Title: Relativity - Lecture 221024

Speakers:

Collection: Relativity (2022/2023)

Date: October 24, 2022 - 9:00 AM

URL: https://pirsa.org/22100084

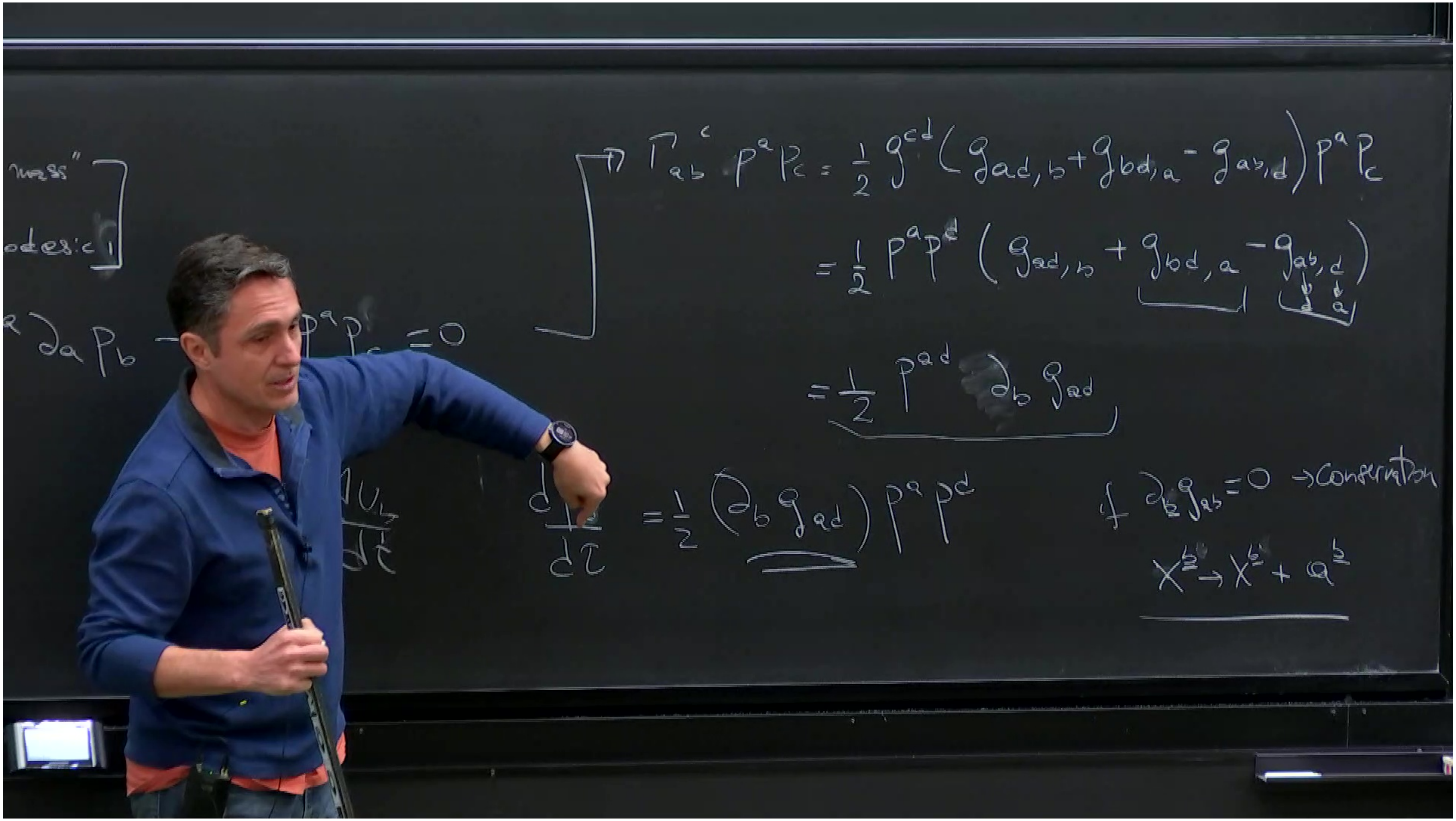
$\boxed{\text{A first word on symmetries}}$

4-momentum of a particle $\rightarrow P^a = m\,U^a$

m: constant "rest ...

U^a tangent to geo...

$\Rightarrow P^a \nabla_a P^b = 0 \rightarrow P^a \nabla_a P_b = 0 \longrightarrow P^a$ ...



$\Gamma_{ab}{}^{c} p^a p_c = \frac{1}{2} g^{cd}(g_{ad,b} + g_{bd,a} - g_{ab,d}) p^a p_c$
$= \frac{1}{2} p^a p^d (g_{ad,b} + g_{bd,a} - g_{ab,d})$
$= \frac{1}{2} p^{ad} \partial_b g_{ad}$
$= \frac{1}{2} (\partial_b g_{ad}) p^a p^d$
if $\partial_b g_{ab} = 0$ → conservation
$x^b \to x^b + a^b$
$\partial_a p_b$
$= 0$

"mass"

odesic

$$^a \partial_a p_b - \Gamma^c_{ab}\, p^a p_c = 0$$

$$\Gamma^c_{ab}\, p^a p_c = \frac{1}{2} g^{cd}\left(g_{ad,b} + g_{bd,a} - g_{ab,d}\right) p^a p_c$$

$$= \frac{1}{2} p^a p^d \left(g_{ad,b} + g_{bd,a} - g_{ab,d}\right)$$

$$= \frac{1}{2} p^a p^d\, \partial_b g_{ad}$$

$$m \frac{du_b}{dt} \qquad \frac{dp_b}{d\tau} = \frac{1}{2}\left(\partial_b g_{ad}\right) p^a p^d$$

if $\partial_b g_{ab} = 0 \rightarrow$ conservation

$$x^b \rightarrow x^b + a^b$$

U^a

$$\left[\begin{array}{l} m: \text{constant "rest mass"} \\ U^a \text{ tangent to geodesic} \end{array}\right]$$

$$\to p^a \nabla_a p_b = 0 \to p^a \partial_a p_b - \Gamma^c_{\ ab}\, p^a p_c = 0$$

$$\to \Gamma_{ab}^{\ \ c}\, p^a p_c = \frac{1}{2} g^{cd} (g_a$$

$$= \frac{1}{2} p^a p^d ($$

$$= \frac{1}{2} p^{ad}\, \partial$$

$$m \frac{dU_b}{d\tau}$$

$$\frac{dp_b}{d\tau} = \frac{1}{2} (\partial_b g_{ad})\, p^a p^d$$

U^a $\left[\begin{array}{l} m: \text{constant "rest mass"} \\ U^a \text{ tangent to geodesic} \end{array} \right]$

