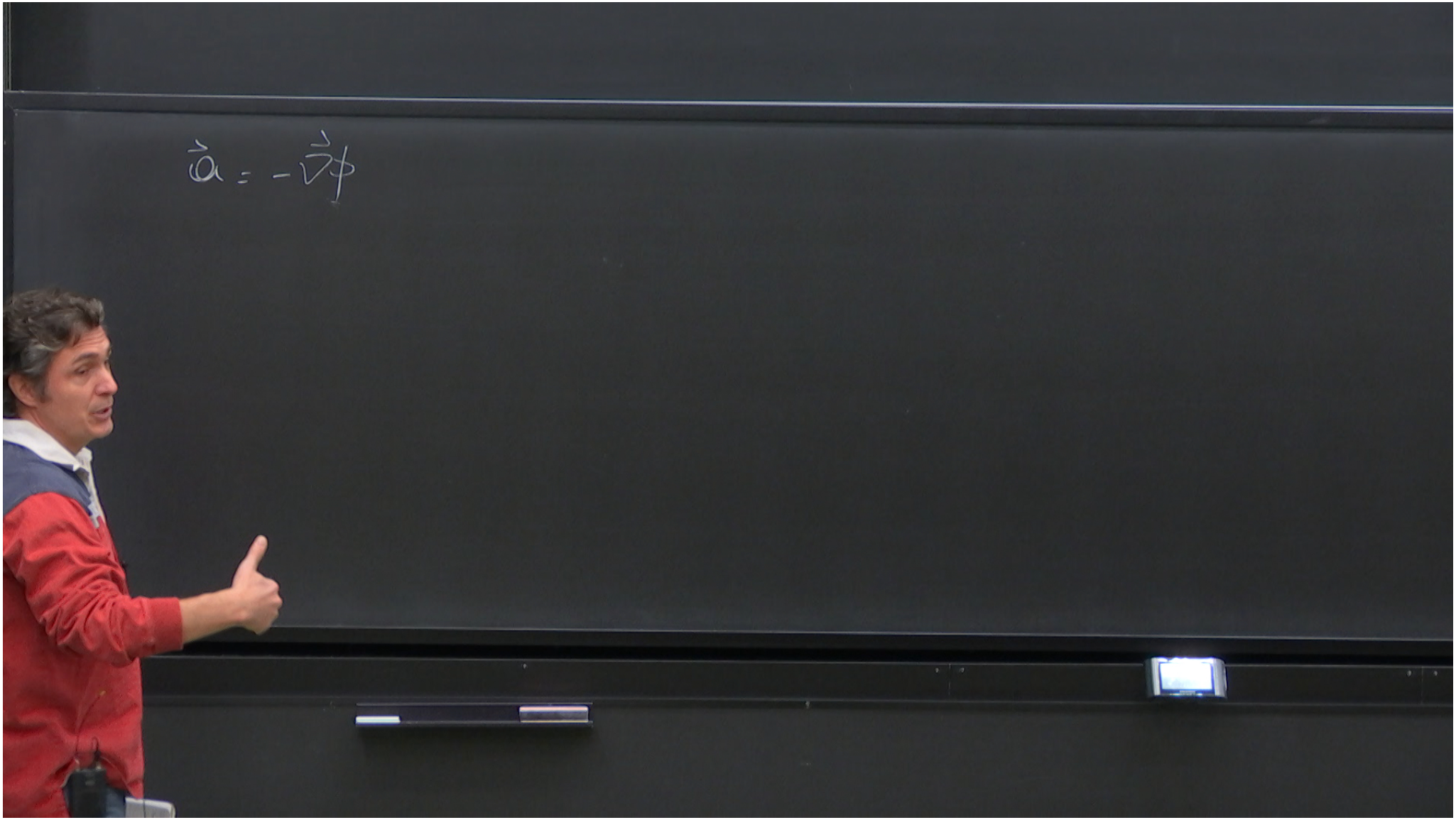
Title: Relativity - Lecture 2210117

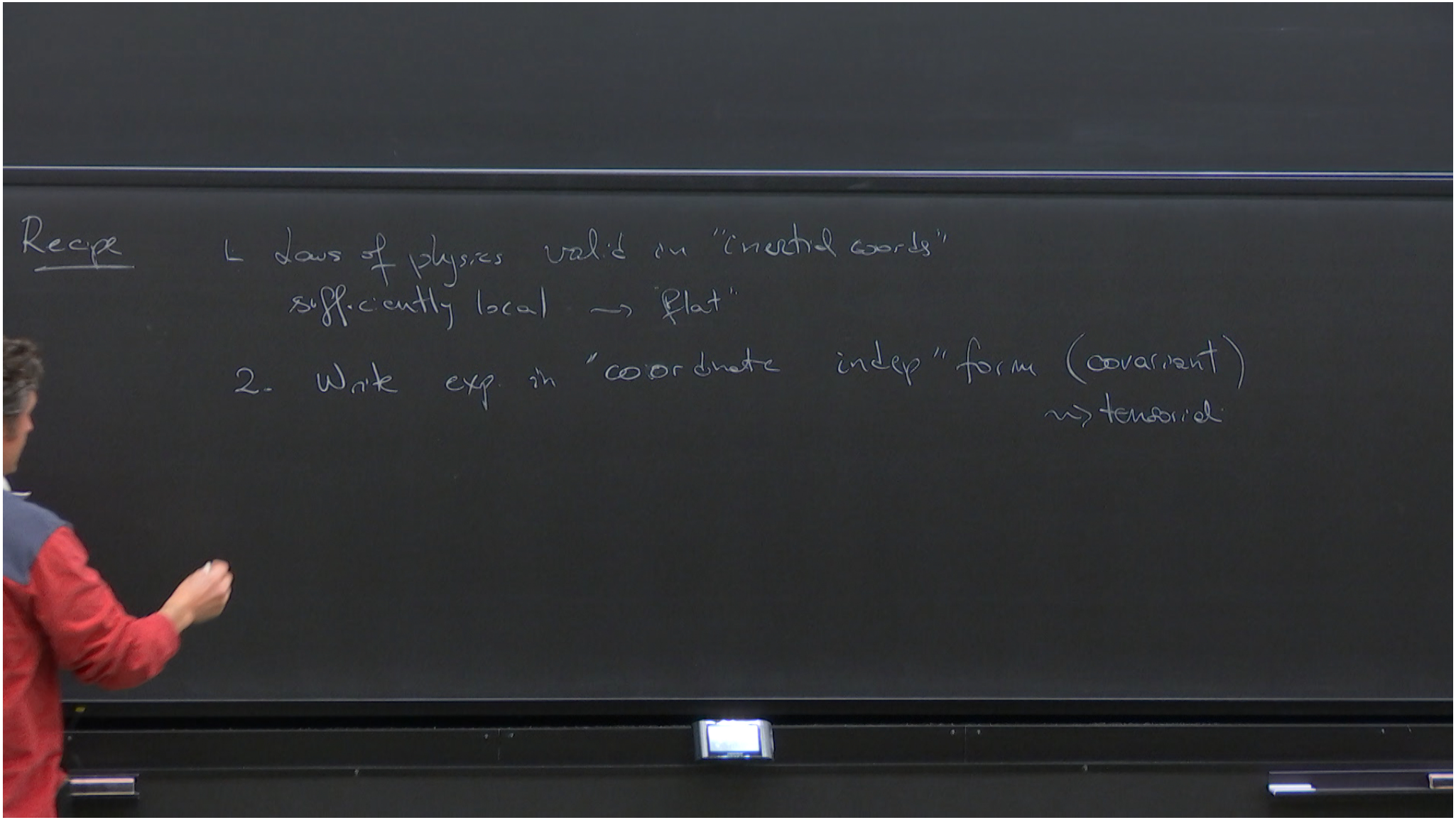
Speakers:

Collection: Relativity (2022/2023)

Date: October 17, 2022 - 9:00 AM

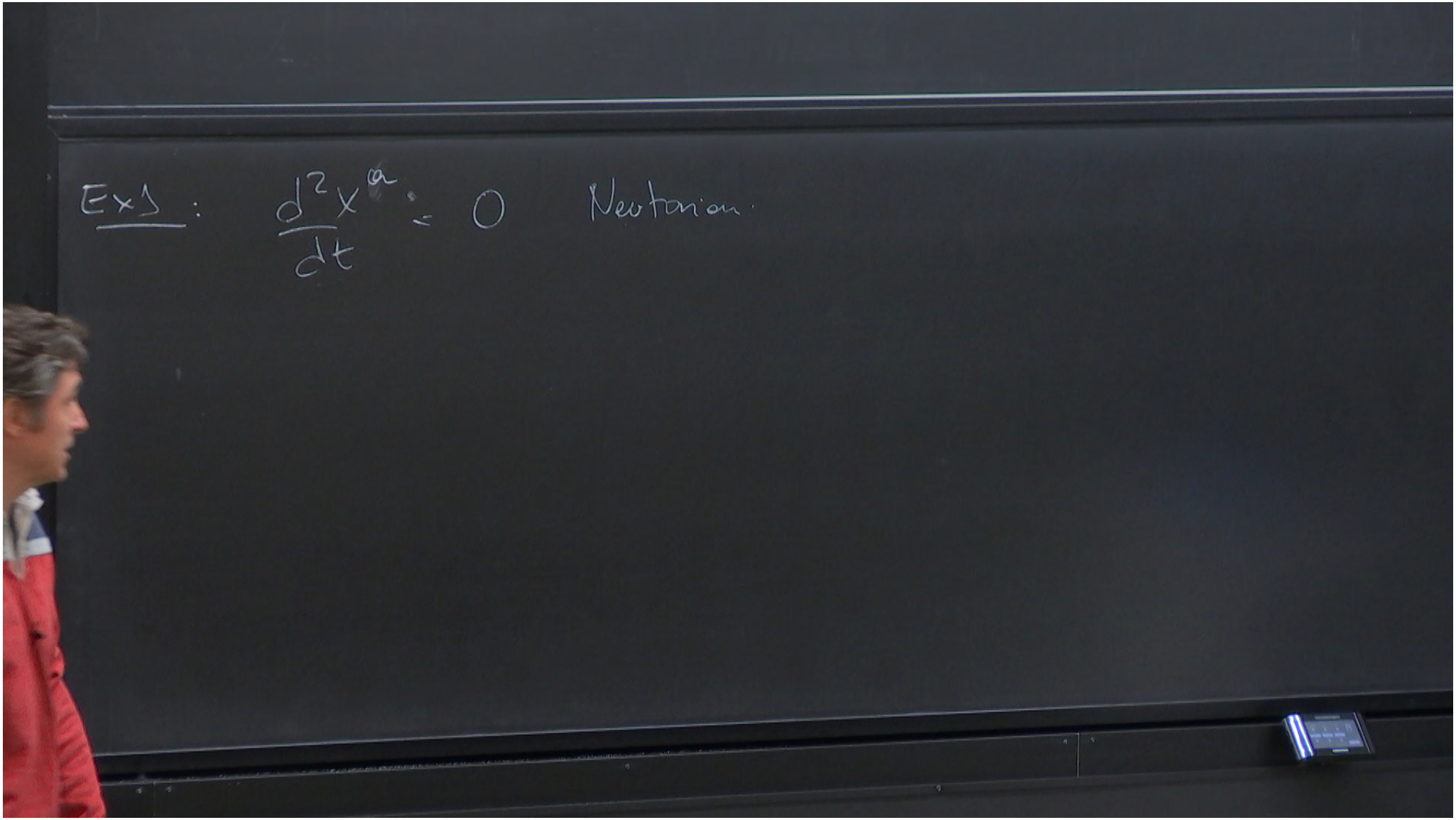
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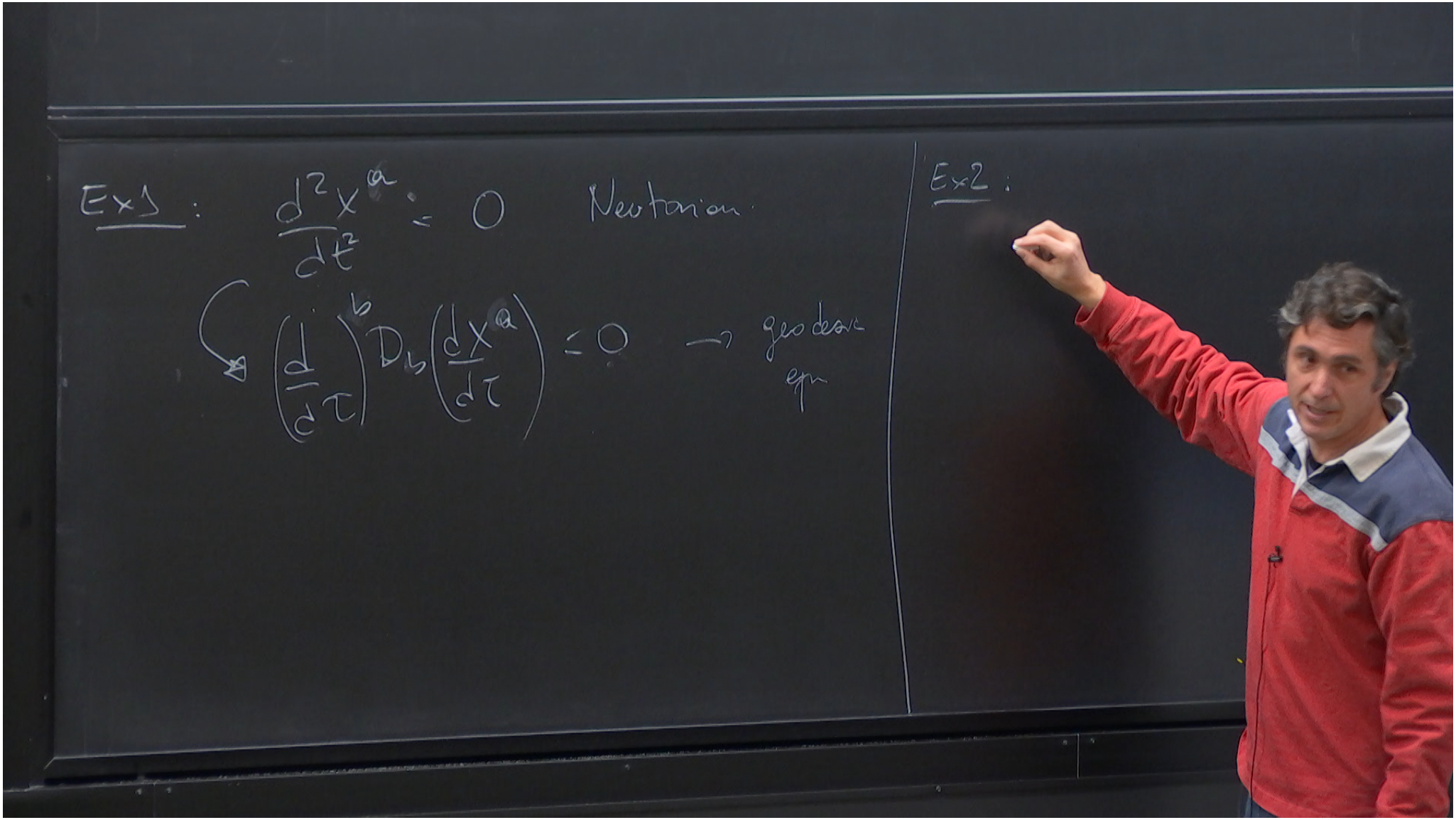

$\vec{a} = -\vec{\nabla}\phi$

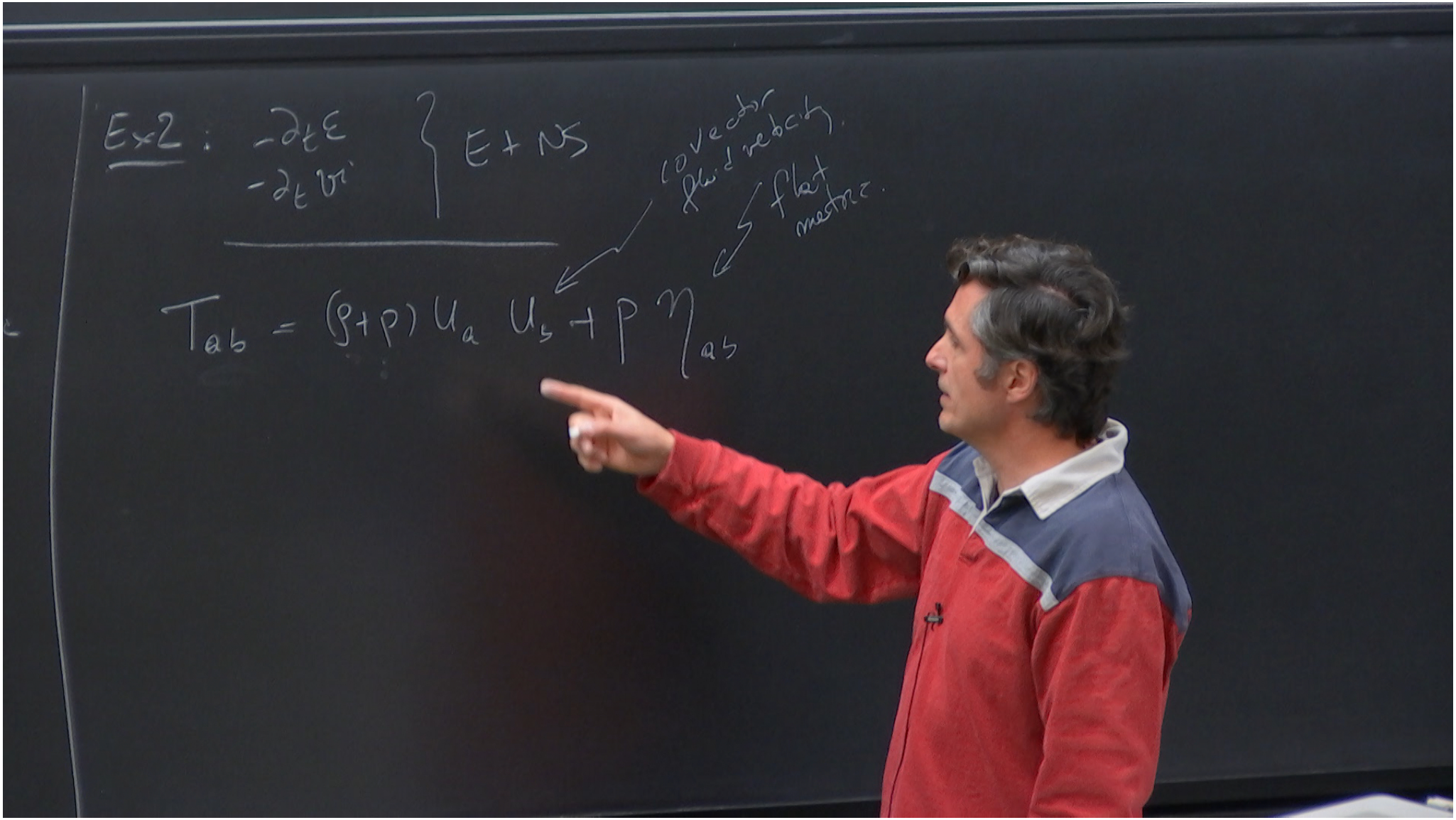

Recipe
1. Laws of physics valid in "inertial coords"
sufficiently local → flat"
2. Write exp. in "coordinate indep" form (covariant)
→ tensorial

