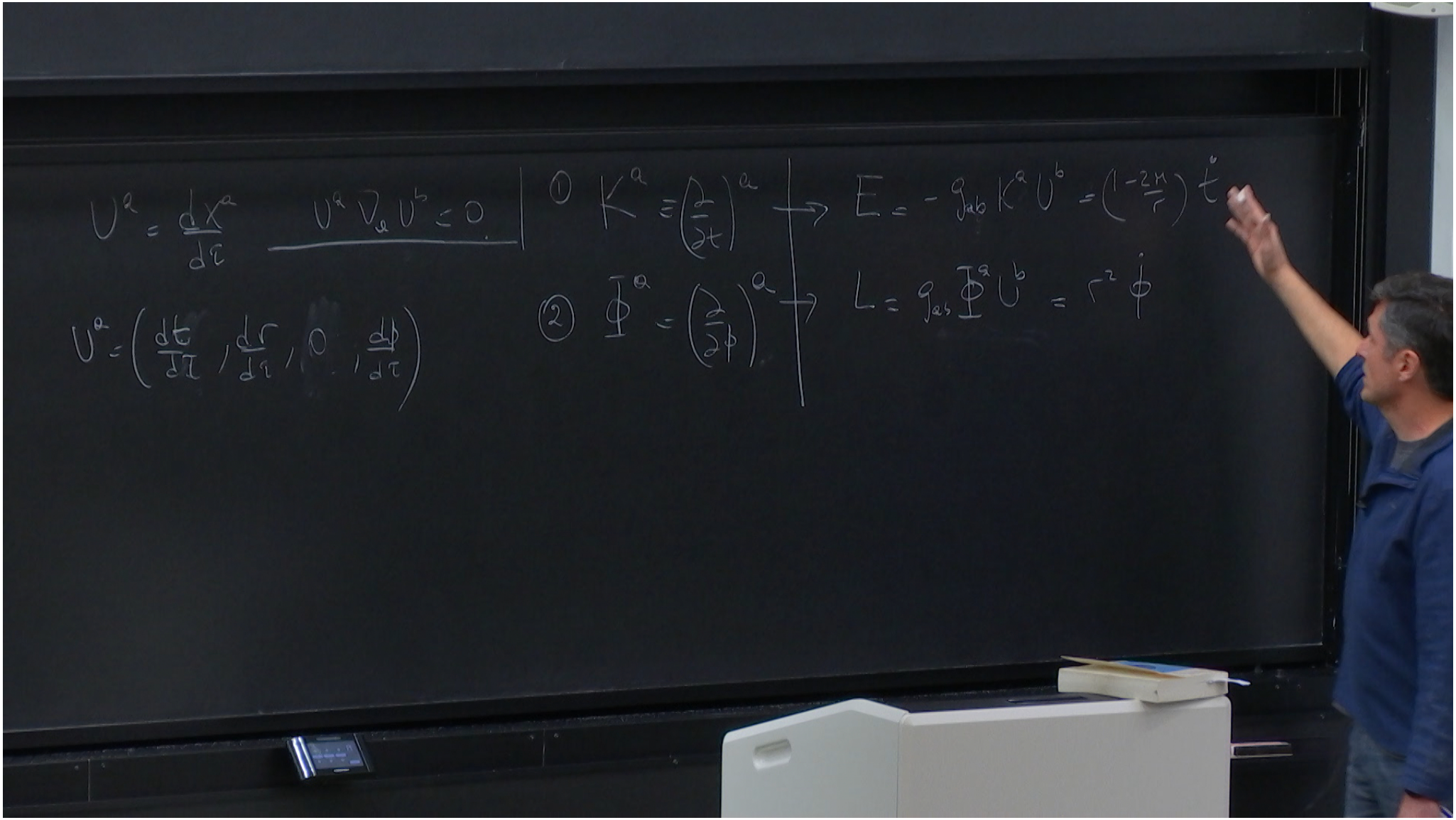

$W_1 \sqrt{1-\frac{2M}{r_1}} = W_2 \sqrt{1-\frac{2M}{r_2}}$
$W_1 \gg W_2$
$W_1 \times \sqrt{1-\frac{2M}{r_1}} = W_2 \times \sqrt{1-\frac{2M}{r_2}}$
$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$
$U^a = \frac{dx^a}{d\tau}$
$U^a \nabla_a U^b = 0$
① $K^a \equiv \left(\frac{\partial}{\partial t}\right)^a$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$
$E = -g_{ab} K^a U^b$

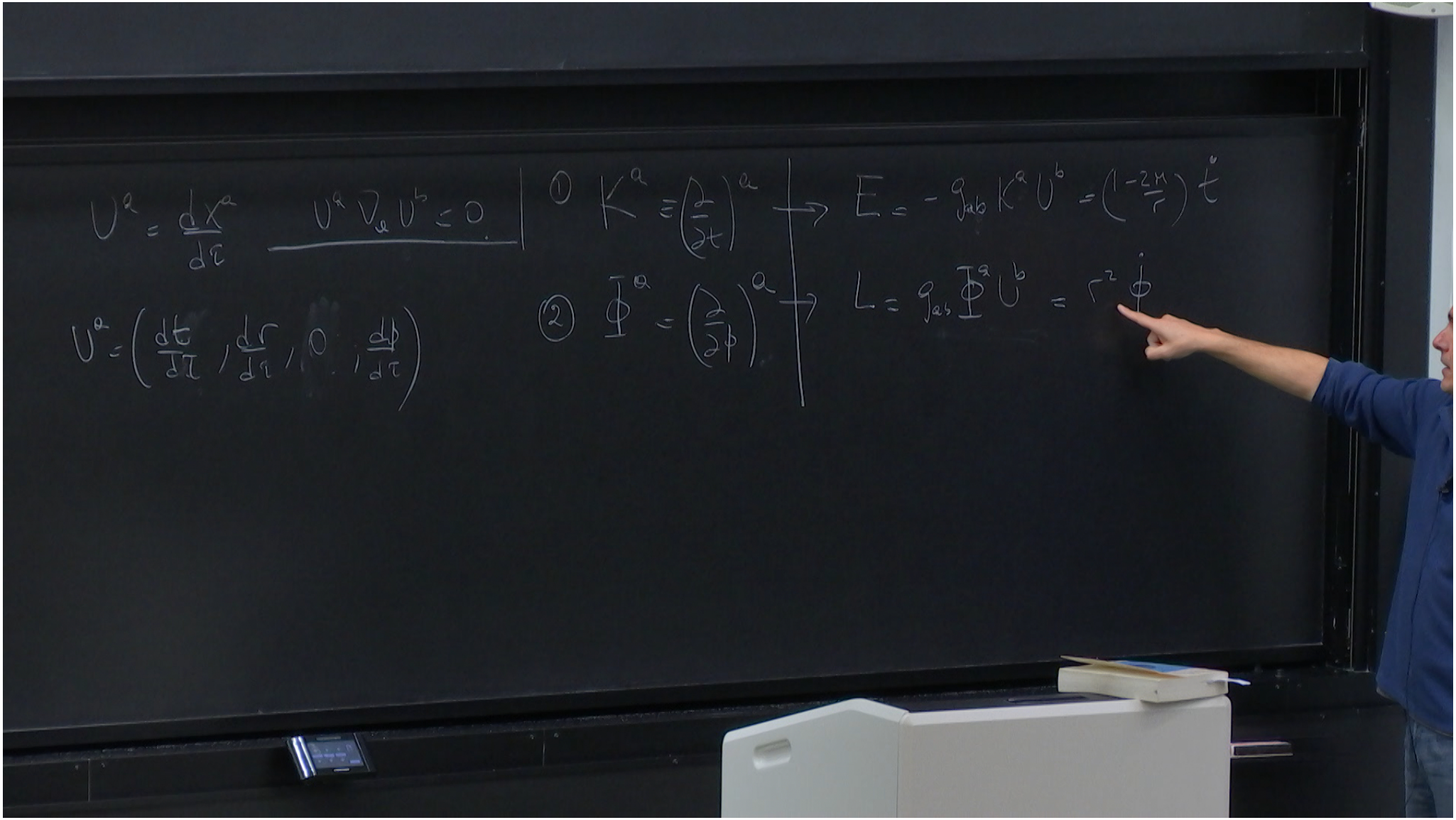

$W_1\sqrt{1-\frac{2M}{r_1}} = W_2\sqrt{1-\frac{2M}{r_2}}$
$W_1 \gg W_2$
$W_1 \times \sqrt{1-\frac{2M}{r_1}} = W_2 \times \sqrt{1-\frac{2M}{r_2}}$
$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$
$U^a = \frac{dx^a}{d\tau}$
$U^a \nabla_a U^b = 0$
① $K^a \equiv \left(\frac{\partial}{\partial t}\right)^a \rightarrow E = -g_{ab} K^a U^b$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \rightarrow L = g_{ab} \Phi^a U^b$

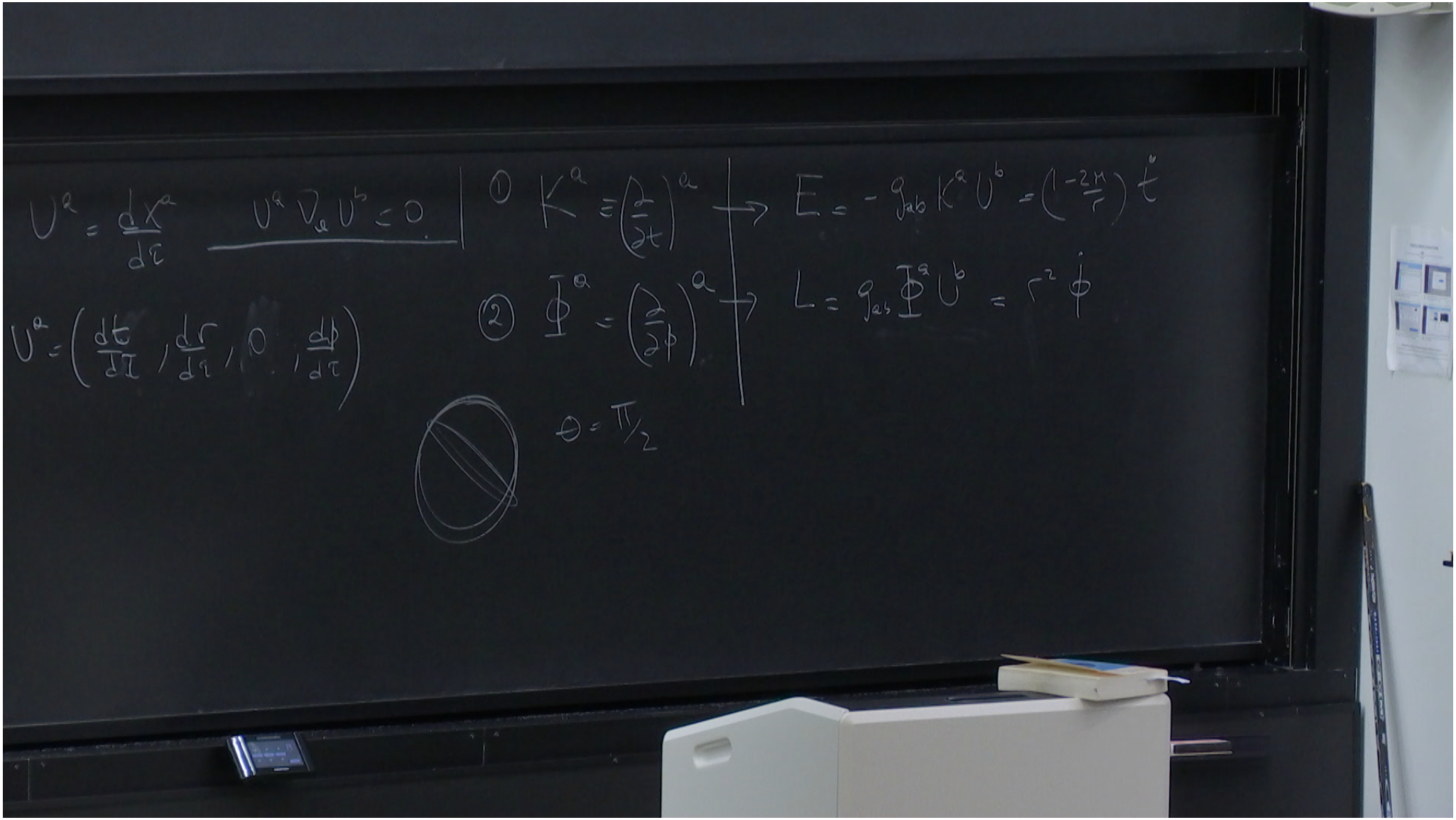

$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$
$U^a = \frac{dx^a}{d\tau}$
$U^a \nabla_a U^b = 0$
① $K^a \equiv \left(\frac{\partial}{\partial t}\right)^a \rightarrow E = -g_{ab} K^a U^b$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \rightarrow L = g_{ab} \Phi^a U^b =$
$U^a = \left(\frac{dt}{d\tau}, d\eta\right.$

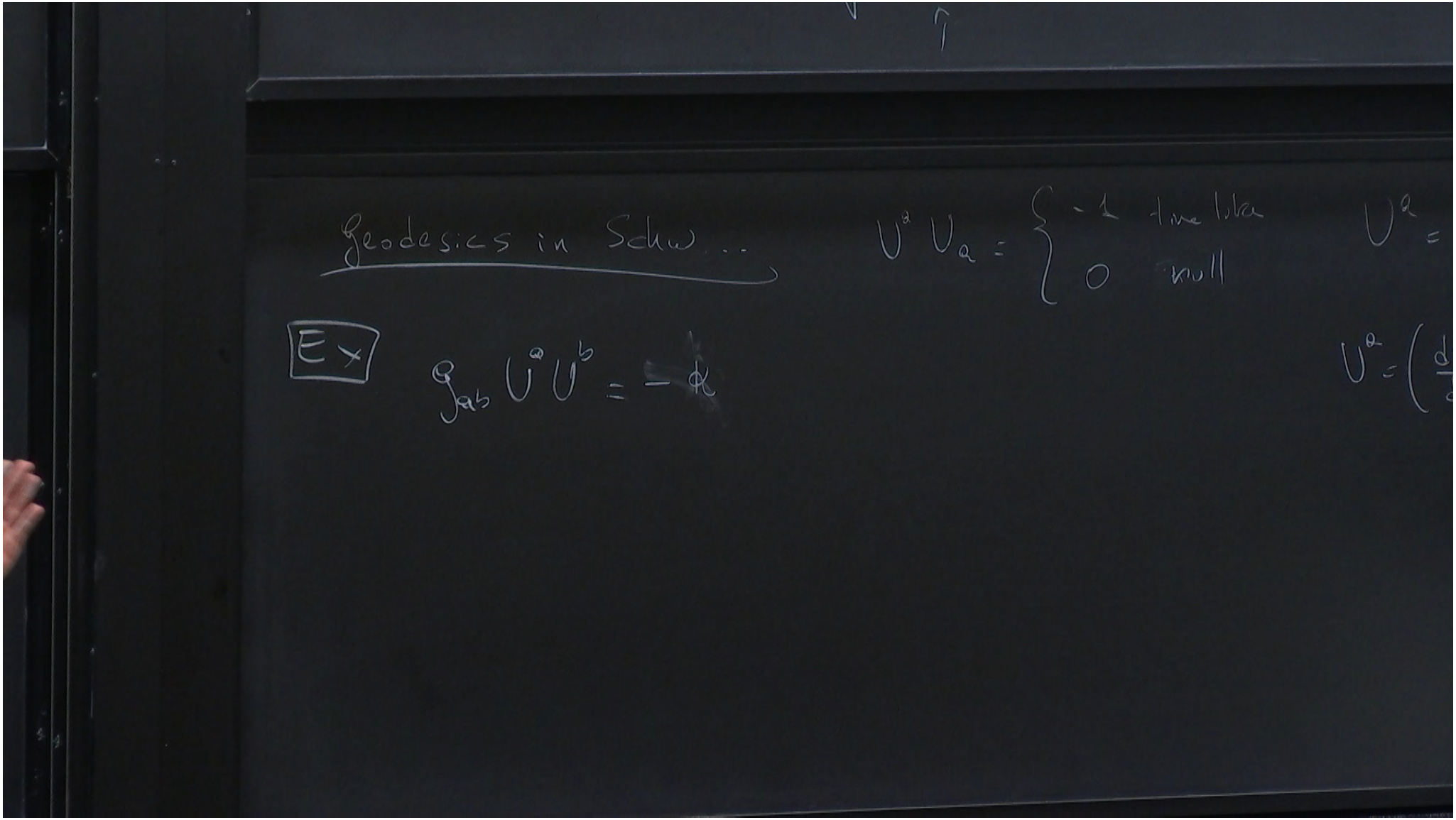

$\sqrt{1-\frac{2M}{r}}$
$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$
$U^a = \frac{dx^a}{d\tau}$
$U^a \nabla_a U^b = 0$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, \frac{d\theta}{d\tau}, \frac{d\phi}{d\tau}\right)$
① $K^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow E = -g_{ab} K^a U^b$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \rightarrow L = g_{ab} \Phi^a U^b =$


$U^a = \frac{dx^a}{d\tau}$ $U^a \nabla_a U^b = 0$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$
① $K^a = \left(\frac{\partial}{\partial t}\right)^a$ → $E = -g_{ab} K^a U^b = \left(1 - \frac{2M}{r}\right)\dot{t}$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a$ → $L = \Phi^a U^b = r^2 \dot{\phi}$


$U^a = \frac{dx^a}{d\tau}$ $U^a \nabla_a U^b = 0$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$
① $K^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow E = -g_{ab} K^a U^b = \left(1 - \frac{2M}{r}\right) \dot{t}$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \rightarrow L = g_{ab} \Phi^a U^b = r^2 \dot{\phi}$


$U^a = \frac{dx^a}{d\tau}$ $U^a \nabla_a U^b = 0$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$
① $K^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow E = -g_{ab} K^a U^b = \left(1 - \frac{2M}{r}\right) \dot{t}$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \rightarrow L = g_{ab} \Phi^a U^b = r^2 \dot{\phi}$


$U^a = \frac{dx^a}{d\tau}$ $U^a \nabla_a U^b = 0$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$
① $K^a = \left(\frac{\partial}{\partial t}\right)^a \rightarrow E = -g_{ab} K^a U^b = \left(1 - \frac{2M}{r}\right) \dot{t}$
② $\Phi^a = \left(\frac{\partial}{\partial \phi}\right)^a \rightarrow L = g_{ab} \Phi^a U^b = r^2 \dot{\phi}$
$\theta = \pi/2$


Geodesics in Schw...
$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$
Ex
$g_{ab} U^a U^b = -1$

Geodesics in Schw...