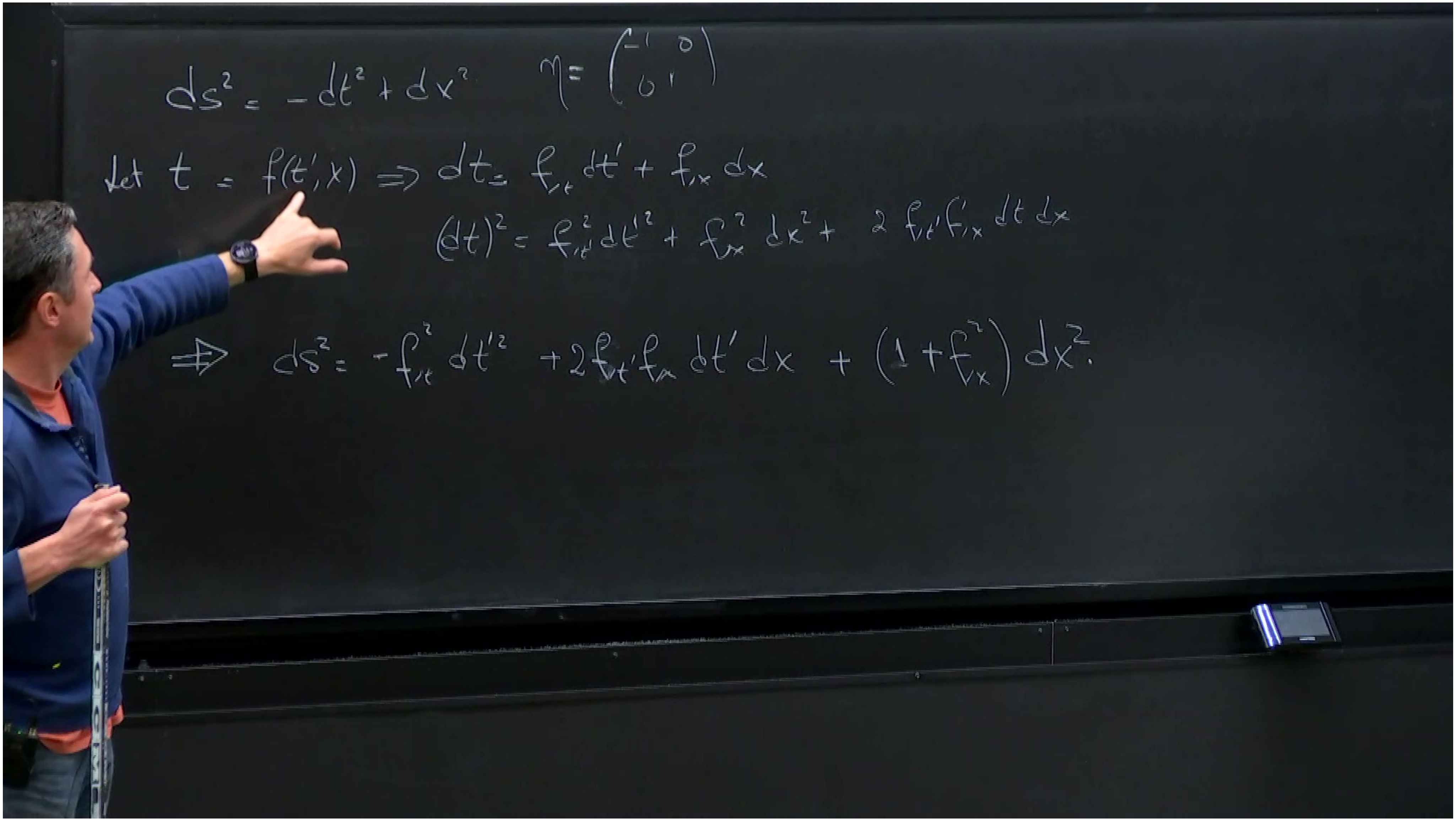
$\rightarrow p^a \nabla_a p_b = 0 \rightarrow \underbrace{p^a}_{mU^a} \partial_a p_b - \Gamma^c_{ab}\, p^a p_c = 0$

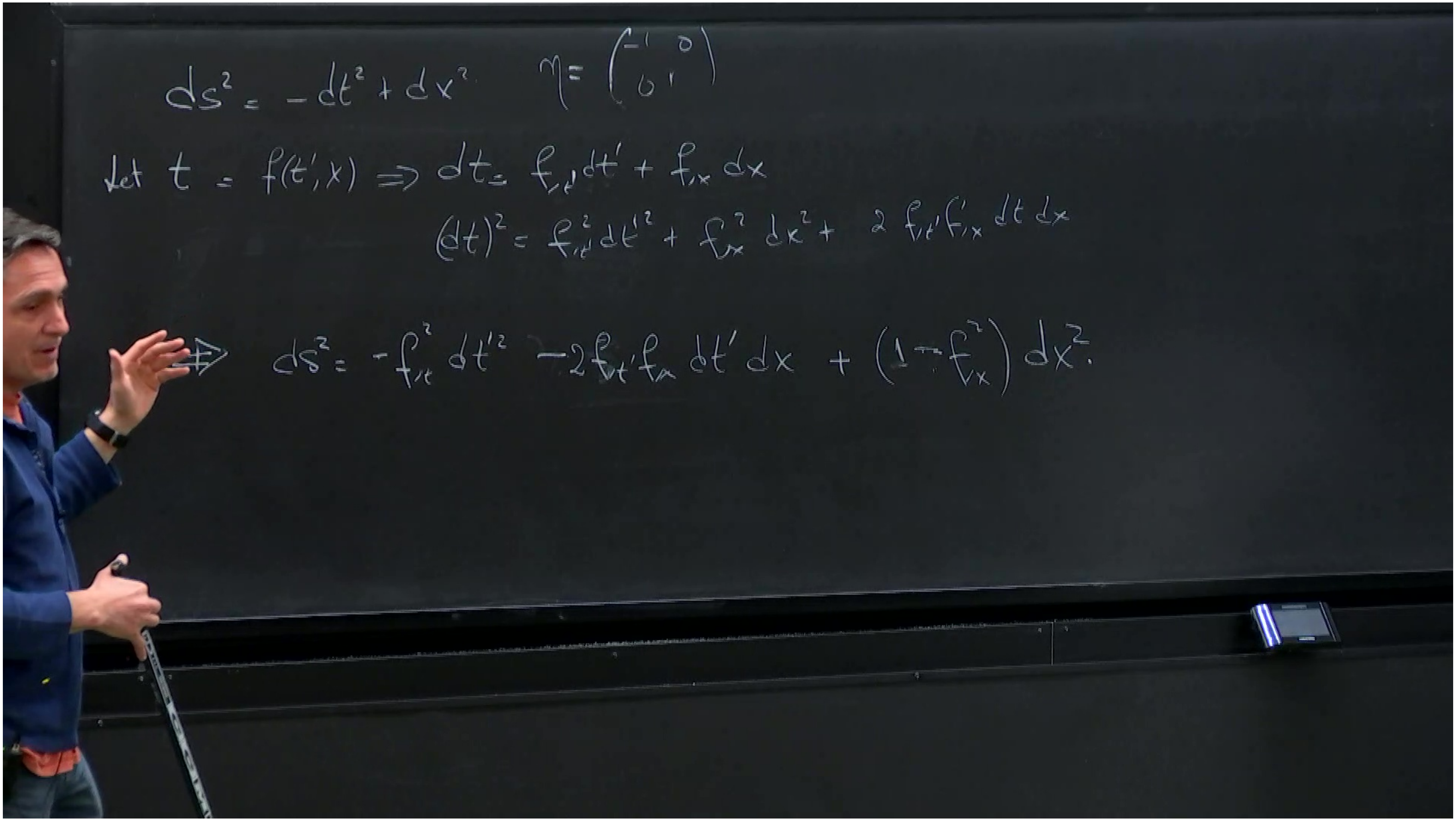
$\rightarrow \Gamma^c_{ab}\, p^a p_c = \frac{1}{2} g^{cd} (g_a$