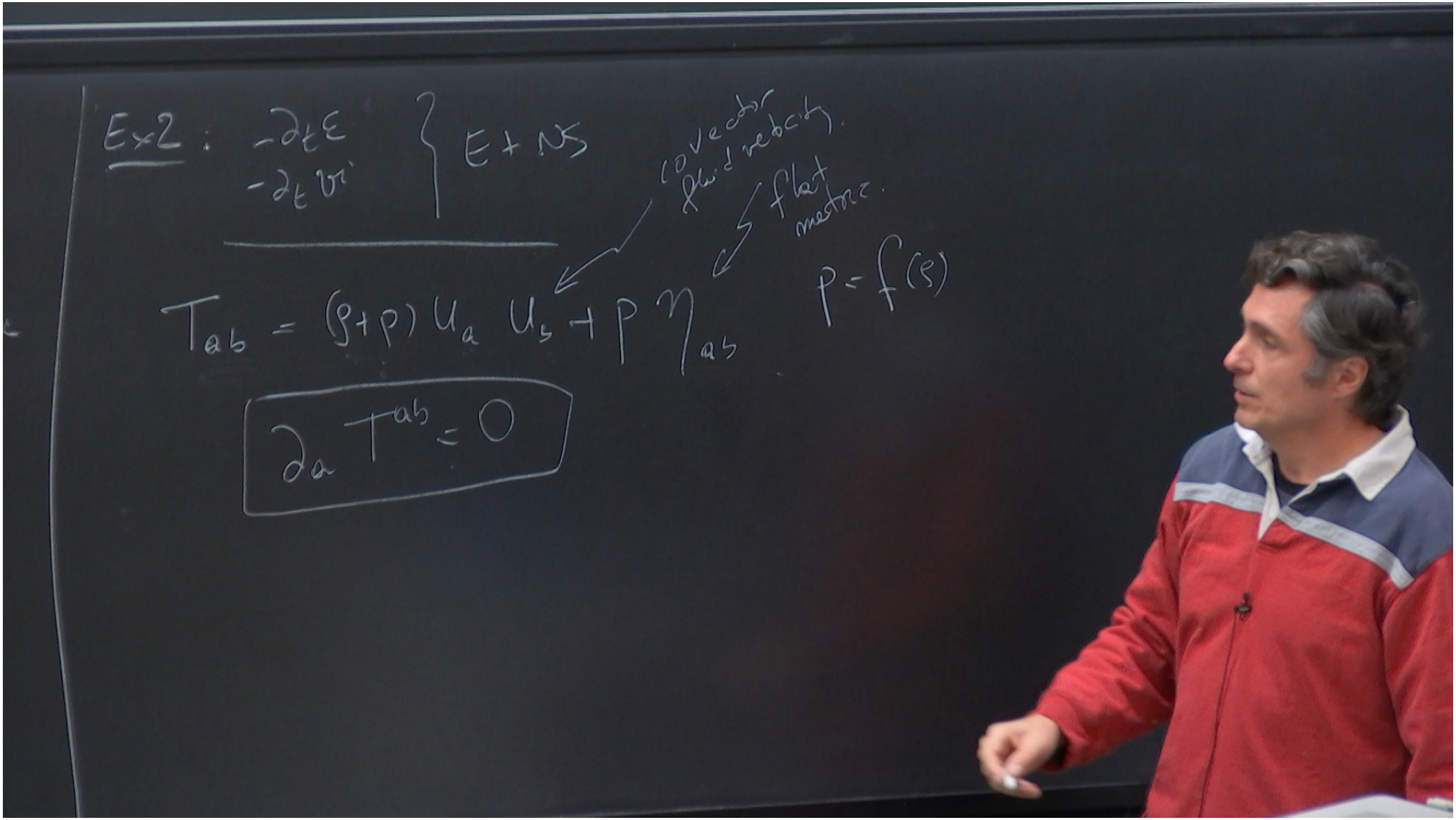
Recipe

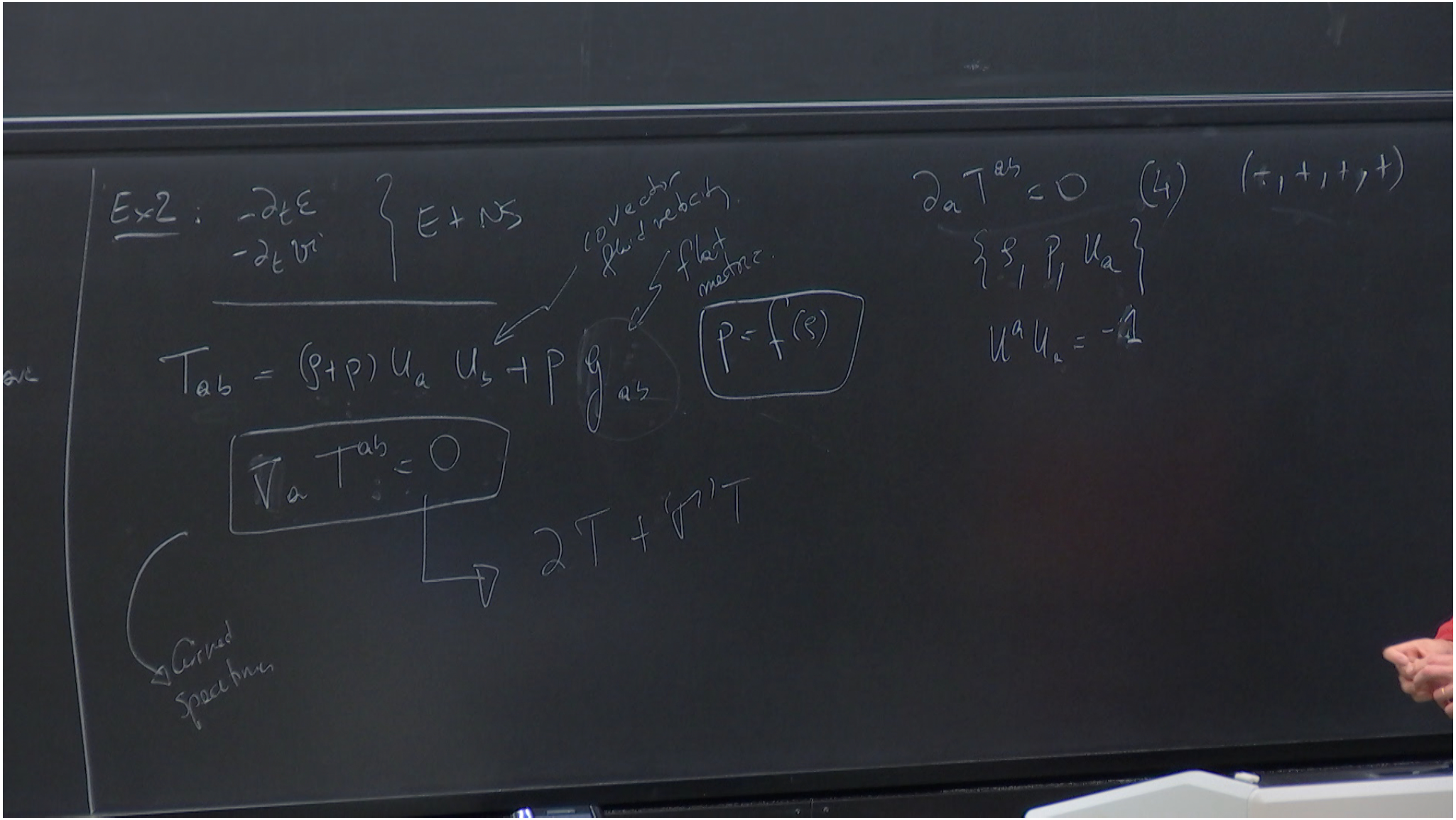
1. Laws of physics valid in "inertial coords"
   sufficiently local → flat"
2. Write exp in "coordinate indep" form (covariant)
   ⇝ tensorial
3. Assert we can mimic traditional laws of physics

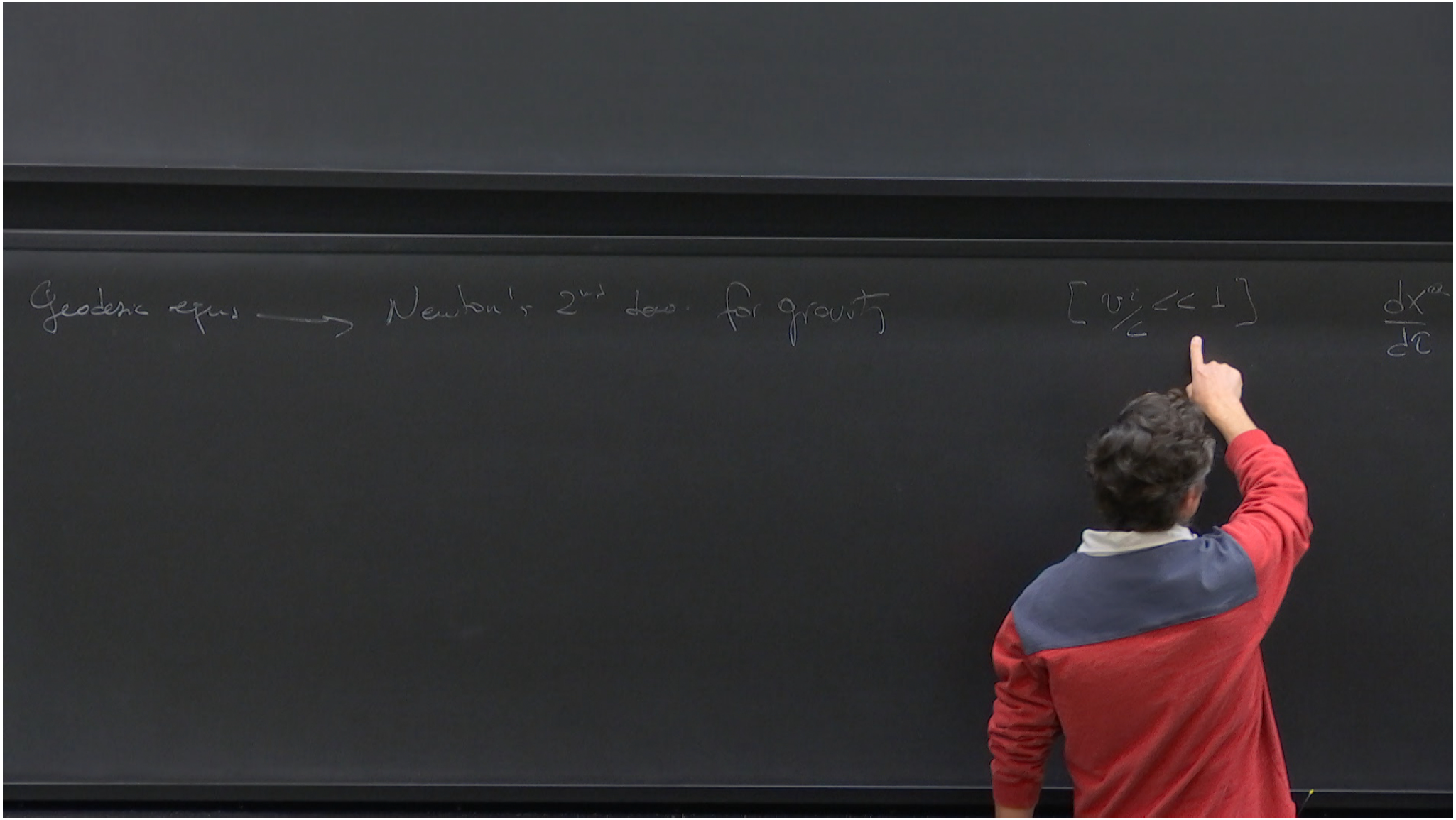

Ex 1 : $\frac{d^2 x^a}{dt} = 0$ Newtonian.

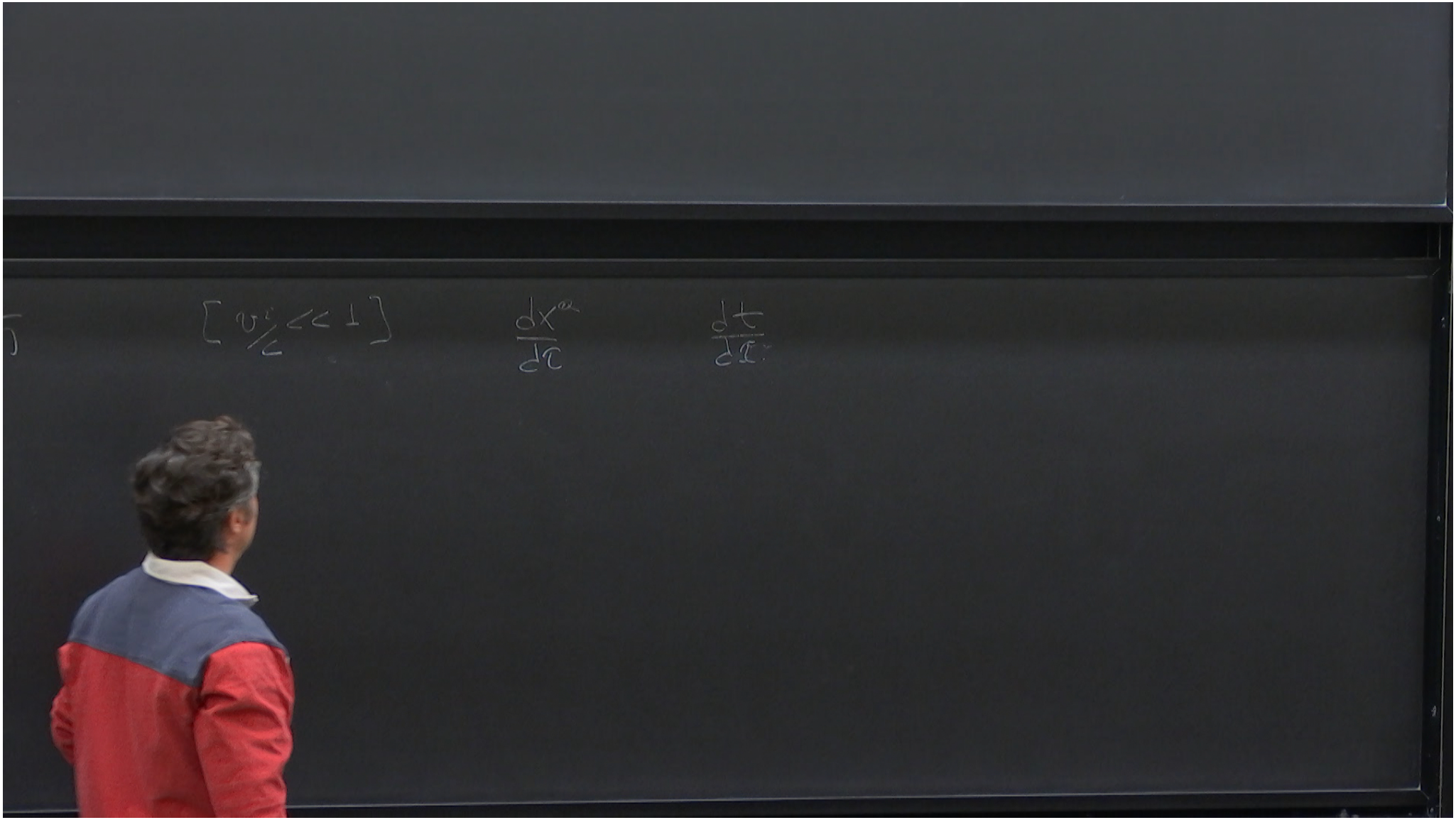

Ex1 : $\frac{d^2x^a}{dt^2} = 0$ Newtonian
$\left(\frac{d}{d\tau}\right)^b D_b\left(\frac{dx^a}{d\tau}\right) = 0$ → geodesic eqn
Ex2 :


Ex 2 : $-\partial_t \mathcal{E}$
$-\partial_t v^i$
} E + NS
covector fluid velocity.
flat metric.
$T_{ab} = (\rho + p) u_a u_b + p \eta_{ab}$


Ex 2 : $-\partial_t E$ $-\partial_t v_i$ } E + NS
covector fluid velocity.
flat metric.
$T_{ab} = (\rho + p) u_a u_b + p \eta_{ab}$
$p = f(\rho)$
$\partial_a T^{ab} = 0$


Ex 2 : $-\partial_t \varepsilon$ $-\partial_t v^i$ } E + NS
covector fluid velocity
flat metric
$T_{ab} = (\rho + p) u_a u_b + p g_{ab}$
$p = f(\rho)$
$\nabla_a T^{ab} = 0$
$\partial T + (\nabla^1) T$
Curved spacetime
$\partial_a T^{ab} = 0$ (4) (+,+,+,+)
$\{\rho, p, u_a\}$
$u^a u_a = -1$

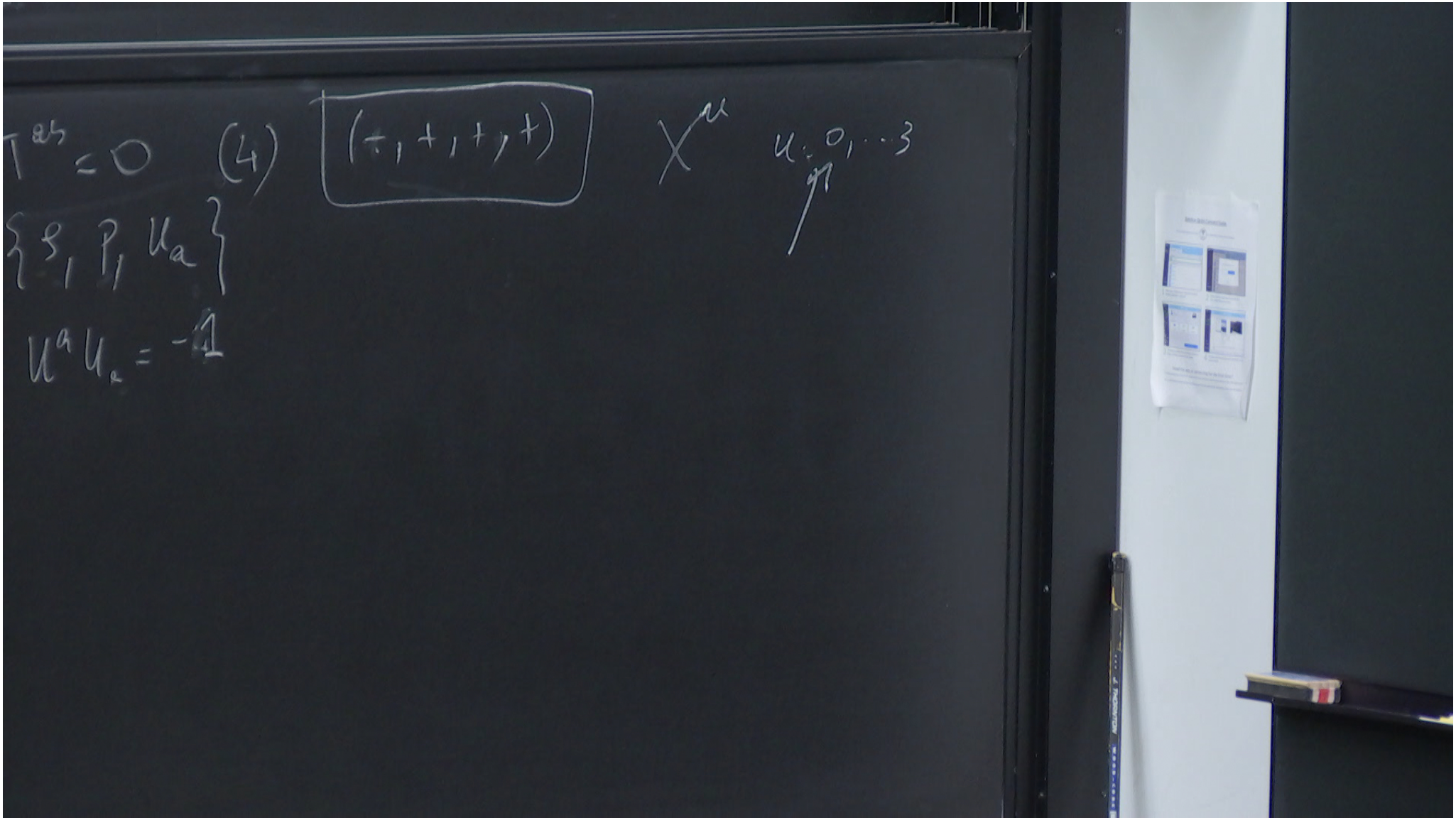

Geodesic eqns ⟶ Newton's 2nd law for gravity
$\left[\frac{v^2}{c^2} << 1\right]$
$\frac{dx^\alpha}{d\tau}$


$\left[\frac{v^2}{c^2} << 1\right]$
$\frac{dx^a}{d\tau}$
$\frac{dt}{d\tau}$

Geodesic eqns ⟶ Newton's 2nd law for gravity $\left[\frac{v_i}{c} << 1\right]$ $\frac{dx^a}{d\tau}$