$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$

$U^a =$

$U^a = \left(\frac{d}{} \right.$

Ex $g_{ab} U^a U^b = -\kappa$

$$\hookrightarrow \boxed{\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} + \kappa\right) = \frac{1}{2}E^2}$$

Geodesics in Schw...

$U^a U_a = \begin{cases} -1 & \text{time like} \\ 0 & \text{null} \end{cases}$

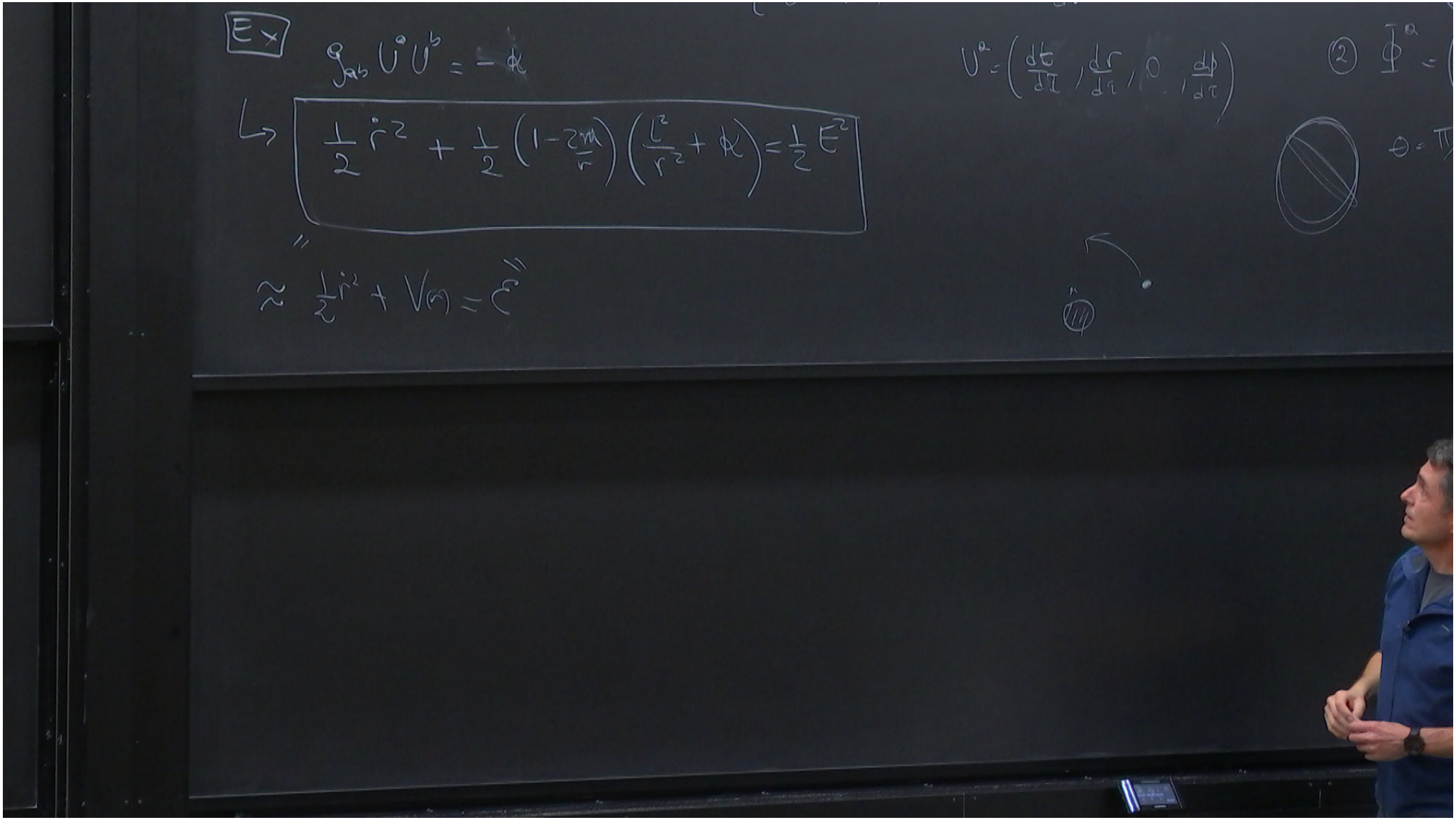
$U^a =$

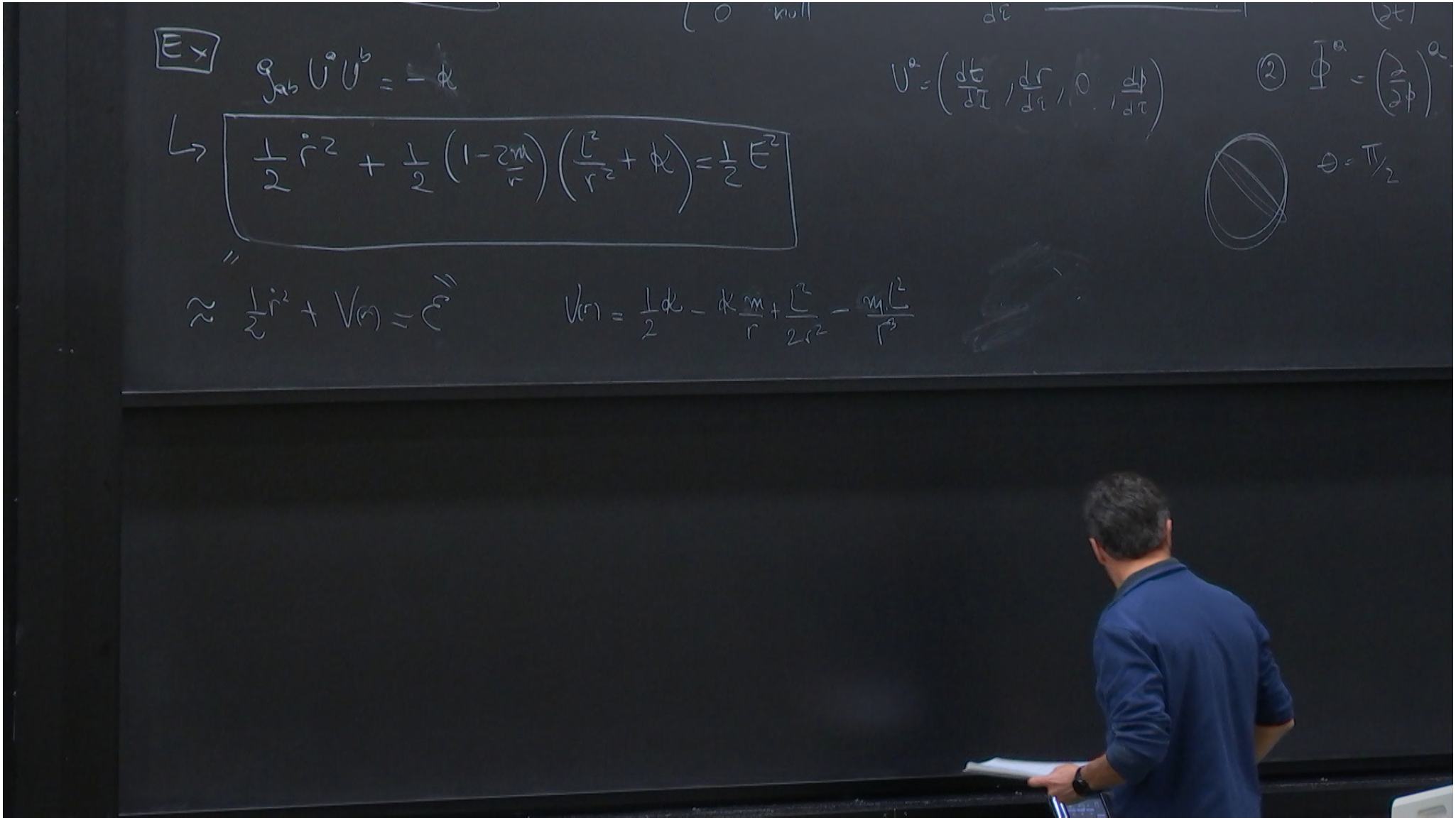
$U^a = \left(\frac{d}{} \right.$

Ex $g_{ab} U^a U^b = -\alpha$

$$\hookrightarrow \boxed{\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} + \alpha\right) = \frac{1}{2}E^2}$$

$$\approx \frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}$$


Ex
$g_{ab} U^a U^b = -\kappa$
$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(\frac{L^2}{r^2}+\kappa\right) = \frac{1}{2}E^2$
$\approx \frac{1}{2}\dot{r}^2 + V_{eff} = \mathcal{E}$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$
② $\vec{\phi}^a$
$\theta = \pi$

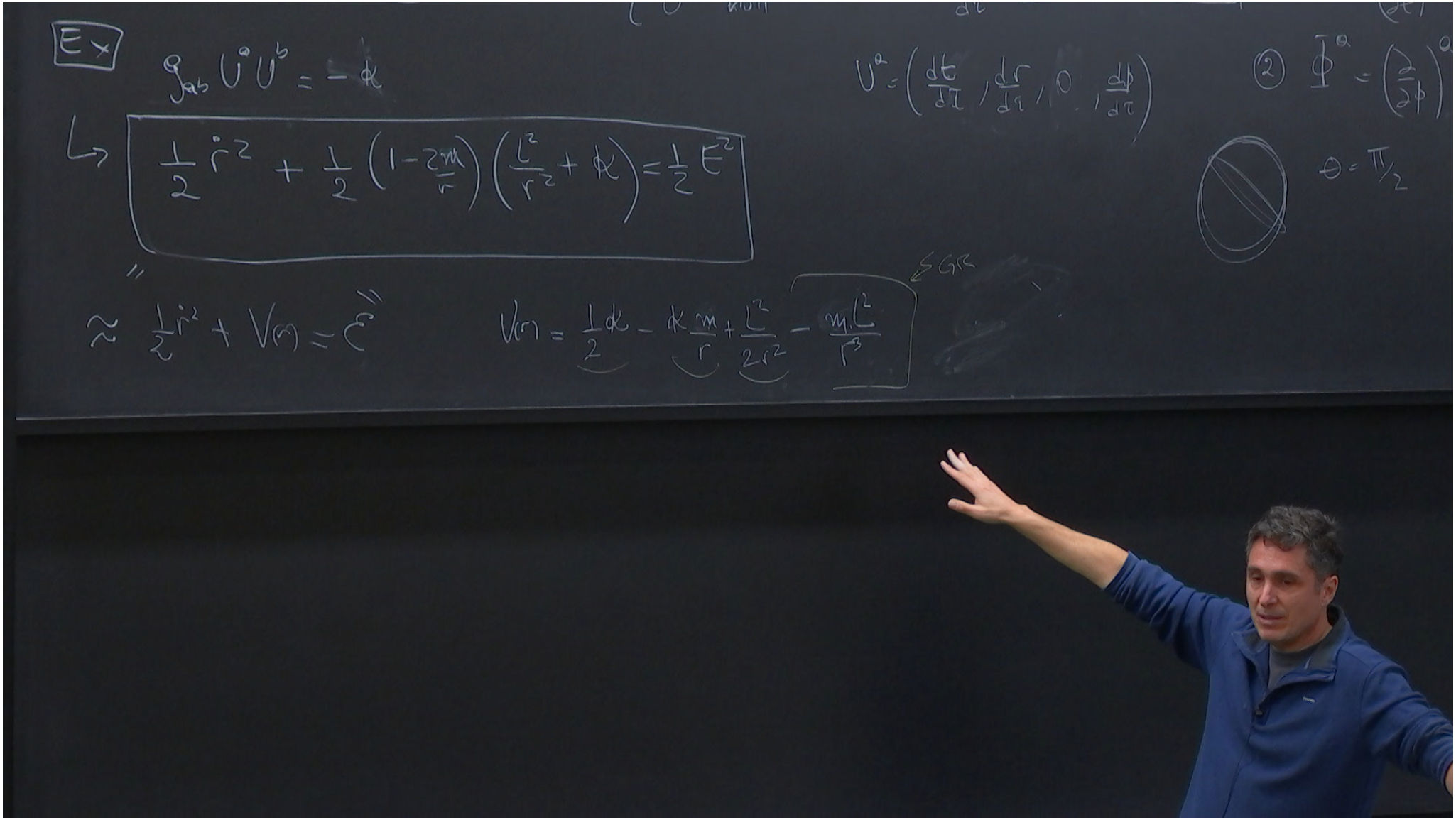

Ex
$g_{ab} U^a U^b = -\alpha$
$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(\frac{L^2}{r^2}+\alpha\right) = \frac{1}{2}E^2$
$\approx \frac{1}{2}\dot{r}^2 + V(r) = E$
$V(r) = \frac{1}{2}\alpha - \alpha\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3}$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$
② $\Phi^a = \left(\frac{\partial}{\partial\phi}\right)^a$
$\theta = \pi/2$

