Title: Relativity - Lecture 221014

Speakers:

Collection: Relativity (2022/2023)

Date: October 14, 2022 - 9:00 AM

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Euclidean case. Basis $= \{\hat{i}, \hat{j}, \hat{k}\}$ $\quad v^a = v^1 \hat{i} + v^2 \hat{j} + v^3 \hat{k}$

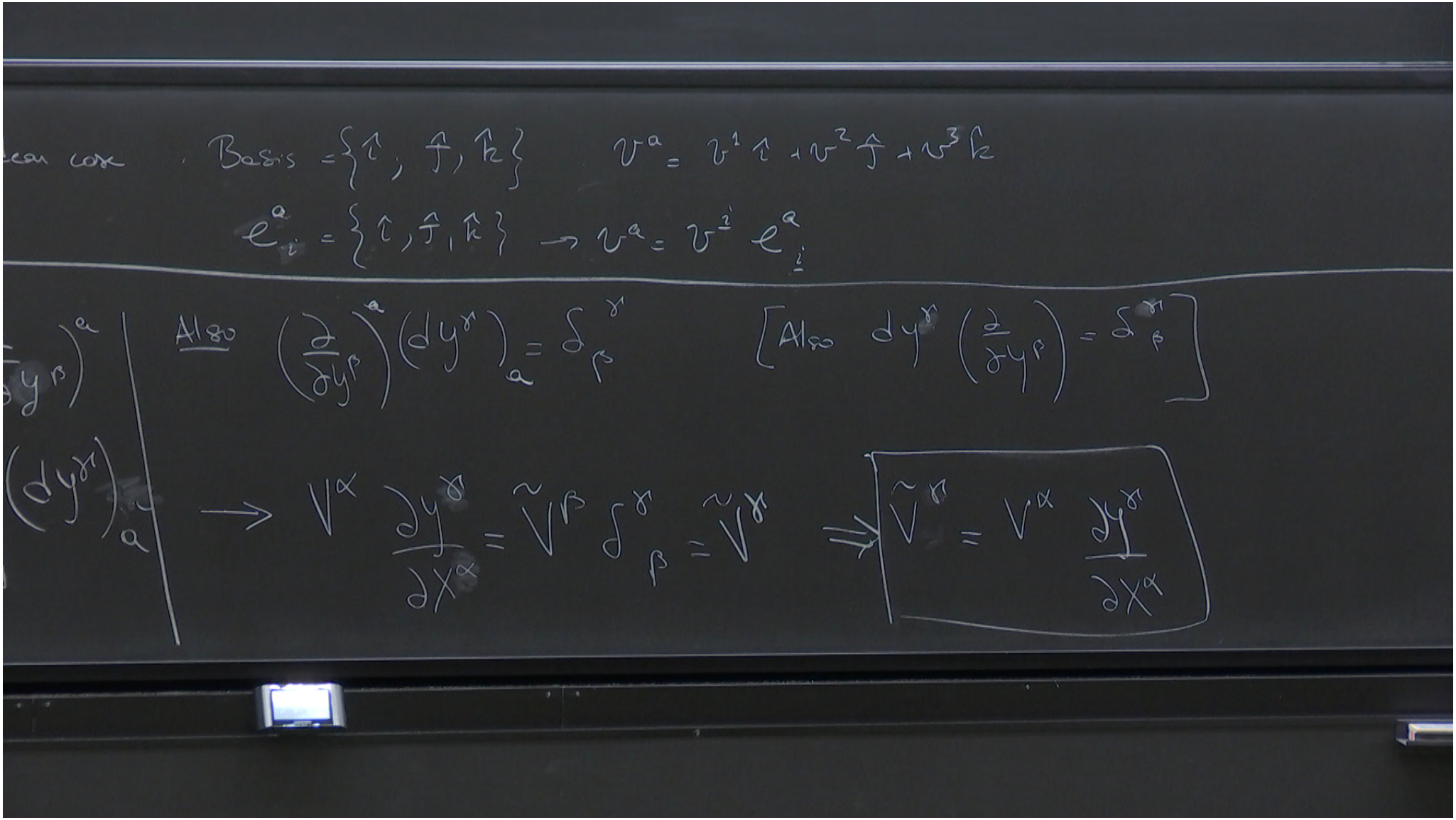
$e^a_i = \{\hat{i}, \hat{j}, \hat{k}\} \rightarrow v^a = v^i e^a_i$

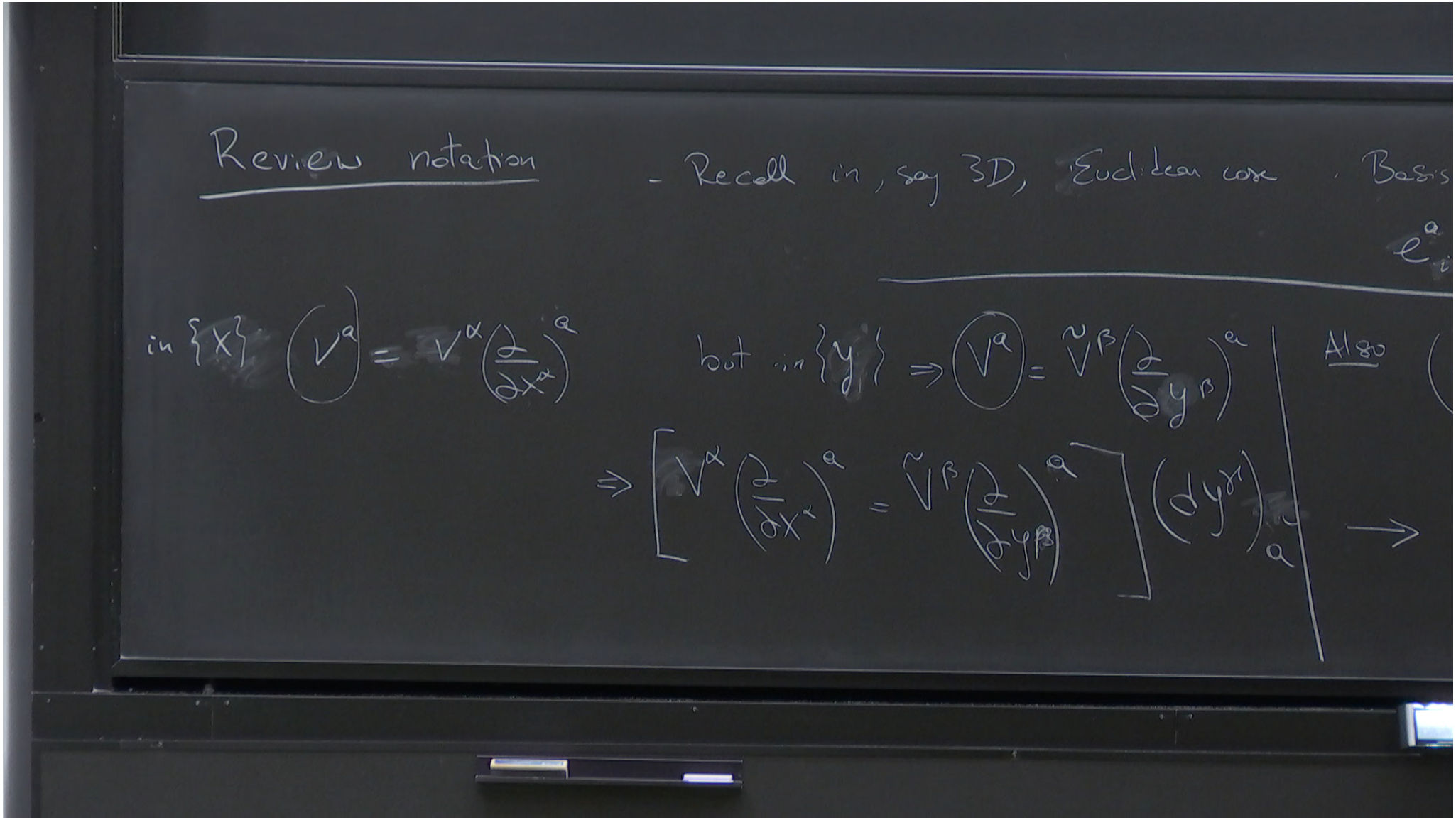
$\beta \left(\frac{\partial}{\partial y^\beta}\right)^a$

Also $\left(\frac{\partial}{\partial y^\beta}\right)(dy^\gamma) = \delta^\gamma_\beta$ $\quad$ [Also $dy^\gamma\left(\frac{\partial}{\partial y^\beta}\right) = \delta^\gamma_\beta$]