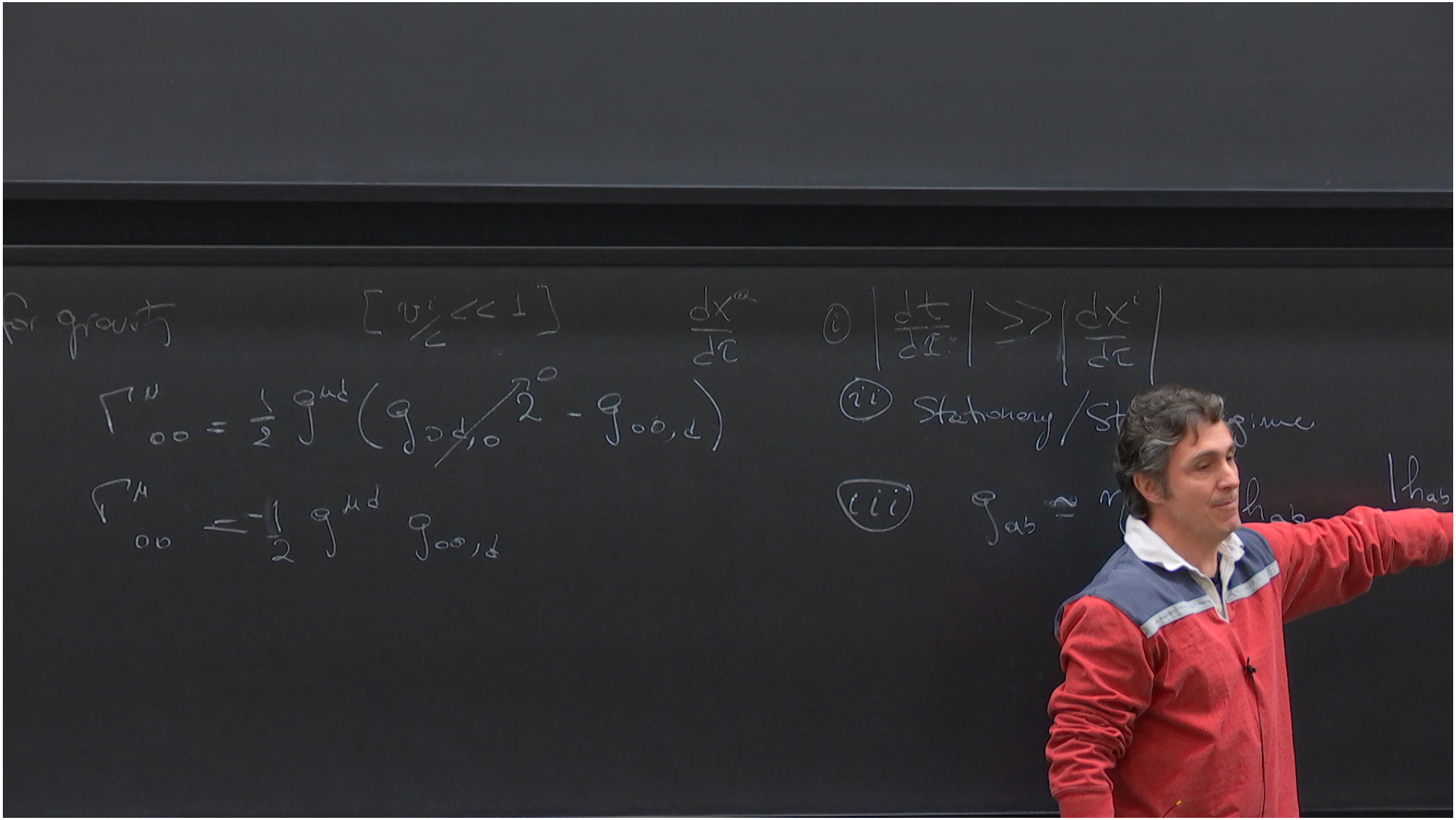
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}$$

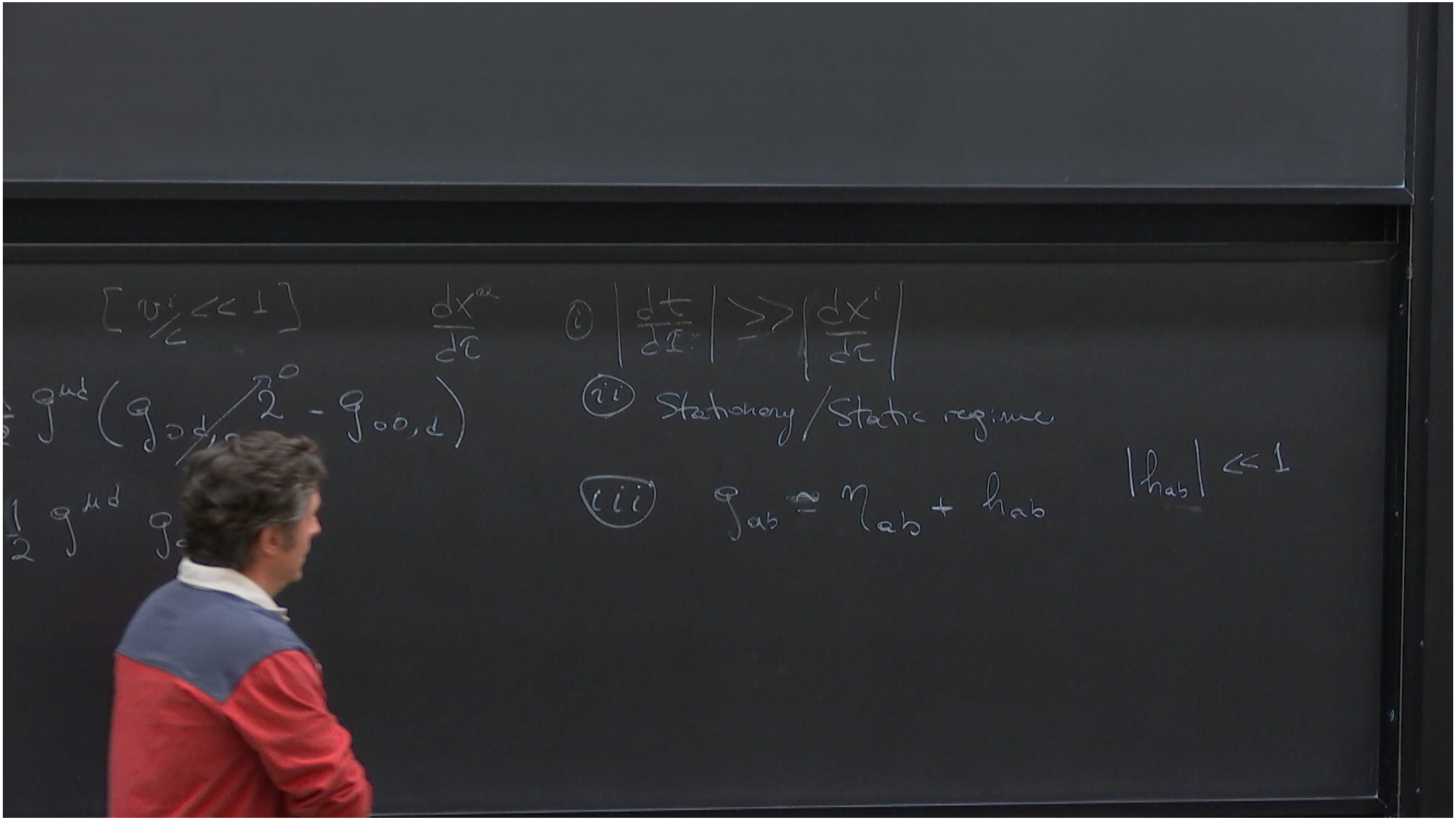
$$\frac{d^2x}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2$$


$T^{ab} = 0$ (4)
$(-,+,+,+)$
X^μ $\mu = 0, \ldots 3$
$\{\rho, P, u_a\}$
$u^a u_a = -1$

for gravity $[v/c << 1]$ $\frac{dx^\alpha}{d\tau}$ (i) $\left|\frac{dt}{d\tau}\right| >> \left|\frac{dx^i}{d\tau}\right|$

$\Gamma^\nu_{00} = \frac{1}{2} g^{\nu d}\left(g_{0d,0}\,2 - g_{00,d}\right)$ (ii) Stationary / Static regime


for gravity
$[v/c << 1]$
$\frac{dx^a}{d\tau}$
(i) $\left|\frac{dt}{d\tau}\right| >> \left|\frac{dx^i}{d\tau}\right|$
$\Gamma^{\mu}_{00} = \frac{1}{2} g^{\mu d}(g_{0d,0} + 2^{0} - g_{00,d})$
(ii) Stationary/St
$\Gamma^{\mu}_{00} = -\frac{1}{2} g^{\mu d} g_{00,d}$
(iii) $g_{ab} \simeq$
h_{ab}
h_{ab}


$\left[\frac{v^i}{c} << 1\right]$
$\frac{dx^\alpha}{d\tau}$
(i) $\left|\frac{dt}{d\tau}\right| >> \left|\frac{dx^i}{d\tau}\right|$
$g^{\mu d}\left(g_{0d,\cancel{2}^{0}} - g_{00,d}\right)$
(ii) Stationary / Static regime
$\frac{1}{2} g^{\mu d}$
(iii) $g_{ab} \simeq \eta_{ab} + h_{ab}$
$|h_{ab}| << 1$

$\left[\frac{v^i}{c} << 1\right]$ $\quad \frac{dx^\alpha}{d\tau}$

$g^{\mu d}\left(g_{0d,0}\cancel{\dot{2}^0} - g_{00,d}\right)$

$\frac{1}{2} g^{\mu d} g_{00,d}$

(i) $\left|\frac{dt}{d\tau}\right| >> \left|\frac{dx^i}{d\tau}\right|$

(ii) Stationary / Static regime

(iii) $g_{ab} \simeq \eta_{ab} + h_{ab} \qquad |h_{ab}| << 1$

$\hookrightarrow g^{ab} \simeq \eta^{ab} - h^{ab}$

$\left[\frac{v^i}{c} << 1\right]$ $\quad \frac{dx^\alpha}{d\tau}$

$g^{\mu d}\left(g_{0d,0}\overset{\rightarrow 0}{2} - g_{00,d}\right)$

$\frac{1}{2} g^{\mu d} h_{00,d}$

(i) $\left|\frac{dt}{d\tau}\right| >> \left|\frac{dx^i}{d\tau}\right|$

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(iii) $g_{ab} \simeq \eta_{ab} + h_{ab} \quad |h_{ab}| << 1$

$\hookrightarrow g^{ab} \simeq \eta^{ab} - h^{ab}$

(iv) Standard Cartesian coords locally $\eta_{ab} = (-1, 1, 1, 1)$

Assert we can mimic traditional laws of physics

for gravity $\left[\frac{v}{c} << 1\right]$ $\frac{dx^\alpha}{d\tau}$

$$\Gamma^\mu_{00} = \frac{1}{2} g^{\mu d}\left(g_{0d,0}^{\,\to 0} - g_{00,d}\right)$$

$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu d} h_{00,d}$$

(i) $\left|\frac{dt}{d\tau}\right| >> \left|\frac{dx^i}{d\tau}\right|$

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↳ $g^{ab} \simeq \eta^{ab} - h^{ab}$

(iv) Standard Cartesian coords locally $\eta_{ab} = (-1, 1, 1, 1)$

Geodesic eqns ⟶ Newton's 2nd law for gravity $\left[\frac{v^2}{c^2} << 1\right]$

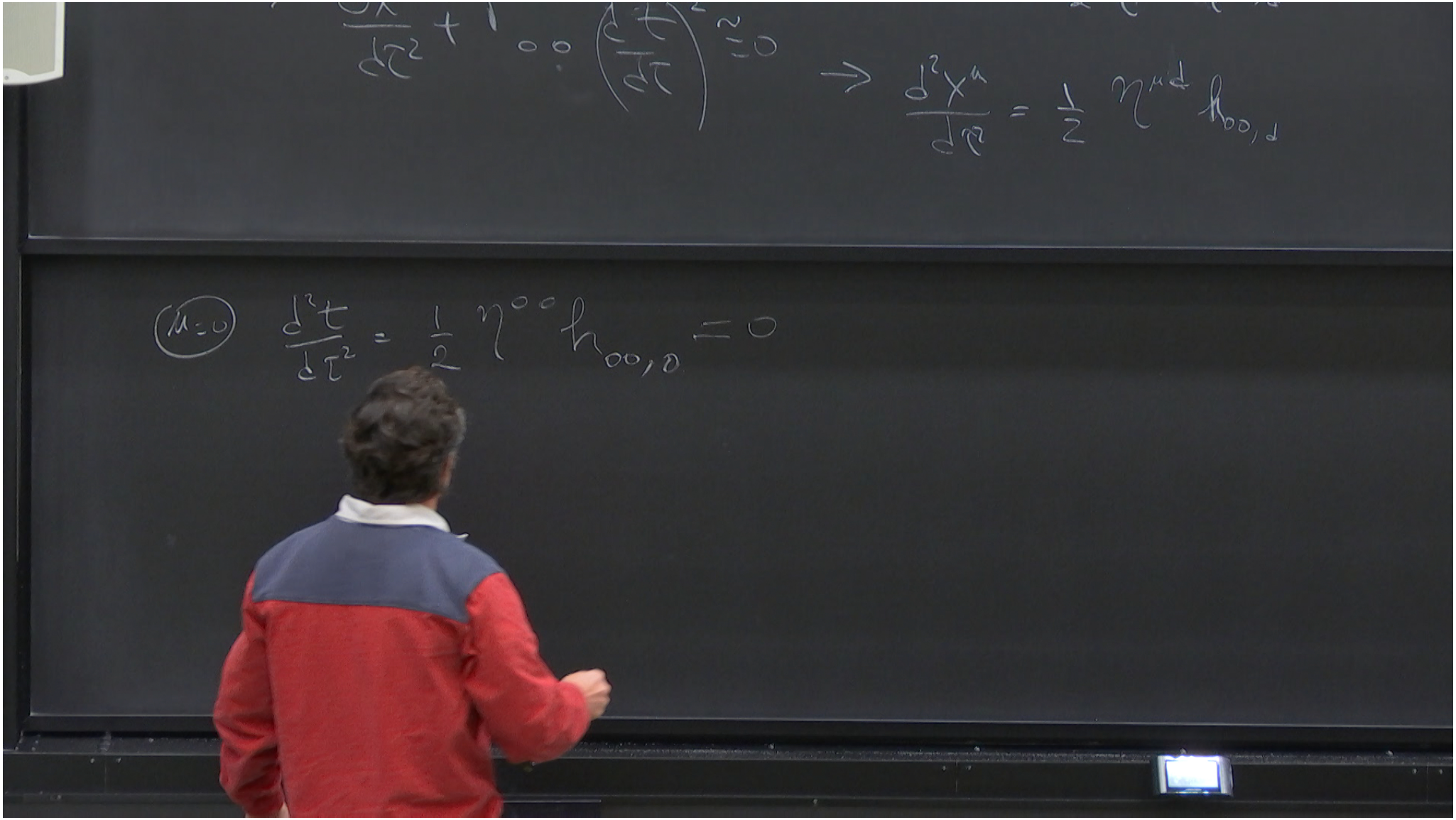
$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}$$

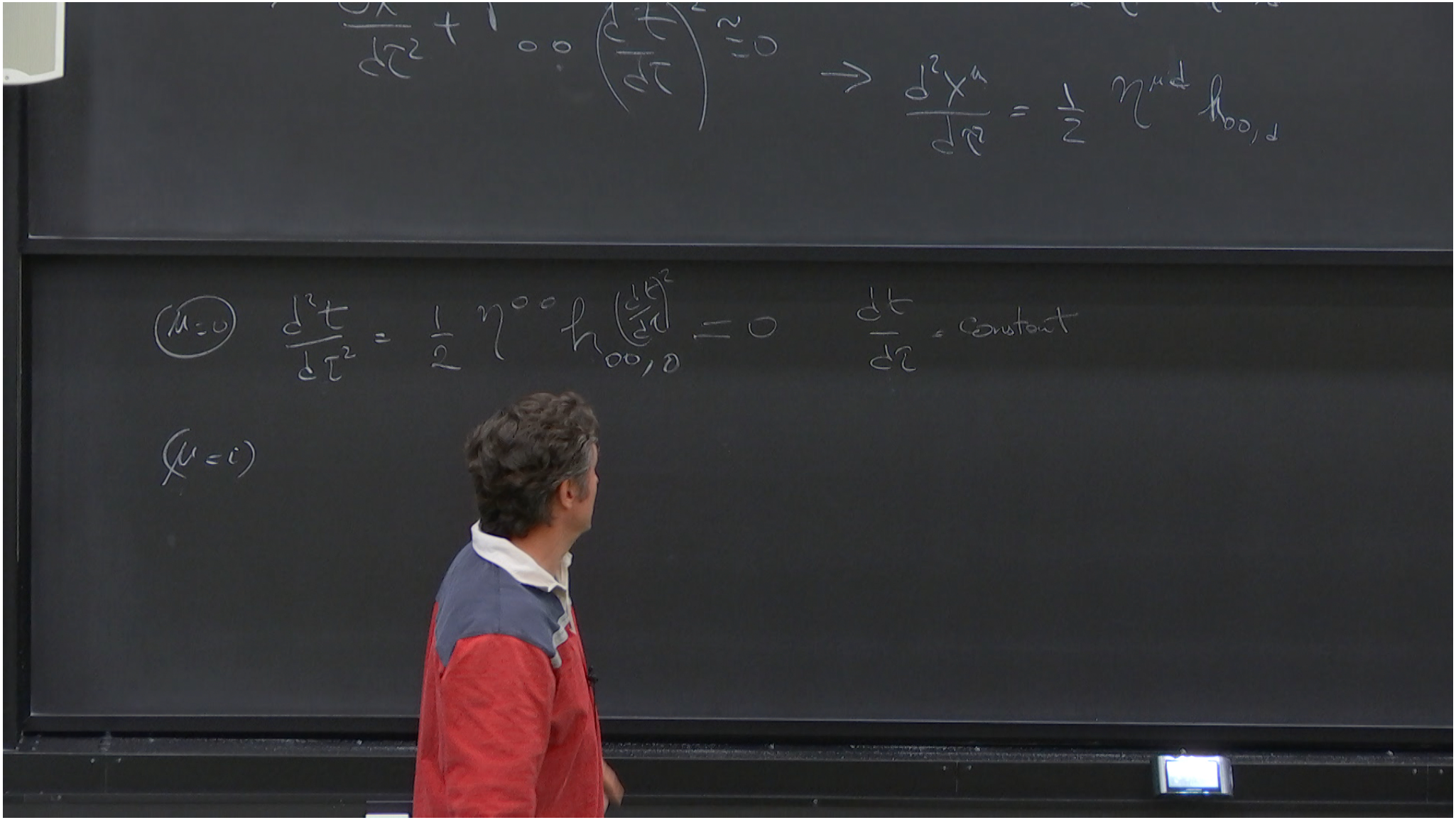
$$\Gamma^\mu_{00} = \frac{1}{2}g^{\mu d}\left(g_{0d,0}^{\,\nearrow 0} - g_{00,d}\right)$$

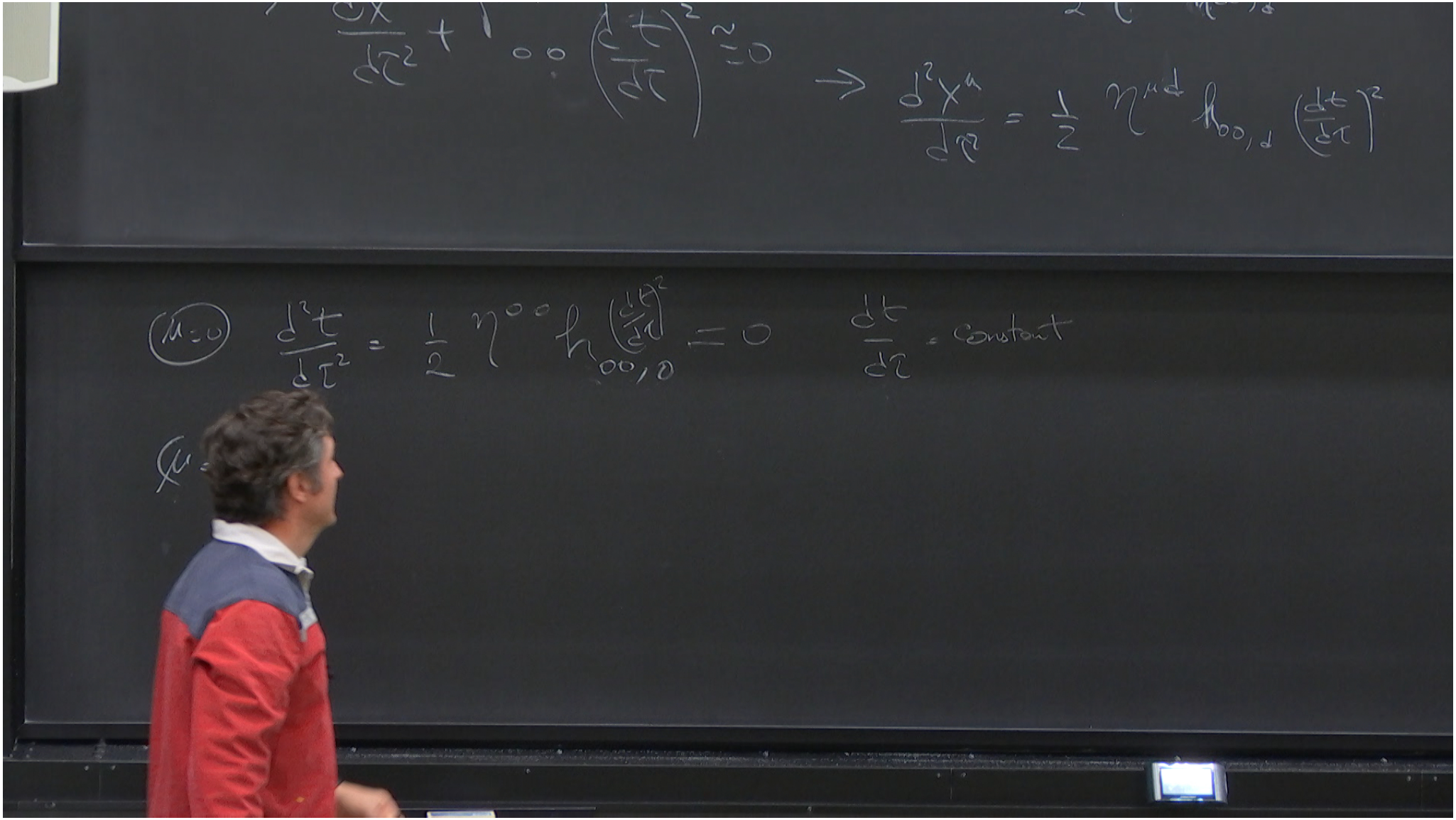
$$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu d}h_{00,d}$$