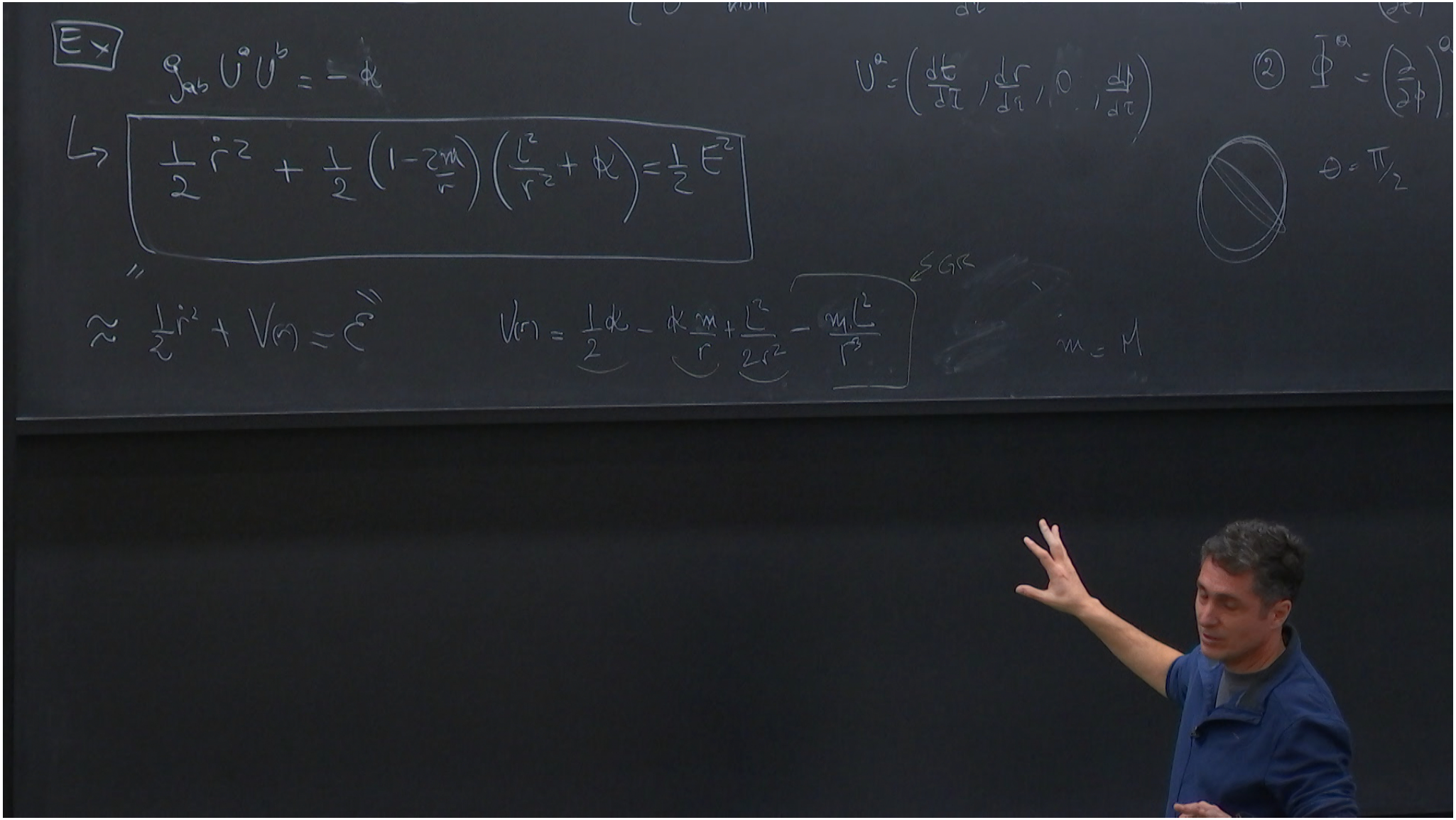
$g_{ab} U^a U^b = -\alpha$

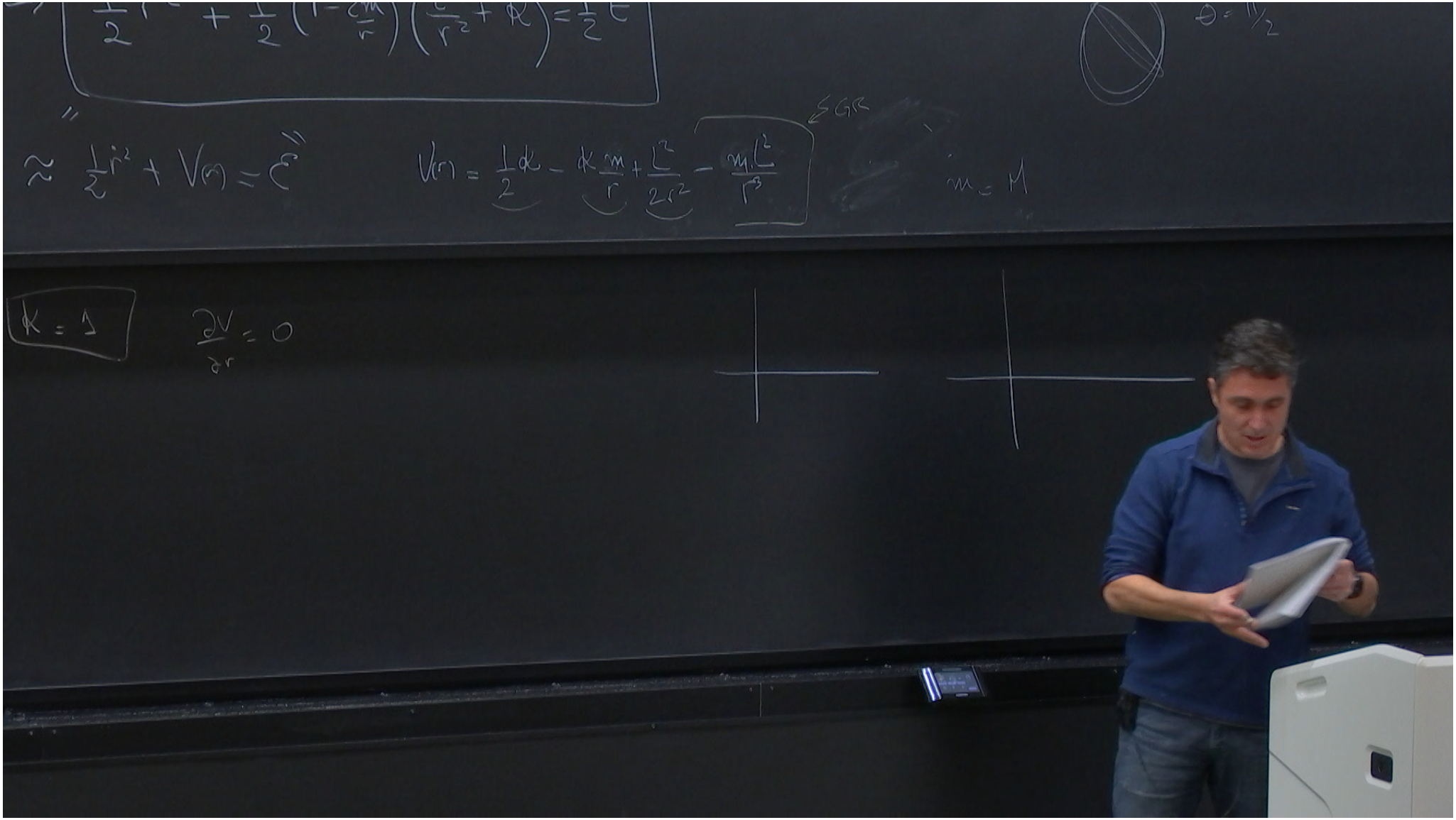
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau} \right)$ (2)

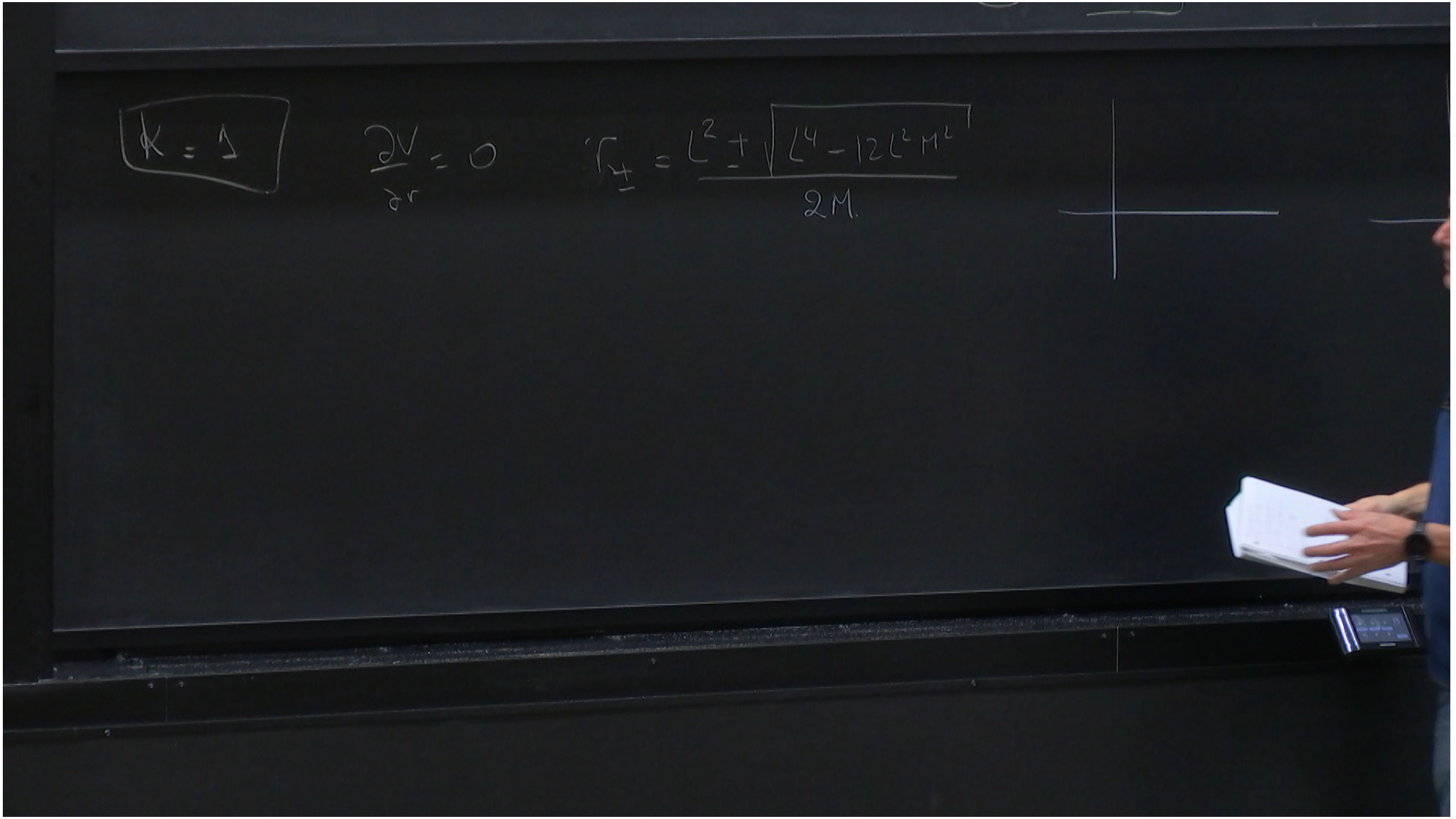
$$\hookrightarrow \boxed{\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} + \alpha\right) = \frac{1}{2}E^2}$$

$$\approx \frac{1}{2}\dot{r}^2 + V(r) = E \qquad \boxed{V(r) = \frac{1}{2}\alpha - \alpha\frac{m}{r} + \frac{L^2}{2r^2} - \frac{m L^2}{r^3}} \leftarrow \text{GR}$$


Ex
$g_{ab} U^a U^b = -\alpha$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$
② $\Phi^a = \left(\frac{\partial}{\partial\phi}\right)^a$
$\theta = \pi/2$
$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} + \alpha\right) = \frac{1}{2}E^2$
$\approx \frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}$
$V(r) = \frac{1}{2}\alpha - \alpha\frac{m}{r} + \frac{L^2}{2r^2} - \frac{m_1 L^2}{r^3}$
$\leftarrow$ GR


Ex
$g_{ab} U^a U^b = -\alpha$
$U^a = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, \frac{d\phi}{d\tau}\right)$
② $\Phi^a = \left(\frac{\partial}{\partial\phi}\right)^a$
$\theta = \pi/2$
$\hookrightarrow \frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1 - \frac{2m}{r}\right)\left(\frac{L^2}{r^2} + \alpha\right) = \frac{1}{2}E^2$
$\approx \frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}$
$V(r) = \frac{1}{2}\alpha - \alpha\frac{m}{r} + \frac{L^2}{2r^2} - \frac{m_1 L^2}{r^3}$
$\leftarrow$ GR
$m = M$

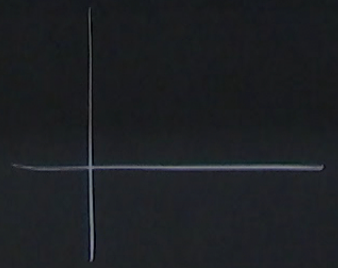

$\frac{1}{2}\dot{r}^2 + V(r) = \mathcal{E}$
$V(r) = \frac{1}{2}\alpha - \alpha\frac{m}{r} + \frac{L^2}{2r^2} - \frac{m_1 L^2}{r^3}$
GR
$m = M$
$\theta = \pi/2$
$\alpha = 1$
$\frac{\partial V}{\partial r} = 0$


k = 1
∂V/∂r = 0
r± = L² ± √(L⁴ − 12L²M²) / 2M

$k = 1$

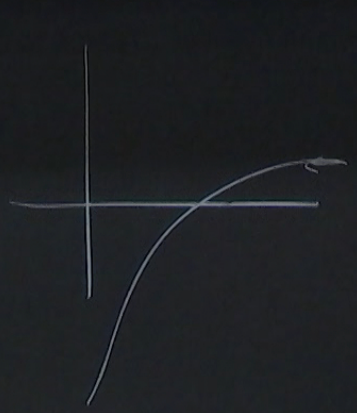
$$\frac{\partial V}{\partial r} = 0 \qquad r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$$

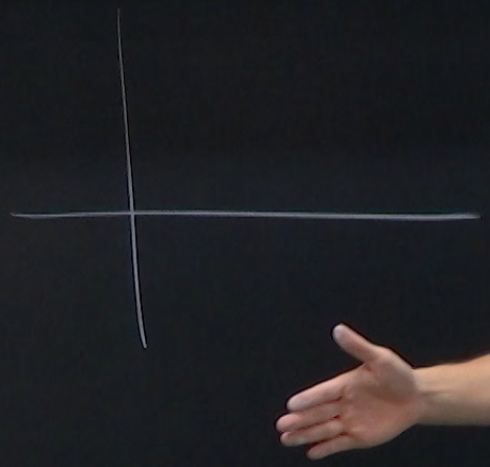
if $L^2 < 12M^2$

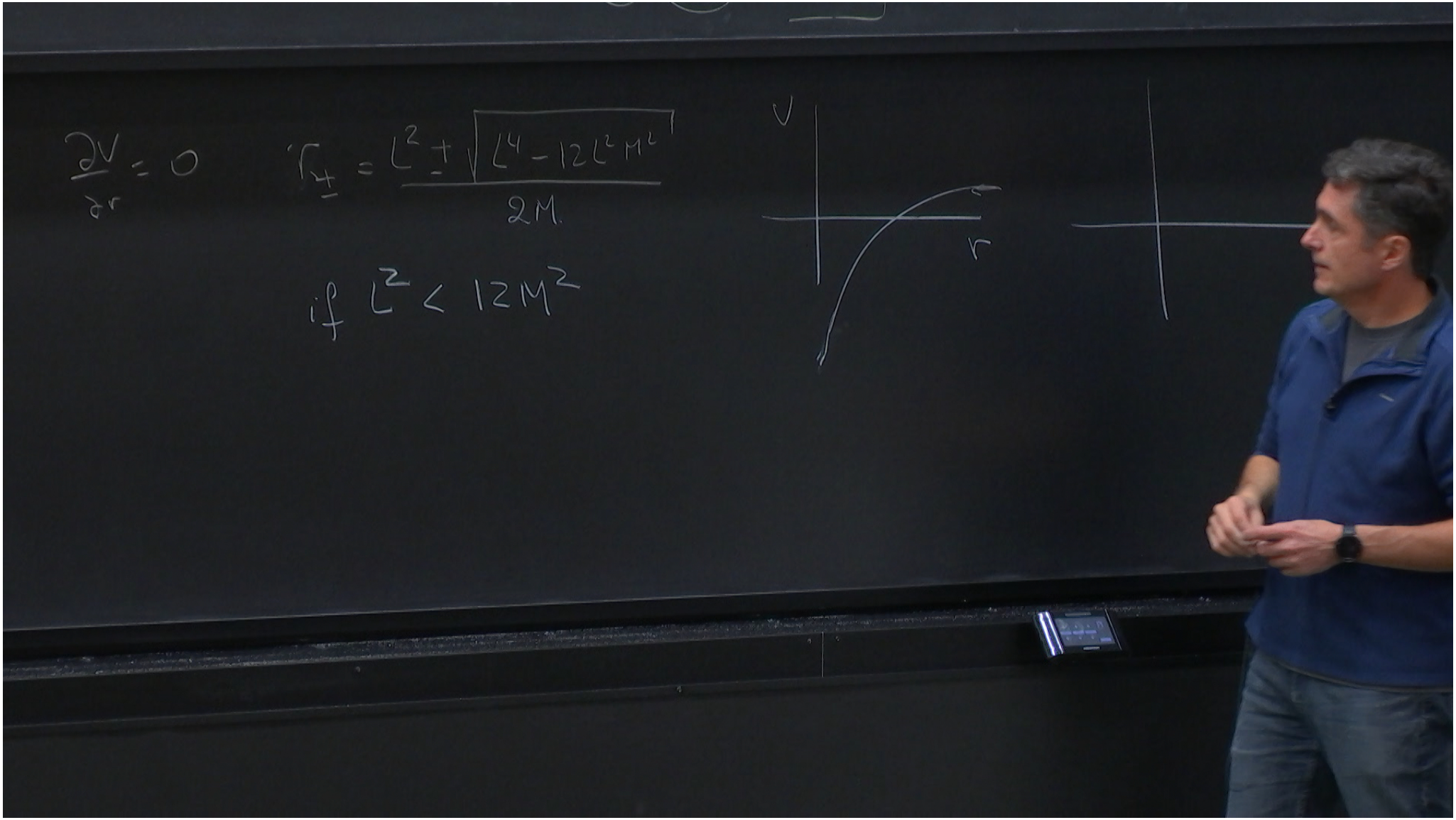


$$\frac{dV}{dr} = 0 \qquad r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$$