$= \frac{1}{2} p^a p^d ($

$= \frac{1}{2} p^{ad}\, \partial$

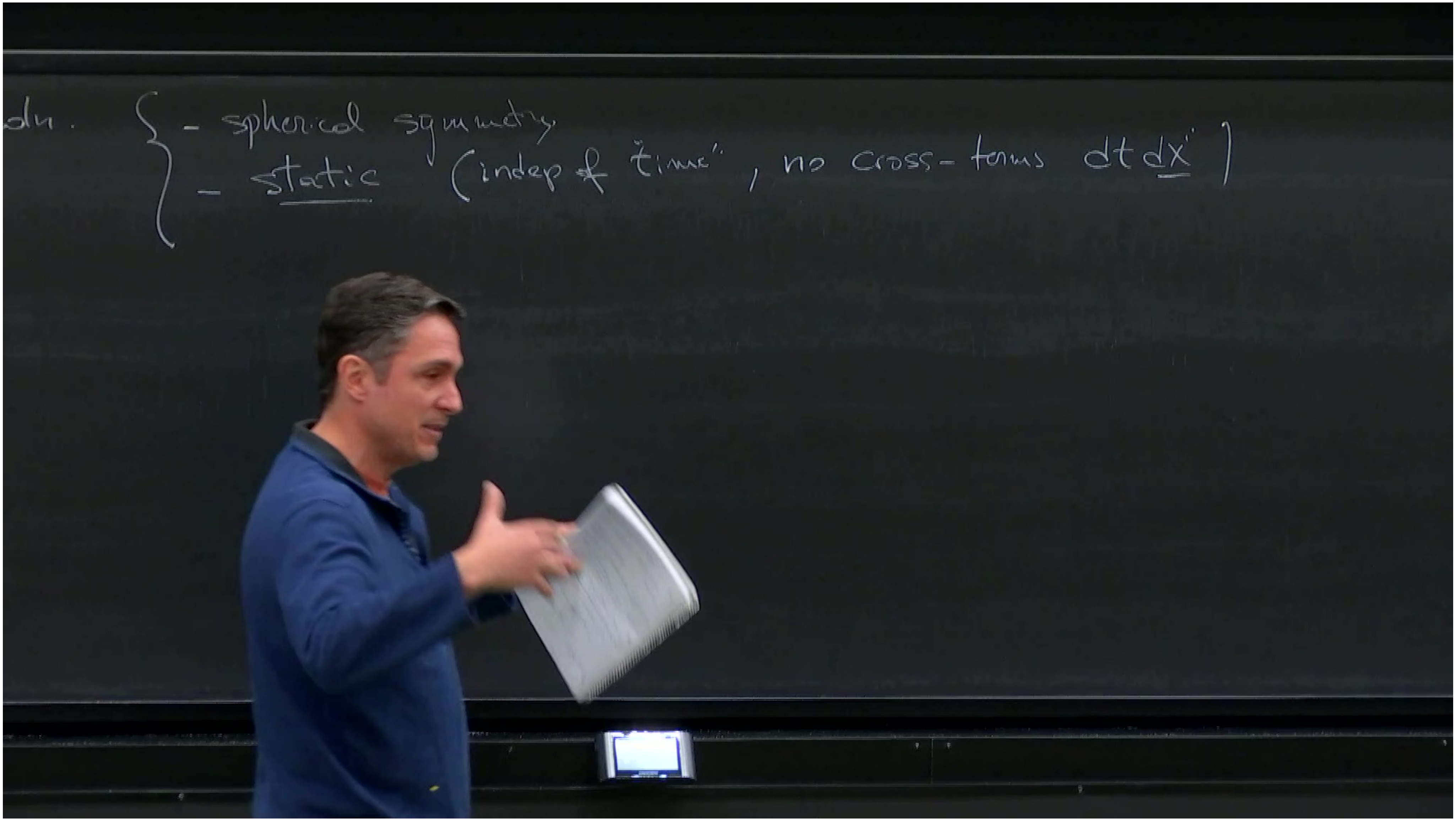
$m^2 \frac{dU_b}{d\tau} \quad m \frac{dp_b}{d\tau} = \frac{1}{2} (\partial_b g_{ad})\, p^a p^d$


$ds^2 = -dt^2 + dx^2 \qquad \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
let $t = f(t', x) \Rightarrow dt = f_{,t'}\, dt' + f_{,x}\, dx$
$(dt)^2 = f_{,t'}^2\, dt'^2 + f_{,x}^2\, dx^2 + 2 f_{,t'} f_{,x}\, dt\, dx$
$\Rightarrow ds^2 = -f_{,t'}^2\, dt'^2 + 2 f_{,t'} f_{,x}\, dt'\, dx + (1 + f_{,x}^2)\, dx^2.$



$ds^2 = -dt^2 + dx^2 \quad \eta = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$
let $t = f(t', x) \Rightarrow dt = f_{,t'} dt' + f_{,x} dx$
$(dt)^2 = f_{,t'}^2 dt'^2 + f_{,x}^2 dx^2 + 2 f_{,t'} f_{,x} dt\, dx$
$\Rightarrow ds^2 = -f_{,t'}^2 dt'^2 - 2 f_{,t'} f_{,x} dt' dx + (1 - f_{,x}^2) dx^2.$