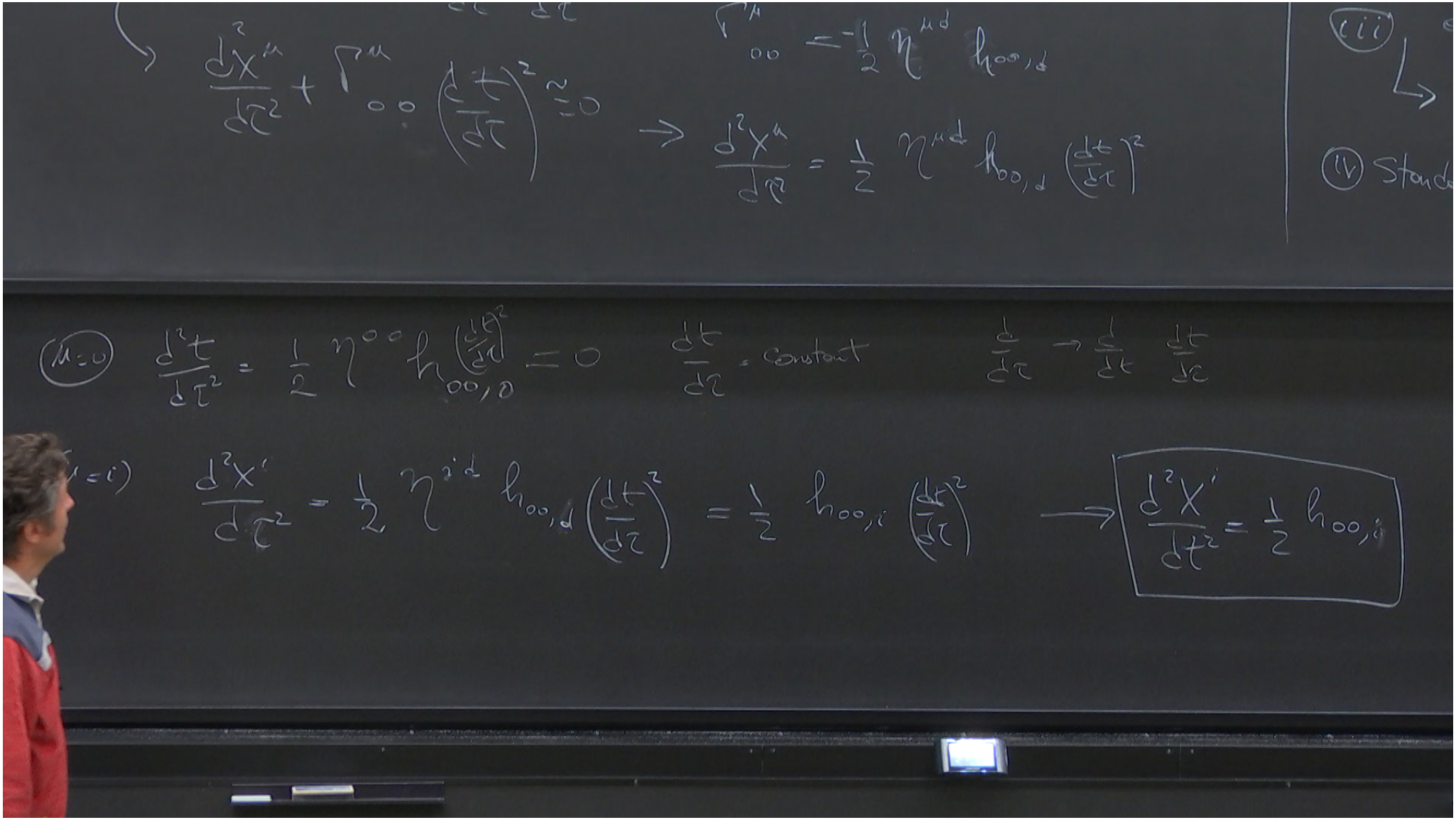
$$\hookrightarrow \frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 \simeq 0 \Rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu d}h_{00,d}$$

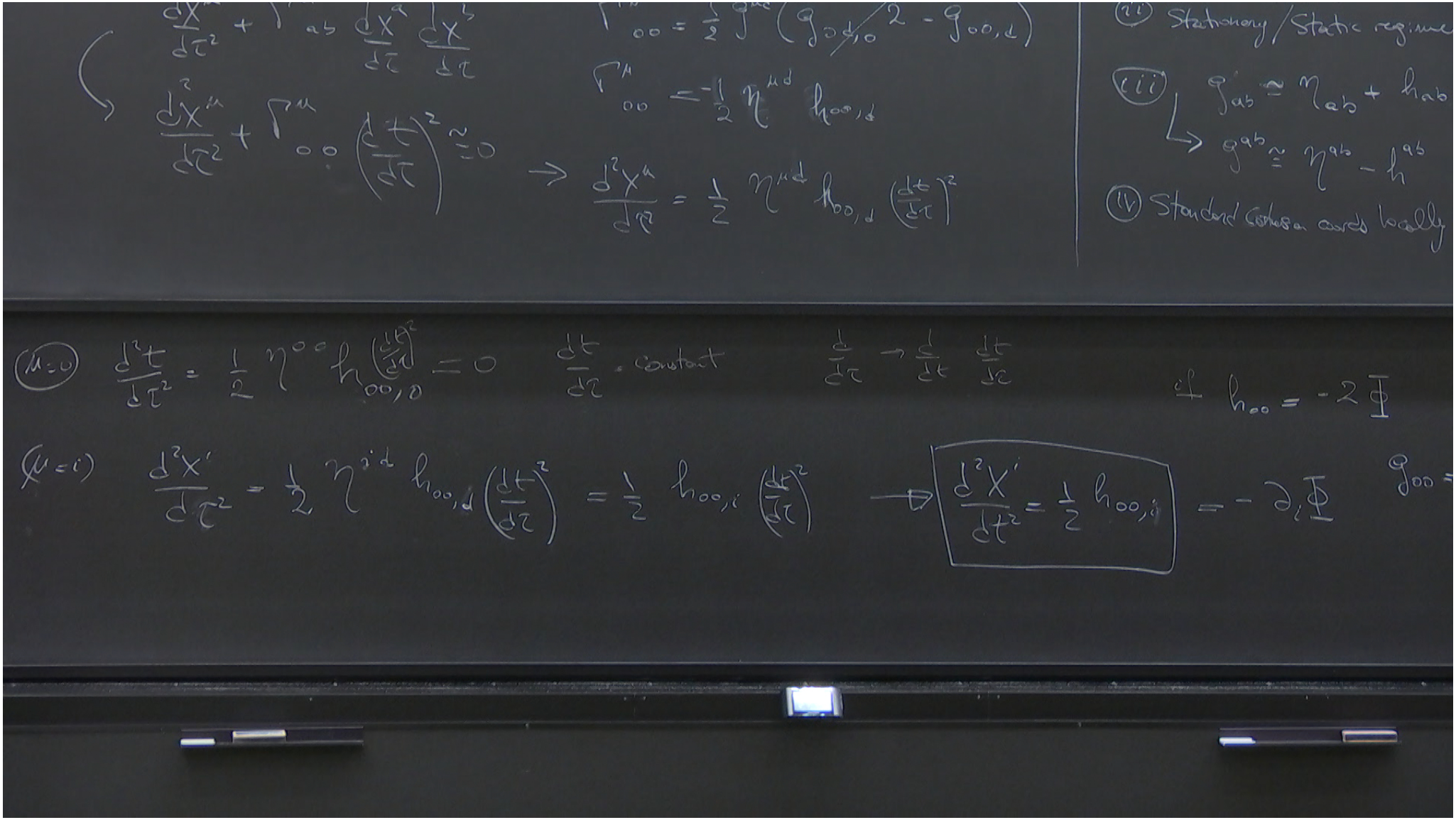
$(\mu = 0)$ $\frac{d^2t}{d\tau^2} =$


$\frac{d^2x}{d\tau^2} + \Gamma^{0}_{00}\left(\frac{dt}{d\tau}\right)^2 \approx 0 \rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\nu}h_{00,\nu}$
$(\mu=0)\ \frac{d^2t}{d\tau^2} = \frac{1}{2}\eta^{00}h_{00,0} = 0$


$\frac{d^2x^i}{d\tau^2} + \Gamma^i_{00}\left(\frac{dt}{d\tau}\right)^2 \simeq 0 \rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\nu} h_{00,\nu}$
$(\mu = 0)$ $\frac{d^2t}{d\tau^2} = \frac{1}{2}\eta^{00} h_{00,0}\left(\frac{dt}{d\tau}\right)^2 = 0$ $\frac{dt}{d\tau}$ = constant
$(\mu = i)$


$\frac{d^2x}{d\tau^2} + \ldots = 0 \Rightarrow \left(\frac{dt}{d\tau}\right)^2 \simeq 0$
$\Rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\alpha} h_{00,\alpha} \left(\frac{dt}{d\tau}\right)^2$
$(\mu=0)\quad \frac{d^2t}{d\tau^2} = \frac{1}{2}\eta^{00} h_{00,0} \left(\frac{dt}{d\tau}\right)^2 = 0 \qquad \frac{dt}{d\tau} = \text{constant}$
$(\mu =$


$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 \simeq 0$
$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu d} h_{00,d}$
$\rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu d} h_{00,d}\left(\frac{dt}{d\tau}\right)^2$
(ii)
(iv) Stand
$(\mu=0)$ $\frac{d^2t}{d\tau^2} = \frac{1}{2}\eta^{00} h_{00,0}\left(\frac{dt}{d\tau}\right)^2 = 0$
$\frac{dt}{d\tau}$ = constant
$\frac{d}{d\tau} \rightarrow \frac{d}{dt}\frac{dt}{d\tau}$
$(\mu=i)$ $\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\eta^{id} h_{00,d}\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{2} h_{00,i}\left(\frac{dt}{d\tau}\right)^2$
$\rightarrow \boxed{\frac{d^2x^i}{dt^2} = \frac{1}{2} h_{00,i}}$


$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{ab}\frac{dx^a}{d\tau}\frac{dx^b}{d\tau}$
$\Gamma^\mu_{00} = \frac{1}{2}\eta^{\mu\alpha}(g_{0\alpha,0} + g_{0\alpha,0} - g_{00,\alpha})$
$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{00}\left(\frac{dt}{d\tau}\right)^2 \simeq 0$
$\Gamma^\mu_{00} = -\frac{1}{2}\eta^{\mu\alpha}h_{00,\alpha}$
$\Rightarrow \frac{d^2x^\mu}{d\tau^2} = \frac{1}{2}\eta^{\mu\alpha}h_{00,\alpha}\left(\frac{dt}{d\tau}\right)^2$
(ii) Stationary / Static regime
(iii) $g_{ab} \simeq \eta_{ab} + h_{ab}$
$g^{ab} \simeq \eta^{ab} - h^{ab}$
(iv) Standard Cartesian coords locally
($\mu = 0$) $\frac{d^2t}{d\tau^2} = \frac{1}{2}\eta^{00}h_{00,0}\left(\frac{dt}{d\tau}\right)^2 = 0$
$\frac{dt}{d\tau}$: constant
$\frac{d}{d\tau} \to \frac{d}{dt}\frac{dt}{d\tau}$
if $h_{00} = -2\Phi$
($\mu = i$) $\frac{d^2x^i}{d\tau^2} = \frac{1}{2}\eta^{i\alpha}h_{00,\alpha}\left(\frac{dt}{d\tau}\right)^2 = \frac{1}{2}h_{00,i}\left(\frac{dt}{d\tau}\right)^2$
$\Rightarrow \boxed{\frac{d^2x^i}{dt^2} = \frac{1}{2}h_{00,i}} = -\partial_i\Phi$
$g_{00} =$

$\frac{1}{2}\eta^{\mu d} h_{00,d}\left(\frac{dt}{d\tau}\right)^2$

$\hookrightarrow g^{ab} \simeq \eta^{ab} - h^{ab}$

(iv) Standard Cartesian coords locally $\eta_{ab} = (-1, 1, 1, 1)$

$\frac{d}{d\tau} \to \frac{d}{dt}\frac{dt}{d\tau}$

if $h_{00} = -2\Phi$

$h_{00,i}\left(\frac{dt}{d\tau}\right)^2 \Rightarrow \boxed{\frac{d^2x^i}{dt^2} = \frac{1}{2} h_{00,i}} = -\partial_i \Phi$

$g_{00} = -(1+2\Phi)$

$\left(\frac{dt}{d\tau}\right)^2$

(iv) Standard Cartesian coords locally $\eta_{\mu\nu} = (-1, 1, 1, 1)$

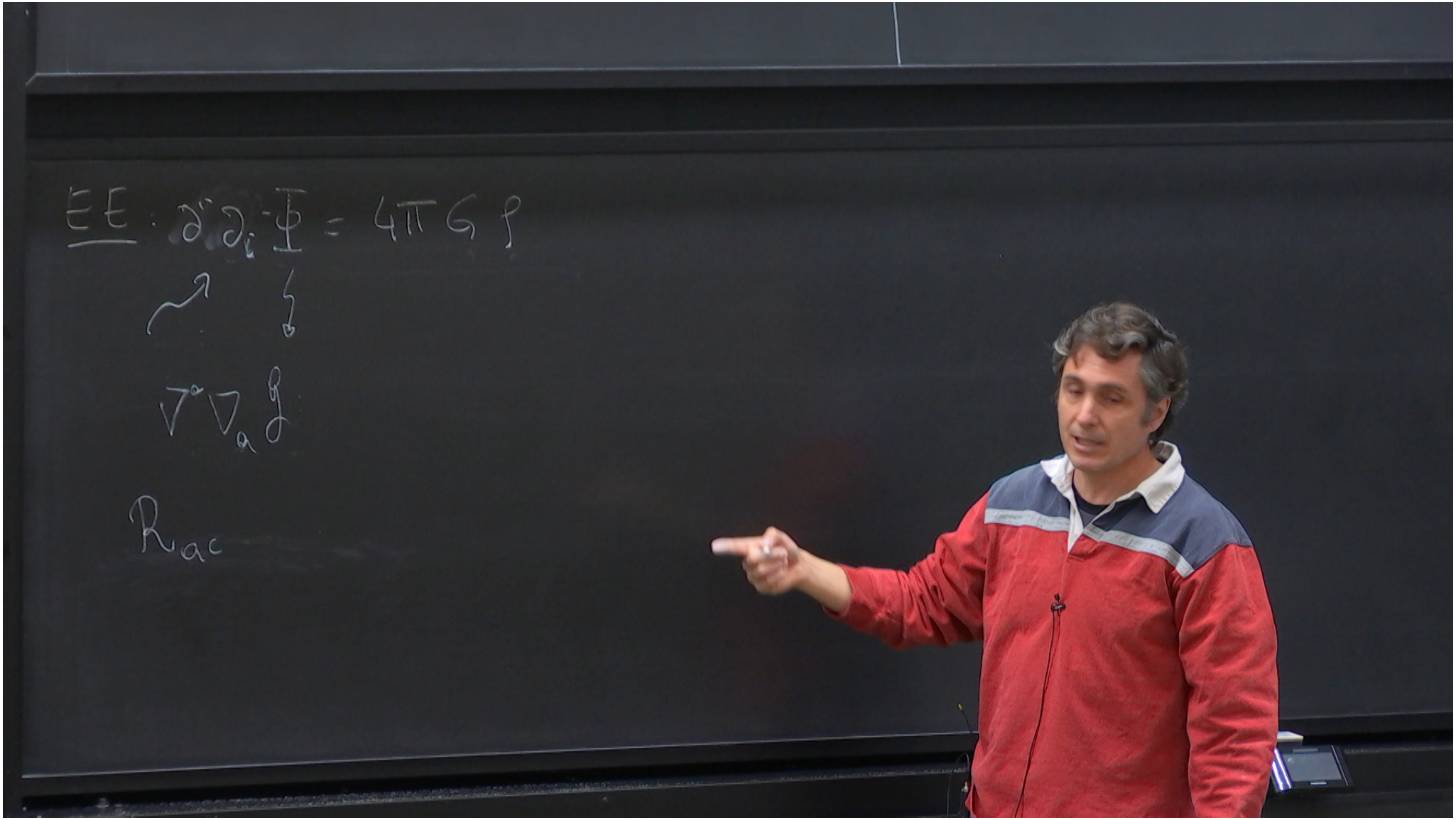
$\frac{d}{d\tau} \to \frac{d}{dt}\frac{dt}{d\tau}$