if $L^2 < 12M^2$

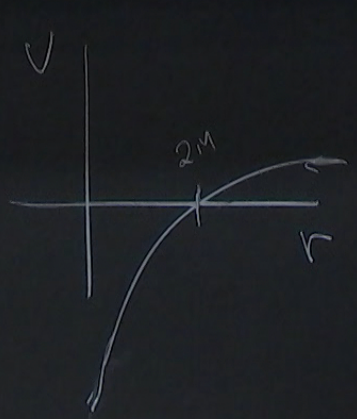


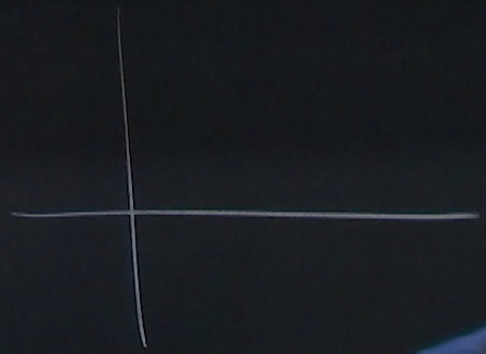


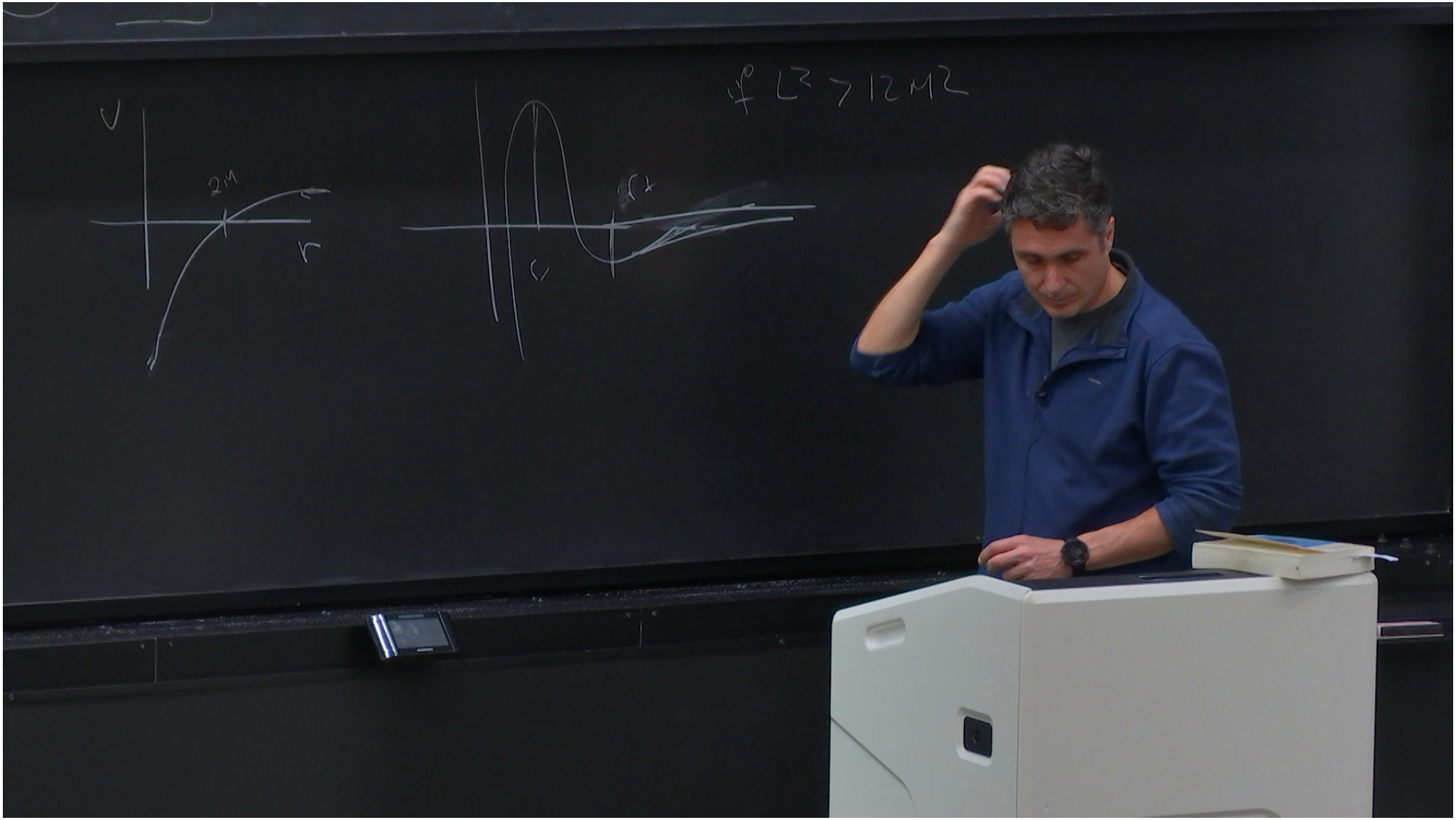

$\frac{dV}{dr} = 0$
$r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
r

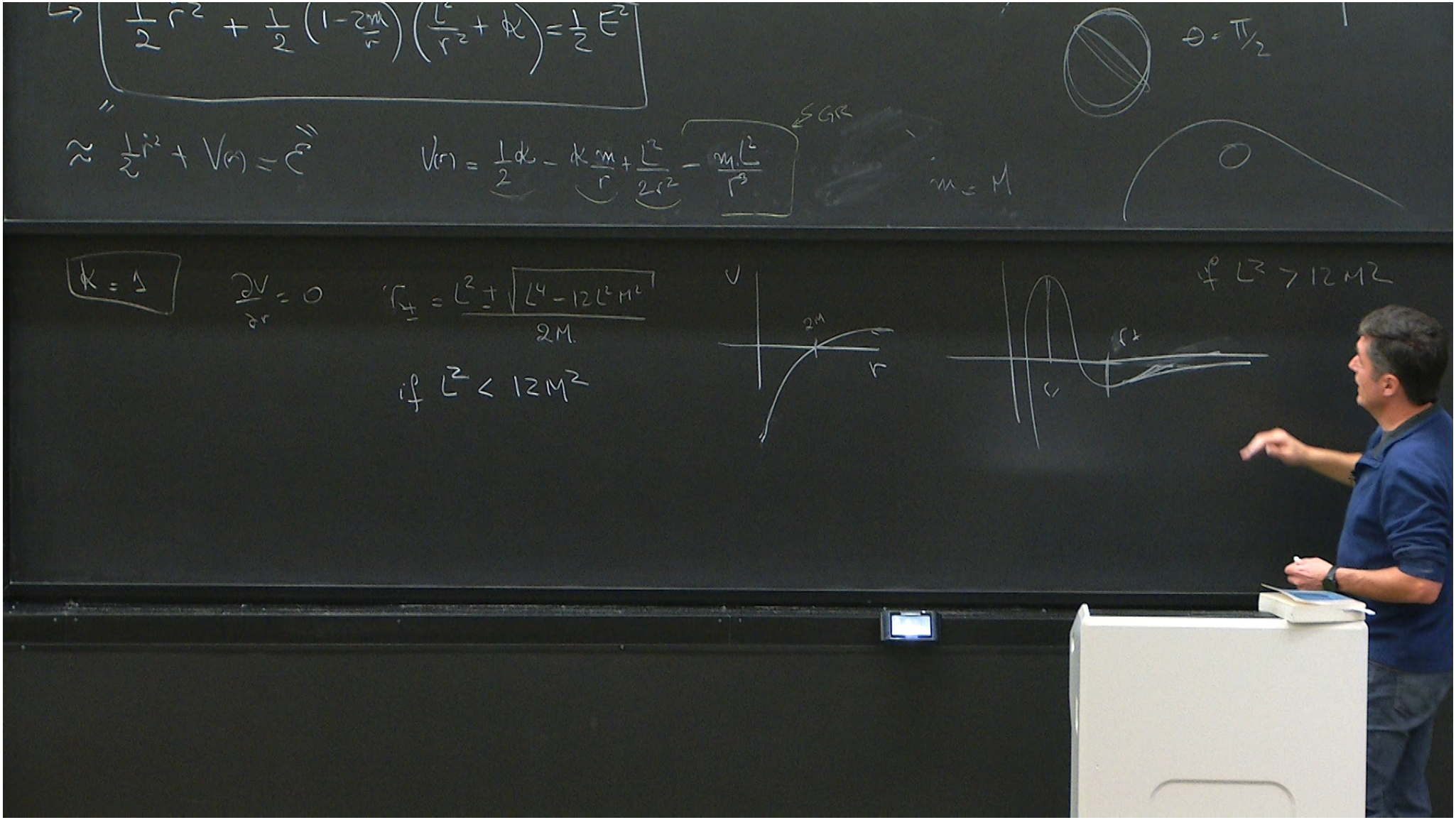
$\frac{dV}{dr} = 0 \qquad r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$