$\rightarrow V^\alpha \frac{\partial y^\gamma}{\partial x^\alpha} = \tilde{V}^\beta \delta^\gamma_\beta = \tilde{V}^\gamma \Rightarrow \tilde{V}^\gamma = V^\alpha \frac{\partial y^\gamma}{\partial x^\alpha}$




Basis $= \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ $v^a = v^1 \hat{\imath} + v^2 \hat{\jmath} + v^3 \hat{k}$
$e^a_i = \{\hat{\imath}, \hat{\jmath}, \hat{k}\} \rightarrow v^a = v^i e^a_i$
Also $\left(\frac{\partial}{\partial y^\beta}\right)^a (dy^\gamma)_a = \delta^\gamma_\beta$
[Also $dy^\gamma \left(\frac{\partial}{\partial y^\beta}\right) = \delta^\gamma_\beta$]
$\rightarrow V^\alpha \frac{\partial y^\gamma}{\partial x^\alpha} = \tilde{V}^\beta \delta^\gamma_\beta = \tilde{V}^\gamma \Rightarrow \tilde{V}^\gamma = V^\alpha \frac{\partial y^\gamma}{\partial x^\alpha}$


Review notation
- Recall in, say 3D, Euclidean coor. Basis
in {x} $V^a = V^\alpha \left(\frac{\partial}{\partial x^\alpha}\right)^a$
but in {y} $\Rightarrow V^a = \tilde{V}^\beta \left(\frac{\partial}{\partial y^\beta}\right)^a$
$\Rightarrow \left[V^\alpha \left(\frac{\partial}{\partial x^\alpha}\right)^a = \tilde{V}^\beta \left(\frac{\partial}{\partial y^\beta}\right)^a\right] \left(\frac{\partial y^\gamma}{\partial x^\alpha}\right)$
Also

Basis $= \{\hat{\imath}, \hat{\jmath}, \hat{k}\}$ $\quad v^a = v^1 \hat{\imath} + v^2 \hat{\jmath} + v^3 \hat{k}$

$e^a_i = \{\hat{\imath}, \hat{\jmath}, \hat{k}\} \rightarrow v^a = v^i e^a_i$

$v^a w^a$ $\quad r^a w_a$

$v^a w^b$

$\left(\frac{\partial}{\partial y^\beta}\right)^a (dy^\gamma)_a = \delta^\gamma_\beta$ $\quad$ [Also $dy^\gamma \left(\frac{\partial}{\partial y^\beta}\right) = \delta^\gamma_\beta$]

$\rightarrow V^\alpha \frac{\partial y^\gamma}{\partial x^\alpha} = \tilde{V}^\beta \delta^\gamma_\beta = \tilde{V}^\gamma \Rightarrow \boxed{\tilde{V}^\gamma = V^\alpha \frac{\partial y^\gamma}{\partial x^\alpha}}$

Ex Using $v_a = v_\alpha \, dx^\alpha_a = \tilde{v}_\beta \, dy^\beta_a$

derive $\tilde{v}_\gamma = v_\alpha \dfrac{\partial x^\alpha}{\partial y^\gamma}$

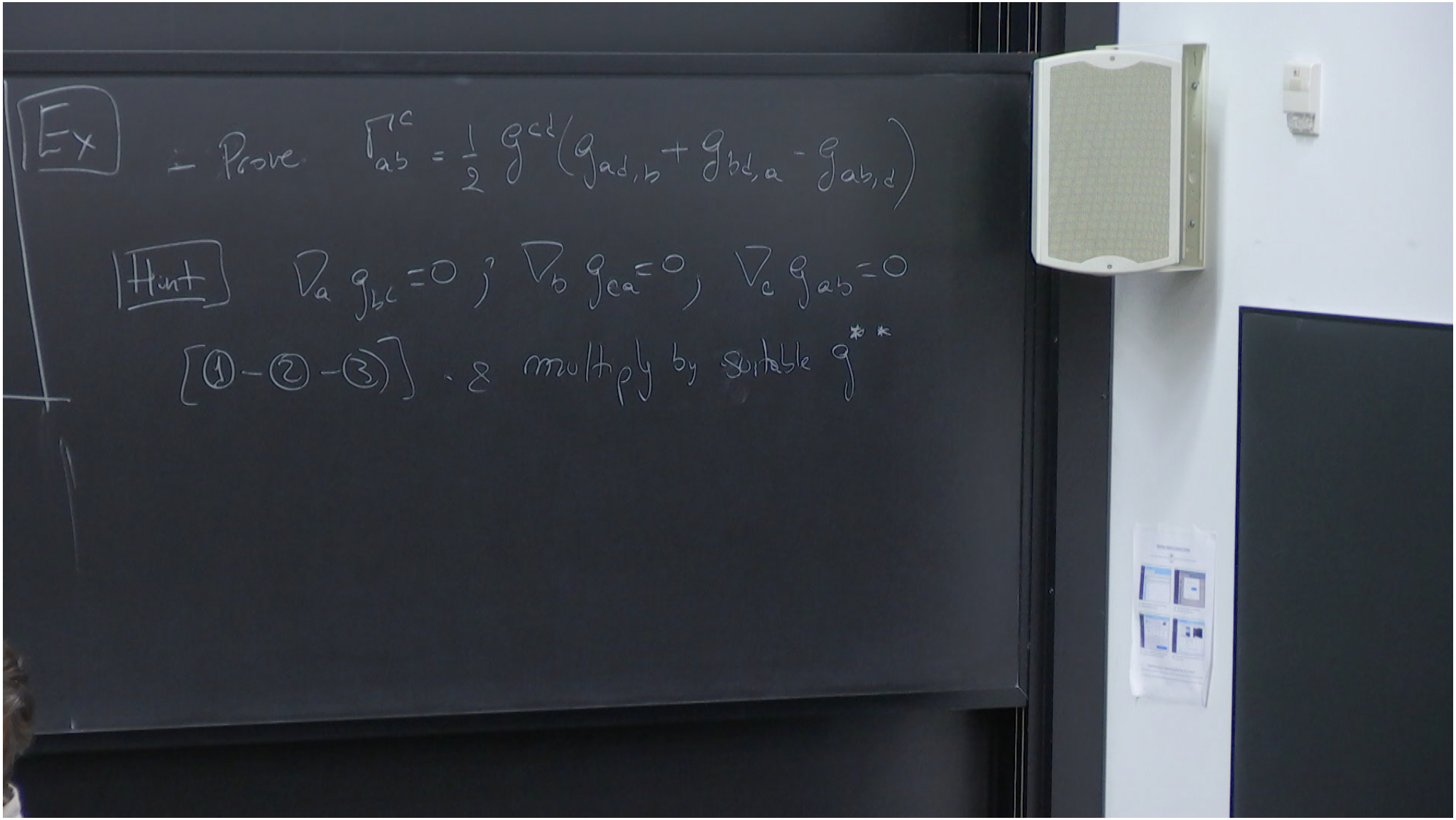
Ex

Derive ex

Riemann

from

obs: Notice where indices are $\dfrac{\partial}{\partial y^\gamma} \rightsquigarrow dy_\gamma$


Ex
- Prove $\Gamma^c_{ab} = \frac{1}{2} g^{cd}(g_{ad,b} + g_{bd,a} - g_{ab,d})$
Hint
$\nabla_a g_{bc} = 0$; $\nabla_b g_{ca} = 0$; $\nabla_c g_{ab} = 0$
[① - ② - ③] . & multiply by suitable g^{**}

$dx^\alpha_a = \tilde{v}_\beta \, dy^\beta_a$

$\frac{\partial x^\alpha}{\partial y^\gamma}$

$\frac{\partial}{\partial y^\gamma} \rightsquigarrow dy_\gamma$

Ex

Derive expression for Riemann tensor from $[\nabla_\mu, \nabla_\nu] V^\rho$

$\nabla_a v_b = \partial_a v_b - \Gamma^c_{ab} v_c$

$\nabla_a v_{bc} = \partial_a v_{bc} - \Gamma^d_{ab} v_{dc} - \Gamma^d_{ac} v_{bd}$