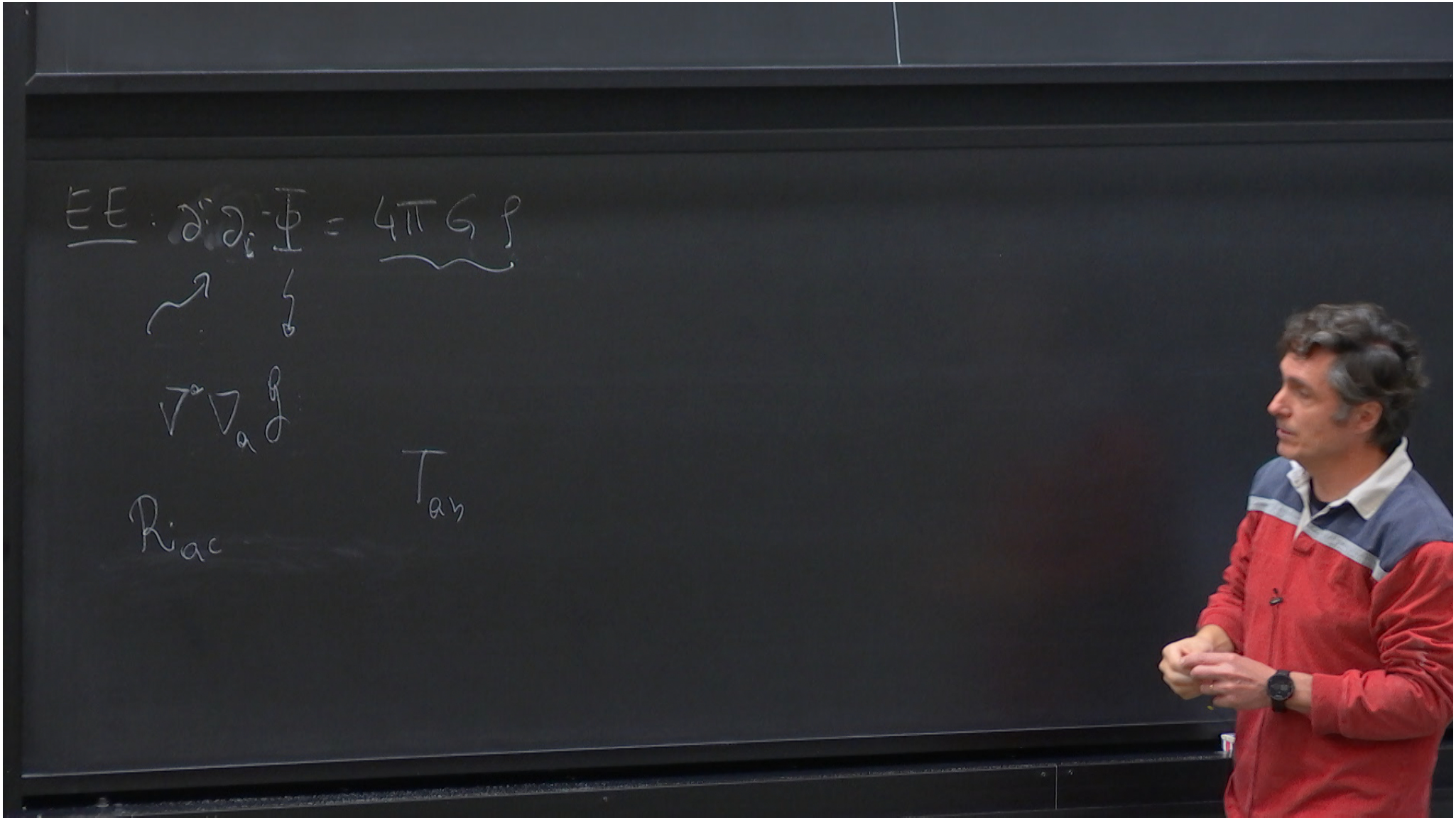
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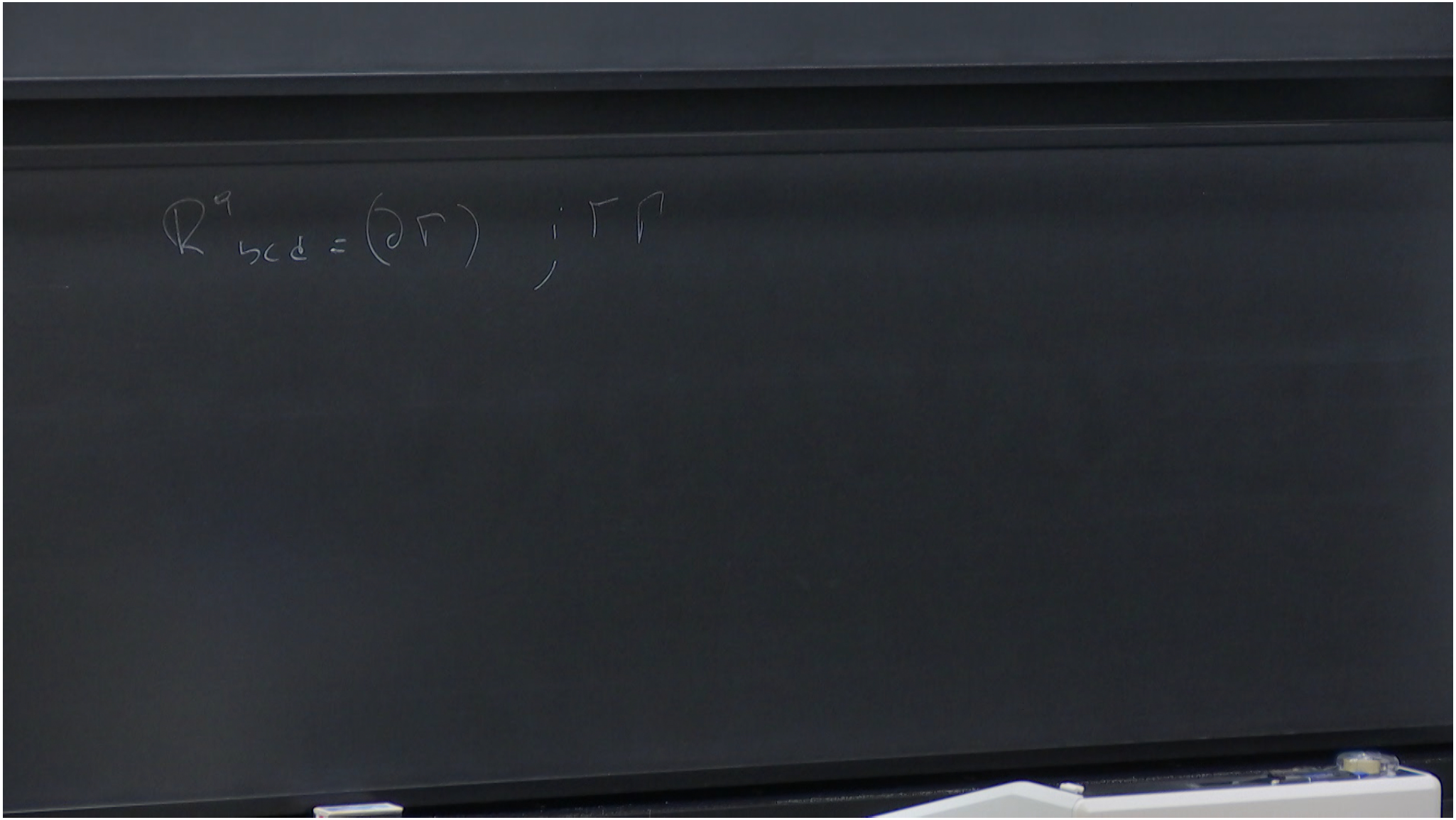
$g_{00} = -(1+2\Phi)$

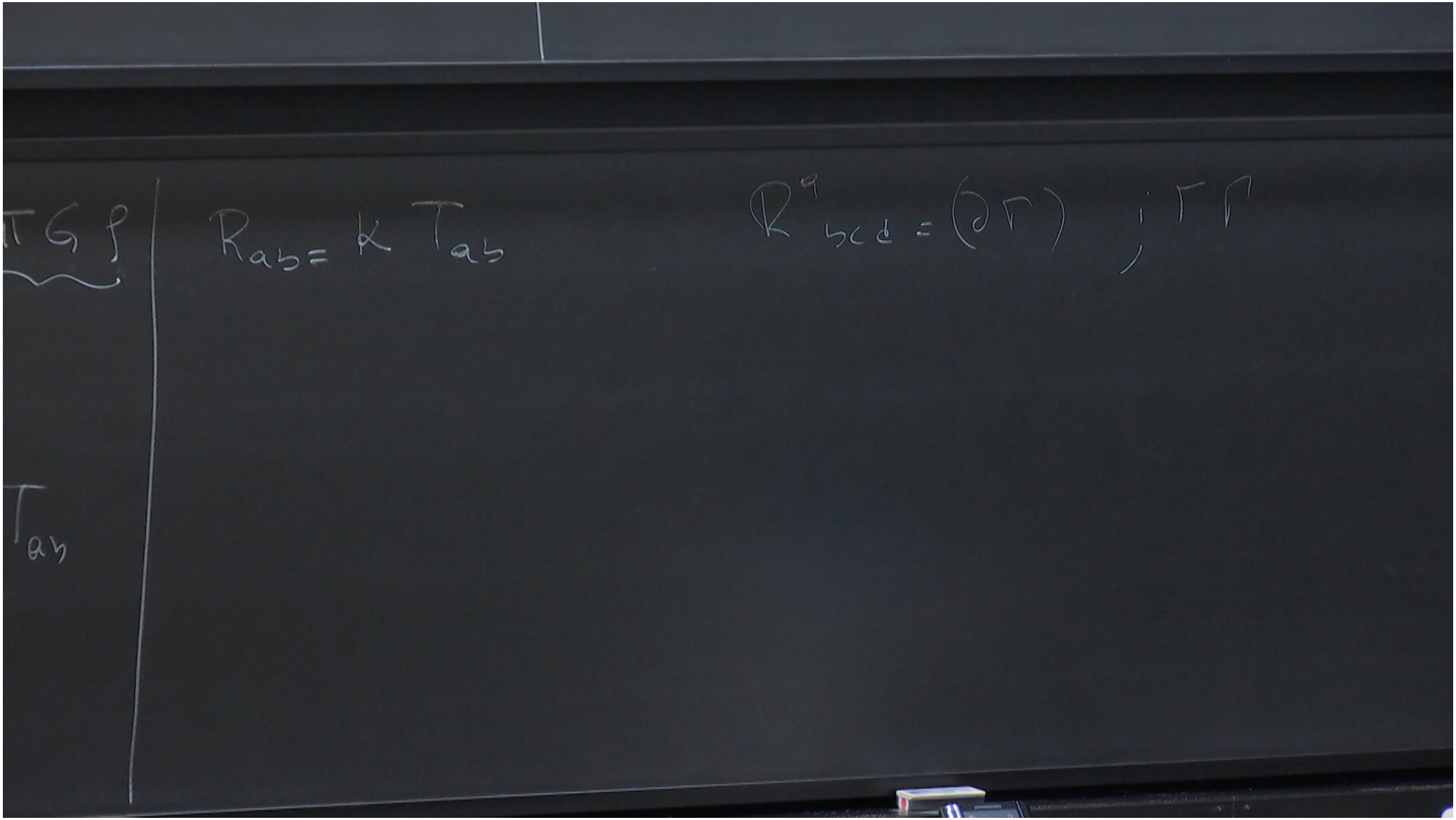
$$\frac{d^2x^i}{dt^2} = \frac{1}{2} h_{00,i} = -\partial_i \Phi$$

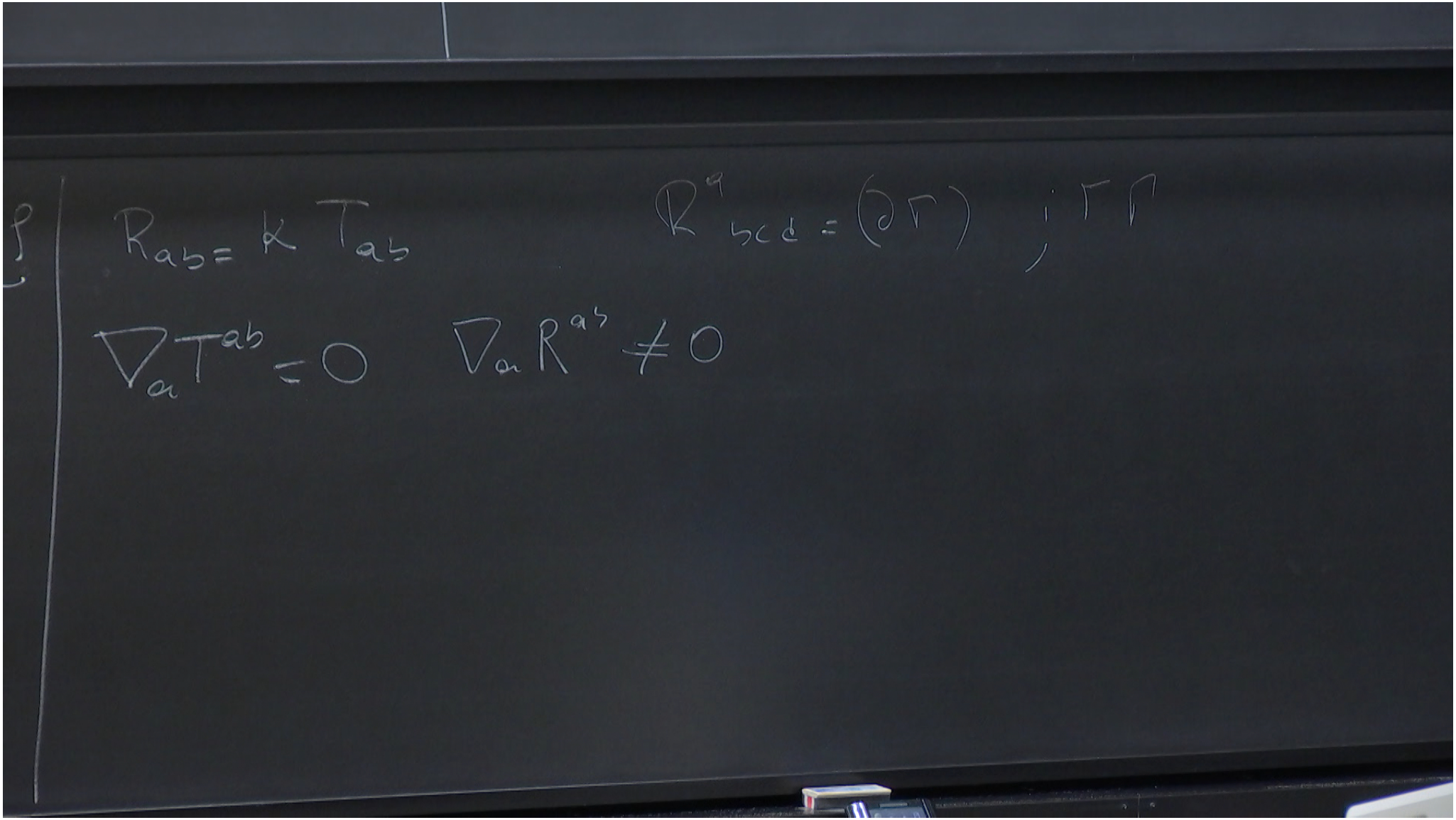
$u^a \nabla_a u^b = a^b$

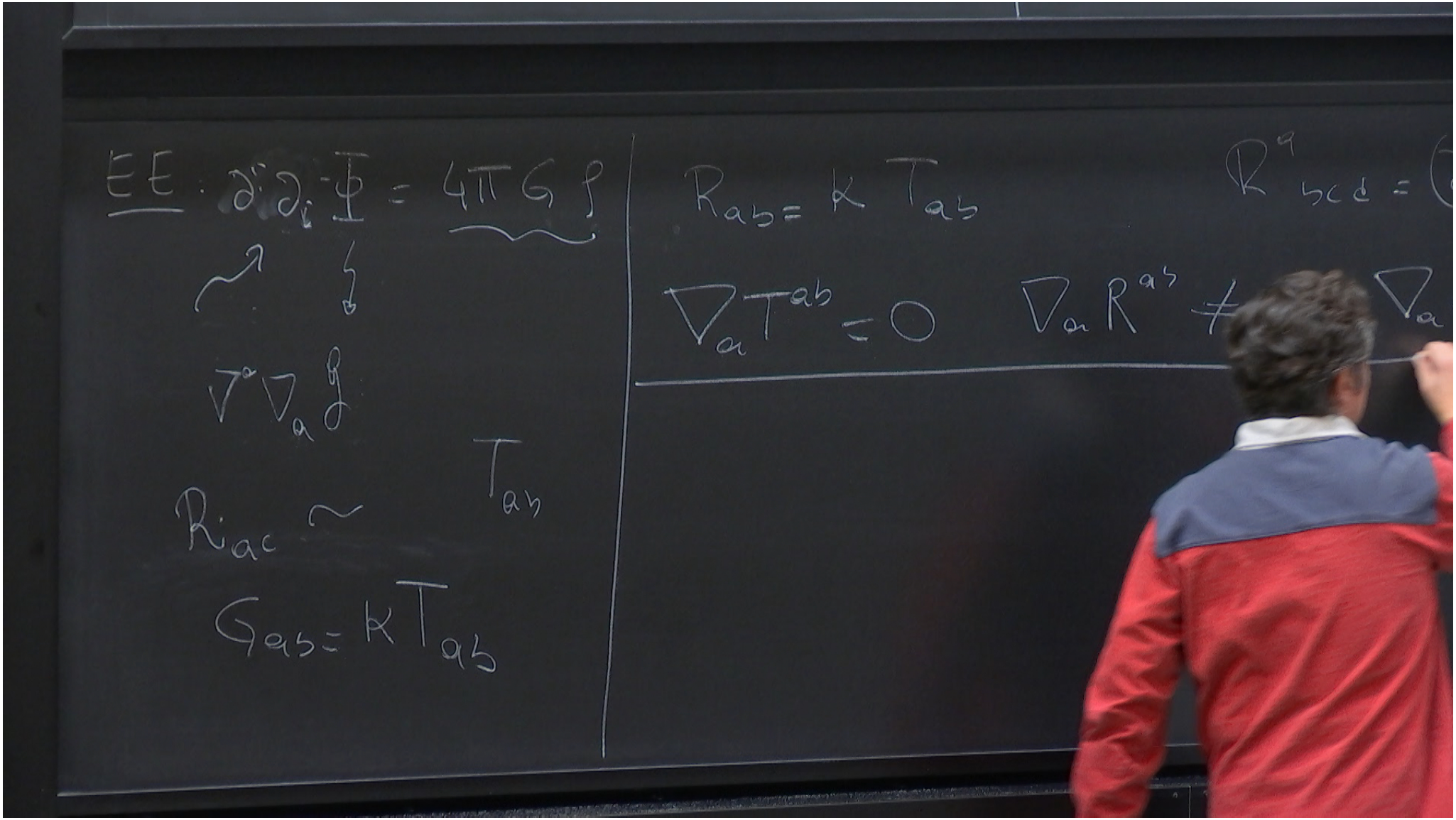

EE · ∂ ∂_i Φ = 4π G ρ
∇^a ∇_a g
R_ac

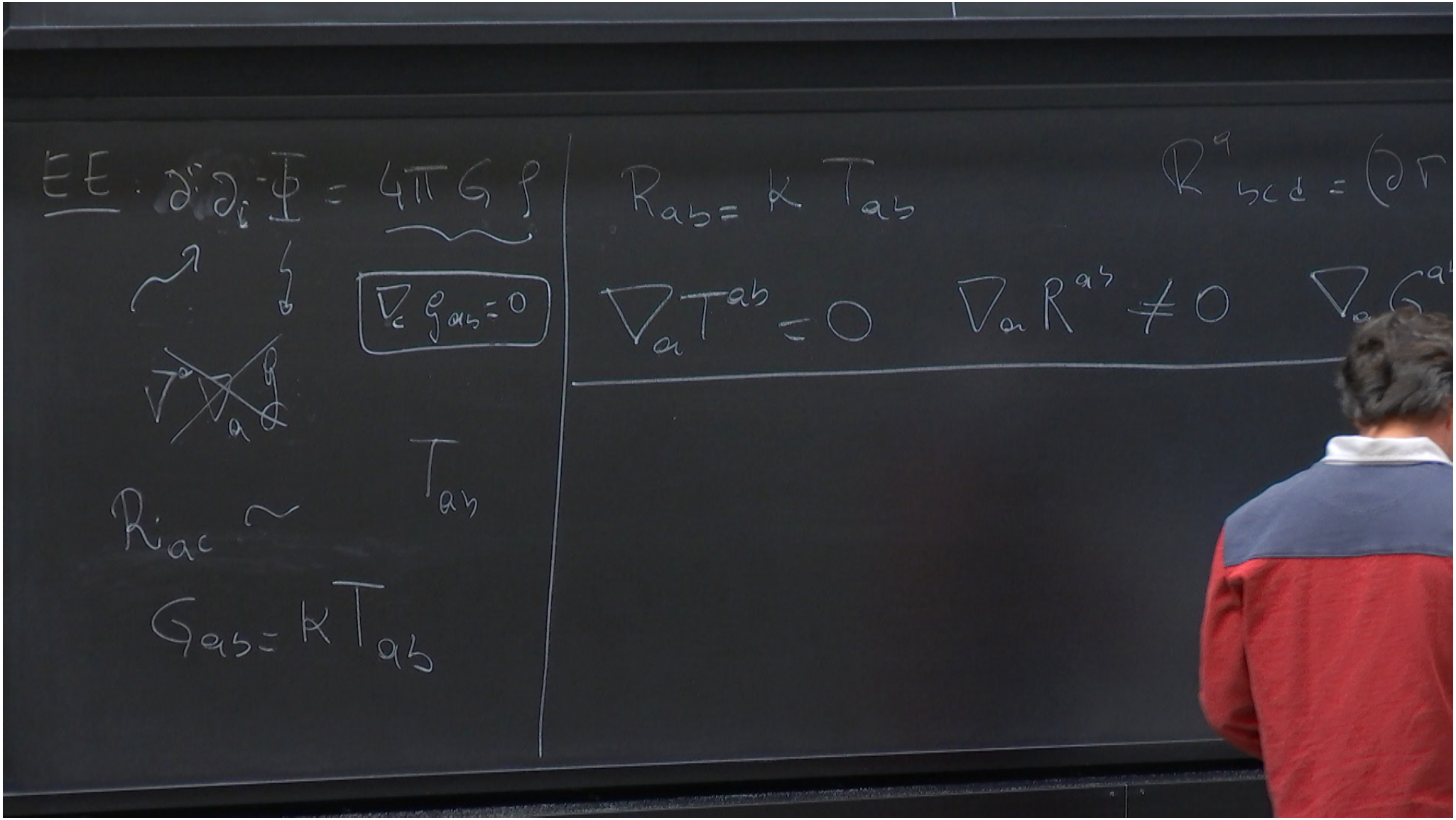

EE · $\partial^i \partial_i \Phi = 4\pi G \rho$
$\nabla^a \nabla_a g$
T_{ab}
R_{iac}