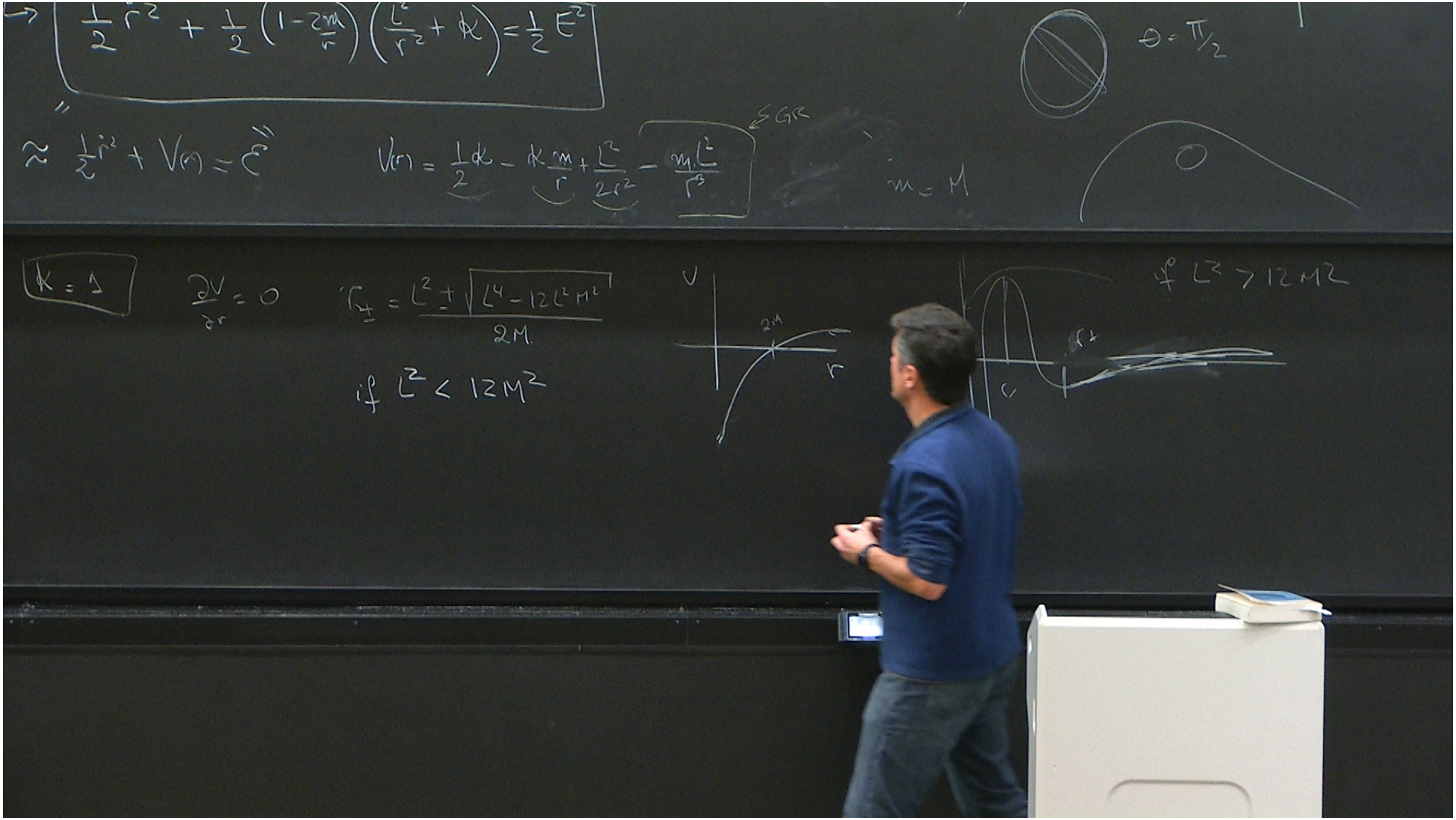
if $L^2 < 12M^2$

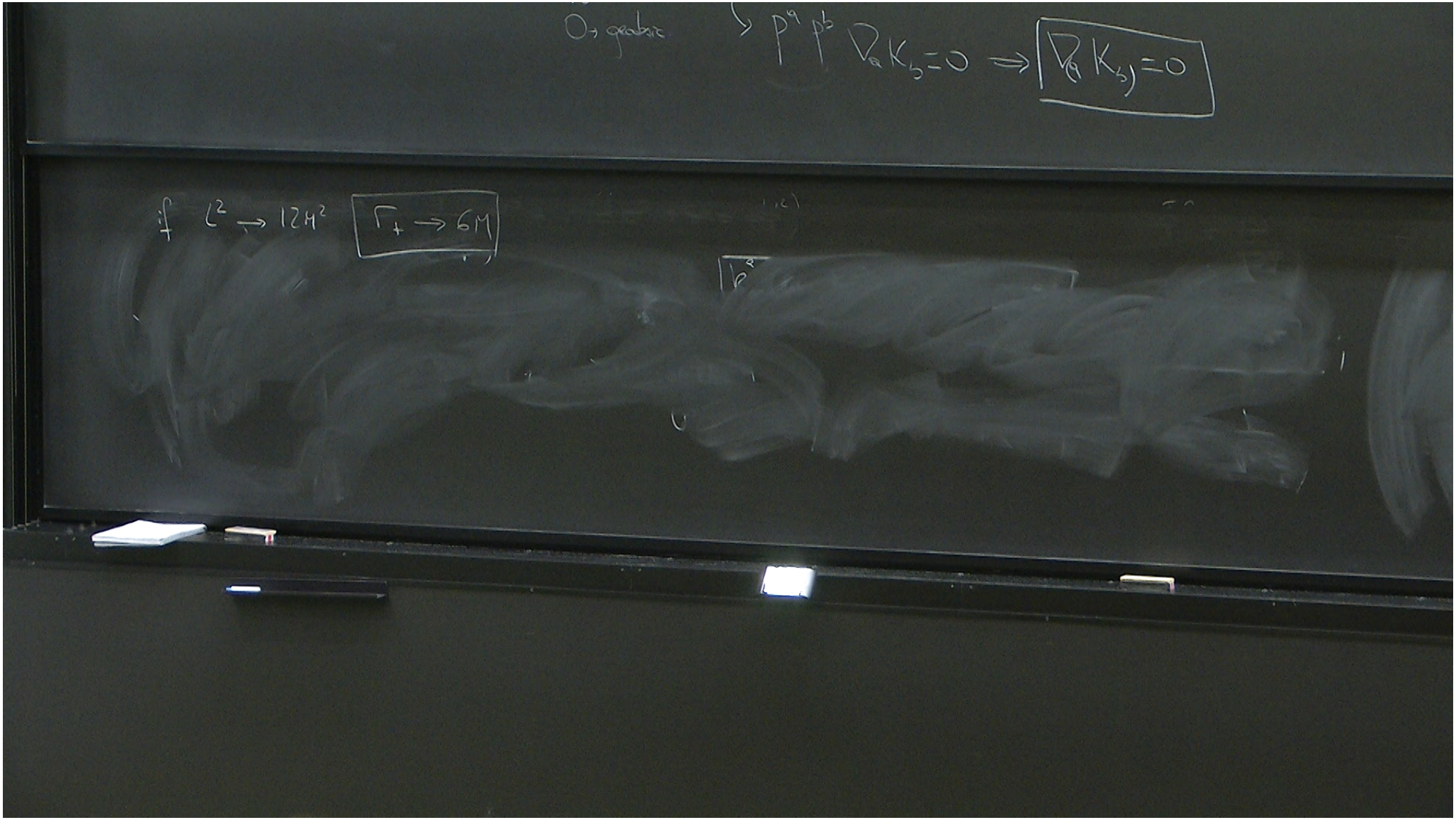


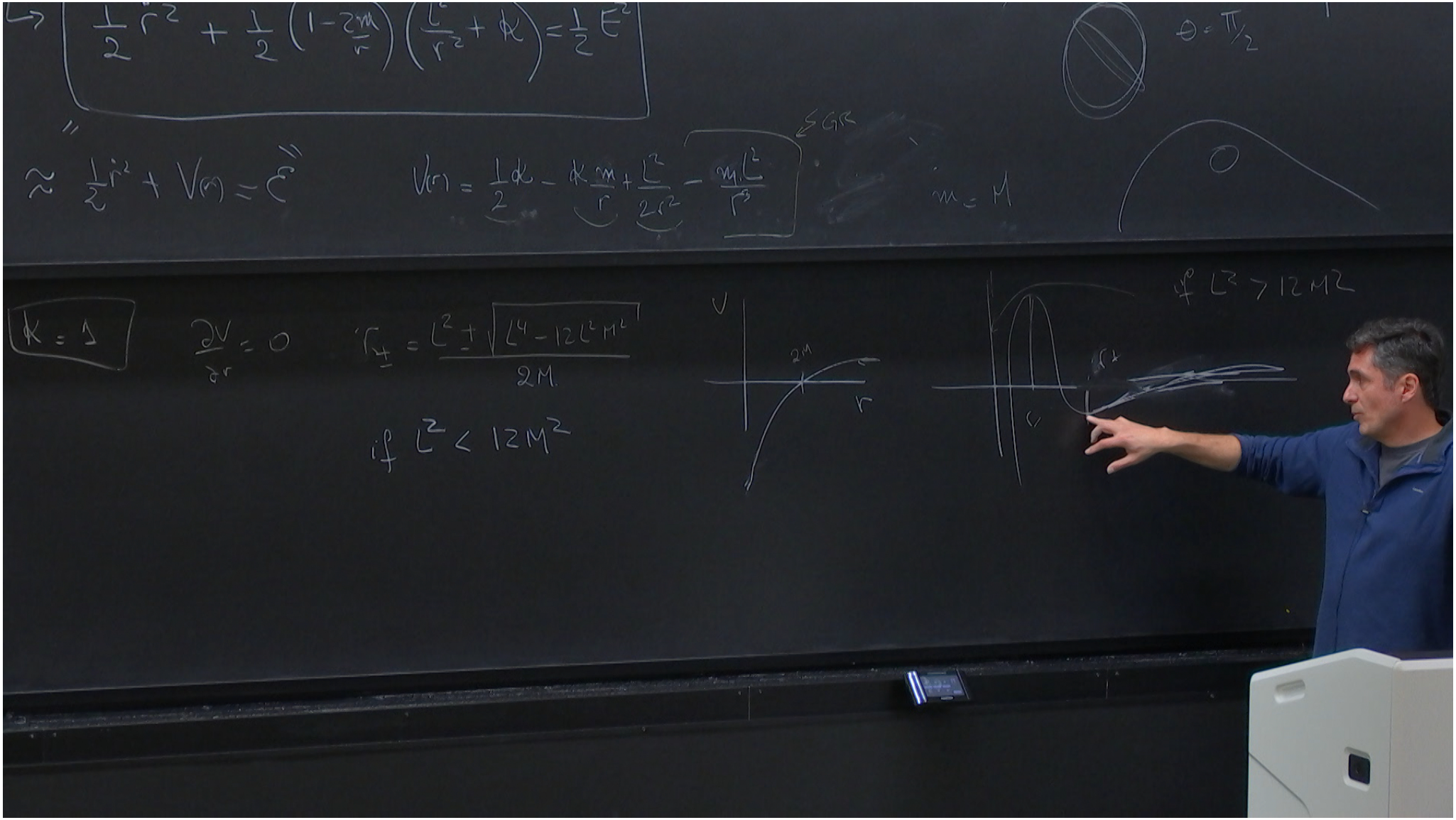


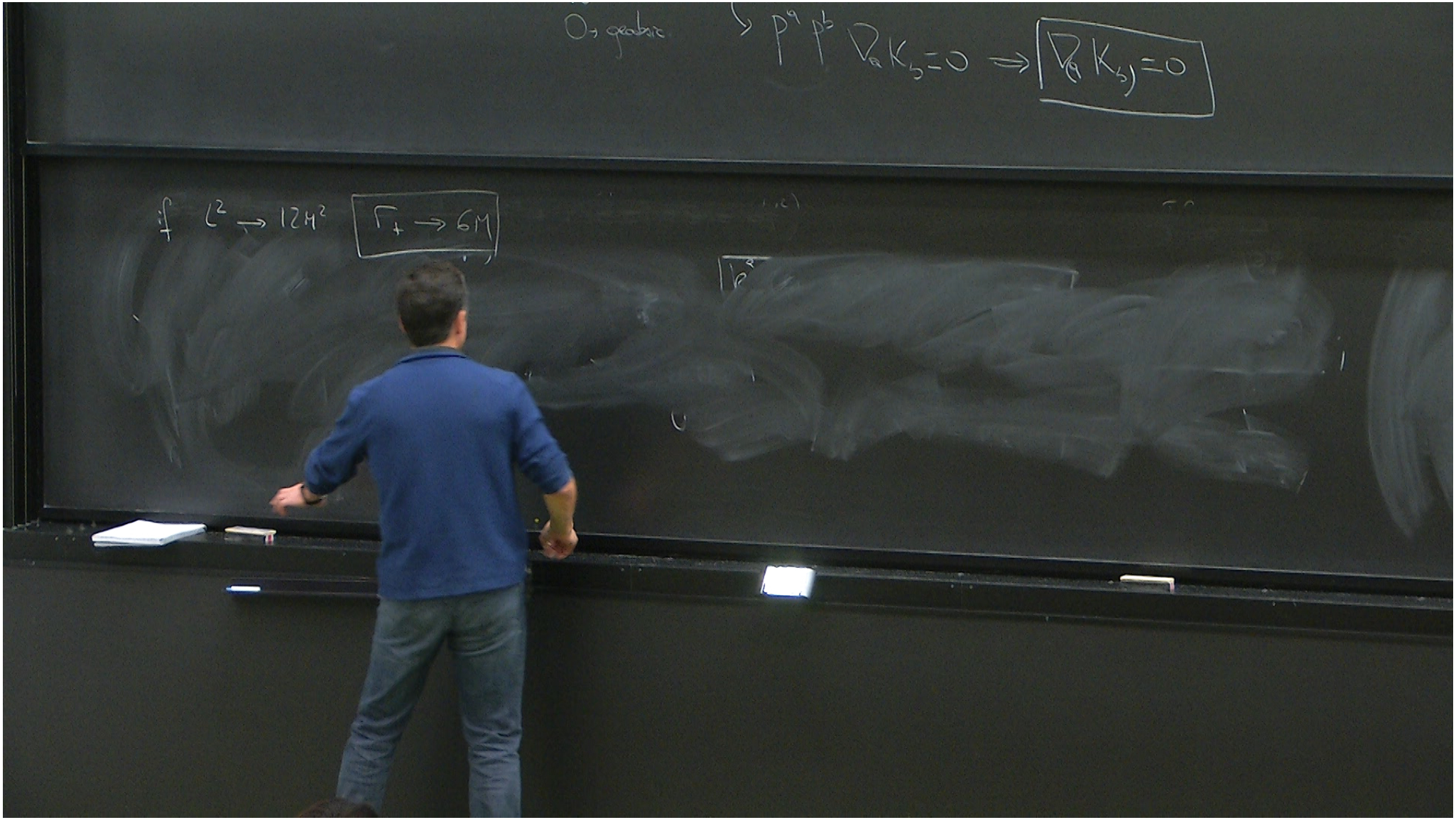

V
2M
r
r
if L² > 12M²

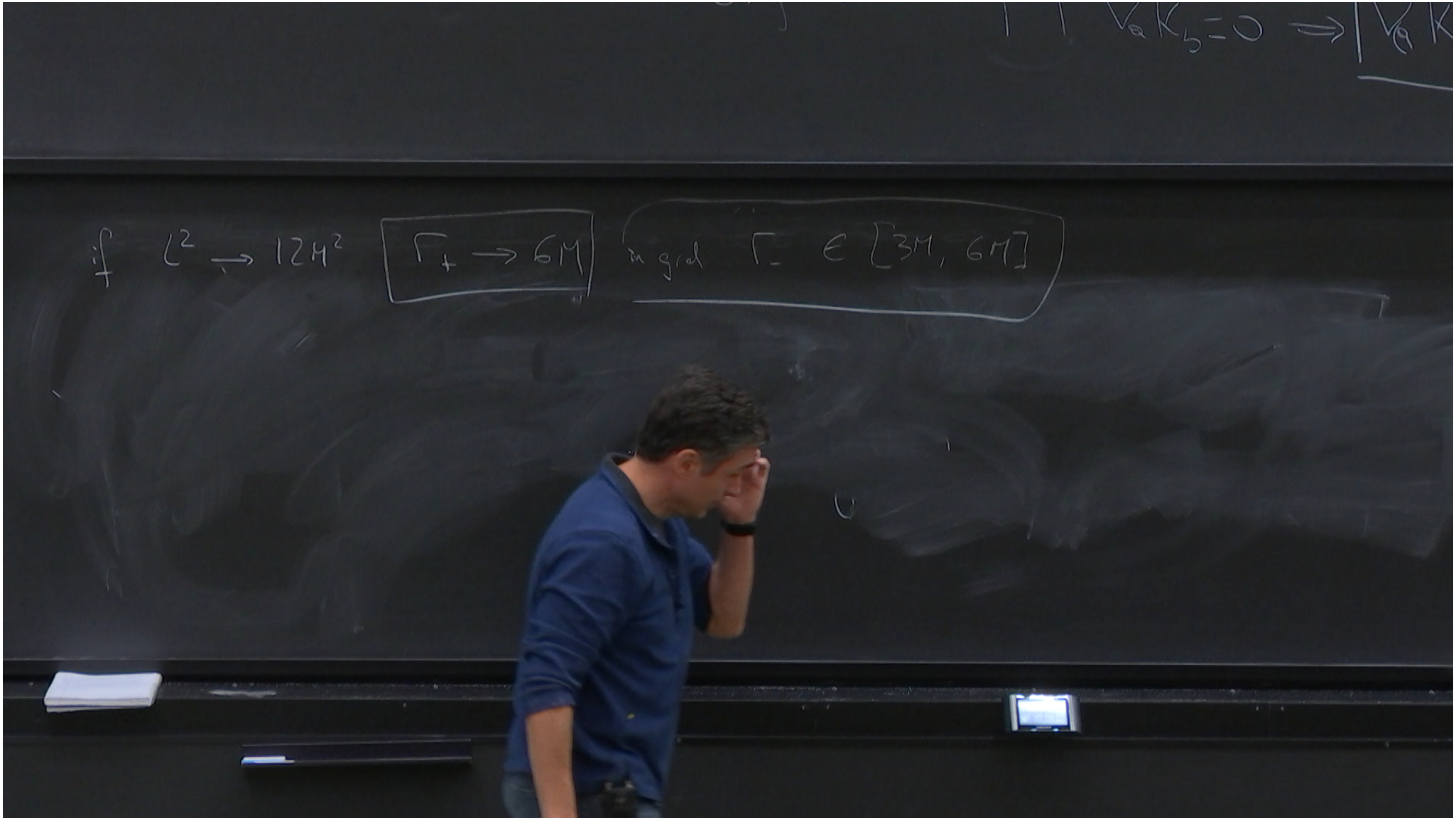

$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(\frac{L^2}{r^2}+k\right) = \frac{1}{2}E^2$
$\theta = \pi/2$
$\approx \frac{1}{2}\dot{r}^2 + V(r) = E^2$
$V(r) = \frac{1}{2}k - k\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3}$
GR
$m = M$
$k = 1$
$\frac{\partial V}{\partial r} = 0$
$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
2M
r
if $L^2 > 12M^2$
r_+
r_-

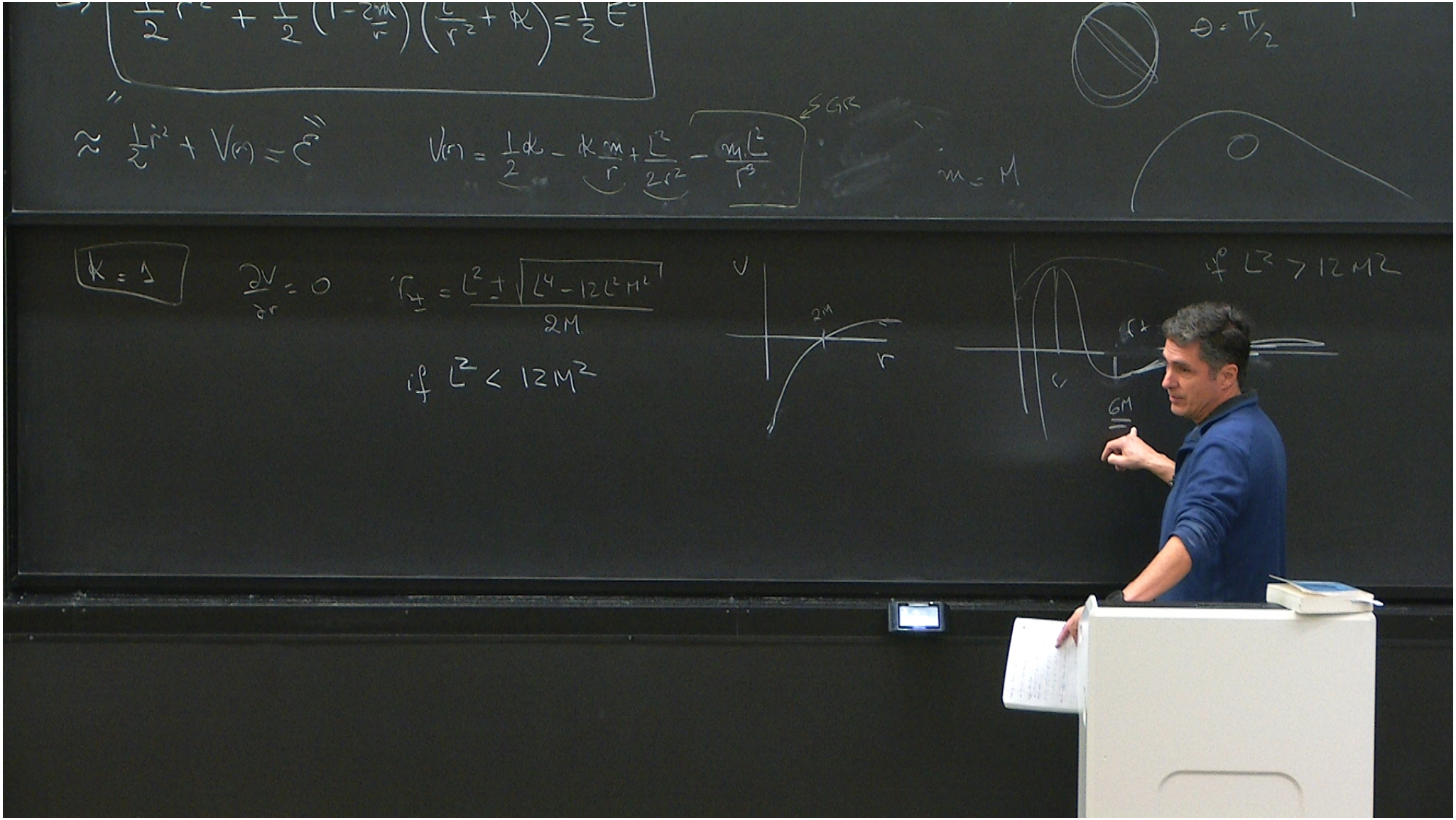

$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L^2}{r^2}+k\right) = \frac{1}{2}E^2$
$\theta = \pi/2$
$\approx \frac{1}{2}\dot{r}^2 + V(r) = E$
$V(r) = \frac{1}{2}k - k\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$
GR
$M = M$
$k = 1$
$\frac{\partial V}{\partial r} = 0$
$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
2M
r
if $L^2 > 12M^2$
r_+
c