Ex – Prove $\Gamma^c_{ab} =$

Hint $\nabla_a g_{bc} = 0$

$[① - ② - ③]$

expression for

enn tensor

$[\nabla_\mu, \nabla_\nu] V^\rho$

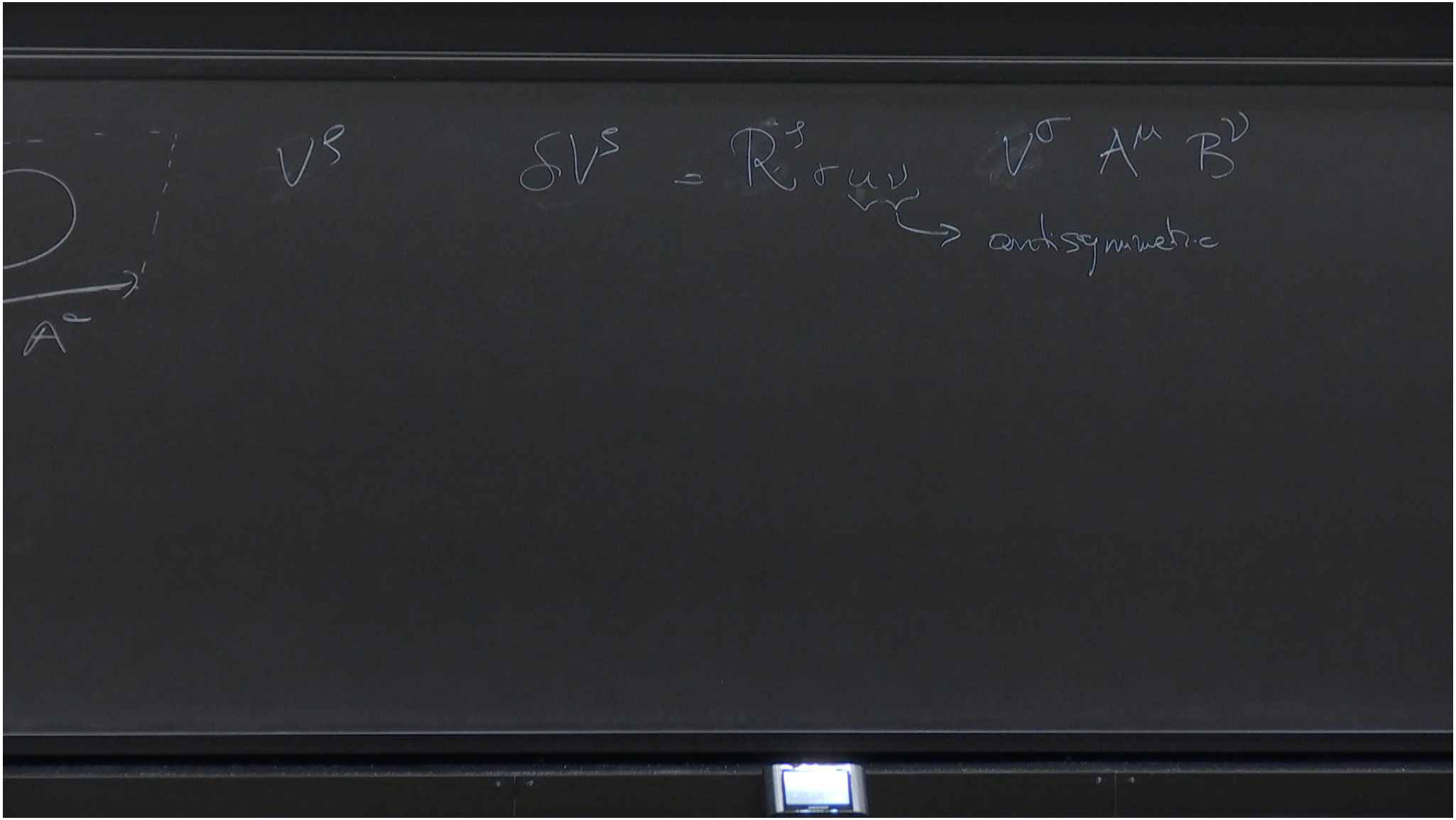
$\nabla_a V_b = \partial_a V_b - \Gamma^c_{ab} V_c$

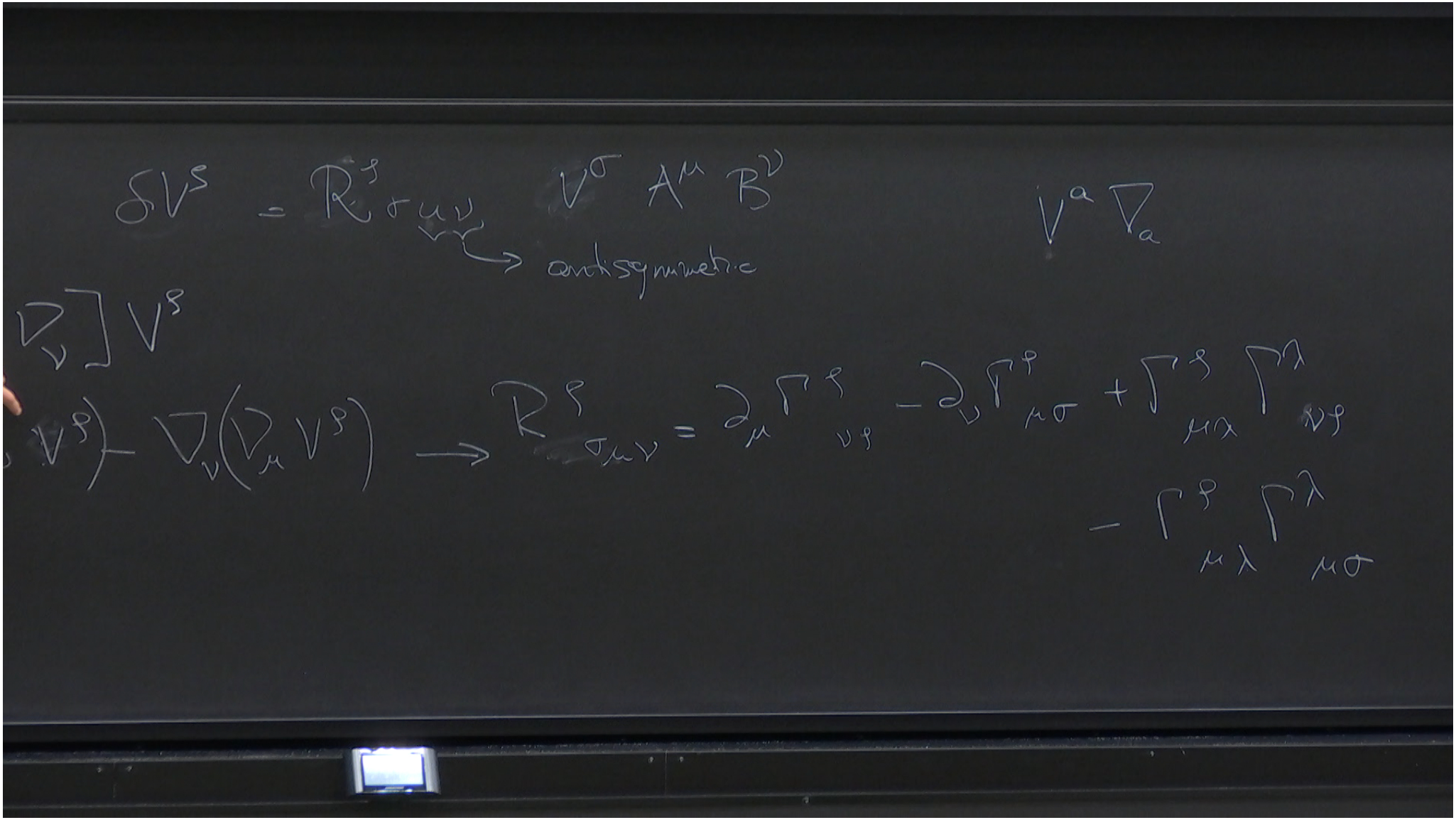
$\nabla_a V_{bc} = \partial_a V_{bc} - \Gamma^d_{ab} V_{dc} - \Gamma^d_{ac} V_{bd}$

Ex – Prove $\Gamma^c_{ab} = \frac{1}{2} g^{cd} (g_{ad,b} + g_{bd,a} - g_{ab,d})$

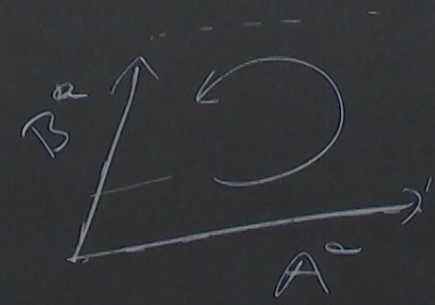
Hint $\nabla_a g_{bc} = 0$; $\nabla_b g_{ca} = 0$, $\nabla_c g_{ab} = 0$