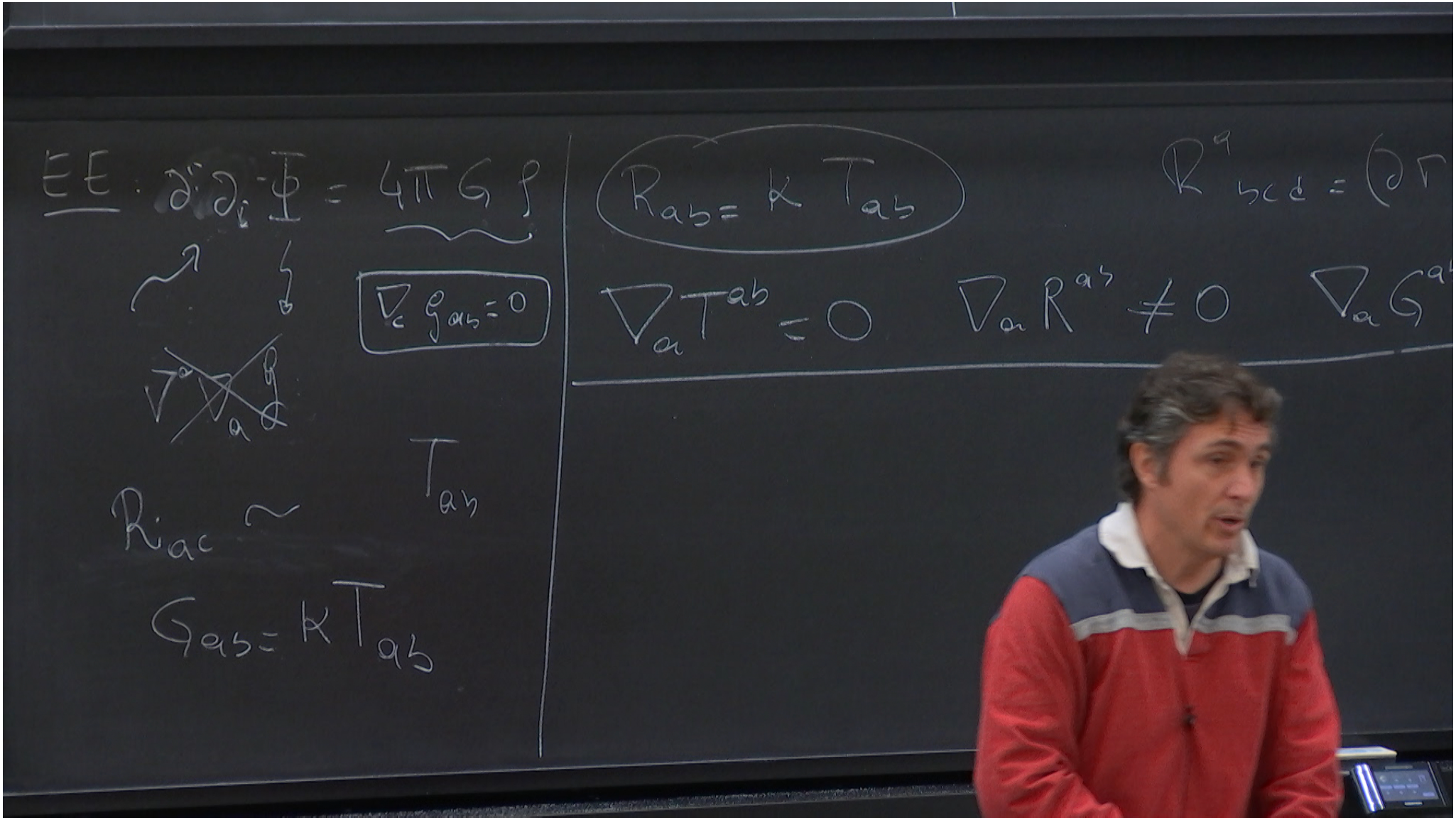

$R^{a}{}_{bcd} = (\partial\Gamma) + \Gamma\Gamma$

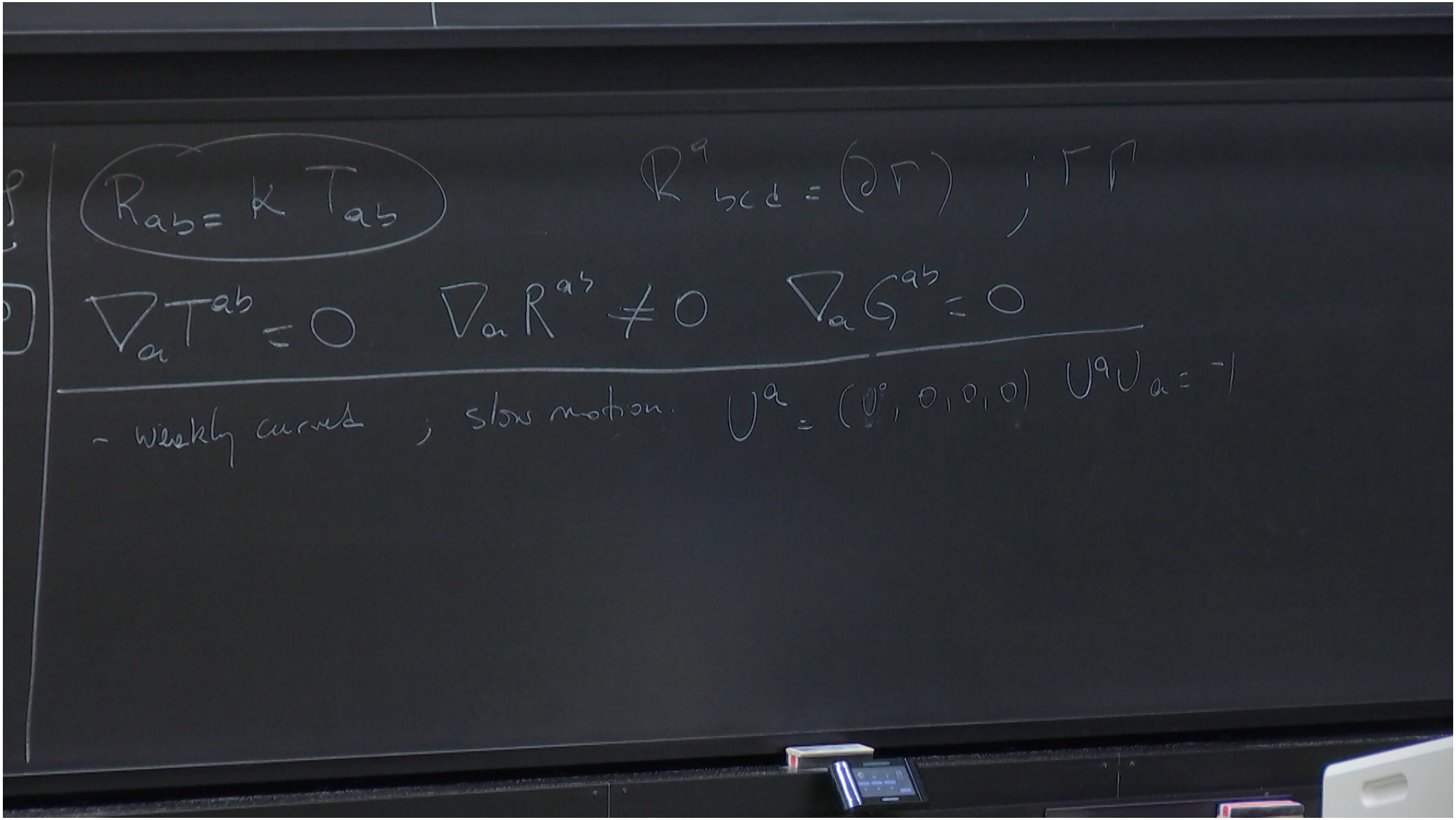

$R_{ab} = \kappa T_{ab}$
T_{ab}

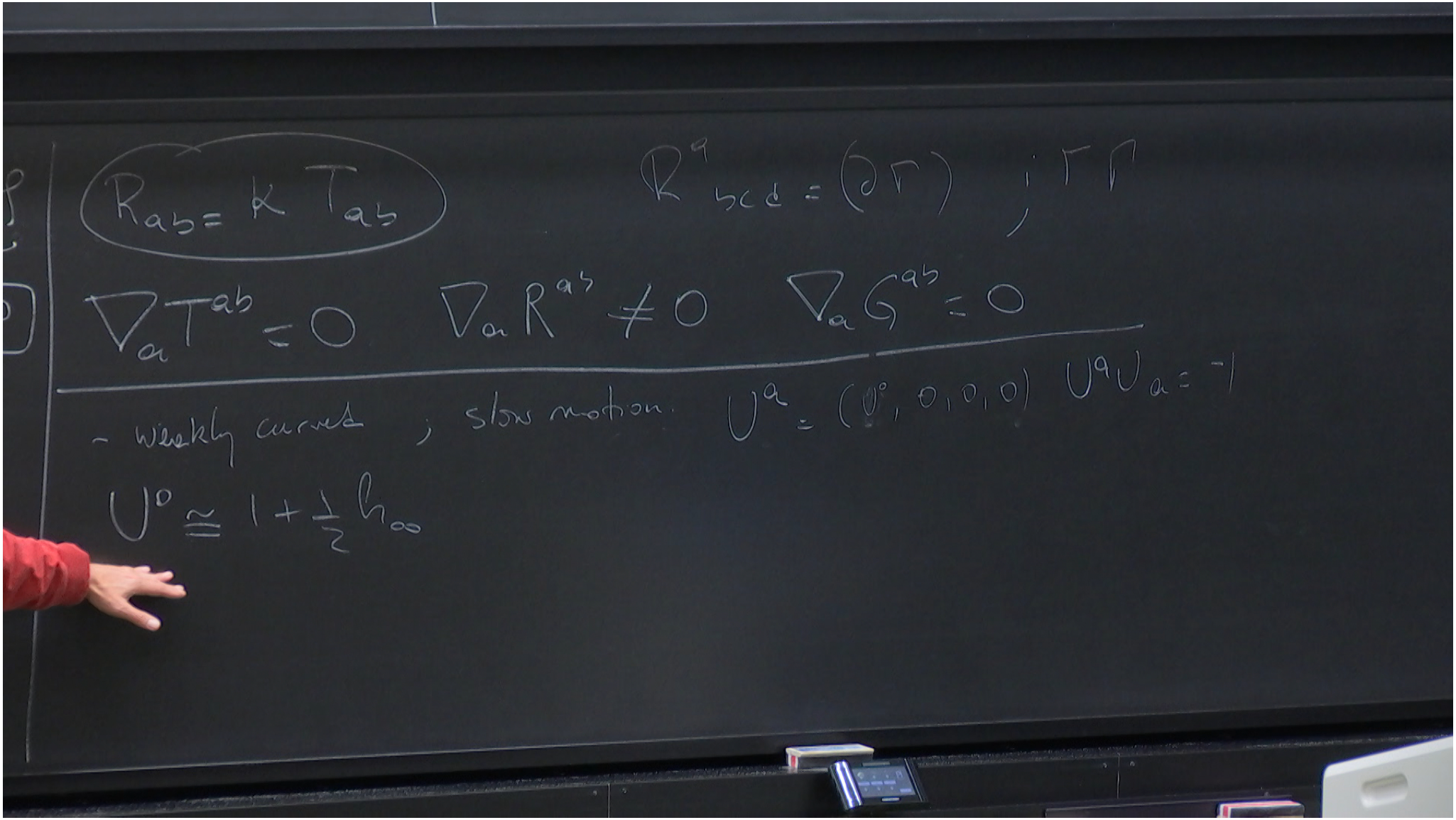

$R_{ab} = \kappa T_{ab}$
$R^{a}{}_{bcd} = (\partial\Gamma) + \Gamma\Gamma$
$\nabla_a T^{ab} = 0$
$\nabla_a R^{ab} \neq 0$


EE: $\partial^i \partial_i \Phi = 4\pi G \rho$
$\nabla^a \nabla_a \rho$
$R_{ac} \sim T_{ab}$
$G_{ab} = \kappa T_{ab}$
$R_{ab} = \kappa T_{ab}$
$\nabla_a T^{ab} = 0$
$\nabla_a R^{ab} \neq$
∇_a
$R^{a}{}_{bcd} =$


EE: $\partial^i \partial_i \Phi = 4\pi G \rho$
$\nabla_c g_{ab} = 0$
$R_{ab} = \kappa T_{ab}$
$R^a{}_{bcd} = (\partial \Gamma$
$\nabla_a T^{ab} = 0$
$\nabla_a R^{ab} \neq 0$
$\nabla_a G^{a}$
$R_{ac} \sim T_{ab}$
$G_{ab} = \kappa T_{ab}$


EE: $\partial^i \partial_i \Phi = 4\pi G \rho$
$R_{ab} = K T_{ab}$
$R^a{}_{bcd} = (\partial \Gamma$
$\nabla_c g_{ab} = 0$
$\nabla_a T^{ab} = 0$
$\nabla_a R^{ab} \neq 0$
$\nabla_a G^{ab}$
$R_{ac} \sim T_{ab}$
$G_{ab} = K T_{ab}$


$R_{ab} = \kappa T_{ab}$
$R^{a}{}_{bcd} = (\partial\Gamma) + \Gamma\Gamma$
$\nabla_a T^{ab} = 0$
$\nabla_a R^{ab} \neq 0$
$\nabla_a G^{ab} = 0$
- weakly curved ; slow motion. $U^a = (1, 0, 0, 0)$ $U^a U_a = -1$


$R_{ab} = \kappa T_{ab}$
$R^{a}{}_{bcd} = (\partial\Gamma) + \Gamma\Gamma$
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- weakly curved ; slow motion. $U^a = (U^0, 0, 0, 0)$ $U^a U_a = -1$
$U^0 \cong 1 + \frac{1}{2} h_{00}$

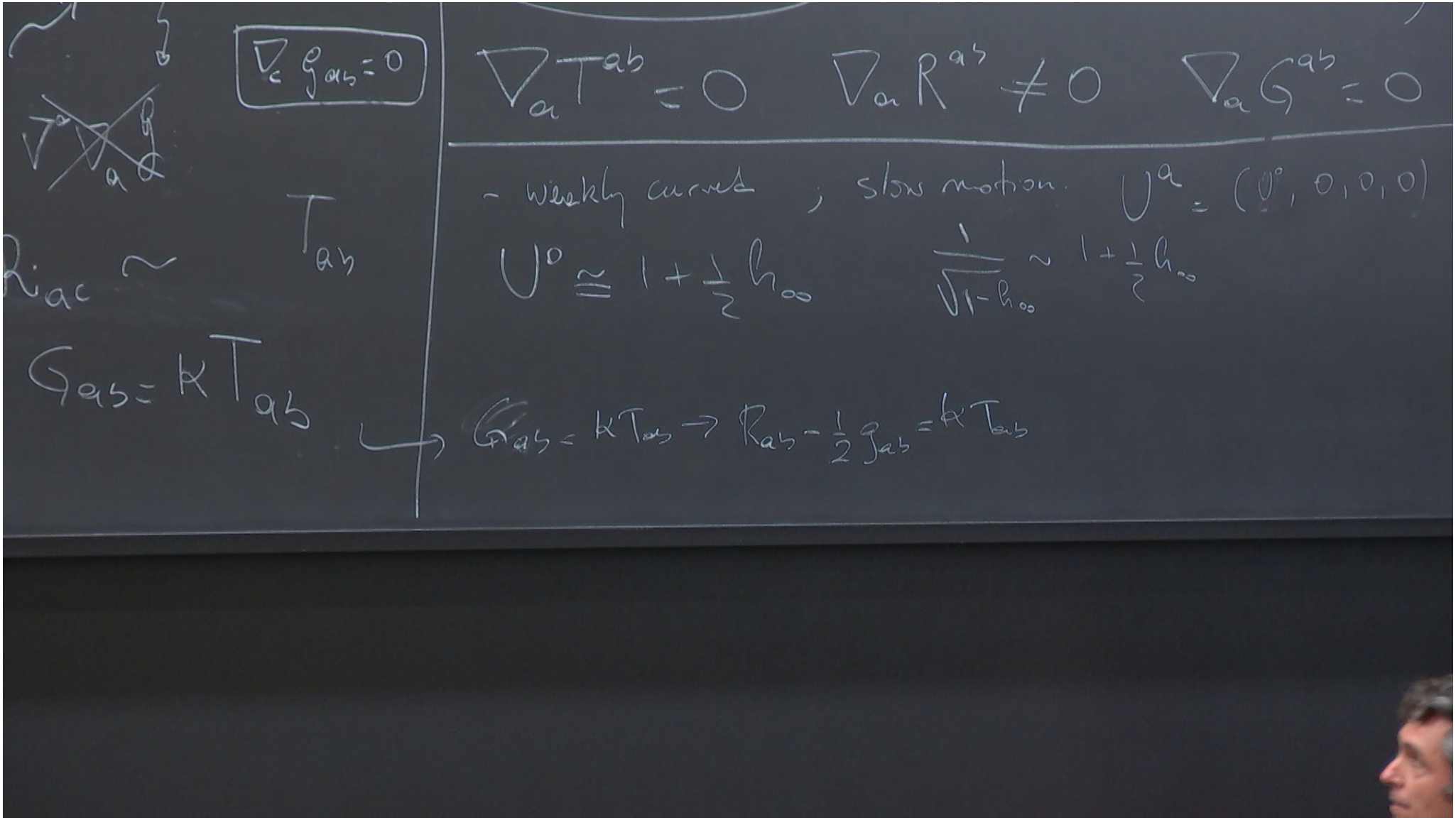
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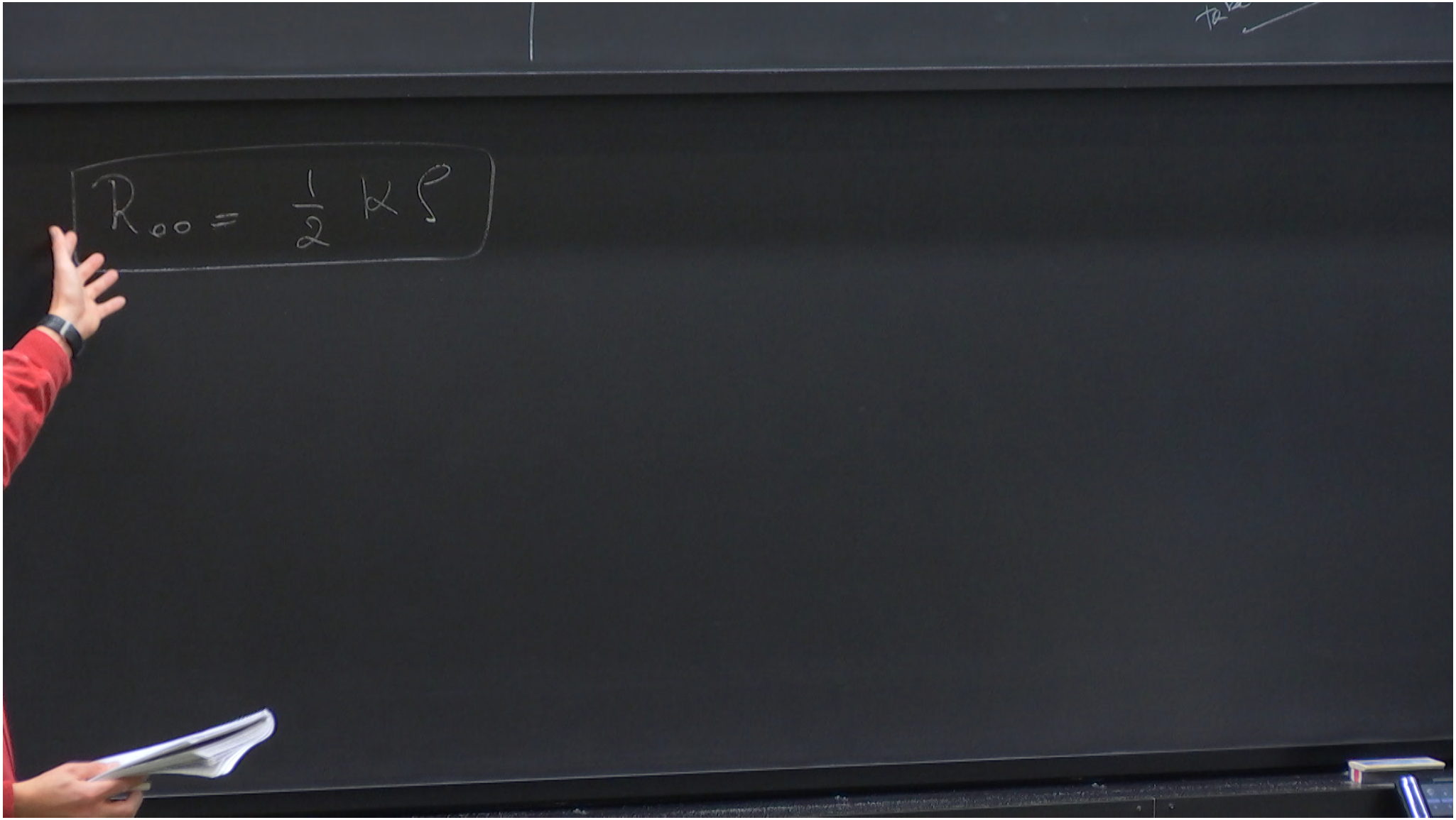