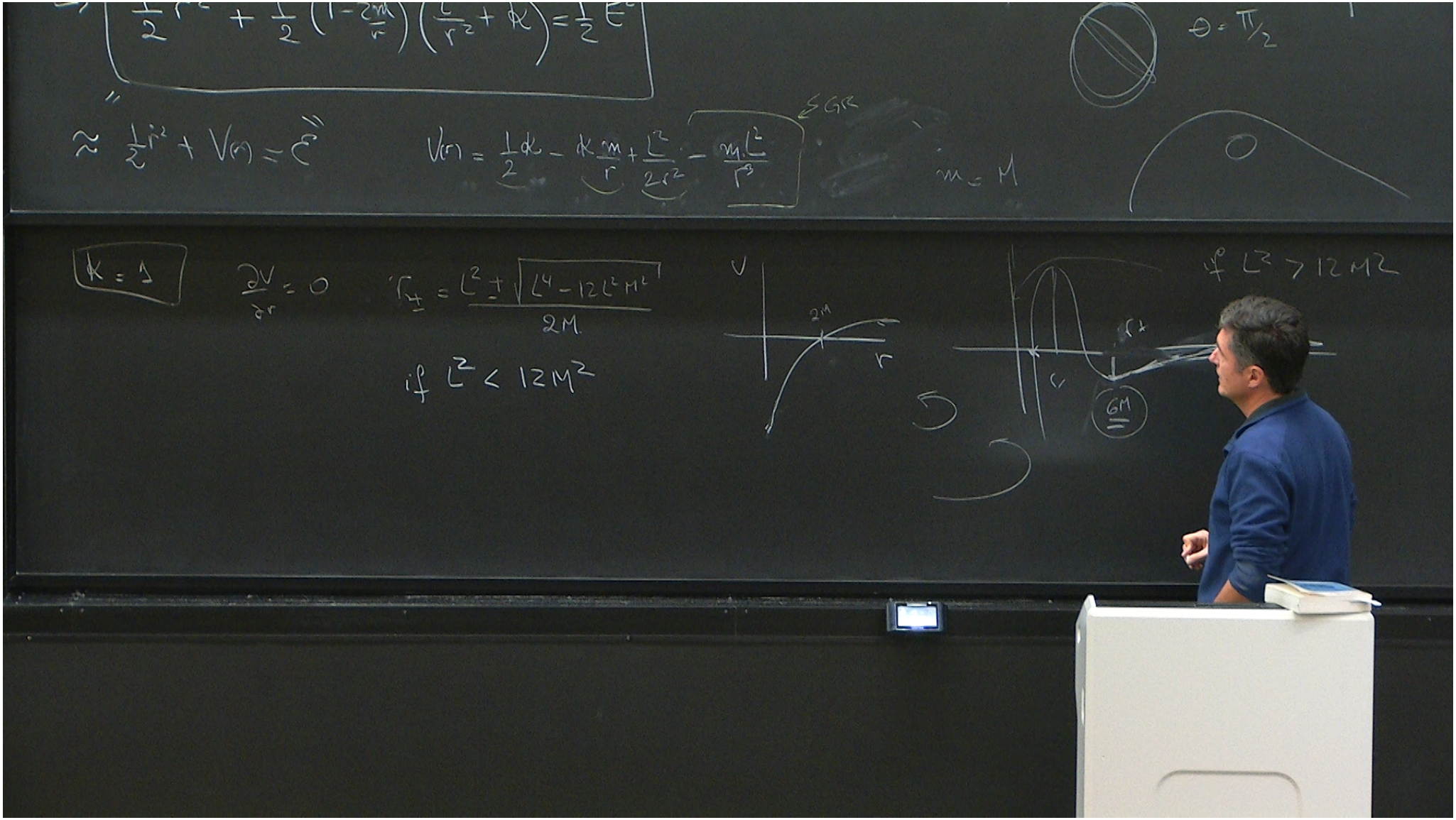

O_3 geodesic
$\Rightarrow p^a p^b \nabla_a K_b = 0 \Rightarrow \boxed{\nabla_{(a} K_{b)} = 0}$
if $L^2 \to 12M^2$ $\boxed{r_+ \to 6M}$

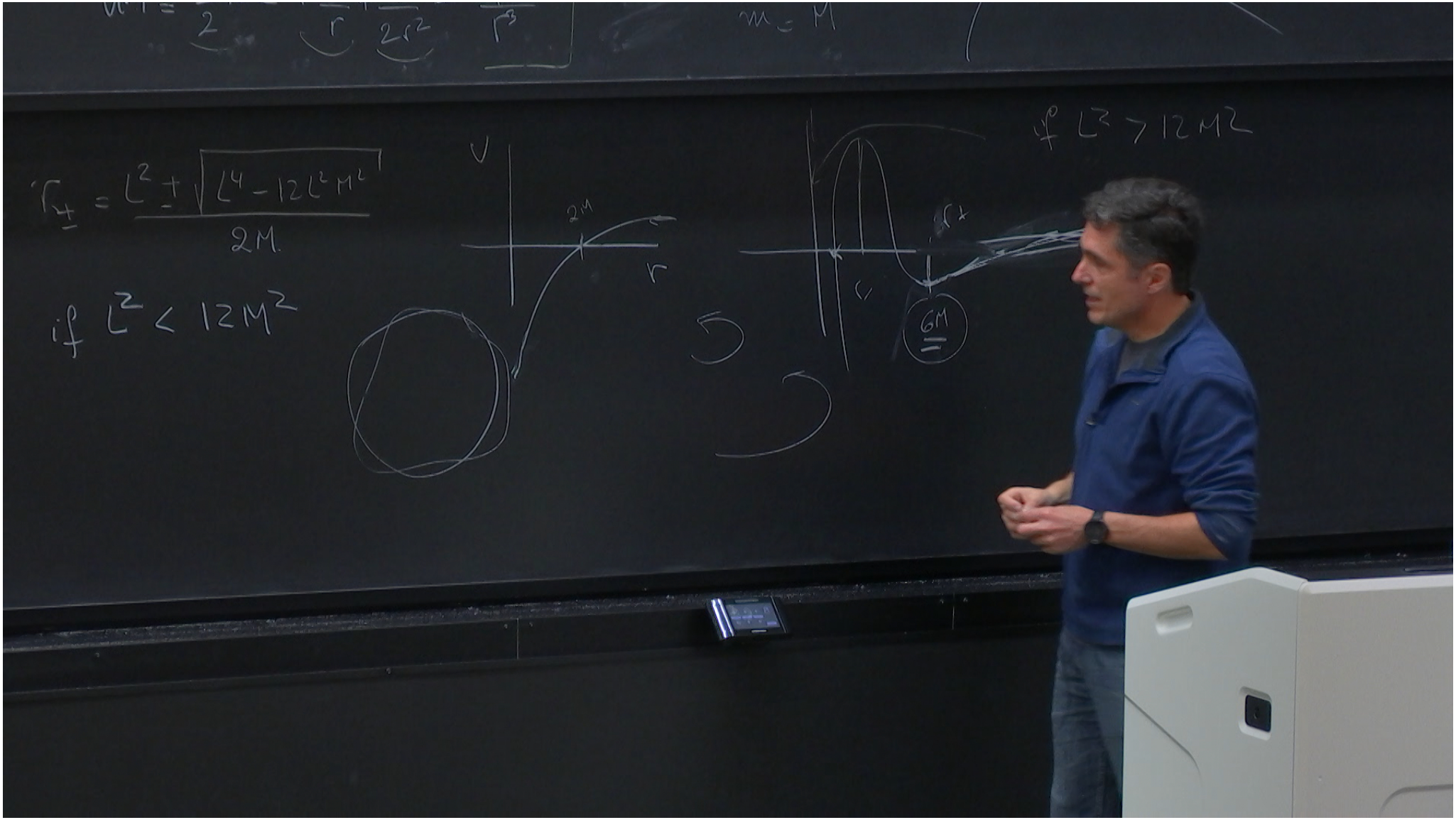

$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2m}{r}\right)\left(\frac{L^2}{r^2}+k\right) = \frac{1}{2}E^2$
$\theta = \pi/2$
$\approx \frac{1}{2}\dot{r}^2 + V(r) = E''$
$V(r) = \frac{1}{2}k - k\frac{m}{r} + \frac{L^2}{2r^2} - \frac{mL^2}{r^3}$
GR
$m = M$
$k = 1$
$\frac{\partial V}{\partial r} = 0$
$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
2M
r
if $L^2 > 12M^2$
r_+

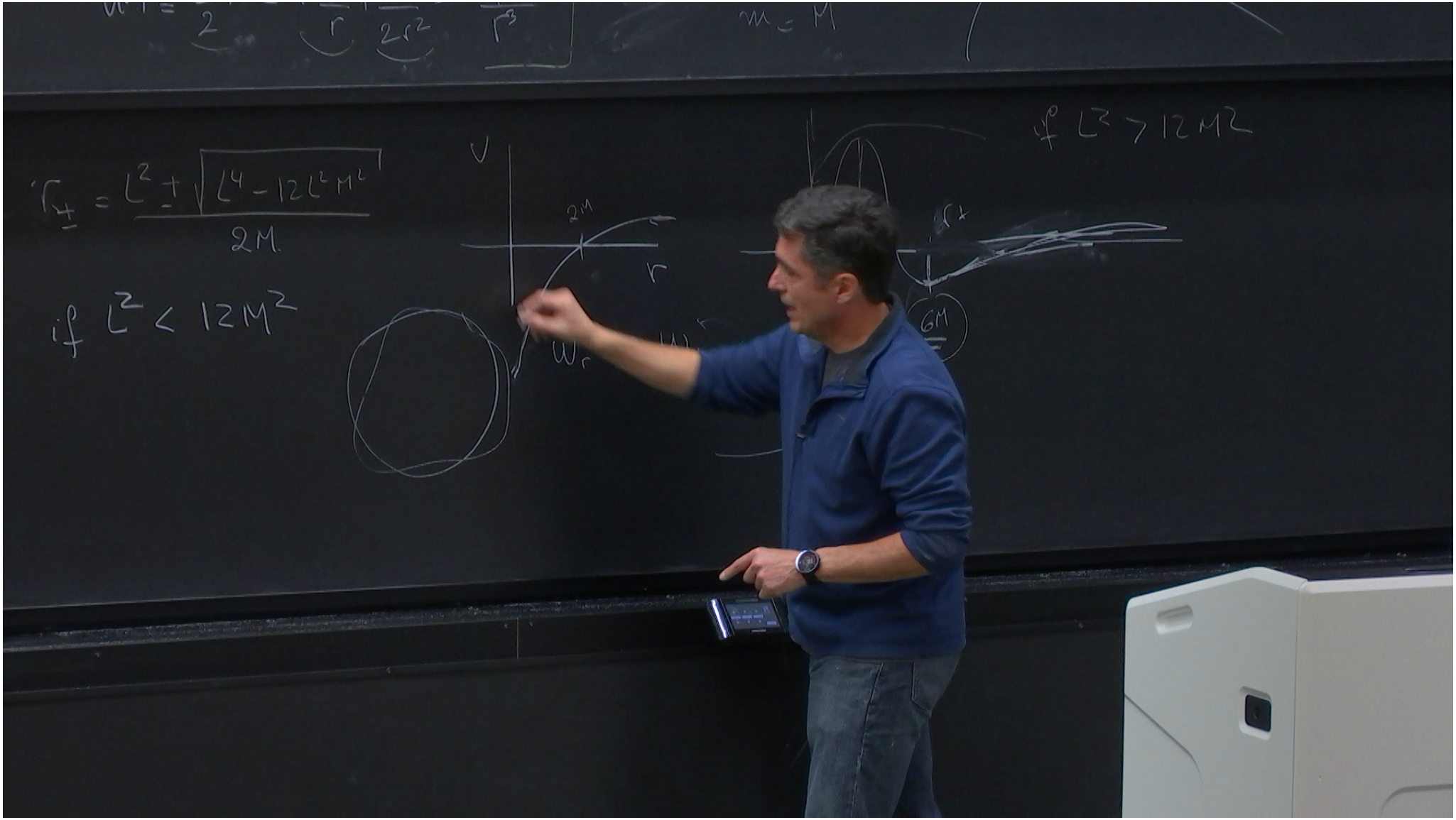

O_3 geodesic
$\hookrightarrow p^a p^b \nabla_a K_3 = 0 \Rightarrow \boxed{\nabla_{(a} K_{3)} = 0}$
if $L^2 \to 12M^2$ $\boxed{r_+ \to 6M}$
p^a