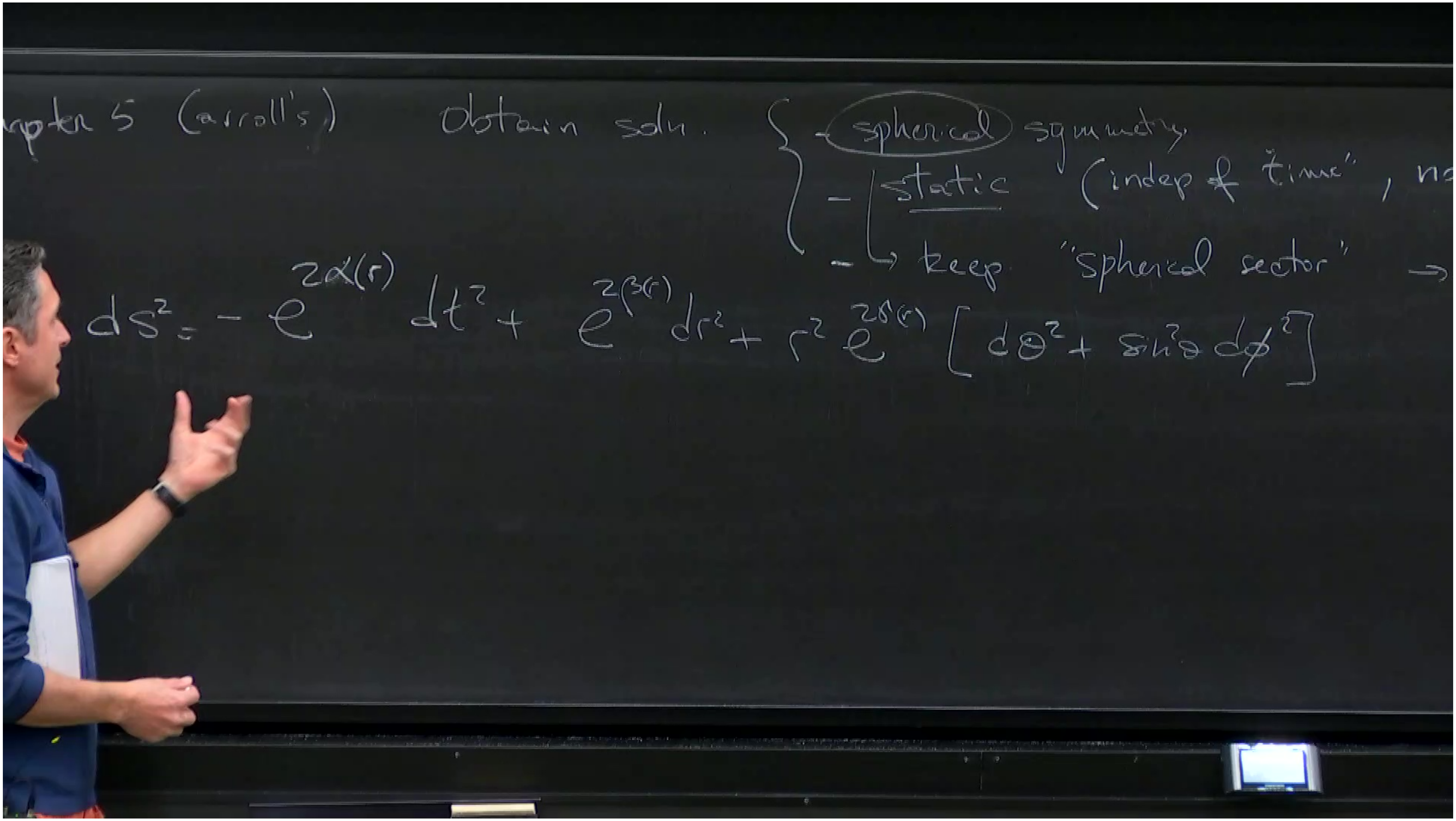
Chapter 5 (Carroll's) Obtain soln. { - spherical symmetry
- static (indep of



- spherical symmetry,
- static (indep of "time", no cross-terms dt dx^i)

soln.

- spherical symmetry
- static ("indep of time", no cross-terms $dt\,d\underline{x}$)
- ↳ keep "spherical sector" → closer "as in flat spacetime"