[① − ② − ③] . & multiply by suitable g^{**}


V^ρ
$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$
antisymmetric
A^ν


$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$
antisymmetric
$V^a \nabla_a$
$\nabla_\nu] V^\rho$
$V^\rho) - \nabla_\nu(\nabla_\mu V^\rho) \rightarrow R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\rho} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\rho}$
$- \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\mu\sigma}$

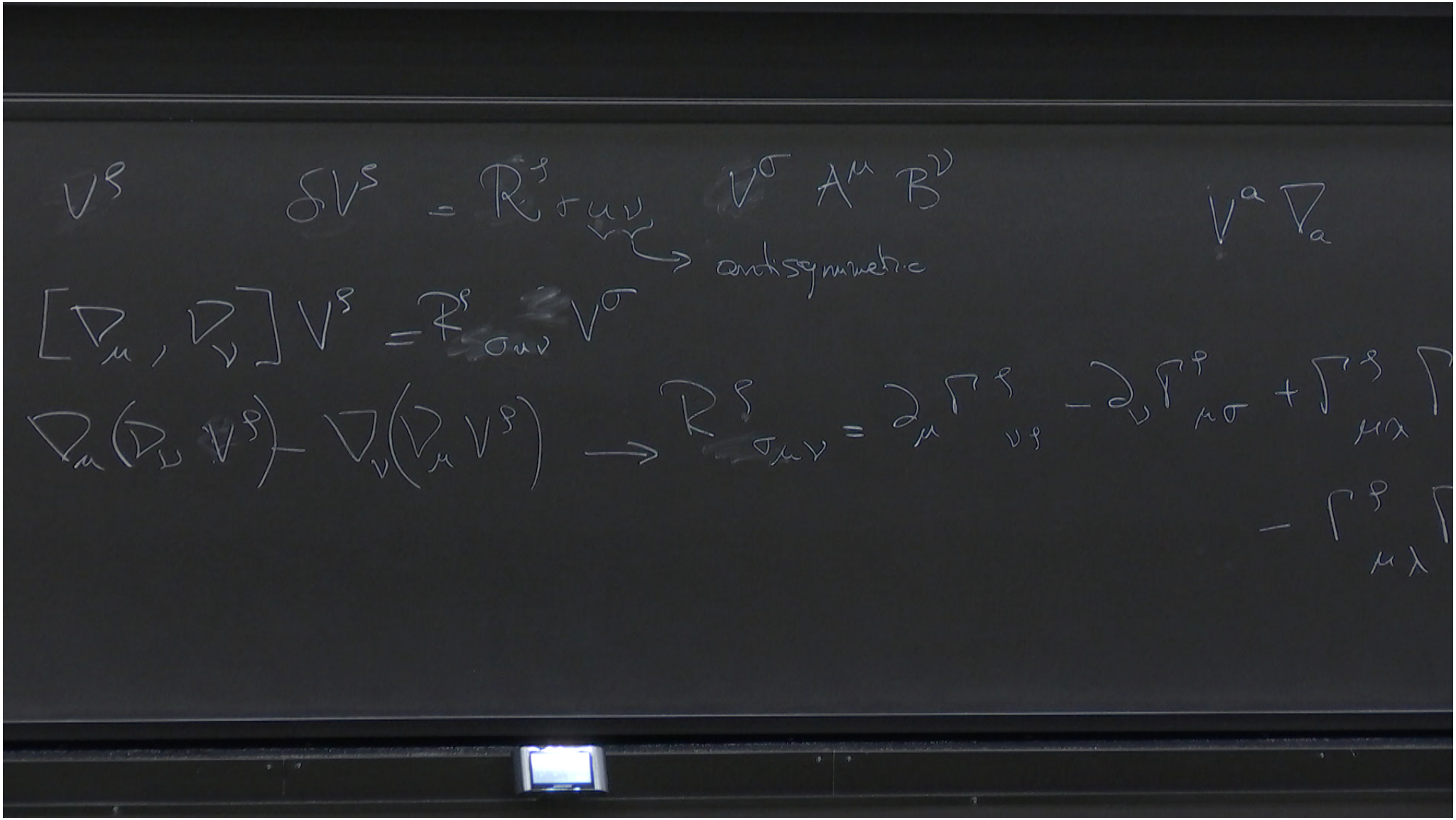
Riemann tensor

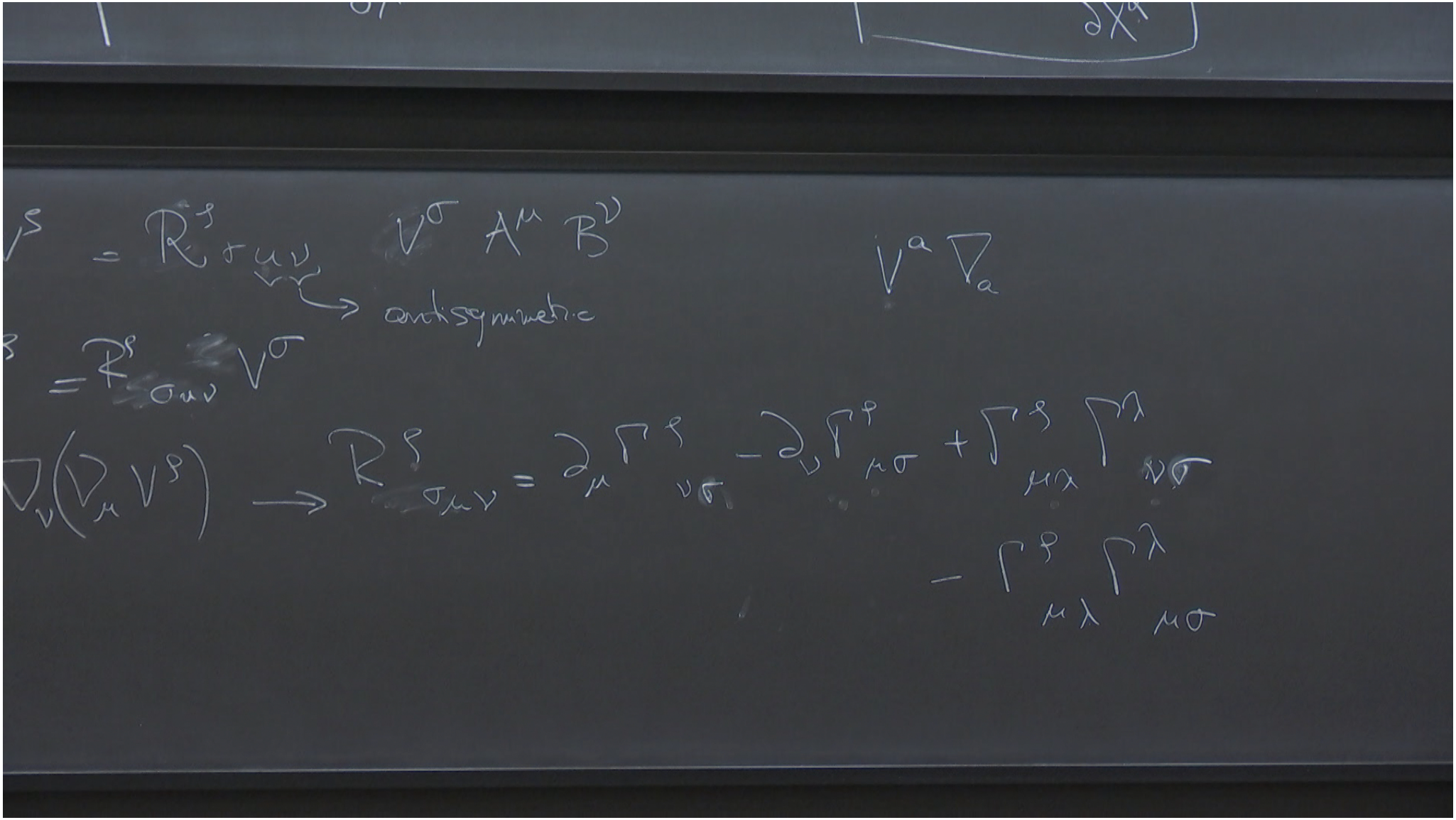


V^ρ $\quad \delta V^\rho = R^\rho{}_{\sigma\mu\nu}$

$[\nabla_\mu, \nabla_\nu] V^\rho$

$\nabla_\mu(\nabla_\nu V^\rho) - \nabla_\nu(\nabla_\mu V^\rho)$ —


V^ρ
$\delta V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma A^\mu B^\nu$
antisymmetric
$V^a \nabla_a$
$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma$
$\nabla_\mu(\nabla_\nu V^\rho) - \nabla_\nu(\nabla_\mu V^\rho) \rightarrow R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}$
$- \Gamma^\rho_{\mu\lambda}$