$\nabla_c g_{ab} = 0$
$\nabla_a T^{ab} = 0$
$\nabla_a R^{ab} \neq 0$
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$R_{ac} \sim T_{ab}$
$G_{ab} = \kappa T_{ab}$
- weakly curved ; slow motion. $U^a = (U^0, 0, 0, 0)$
$U^0 \cong 1 + \frac{1}{2} h_{00}$
$\frac{1}{\sqrt{1 - h_{00}}} \sim 1 + \frac{1}{2} h_{00}$
$G_{ab} = \kappa T_{ab} \rightarrow R_{ab} - \frac{1}{2} g_{ab} = \kappa T_{ab}$

$\nabla_a T^{ab} = 0 \quad \nabla_a R^{ab} \neq 0 \quad \nabla_a G^{ab} = 0$

- weakly curved ; slow motion. $U^a = (0, 0, 0, 0) \quad U^a U_a = -1$

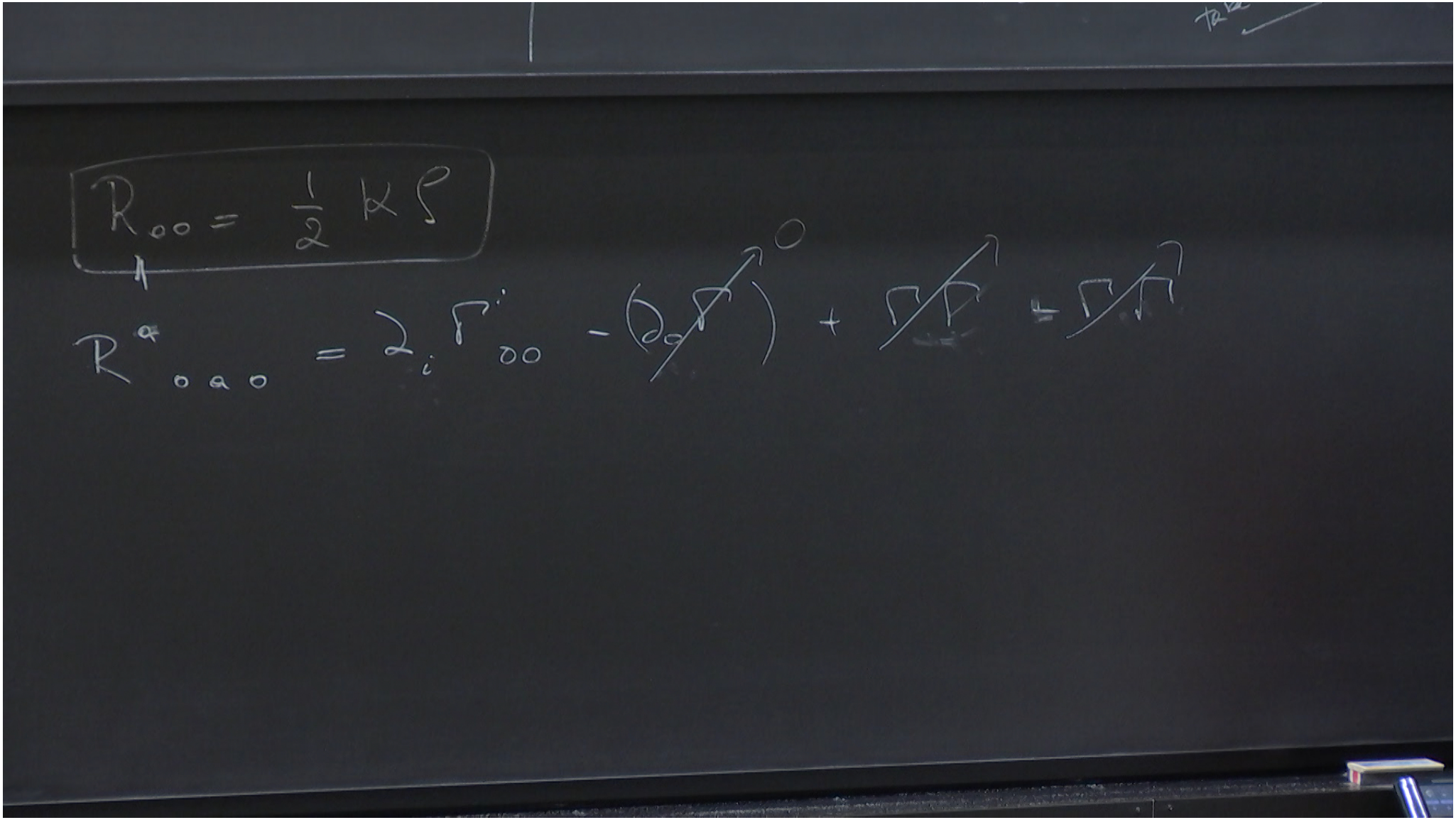
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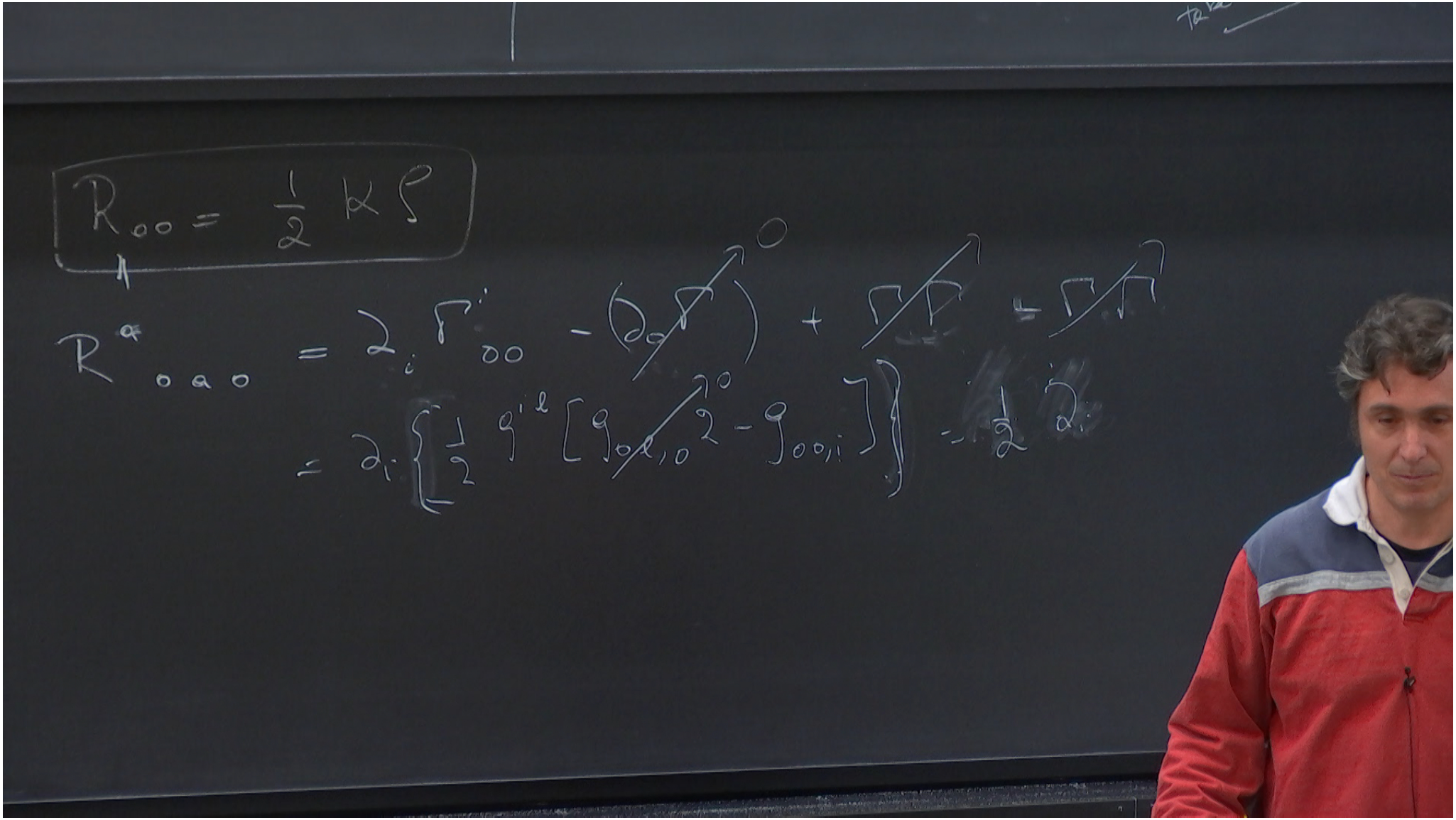
$G_{ab} = \kappa T_{ab} \rightarrow R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab}$ take trace: $R - 2R = \kappa T$, $-R = \kappa T$ $\Big\} \quad R_{ab} = \kappa \left[T_{ab} - \frac{1}{2} g_{ab} T \right]$


$R_{00} = \frac{1}{2} k \rho$

$$R_{00} = \frac{1}{2}\kappa\rho$$

$$R^{a}{}_{0a0} = \partial_i \Gamma^{i}_{00} - (\partial_0 \Gamma) + \Gamma\Gamma - \Gamma\Gamma$$


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$R_{00} = \frac{1}{2} \kappa \rho$
$R^{a}{}_{0a0} = \partial_i \Gamma^{i}_{00} - (\partial_a \Gamma^{a}_{00}) + \Gamma\Gamma - \Gamma\Gamma$
$= \partial_i \left\{ \frac{1}{2} g^{il} [\partial_0 g_{l0} + \partial_0 g_{0l} - \partial_l g_{00}] \right\} = -\frac{1}{2} \partial_i$

$G_{ab} = \kappa T_{ab} \rightarrow R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab}$

take trace:

$R - 2R = \kappa T$

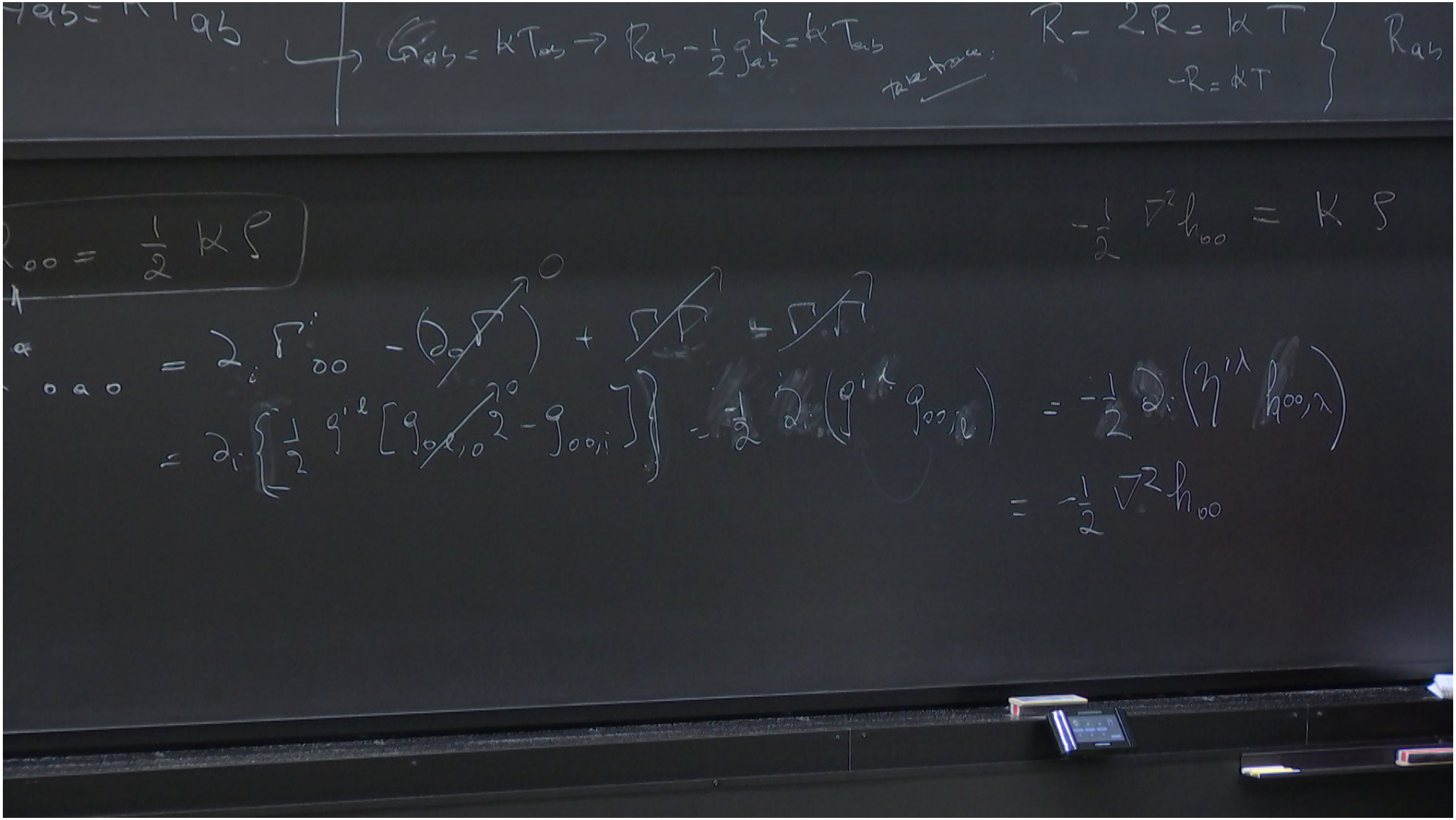
$-R = \kappa T$

$$R_{00} = \frac{1}{2} \kappa \rho$$

$$R^{a}{}_{0a0} = \partial_i \Gamma^{i}_{00} - (\partial_a \Gamma) + \Gamma\Gamma = \Gamma\Gamma$$

$$= \partial_i \left\{ \left[\frac{1}{2} g^{il} \left[g_{0l,0} + g_{00,l} - g_{00,l} \right] \right] \right\} - \frac{1}{2} \partial_i \left(g^{il} g_{00,l} \right) = -\frac{1}{2} \partial_i \left(\eta^{il} h_{00,l} \right)$$

$$= -\frac{1}{2} \nabla^2 h_{00}$$


$R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab}$
take trace:
$R - 2R = \kappa T$
$-R = \kappa T$
R_{ab}
$= \frac{1}{2} \kappa \rho$
$-\frac{1}{2} \nabla^2 h_{00} = \kappa \rho$
$= 2_i \Gamma^i_{00} - (\partial_a \Gamma^a_{00}) + \Gamma\Gamma - \Gamma\Gamma$
$= \partial_i \left\{ \frac{1}{2} \eta^{il} [g_{0l,0} + g_{0l,0} - g_{00,i}] \right\} - \frac{1}{2} \partial_i (\eta^{il} g_{00,l})$
$= -\frac{1}{2} \partial_i (\eta^{il} h_{00,l})$
$= -\frac{1}{2} \nabla^2 h_{00}$

κT_{ab} take trace: $R - 2R = \kappa T$, $-R = \kappa T$ $\Big\}$ $R_{ab} = \kappa\left[T_{ab} - \frac{1}{2} g_{ab} T\right]$

$$-\frac{1}{2}\nabla^2 h_{00} = \frac{1}{2}\kappa\rho$$

$h_{00} = -2\Phi$

$\nabla^2(2\Phi) = \kappa\rho$

$\kappa = 8\pi$

$$\partial_i\left(\eta^{ik} g_{00,k}\right) = -\frac{1}{2}\partial_i\left(\eta^{ik} h_{00,k}\right)$$