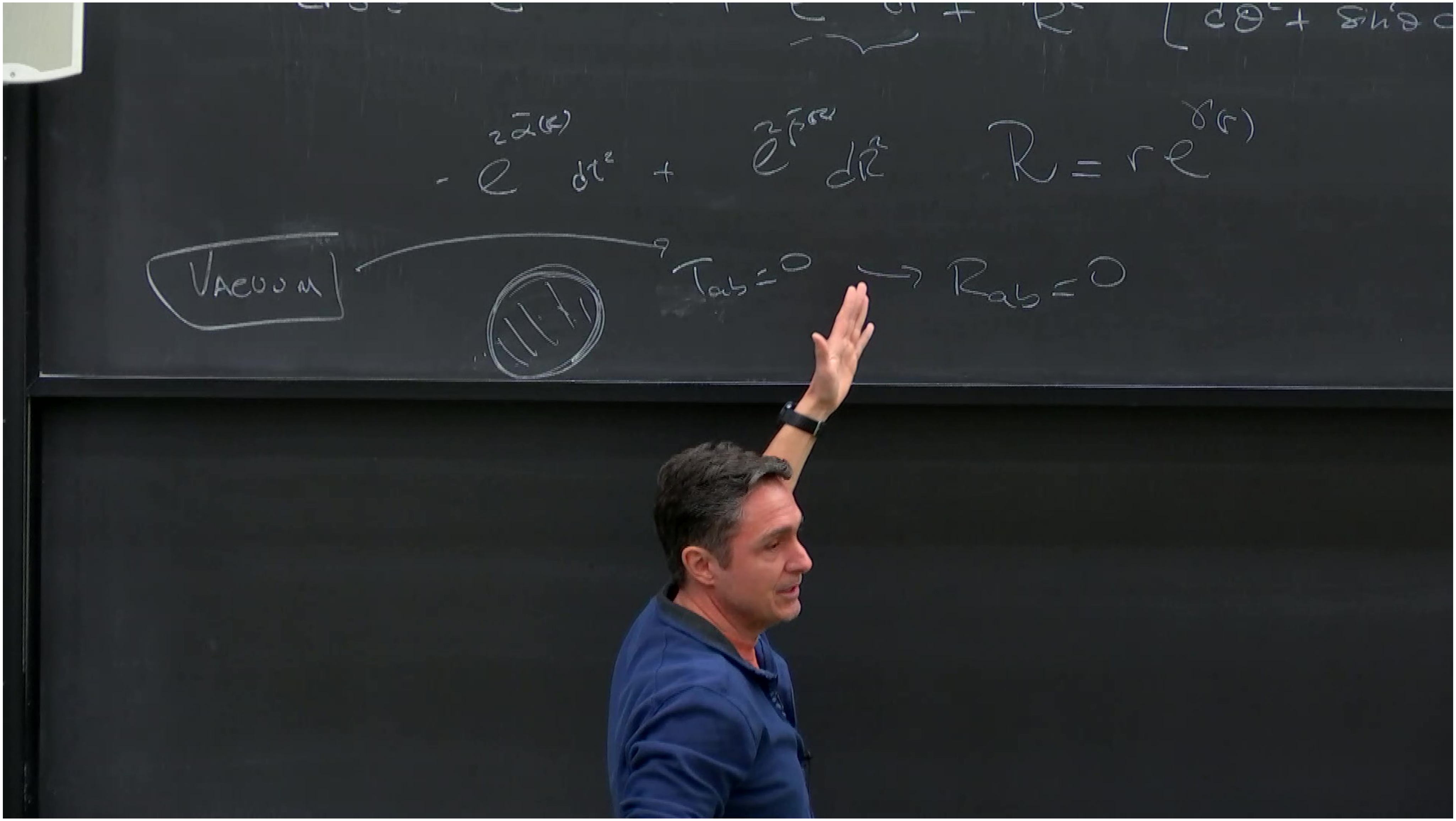


apter 5 (Carroll's)
Obtain soln.
- spherical symmetry
- static (indep of "time", n
- keep "spherical sector" →
$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 e^{2\gamma(r)} [d\theta^2 + \sin^2\theta \, d\phi^2]$

pter 5 (Carroll's)   obtain soln. { - spherical symmetry
- static (indep of "time", no
- keep "spherical sector" →

$$ds^2 = -e^{2\bar{\alpha}(R)} dt^2 + \underbrace{e^{2\bar{\beta}(R)} dr^2}_{} + R^2 \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right]$$

$$-e^{2\bar{\alpha}(R)} \quad + \quad e^{2\bar{\beta}(R)} dR^2 \qquad R = r e^{\gamma(r)}$$


R = re^{γ(r)}
VACUUM
Tab = 0
Rab = 0

$\sin\theta \, d\phi \,]$

$$g_{ab} = \begin{pmatrix} -e^{2\alpha(r)} & & & \\ & e^{2\beta(r)} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \qquad g^{ab} = \begin{pmatrix} -e^{-2\alpha} & & & \\ & e^{-2\beta} & & \\ & & 1/r^2 & \\ & & & 1/r^2\sin^2\theta \end{pmatrix}$$

$$\Gamma^t{}_{rr} = \partial_r \alpha \; ; \quad \Gamma^r{}_{tt} = e^{2(\alpha-\beta)}\partial_r\alpha \, , \quad \Gamma^r{}_{rr} = \partial_r \beta$$

$$\Gamma^\theta{}_{r\theta} = 1/r \, , \quad \Gamma^r{}_{\theta r} = -r e^{-2\beta} \, , \quad \Gamma^\phi{}_{r\phi} = 1/r$$

$$\Gamma^r{}_{\phi\phi} = -r e^{-2\beta}\sin^2(\theta) \, , \quad \Gamma^\theta{}_{\phi\phi} = -\sin(\theta)\cos(\theta) \, , \quad \Gamma^\phi{}_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)}$$

$\partial_r \alpha \equiv \alpha'$

$r\beta$

$\phi = \frac{\cos(\theta)}{\sin(\theta)}$

$$R^t{}_{rtr} = \alpha'\beta' - \alpha'' - (\alpha')^2$$

$$R^t{}_{\theta t \theta} = -r e^{-2\beta} \alpha'$$

$\vdots$

$$R_{tt} = e^{2(\alpha-\beta)}\left[\alpha'' + \alpha'^2 - \alpha'\beta' + \frac{2}{r}\alpha'\right]$$

$$R_{rr} = -\alpha'' - \alpha'^2 + \alpha'\beta' + \frac{2}{r}\beta'$$

$$R_{\theta\theta} = e^{-2\beta}\left[r(\beta' - \alpha') - 1\right] + 1$$

$$\rightsquigarrow\ e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0 = \frac{2}{r}(\alpha' + \beta') \rightsquigarrow \alpha = -\beta + \mathrm{con}$$

- static (indep of
- keep "spherical

$$ds^2 = -e^{2\bar{\alpha}(R)} dt^2 + \underbrace{e^{2\bar{\beta}(R)} dr^2}_{} + R^2 \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right.$$

$$-e^{2\bar{\alpha}(R)} dt^2 + e^{2\bar{\beta}(R)} dR^2 \qquad R = r e^{\gamma(r)}$$

$\boxed{\text{VACUUM}} \longrightarrow T_{ab} = 0 \longrightarrow R_{ab} = 0$

$$\Gamma^{\theta}_{r\theta} = 1/r, \quad \Gamma^{r}_{\theta r} = -r e^{-2\beta}, \quad \Gamma^{\phi}_{r\phi} = 1/r$$

$$\Gamma^{r}_{\phi\phi} = -r e^{-2\beta} \sin^2(\theta), \quad \Gamma^{\theta}_{\phi\phi} = -\sin(\theta)\cos(\theta)$$

$$-e^{2\bar{\alpha}(R)}\,dt^2 + e^{2\bar{\beta}(R)}\,dR^2 \qquad R = r\,e^{\gamma(r)}$$

$\boxed{\text{VACUUM}}$

$$T_{ab}=0 \longrightarrow R_{ab}=0$$

$$e^{2\alpha}\,dt^2$$

$$\Gamma^{\theta}_{r\theta} = 1/r\,, \quad \Gamma^{r}_{\theta r} = -r\,e^{-2\beta}\,, \quad \Gamma^{\phi}_{r\phi} = 1/$$

$$\Gamma^{r}_{\phi\phi} = -r\,e^{-2\beta}\sin^2(\theta)\,, \quad \Gamma^{\theta}_{\phi\phi} = -\sin(\theta)\cos(\theta)$$

$$dt = dt'\,e^{-c}$$

$$dt^2 = (dt')^2\,e^{-2c}$$

$$\rightsquigarrow \quad e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0 = \frac{2}{r}(\alpha' + \beta') \rightsquigarrow \alpha = -\beta$$