$V^\sigma A^\mu B^\nu$
$V^a \nabla_a$
antisymmetric
$\nabla_\nu(\nabla_\mu V^\rho) \longrightarrow R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma}$
$- \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\mu\sigma}$

Properties

① $R_{abcd} = -R_{bacd}$

② $R_{abcd} = -R_{abdc}$

③ $R_{(ab)(cd)} = R_{cdab}$

④ $R_{[abcd]} = 0$

M_{AB} $\frac{n(n-1)}{2}$ An

N_{CD} $\frac{m(m+1)}{2}$ Sy

$\frac{1}{2}\left[\frac{n(n-1)}{2}\right]\left[\frac{n(n-1)}{2}+1\right]$

$\rightarrow \frac{n(n-1)(n-2)(n-3)}{4!} = 0$

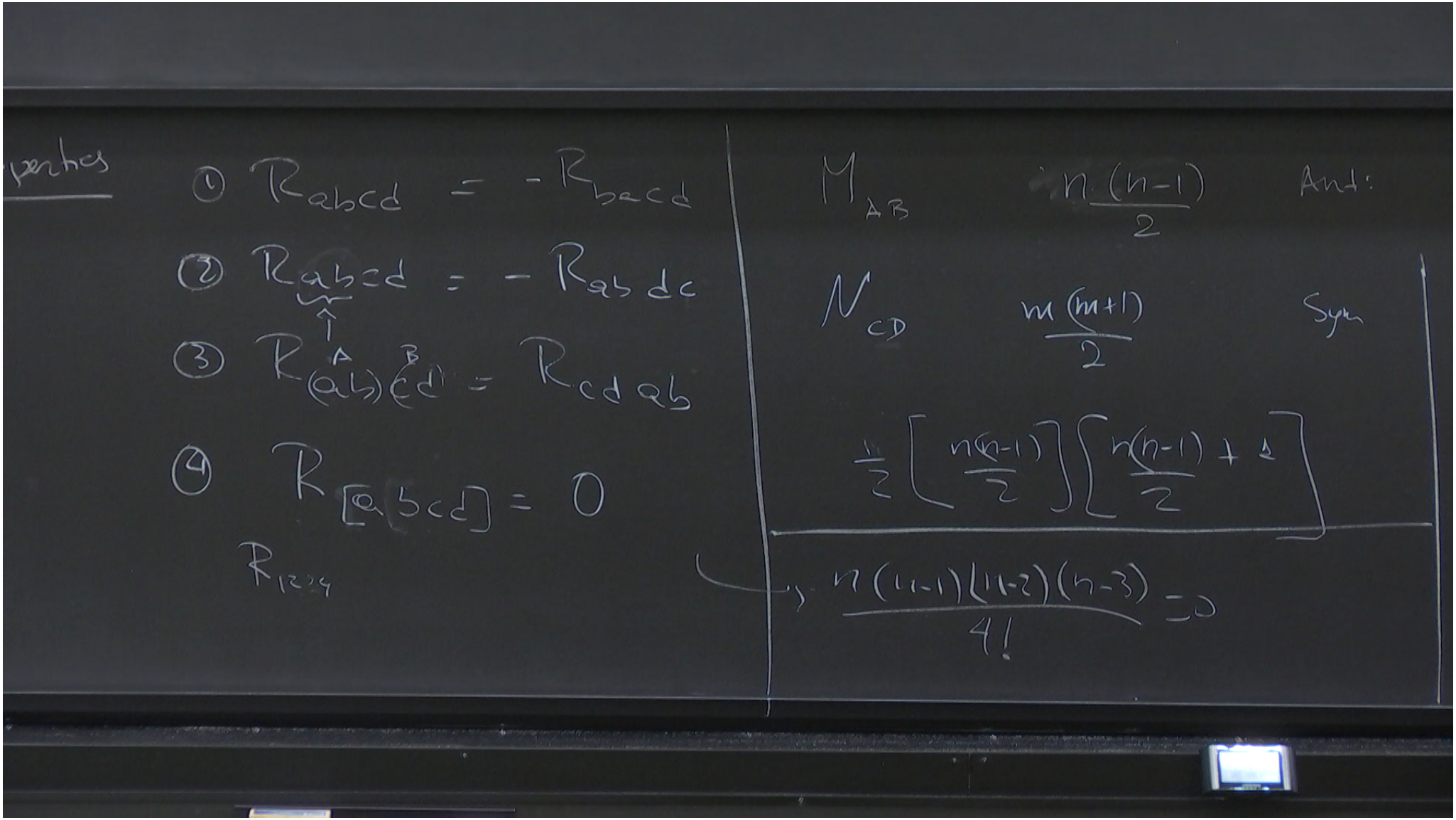
M_{AB} $\frac{n(n-1)}{2}$ And:

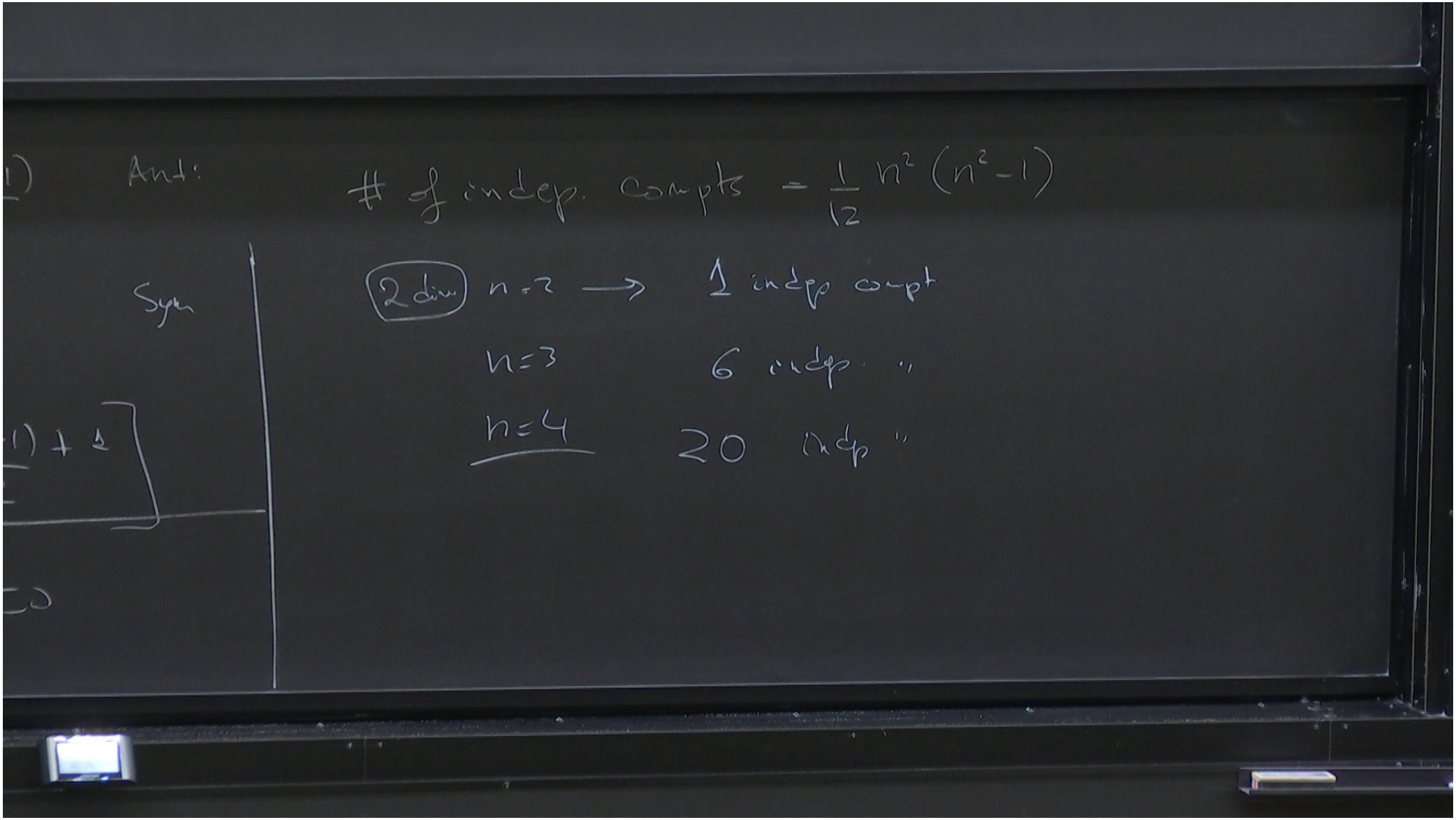
N_{CD} $\frac{m(m+1)}{2}$ Sym

$$\frac{1}{2}\left[\frac{n(n-1)}{2}\right]\left[\frac{n(n-1)}{2}+1\right]$$

$$\Rightarrow \frac{n(n-1)(n-2)(n-3)}{4!}$$

# of indep. compts $= \frac{1}{12} n^2 (n$


perties
① $R_{abcd} = -R_{bacd}$
② $R_{abcd} = -R_{abdc}$
③ $R_{(ab)(cd)} = R_{cdab}$
④ $R_{[abcd]} = 0$
R_{1234}
M_{AB} $\frac{n(n-1)}{2}$ And:
N_{CD} $\frac{m(m+1)}{2}$ Sym
$\frac{1}{2}\left[\frac{n(n-1)}{2}\right]\left[\frac{n(n-1)}{2}+1\right]$
$\frac{n(n-1)(n-2)(n-3)}{4!} = 0$


And:
# of indep. compts = $\frac{1}{12} n^2 (n^2-1)$
Sym
2 dim $n=2 \longrightarrow$ 1 indep compt
$n=3$ 6 indep "
$n=4$ 20 indep "

$$T_{ab} = T^{TF}_{(ab)} + (\text{Trace}) + T_{[ab]} \qquad \underline{\text{Trace}}\ \ g_{ab} T^{ab} = T_b{}^b$$

$$= \frac{1}{2}\left(T_{ab} + T_{ba}\right)$$

$$T_{ab} = T^{TF}_{(ab)} + (\text{Trace}) + T_{[ab]} \qquad \underline{\text{Trace}}\ \ g_{ab} T^{ab} = T_b{}^b$$

$$\underline{\text{Ricci tensor}}: \quad R_{ab} = R^{\lambda}{}_{a\lambda b} = R^{\lambda}{}_{acb}\, g_{\lambda}{}^{c}$$

$$T_{ab} = T^{TF}_{(ab)} + (\text{Trace}) + T_{[ab]} \qquad \underline{\text{Trace}}\ \ g_{ab} T^{ab} = T_b{}^b$$

Ricci tensor $R_{ab} = R^{\lambda}{}_{a\lambda b} = R^{\lambda}{}_{acb}\, g_{\lambda}{}^{c}$

Ricci scalar $R = g^{ab} R_{ab}$

Trace free part $C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}\left[g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho} \right.$

Weyl tensor $n \geq 3$

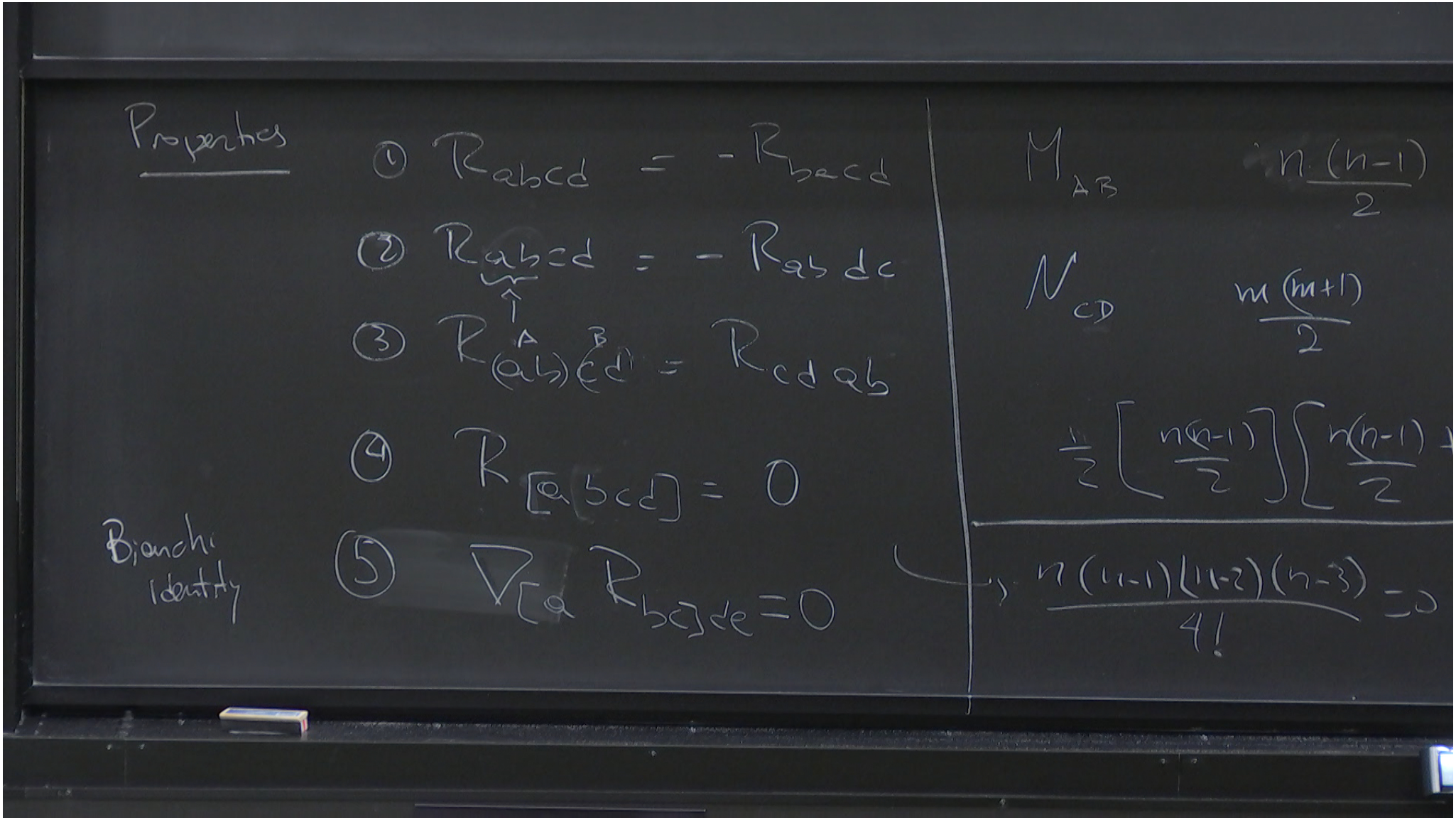
$+(\text{Trace}) + T_{[ab]}$ $\quad$ Trace $\quad g_{ab} T^{ab} = T_b{}^b$