$$= -\frac{1}{2}\nabla^2 h_{00}$$

$G_{ab} = 8\pi G T_{ab}$ Einstein eqn. $\{M, g_{ab}\}$

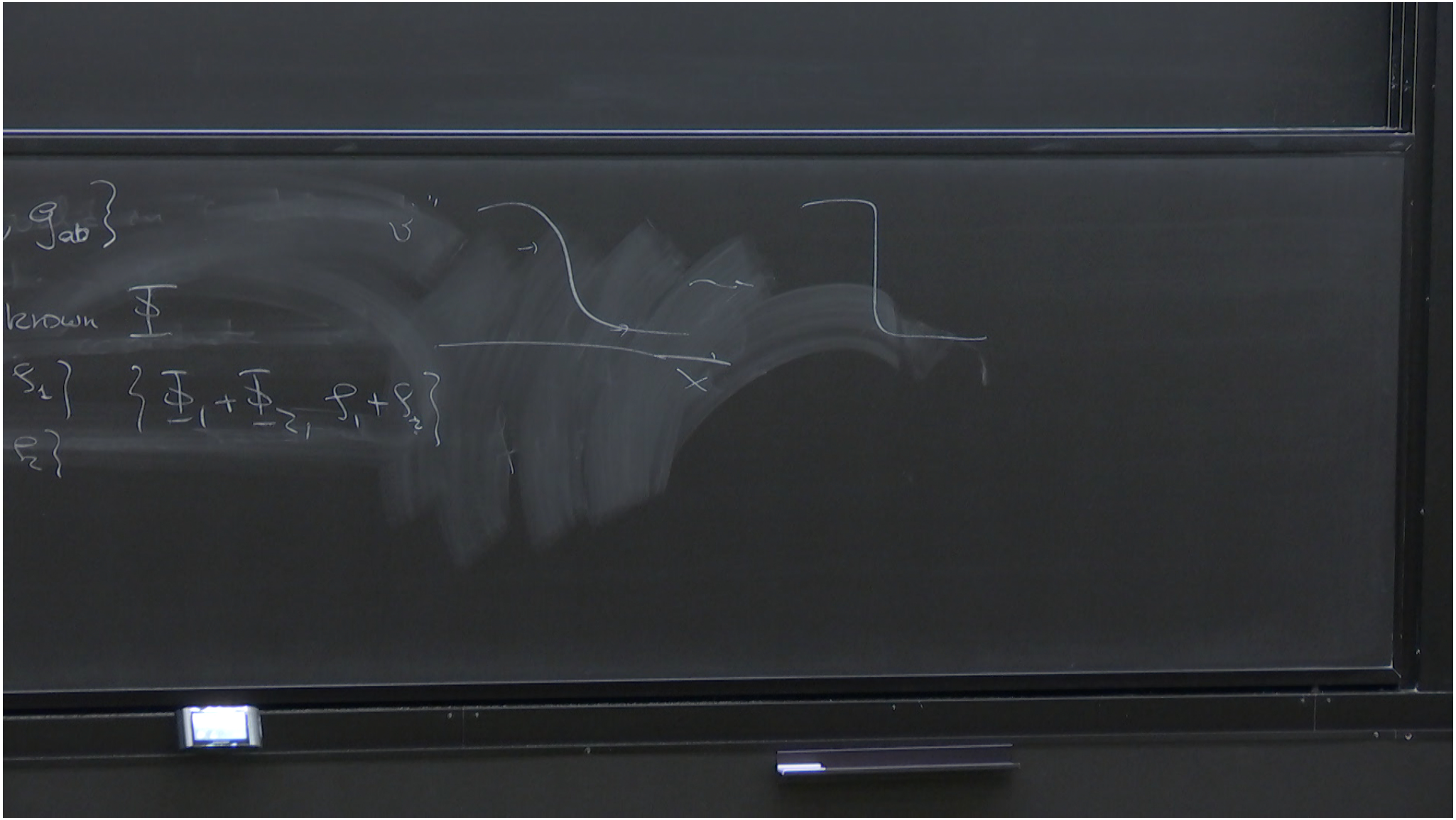
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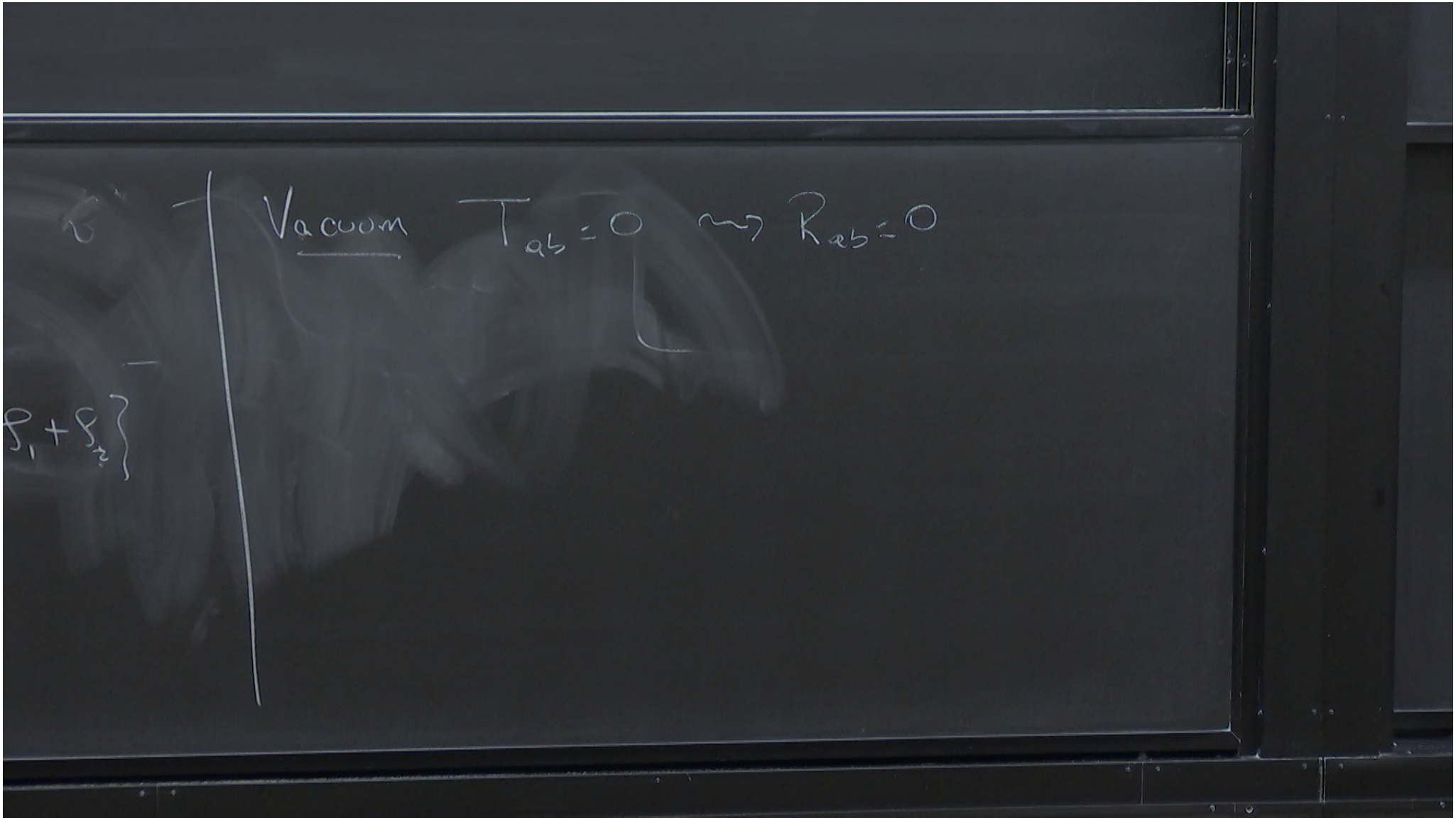
10 eqns $\longrightarrow$ Newtonian gravity 1 unknown Φ

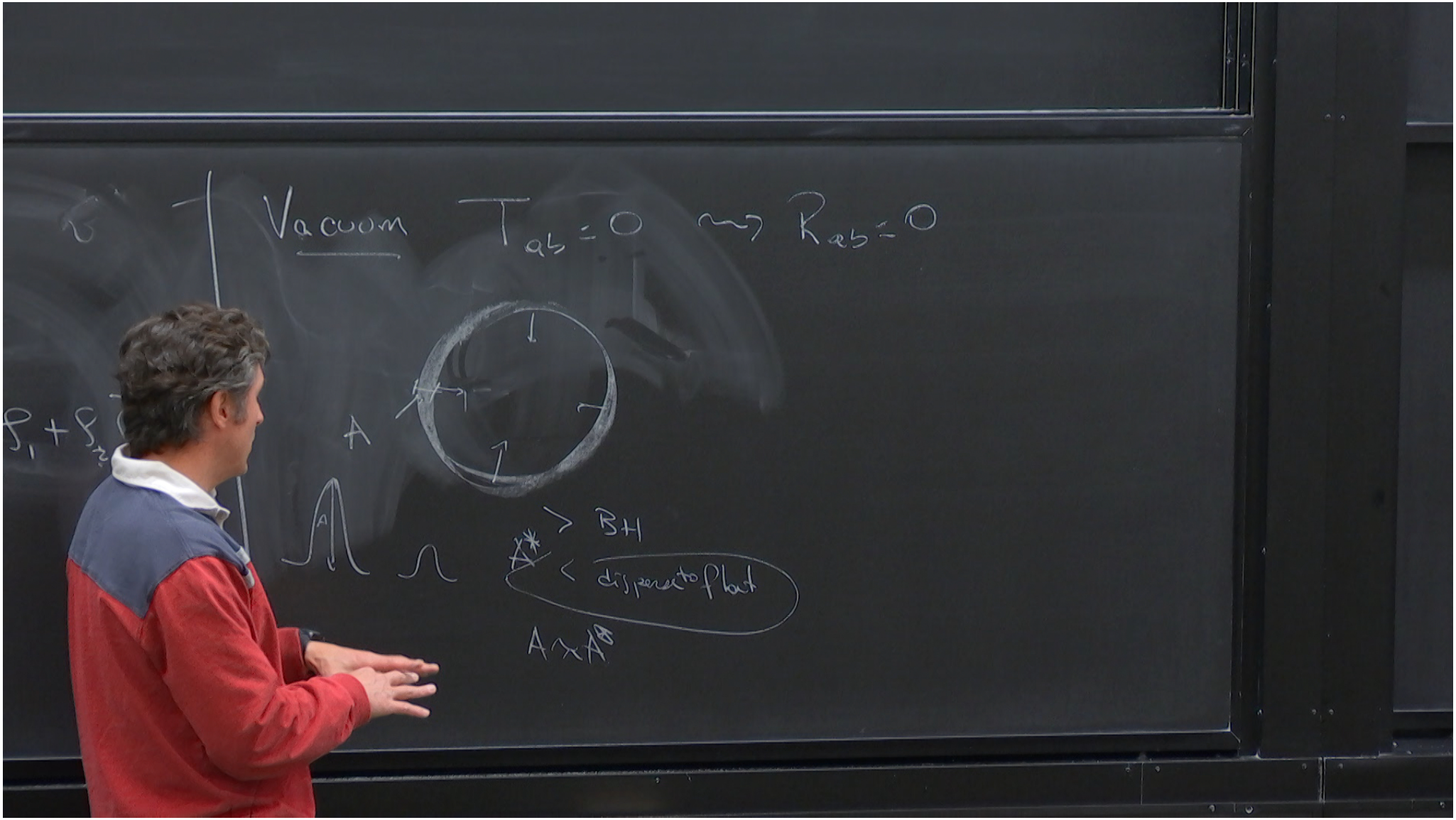
$\nabla^a G_{ab} = 0$ (4) - linear $\{\Phi_1, \rho_1\}$ $\{\Phi_1 + \Phi_2, \rho_1 + \rho_2\}$

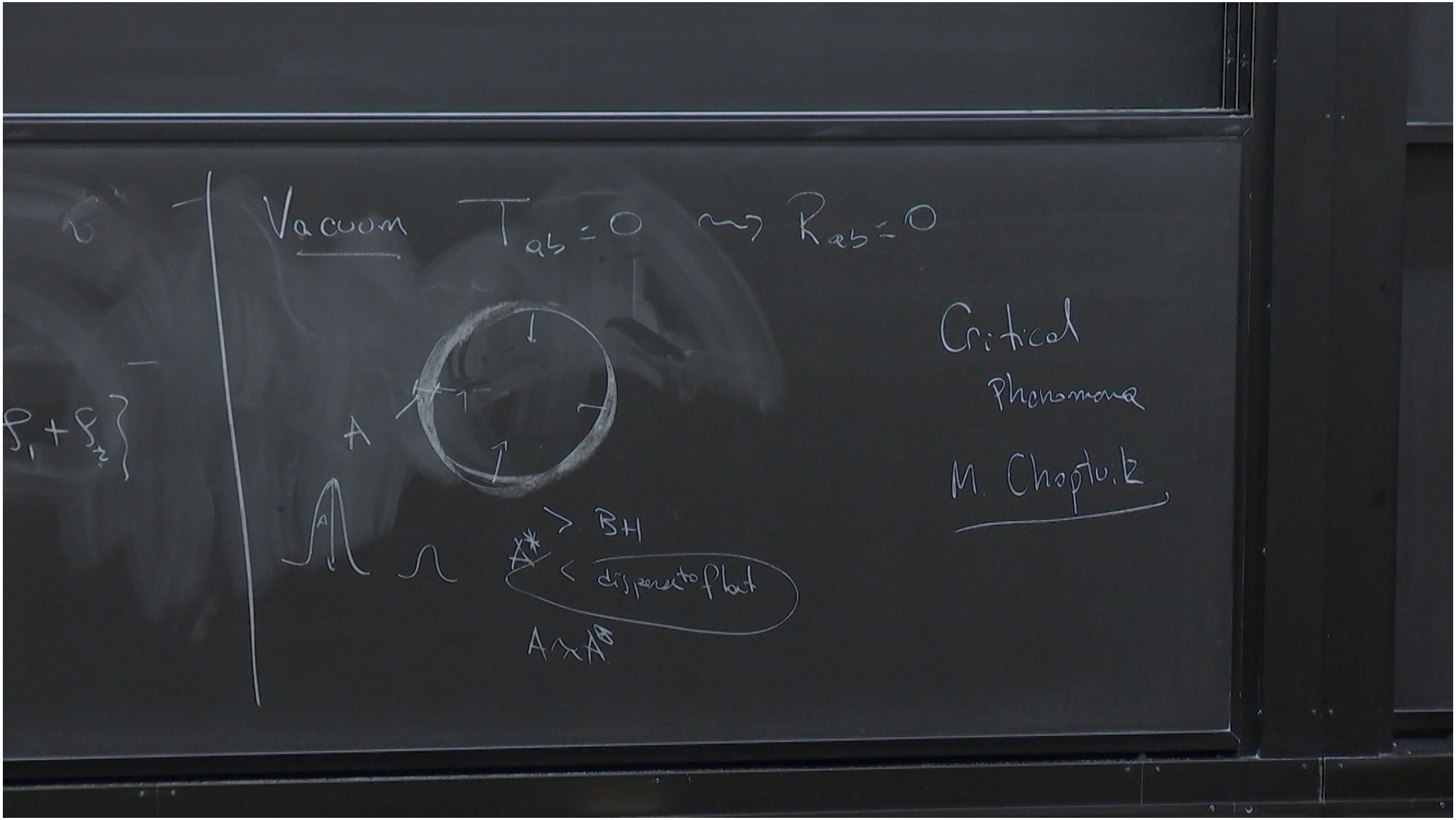
$\{\Phi_2, \rho_2\}$

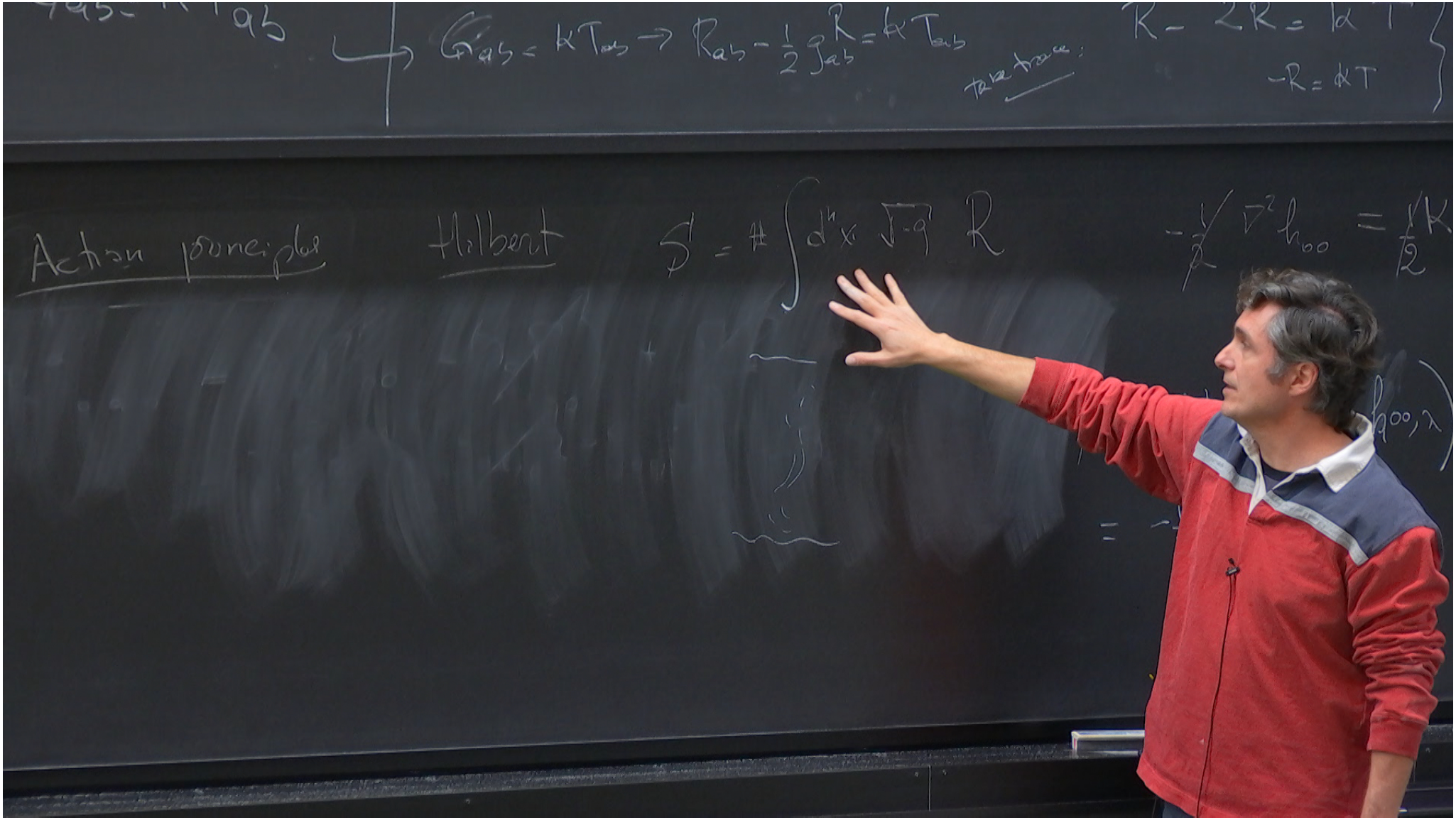
(6 eqns) "non linear"

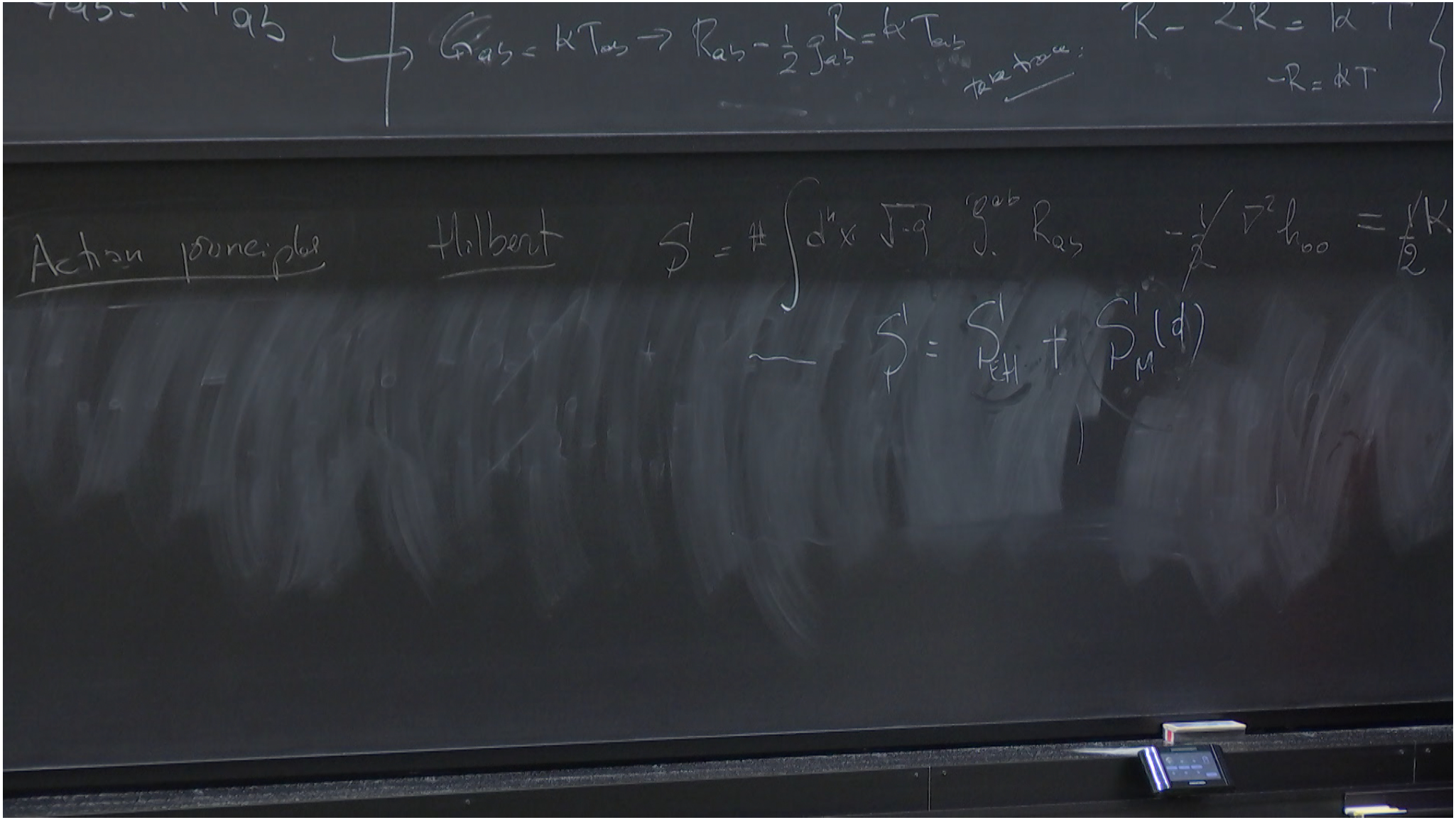

g_{ab}
known
$\mathcal{I}$
ρ_1
ρ_2
$\mathcal{I}_1 + \mathcal{I}_2$, $\rho_1 + \rho_2$
$\mathcal{I}^+$
i^+


Vacuum $T_{ab} = 0 \rightsquigarrow R_{ab} = 0$
$\rho_1 + \rho_2$


Vacuum $T_{ab}=0$ $\rightsquigarrow$ $R_{ab}=0$
A
A
A^*
> B+1
< dispersion of ϕ out
$A \times A^*$


Vacuum $T_{ab}=0 \rightsquigarrow R_{ab}=0$
A
$A^* > B+1$
$< \text{dispersion of } \phi$
$A \times A^*$
Critical
Phenomena
M. Choptuik


Action principle
Hilbert
trace


Action principle
Hilbert
$S' = \# \int d^4x \sqrt{-g}\, g^{ab} R_{ab}$
$S' = S'_{EH} + S'_M(q)$
take trace:
$R = \kappa T$