$R_s =$ con

$$R_{\theta\theta} = 0 \Rightarrow e^{2\alpha}(2r\alpha' + 1) = 1$$

$$\partial_r\left[r e^{2\alpha}\right] = 1$$

$$\boxed{e^{2\alpha} = 1 - \frac{R_s}{r}}$$

$$\partial_r\left[r e^{2\alpha}\right] = 1$$

$$\boxed{e^{2\alpha} = 1 - \frac{R_s}{r}}$$

$$\boxed{ds^2 = -\left(1 - \frac{R_s}{r}\right)dt^2 + \frac{1}{\left(1 - \frac{R_s}{r}\right)}dr^2 + r^2 d\Omega^2}$$

$\rightsquigarrow \alpha = -\beta$

$R_s =$ constant

$$\boxed{e^{2\alpha} = 1 - \frac{R_s}{r}}$$

ds^2

$R_s = 2GM$

$g_{00} = -(1 + 2\Phi)$

$r = 0$

$$R_s = \frac{2GM}{c^2} \simeq 3\left(\frac{M}{M_\odot}\right)\text{km}$$

→ Curvature Invariants: $$\boxed{R_{abcd}R^{abcd} = \frac{48\,G^2 M^2}{r^6}}$$

km

$$\ldots_{cd} R^{abcd} = \boxed{\frac{48 G^2 M^2}{r^6}}$$

$$\left(\nabla_e R_{abcd}\right)\left(\nabla^e R^{abcd}\right)$$

$\rightsquigarrow e^{\dots}\; R_{tt} + R_{rr} = 0 = \frac{2}{r}(\alpha' + \beta') \rightsquigarrow \alpha = -\beta$

$R_s = \text{const}$

$R_{\theta\theta} = 0 \Rightarrow e^{2\alpha}(2r\alpha' + 1) = 1$

$$\boxed{e^{2\alpha} = 1 - \frac{R_s}{r}}$$

$\partial_r \left[r e^{2\alpha} \right] = 1$

$$\boxed{ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 + r^2 d\Omega^2}$$

$R_s = 2G\dots$

$r = 2GM$

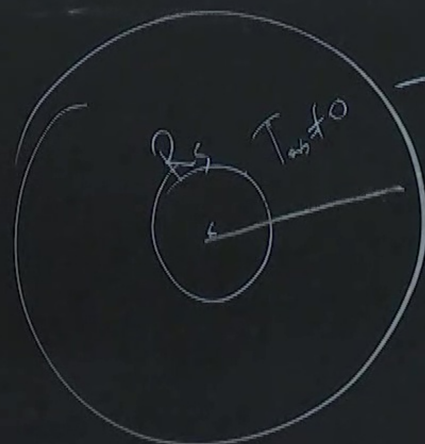
$r = 0$

$R_s = \frac{2GM}{\dots} \simeq 3 \left(\frac{M}{M_{\dots}}\right) \mathrm{km}$

$$R_s = \frac{2GM}{c^2} \simeq 3\left(\frac{M}{M_\odot}\right) \text{km} \ll R_\odot$$

→ Curvature Invariants: $\boxed{R_{abcd}R^{abcd} = \frac{48 G^2 M^2}{r^6}}$

$\left(R_e \, R_{abc} \right.$



$\rightsquigarrow \; e^{\ldots} \; R_{tt} + R_{rr} = 0 = \frac{2}{r}(\alpha' + \beta') \rightsquigarrow \alpha = -\beta$

$R_s = \text{con}\ldots$

$R_{00} = 0 \Rightarrow e^{2\alpha}(2r\alpha' + 1) = 1$

$\partial_r \left[r e^{2\alpha} \right] = 1$

$$\boxed{e^{2\alpha} = 1 - \frac{R_s}{r}}$$

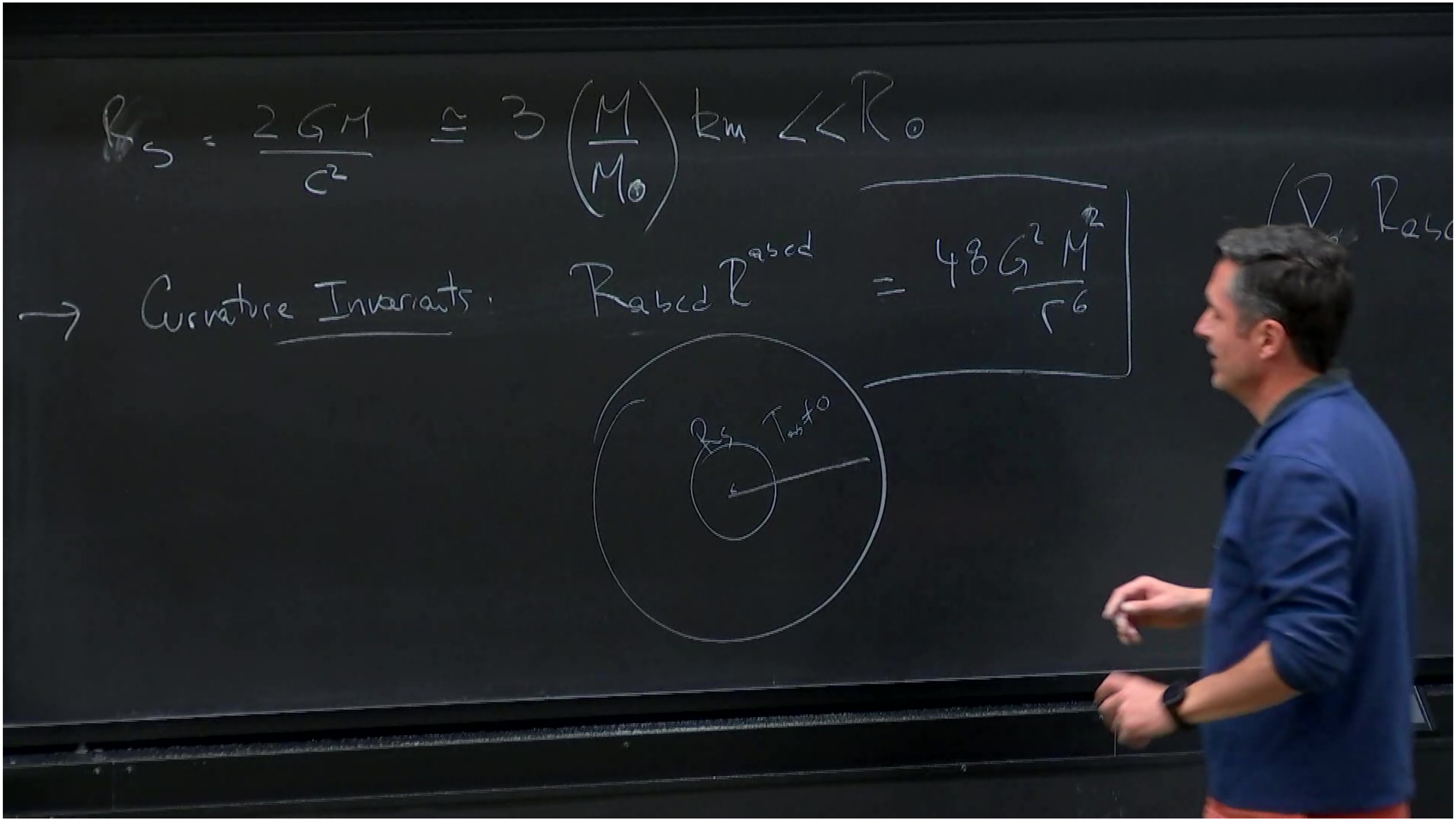
$$\boxed{ds^2 = -\left(1 - \frac{2MG}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r}\right)} dr^2 + r^2 d\Omega^2}$$