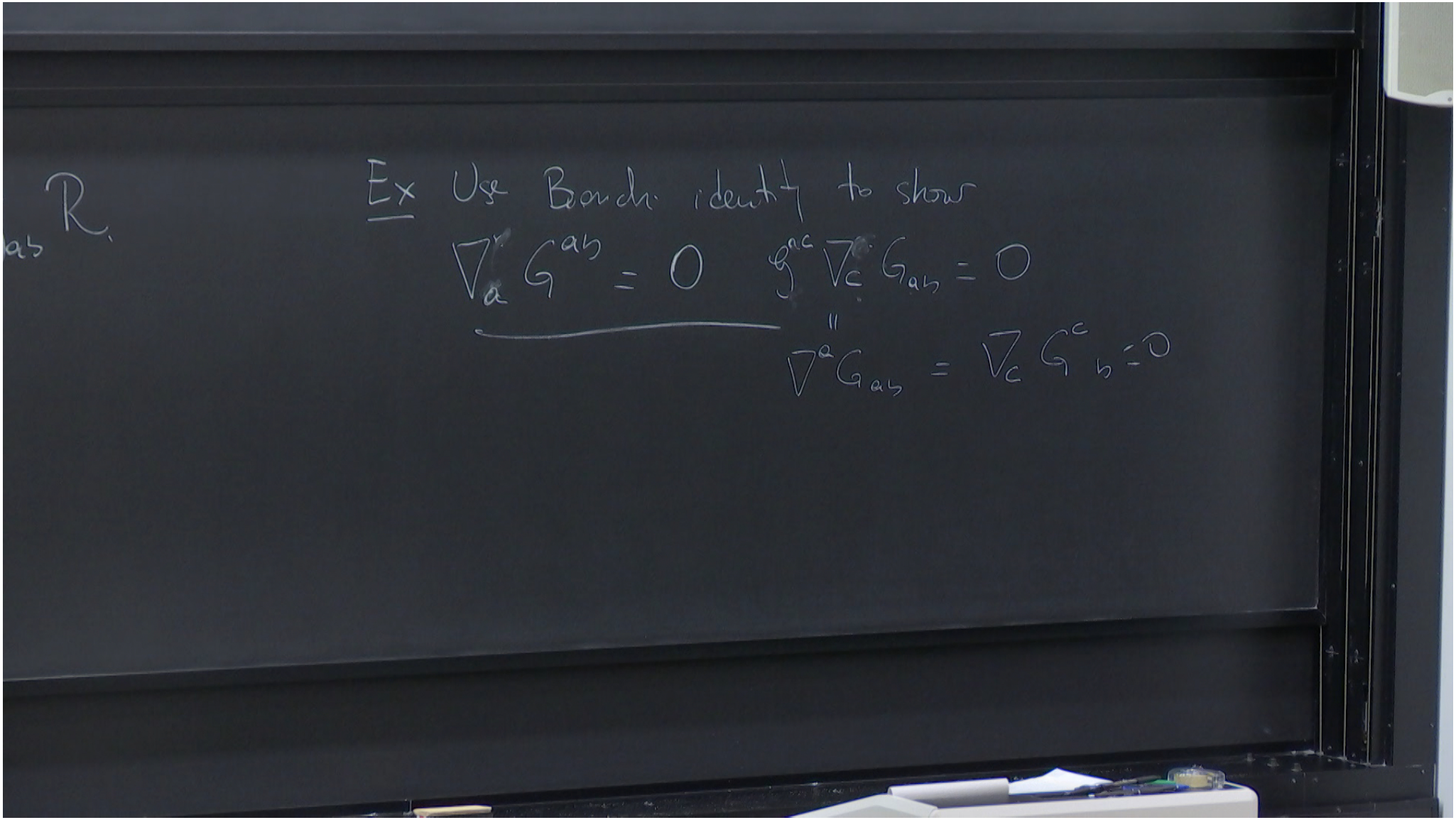
$$R_{ab} = R^{\lambda}{}_{a\lambda b} = R^{\lambda}{}_{acb}\, g_{\lambda}{}^{c}$$

$$R = g^{ab} R_{ab}$$

$$C_{\rho\sigma\mu\nu} = R_{\rho\sigma\mu\nu} - \frac{2}{(n-2)}\left[g_{\rho[\mu} R_{\nu]\sigma} - g_{\sigma[\mu} R_{\nu]\rho} \right] + \frac{2}{(n-1)(n-2)} g_{\rho[\mu} g_{\nu]\sigma} R$$

for $n \geq 3$

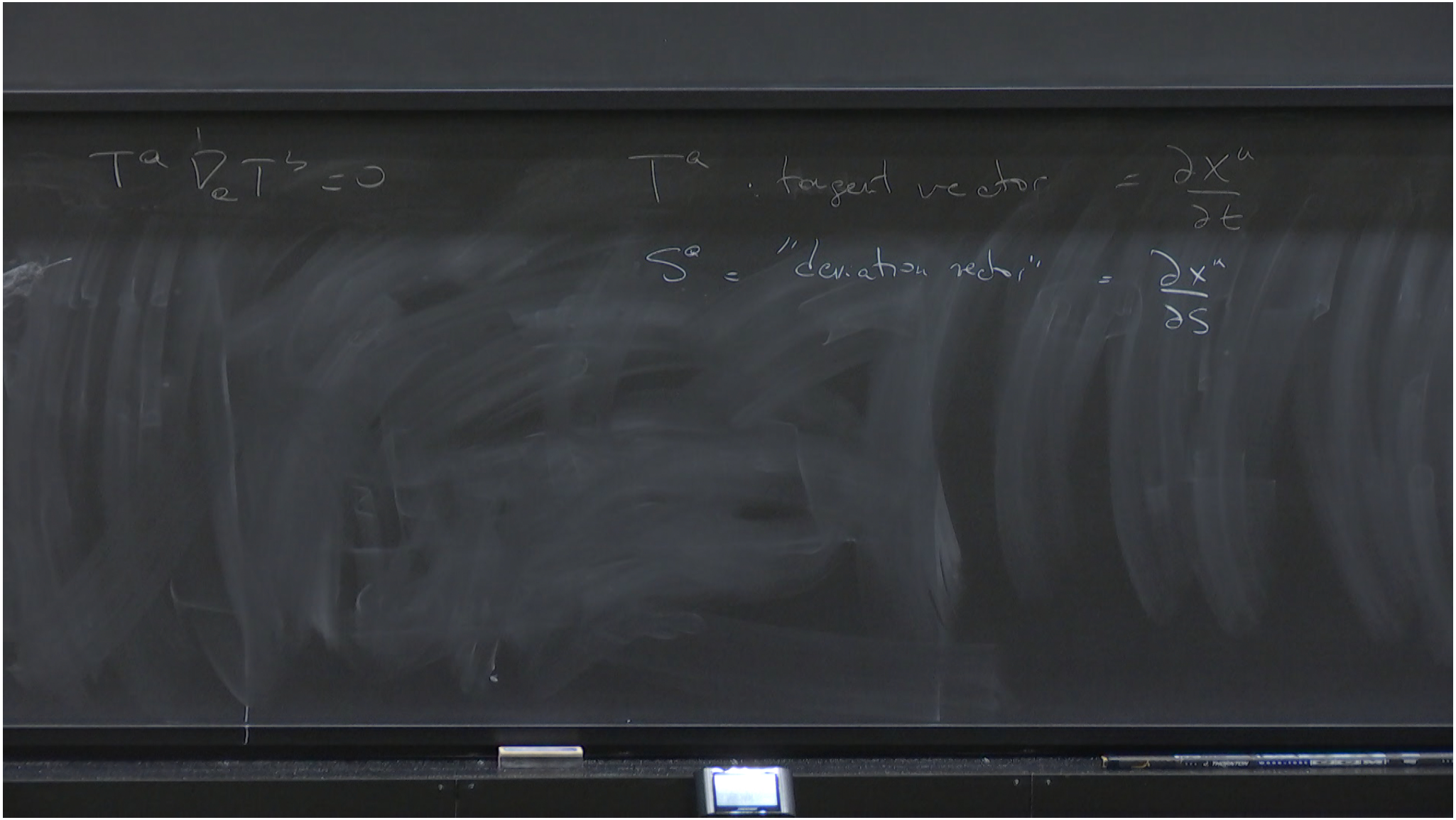

Properties
① $R_{abcd} = -R_{bacd}$
② $R_{abcd} = -R_{abdc}$
③ $R_{(ab)(cd)} = R_{cdab}$
④ $R_{[abcd]} = 0$
Bianchi identity
⑤ $\nabla_{[a} R_{bc]de} = 0$
M_{AB} $\frac{n(n-1)}{2}$
N_{CD} $\frac{m(m+1)}{2}$
$\frac{1}{2}\left[\frac{n(n-1)}{2}\right]\left[\frac{n(n-1)}{2}+\right.$
$\frac{n(n-1)(n-2)(n-3)}{4!} = 0$