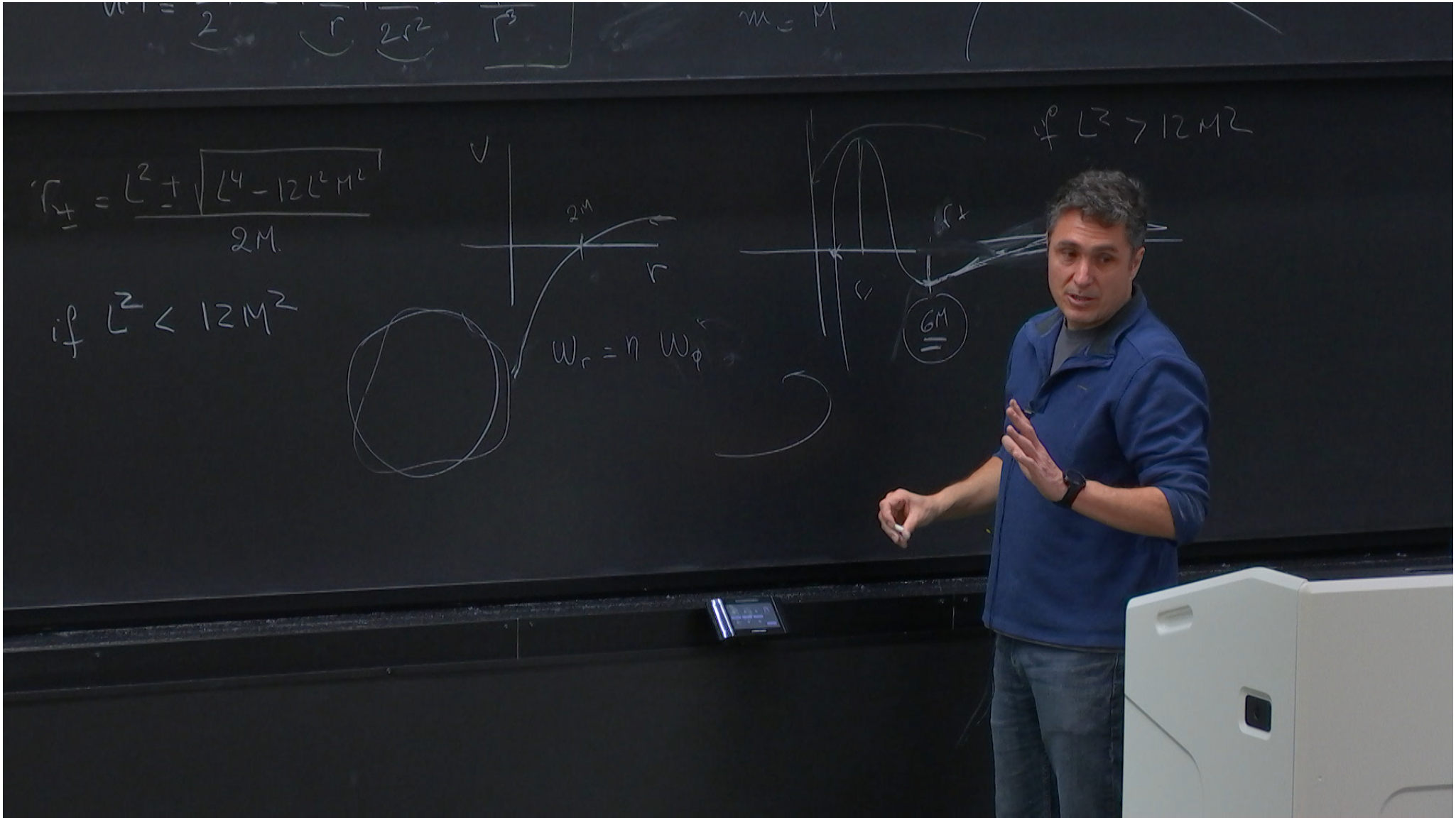

if $L^2 \to 12M^2$ $\Gamma_+ \to 6M$ in grid $\Gamma_- \in [3M, 6M]$

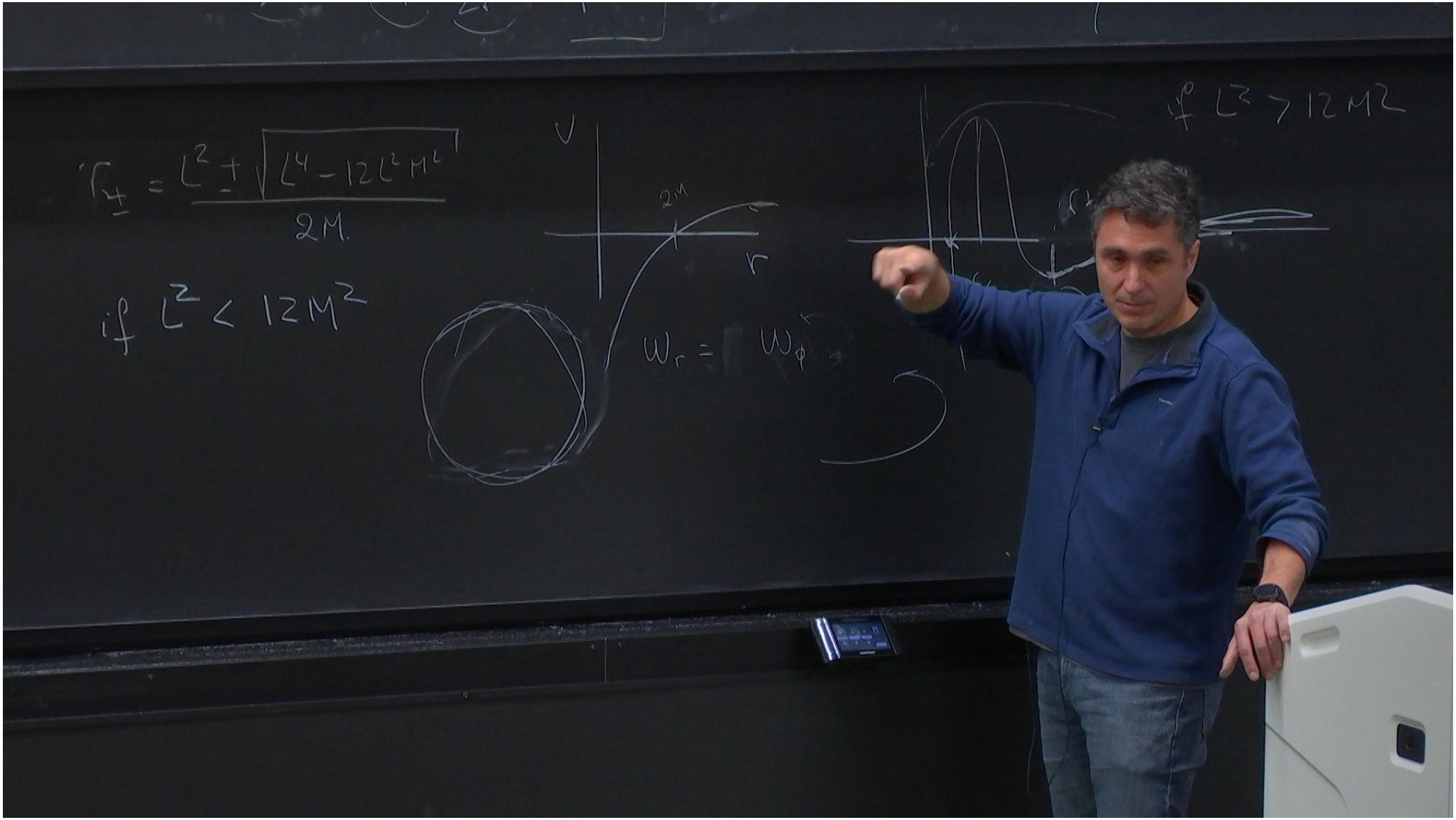

$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L}{r^2}+k\right) = \frac{1}{2}E^2$
$\theta = \pi/2$
$\approx \frac{1}{2}\dot{r}^2 + V(r) = E$
$V(r) = \frac{1}{2}k - k\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$
GR
$M = M$
$k = 1$
$\frac{\partial V}{\partial r} = 0$
$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
2M
r
if $L^2 > 12M^2$
r_+
r_-
6M

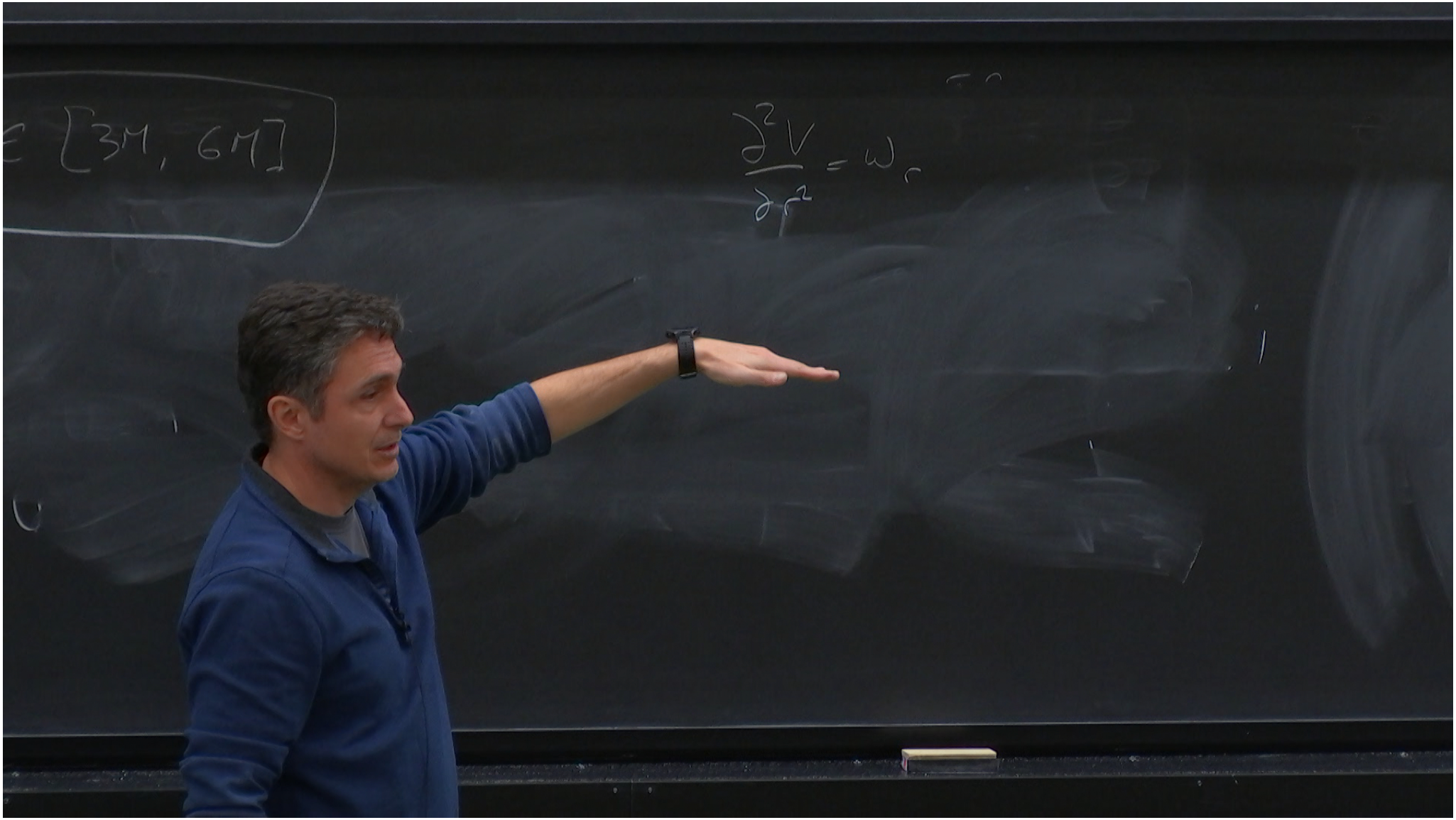

$\frac{1}{2}\dot{r}^2 + \frac{1}{2}\left(1-\frac{2M}{r}\right)\left(\frac{L}{r^2}+k\right) = \frac{1}{2}E^2$
$\theta = \pi/2$
$\approx \frac{1}{2}\dot{r}^2 + V(r) = E$
$V(r) = \frac{1}{2}k - k\frac{M}{r} + \frac{L^2}{2r^2} - \frac{ML^2}{r^3}$
GR
$M = M$
$k = 1$
$\frac{\partial V}{\partial r} = 0$
$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
2M
r
if $L^2 > 12M^2$
r_+
r_-
GM

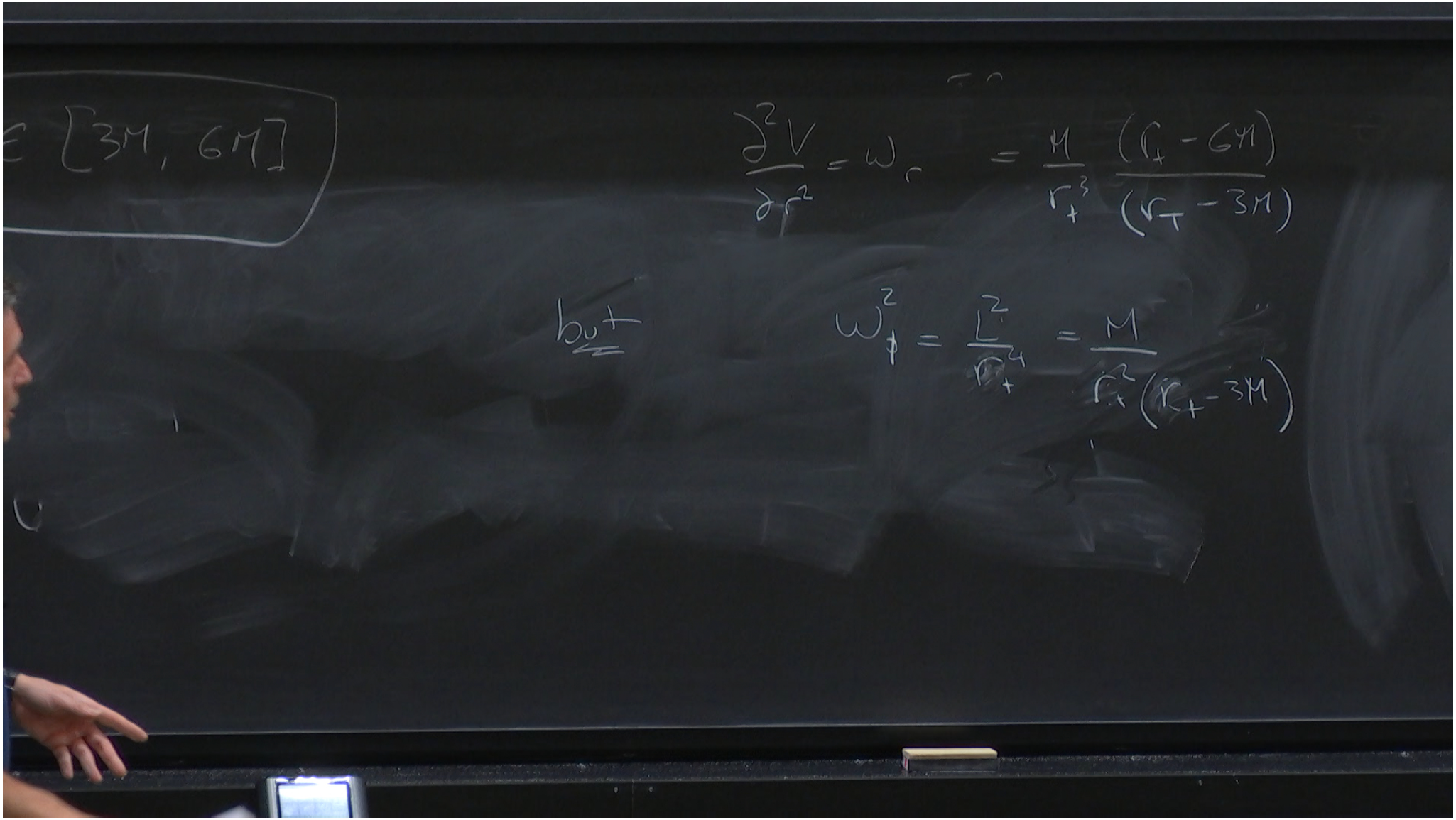

$r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
2M
r
if $L^2 > 12M^2$
6M

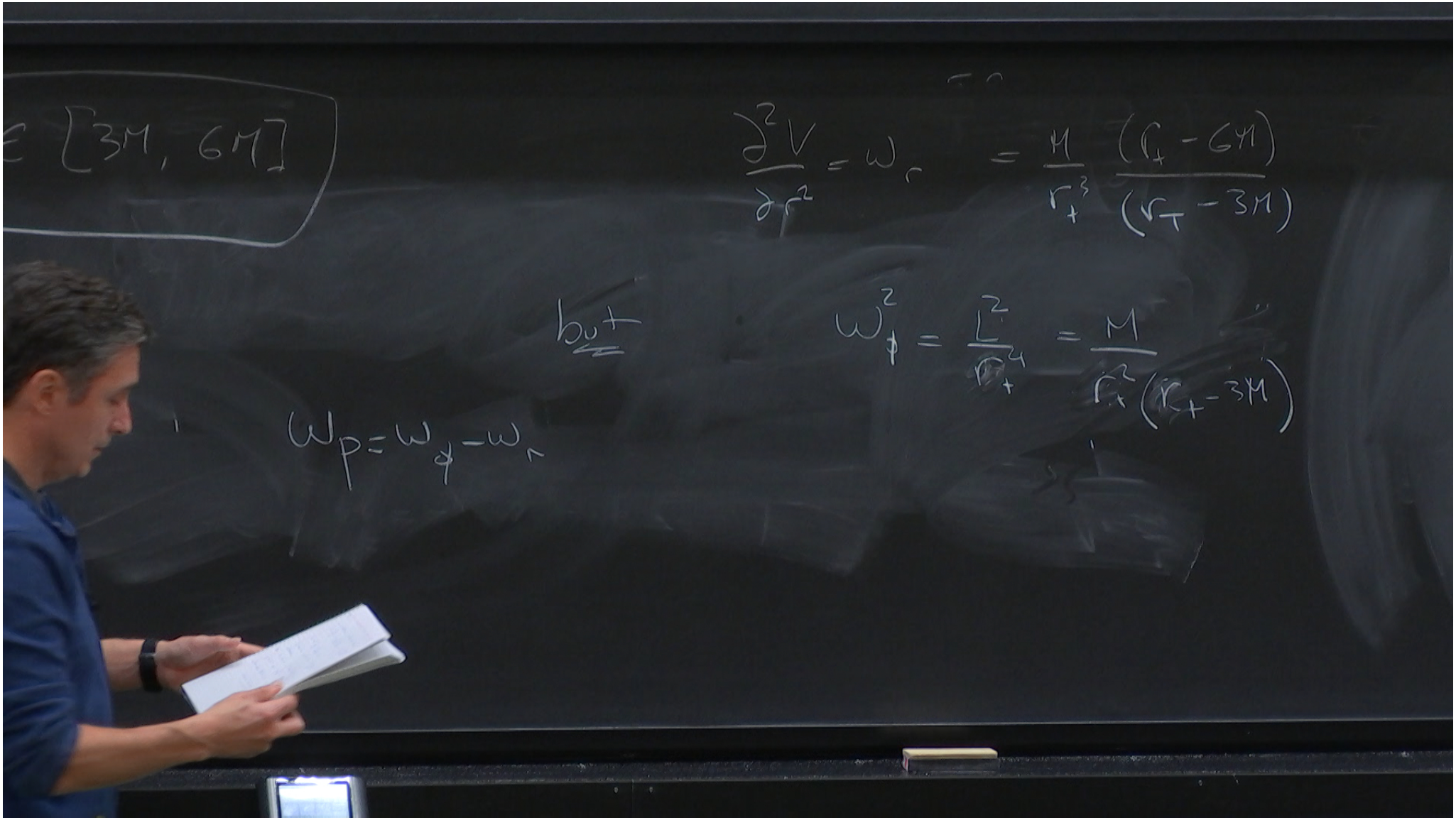

$r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
if $L^2 > 12M^2$
V
2M
r
GM


$r_{\pm} = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
2M
r
$w_r = n\, w_\phi$
if $L^2 > 12M^2$
6M


$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2M^2}}{2M}$
if $L^2 < 12M^2$
V
2M
r
$W_r = W_\phi$
if $L^2 > 12M^2$


[3M, 6M]
$\frac{\partial^2 V}{\partial r^2} = \omega_c$


$\in [3M, 6M]$
$\frac{\partial^2 V}{\partial r^2} = \omega_r^2 = \frac{M}{r_+^3} \frac{(r_+ - 6M)}{(r_+ - 3M)}$
but
$\omega_\phi^2 = \frac{L^2}{r_+^4} = \frac{M}{r_+^3 (r_+ - 3M)}$


$\in [3M, 6M]$
$\frac{\partial^2 V}{\partial r^2} = \omega_r^2 = \frac{M}{r_+^3} \frac{(r_+ - 6M)}{(r_+ - 3M)}$
but
$\omega_\phi^2 = \frac{L^2}{r_+^4} = \frac{M}{r_+^3}$
$\omega_p = \omega_\phi - \omega_r$