$R_s = 2G\ldots$

$r = 2GM$

$r = 0$

$R_s = \frac{2GM}{c^2} \cong 3 \left(\frac{M}{M_\odot}\right) \text{km} << R_\odot$

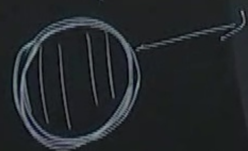

$R_s = \frac{2GM}{c^2} \cong 3\left(\frac{M}{M_\odot}\right) \text{km} \ll R_\odot$
Curvature Invariants:
$R_{abcd}R^{abcd} = 48\frac{G^2M^2}{r^6}$
R_s
$T_{ab} \neq 0$

$$R_s = \frac{2GM}{c^2} \simeq 3\left(\frac{M}{M_\odot}\right)\text{km} \ll R_\odot$$

→ Curvature Invariants. $\boxed{R_{abcd}R^{abcd} = 48\frac{G^2M^2}{r^6}}$

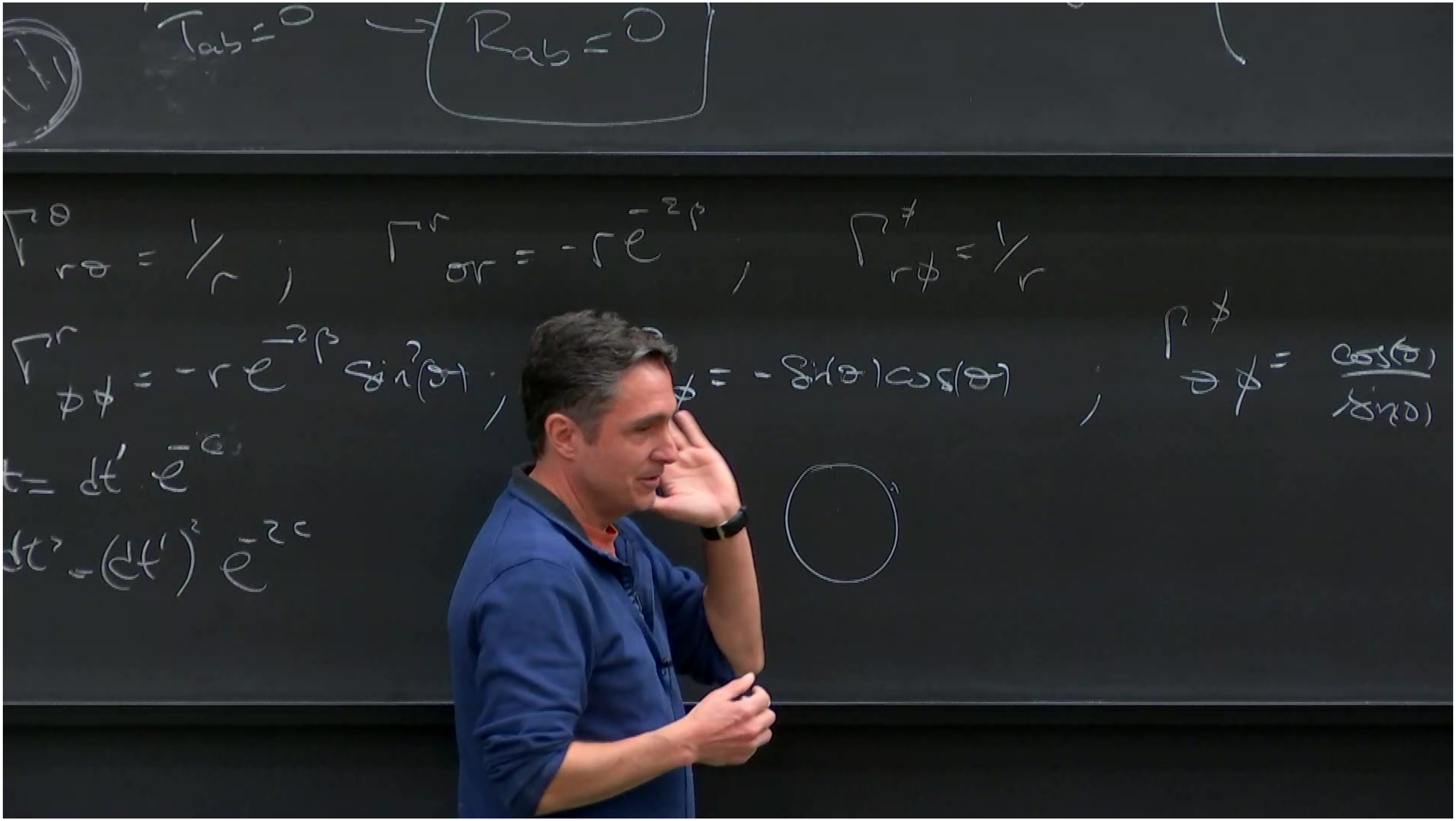
$R_s \sim M$

M

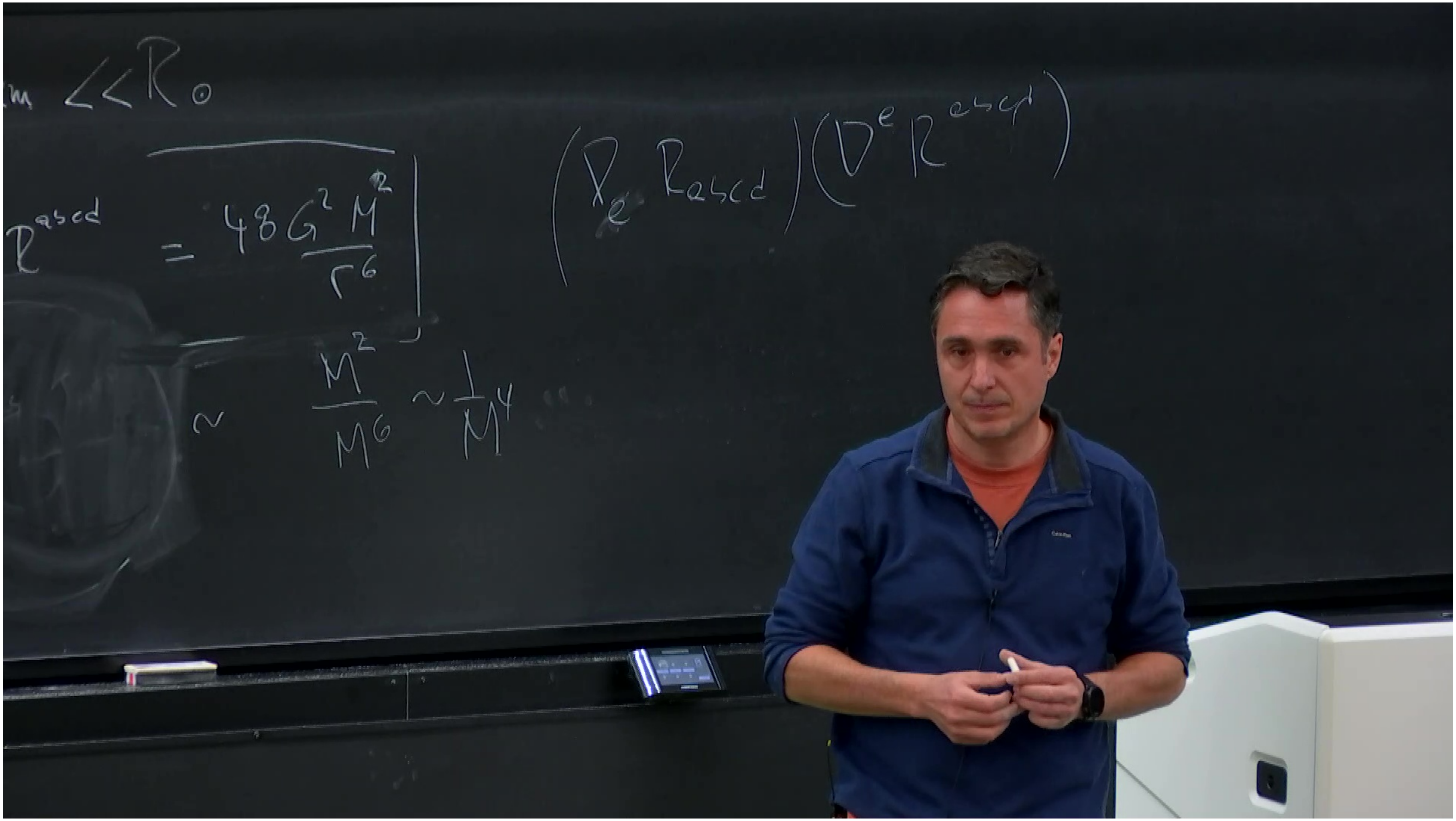


$$\sim \frac{M^2}{M^6} \sim \frac{1}{M^4}$$

$\left(R_e \; R_{abc}\right.$



$T_{ab} = 0 \to \boxed{R_{ab} = 0}$
$\Gamma^\theta_{r\theta} = 1/r$, $\Gamma^r_{\theta r} = -r e^{-2\beta}$, $\Gamma^\phi_{r\phi} = 1/r$
$\Gamma^r_{\phi\phi} = -r e^{-2\beta} \sin^2(\theta)$, $\phi = -\sin(\theta)\cos(\theta)$; $\Gamma^\phi_{\theta\phi} = \frac{\cos(\theta)}{\sin(\theta)}$
$t = dt' e^{-c}$
$dt^2 = (dt')^2 e^{-2c}$


$<< R_\odot$
$R^{abcd} = 48 \frac{G^2 M^2}{r^6}$
$(\nabla_e R_{abcd})(\nabla^e R^{abcd})$
$\sim \frac{M^2}{M^6} \sim \frac{1}{M^4}$

5 (Carroll's) Obtain soln. $\left\{ \begin{array}{l} \text{- spherical symmetry} \\ \text{- static (indep of "time")} \\ \text{- keep "spherical sector"} \end{array} \right.$

$$ds^2 = -e^{2\bar{\alpha}(R)} dt^2 + \underbrace{e^{2\bar{\beta}(R)} dr^2} + R^2 \left[d\theta^2 + \sin^2\theta \, d\phi^2 \right]$$

$$-e^{2\bar{\alpha}(R)} dt^2 + e^{2\bar{\beta}(R)} dR^2 \qquad R = r e^{\gamma(r)}$$

$g_{ab} =$

$T_{ab} = 0 \rightarrow \boxed{R_{ab} = 0}$

$$\Gamma^{\theta}_{r\theta} = 1/r, \quad \Gamma^{r}_{\theta r} = -r e^{-2\beta}, \quad \Gamma^{\phi}_{r\phi} = 1/r$$

$$e^{2(\beta-\alpha)} R_{tt} + R_{rr} = 0 = \frac{2}{r}(\alpha' + \beta') \rightsquigarrow \alpha = -\beta$$

R_s = constant

$$R_{\theta\theta} = 0 \Rightarrow e^{2\alpha}(2r\alpha' + 1) = 1$$

$$\partial_r\left[r e^{2\alpha}\right] = 1$$

$$\boxed{e^{2\alpha} = 1 - \frac{R_s}{r}}$$

$$R_s = 2GM$$

$$\boxed{ds^2 = -\left(1 - \frac{2MG}{r^p}\right) dt^2 + \frac{1}{\left(1 - \frac{2GM}{r^p}\right)} dr^2 + r^2 d\Omega^2_{(n)}}$$

$r = 2GM$

$r = 0$

1) Take soln. compute geodesics → timelike / → null like

$$U^a \nabla_a U^b = 0 \quad \rightarrow U^a U_a = \{-1, 0\} = \epsilon$$

2) Exploit symmetries

1) Take soln. compute geodesics $\nearrow$ timelike, $\searrow$ null like

$$U^a \nabla_a U^b = 0 \qquad \rightarrow U^a U_a = \{-1, 0\} = \epsilon$$

2) Exploit symmetries

$$\frac{d p_{\underline{b}}}{d\tau} = 0 \quad \text{if} \quad \partial_{\underline{b}} g_{ab} = 0$$