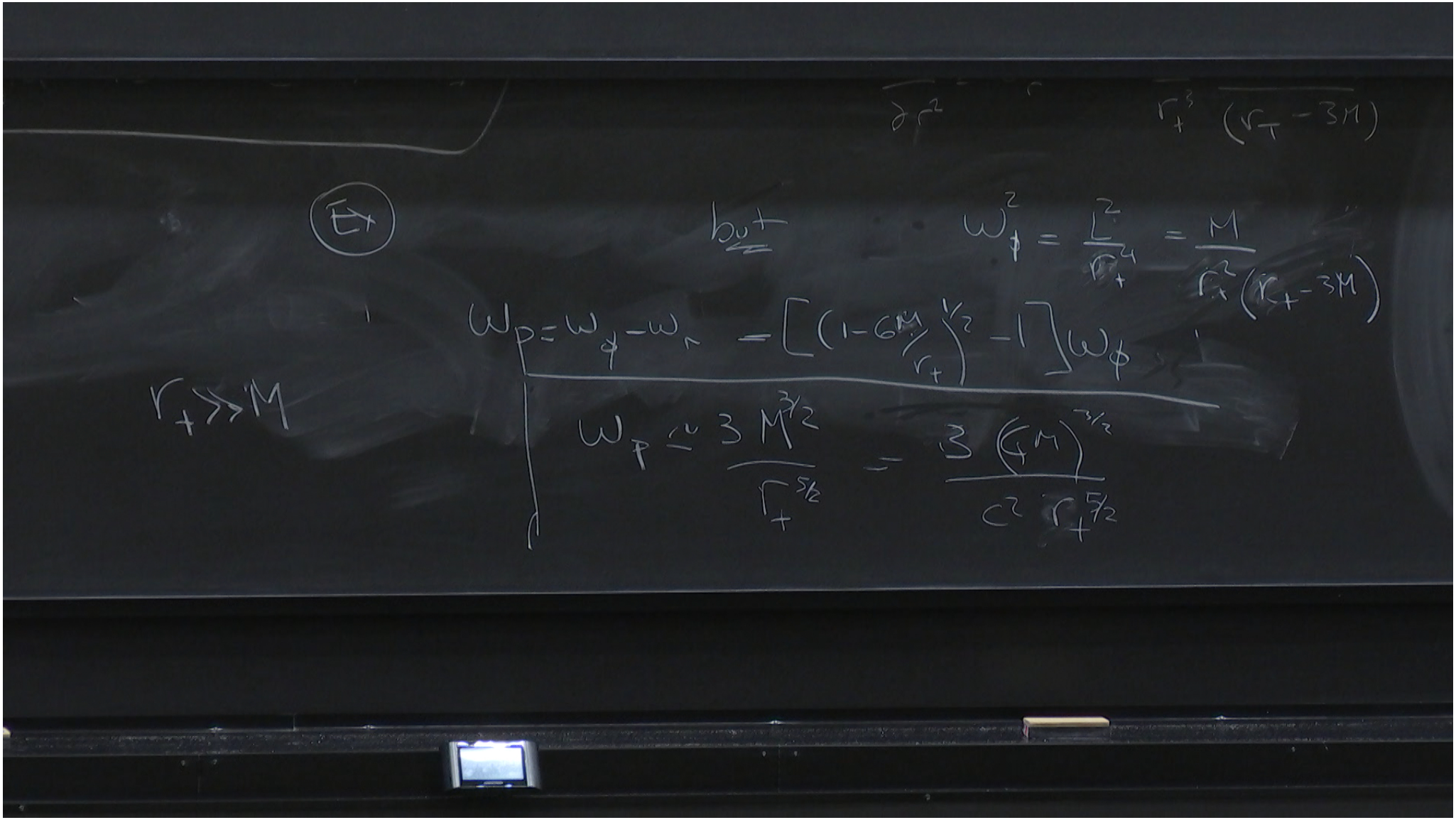
$\in [3M, 6M]$

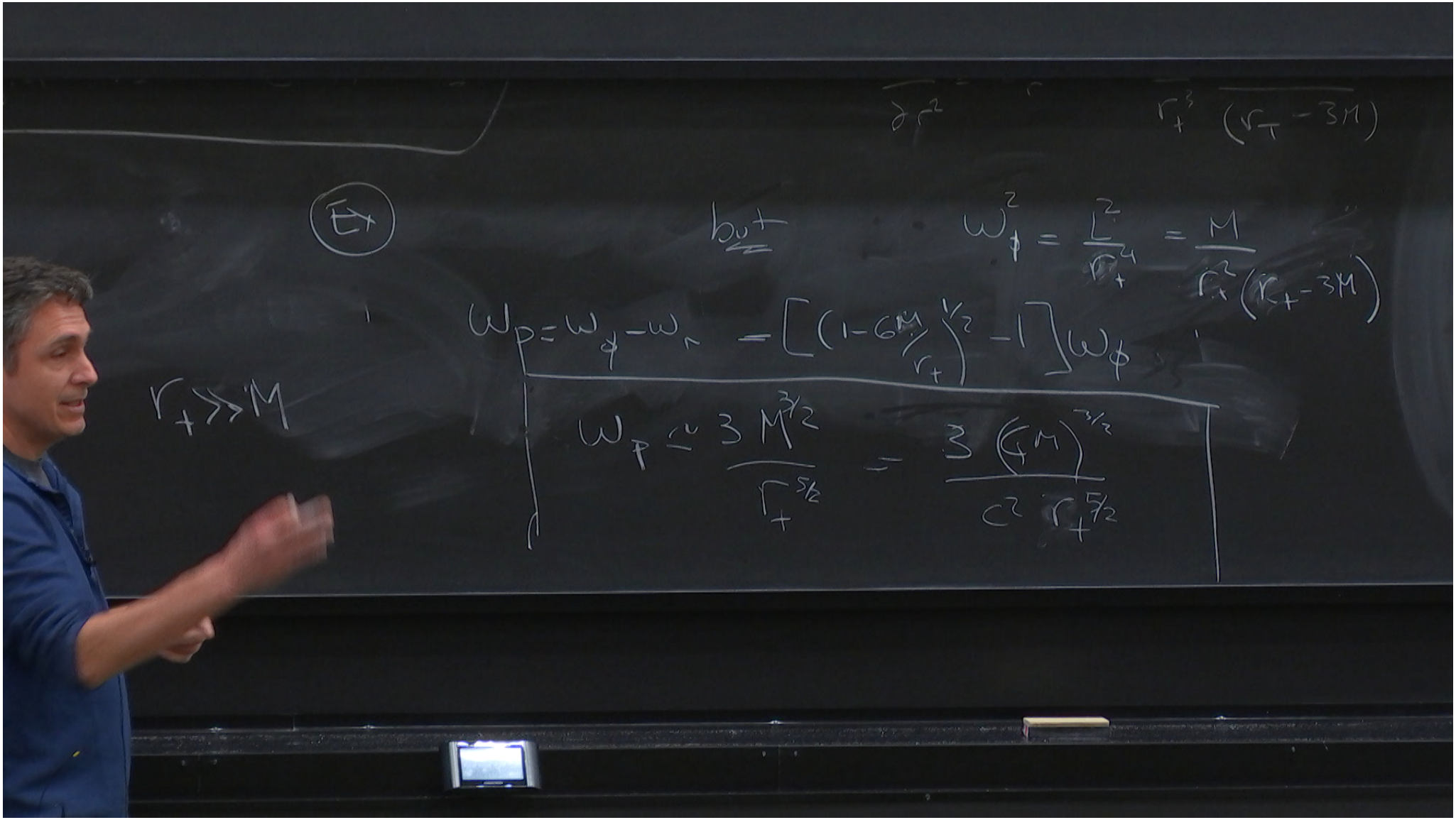
$$\frac{\partial^2 V}{\partial r^2} = \omega_r^2 = \frac{M}{r_+^3}\,\frac{(r_+ - 6M)}{(r_+ - 3M)}$$

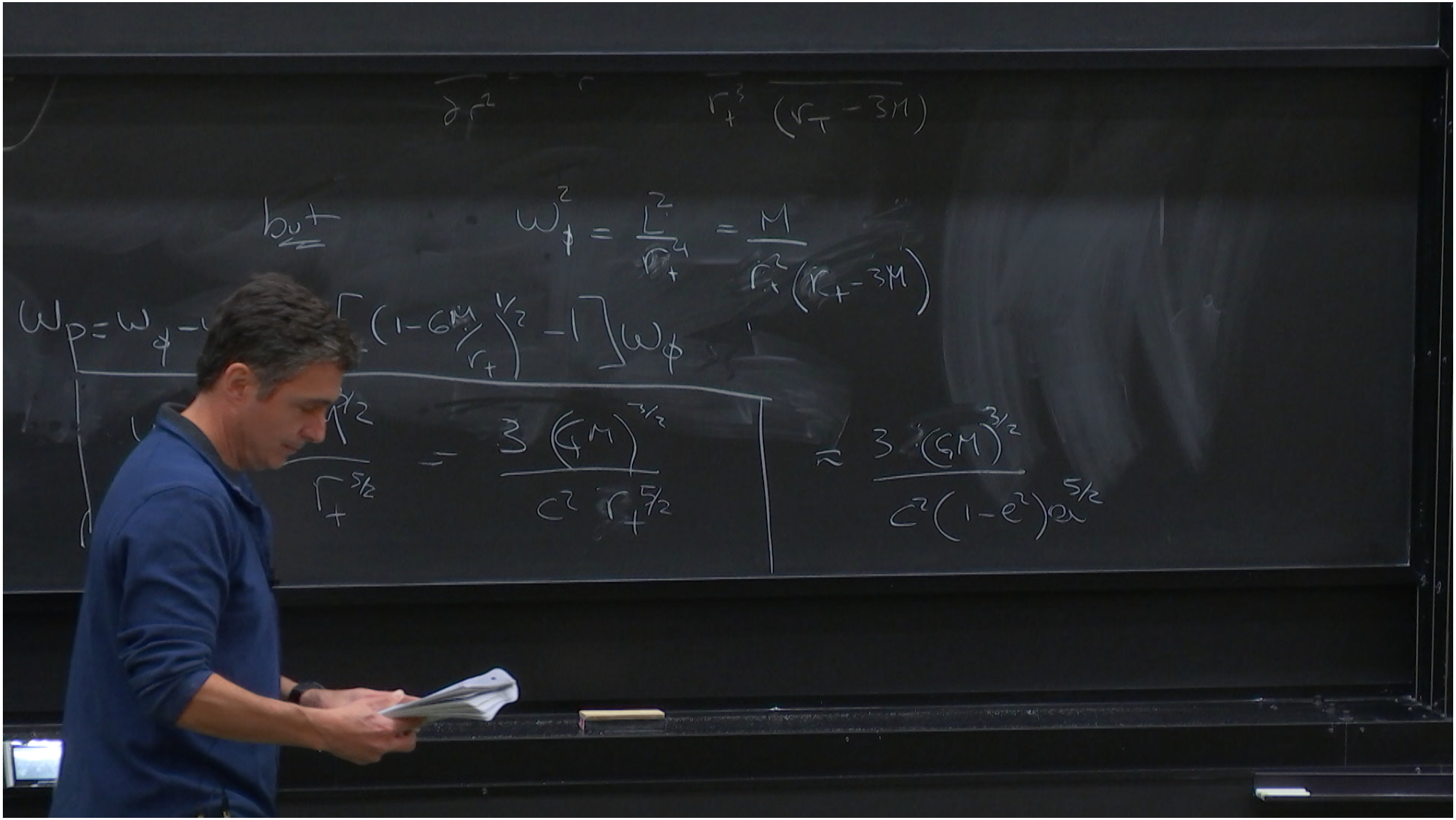
but

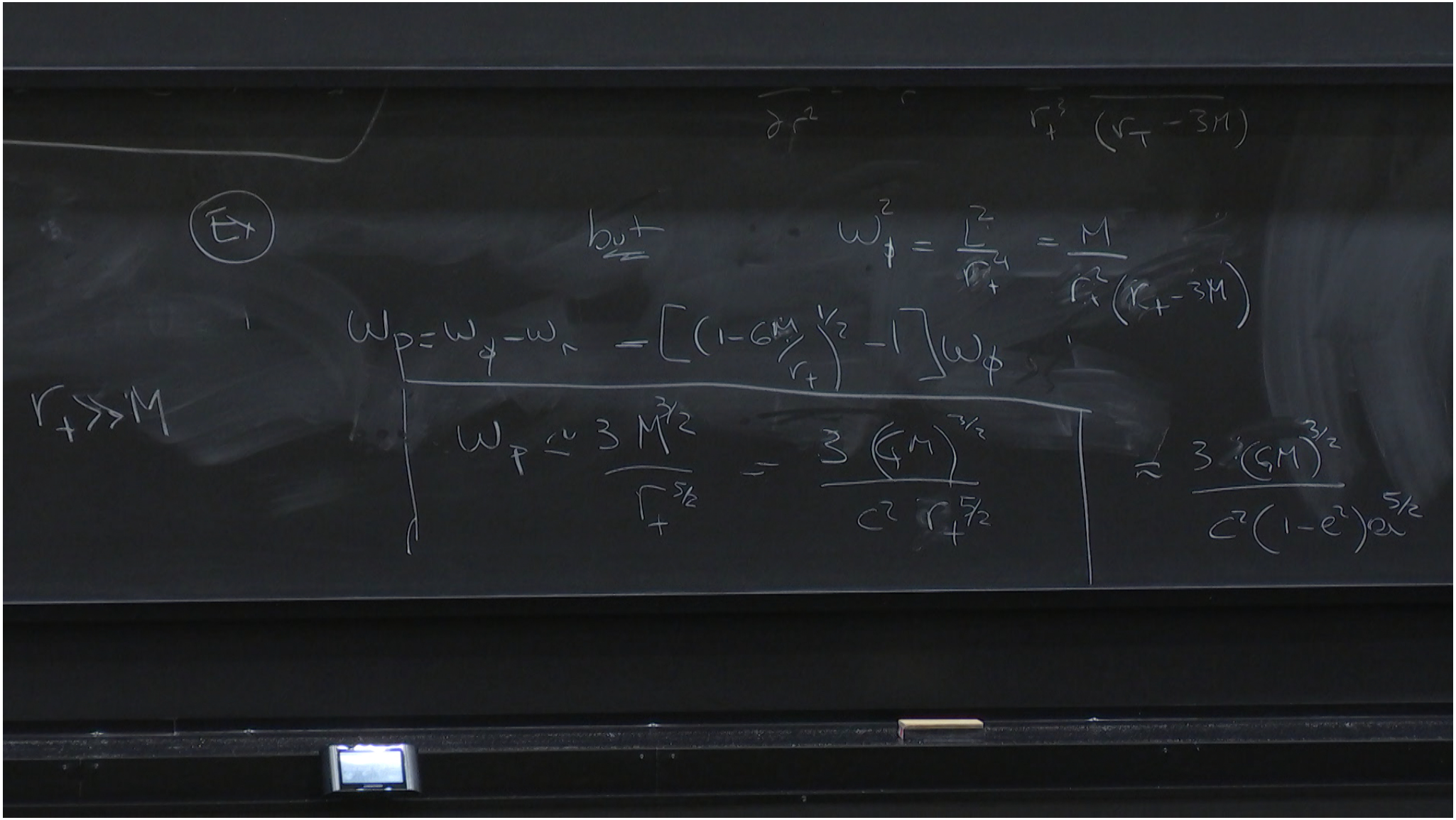
$$\omega_\phi^2 = \frac{L^2}{r_+^4} = \frac{M}{r_+^2(r_+ - 3M)}$$

$$\omega_p = \omega_\phi - \omega_r = \left[\left(1 - \frac{6M}{r_+}\right)^{1/2} - 1\right]\omega_\phi$$


Ex
but
$\omega_\phi^2 = \frac{L^2}{r_+^4} = \frac{M}{r_+^3}$
$\omega_p = \omega_\phi - \omega_r = \left[\left(1-\frac{6M}{r_+}\right)^{1/2} - 1\right]\omega_\phi$
$r_+ \gg M$
$\omega_p \simeq \frac{3M^{3/2}}{r_+^{5/2}} = \frac{3(GM)^{3/2}}{c^2 r_+^{5/2}}$


$r_+ \gg M$
$\omega_p = \omega_\phi - \omega_r = -\left[\left(1 - 6\frac{M}{r_+}\right)^{1/2} - 1\right]\omega_\phi$
$\omega_p \simeq \frac{3M^{3/2}}{r_+^{5/2}} = \frac{3(GM)^{3/2}}{c^2 r_+^{5/2}}$
$\omega_\phi^2 = \frac{L^2}{r_+^4} = \frac{M}{r_+^3(r_+ - 3M)}$


$\omega_\phi^2 = \frac{L^2}{r_+^4} = \frac{M}{r_+^3}$
$\omega_p = \omega_\phi - \left[\left(1 - 6M/r_+\right)^{1/2} - 1\right]\omega_\phi$
$\frac{3(GM)^{3/2}}{r_+^{5/2}} = \frac{3(GM)^{3/2}}{c^2 r_+^{5/2}}$
$\approx \frac{3(GM)^{3/2}}{c^2(1-e^2)a^{5/2}}$


Ex
but
$\omega_\phi^2 = \frac{L^2}{r_+^4} = \frac{M}{r_+^3(r_+ - 3M)}$
$\omega_p = \omega_\phi - \omega_r = -\left[\left(1 - 6\frac{M}{r_+}\right)^{1/2} - 1\right]\omega_\phi$
$r_+ \gg M$
$\omega_p \simeq 3\frac{M^{3/2}}{r_+^{5/2}} = \frac{3(GM)^{3/2}}{c^2 r_+^{5/2}}$
$\simeq \frac{3(GM)^{3/2}}{c^2(1-e^2)a^{5/2}}$