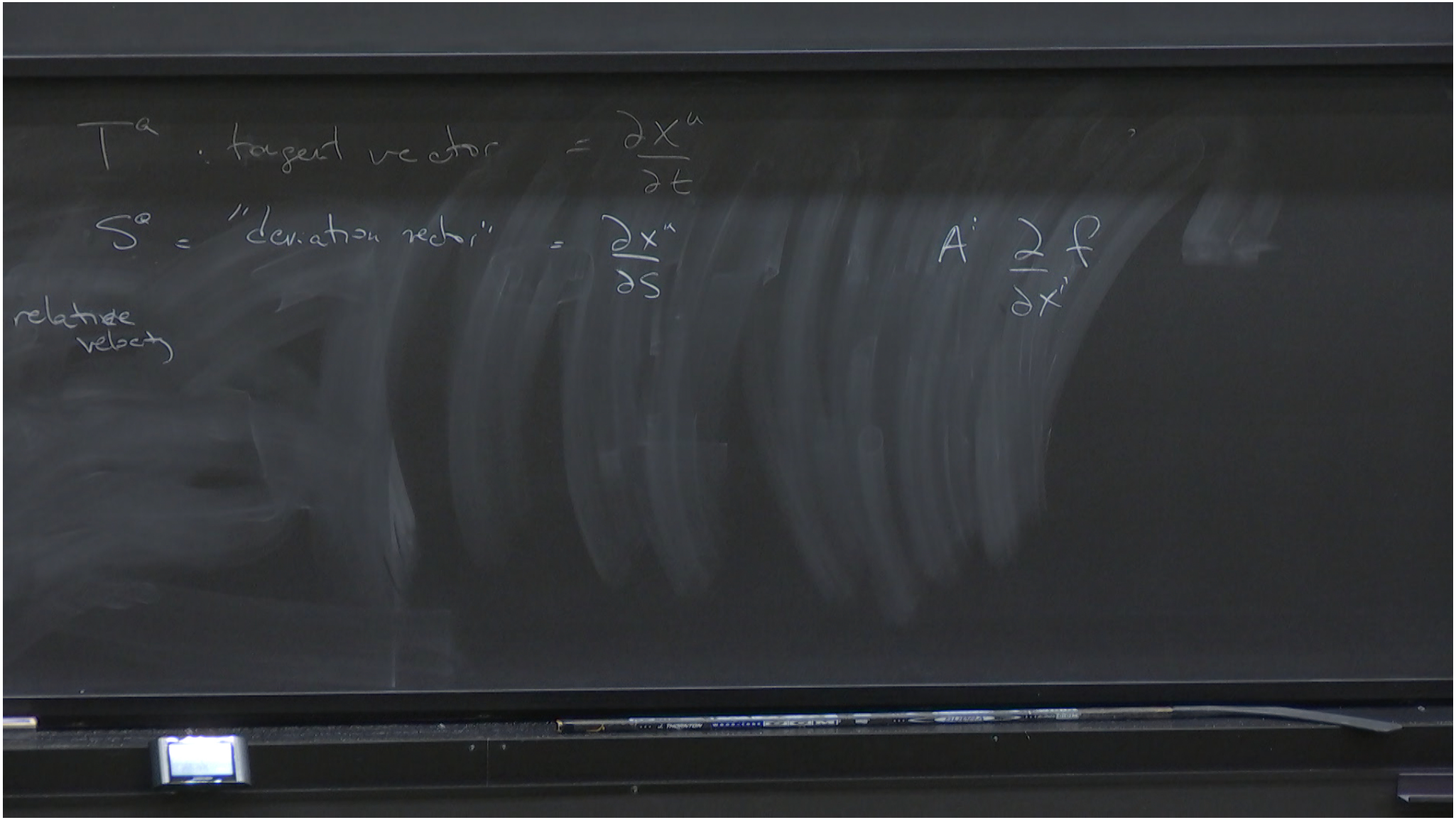

R
Ex Use Bianchi identity to show
$\nabla_a G^{ab} = 0$
$g^{ac}\nabla_c G_{ab} = 0$
$\nabla^a G_{ab} = \nabla_c G^c{}_b = 0$

Einstein tensor $G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R$

Ex U

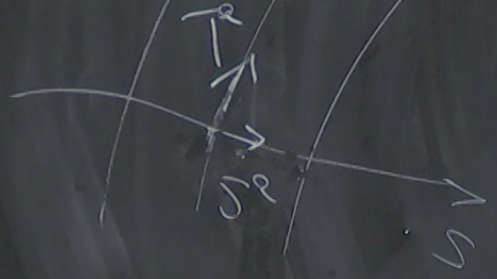
∇


$T^a \nabla_a T^b = 0$
T^a : tangent vector $= \frac{\partial x^a}{\partial t}$
S^a = "deviation vector" $= \frac{\partial x^a}{\partial s}$


T^a : tangent vector $= \frac{\partial x^a}{\partial t}$
S^a = "deviation vector" $= \frac{\partial x^a}{\partial s}$
relative velocity
$A: \frac{\partial}{\partial x^a} f$

Geodesic deviation

$$T^a \nabla_a T^b = 0$$



$$A^\mu = T^\beta \nabla_\beta \left(T^\alpha \nabla_\alpha S^\mu \right)$$

relative velocity

$$= \left(T^\beta \nabla_\beta T^\alpha \right) \nabla_\alpha S^\mu + T^\alpha T^\beta \left(\nabla_\beta \nabla_\alpha S^\mu \right)$$

$T^a \nabla_a T^b = 0$

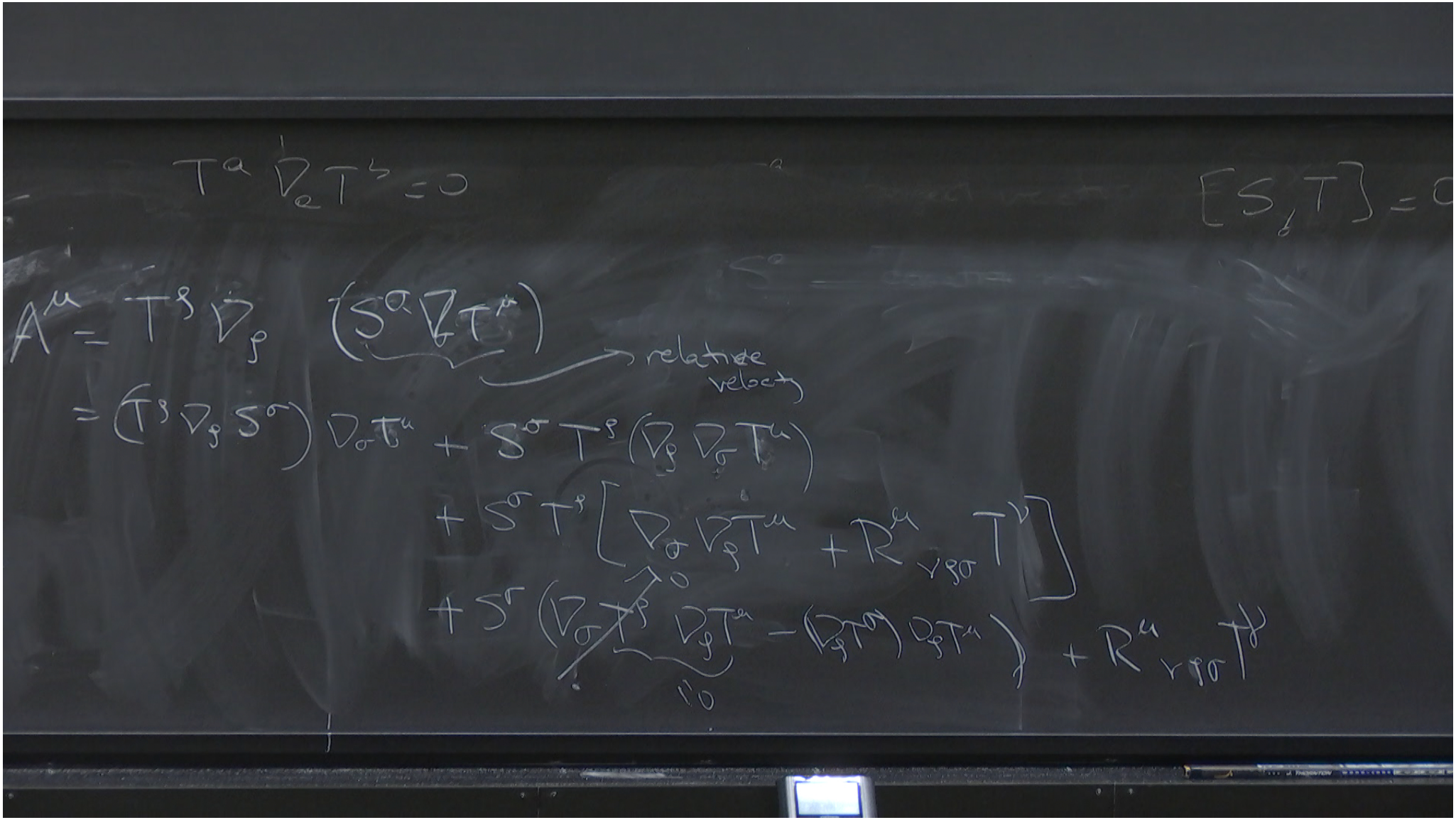
$[S, T] = 0$

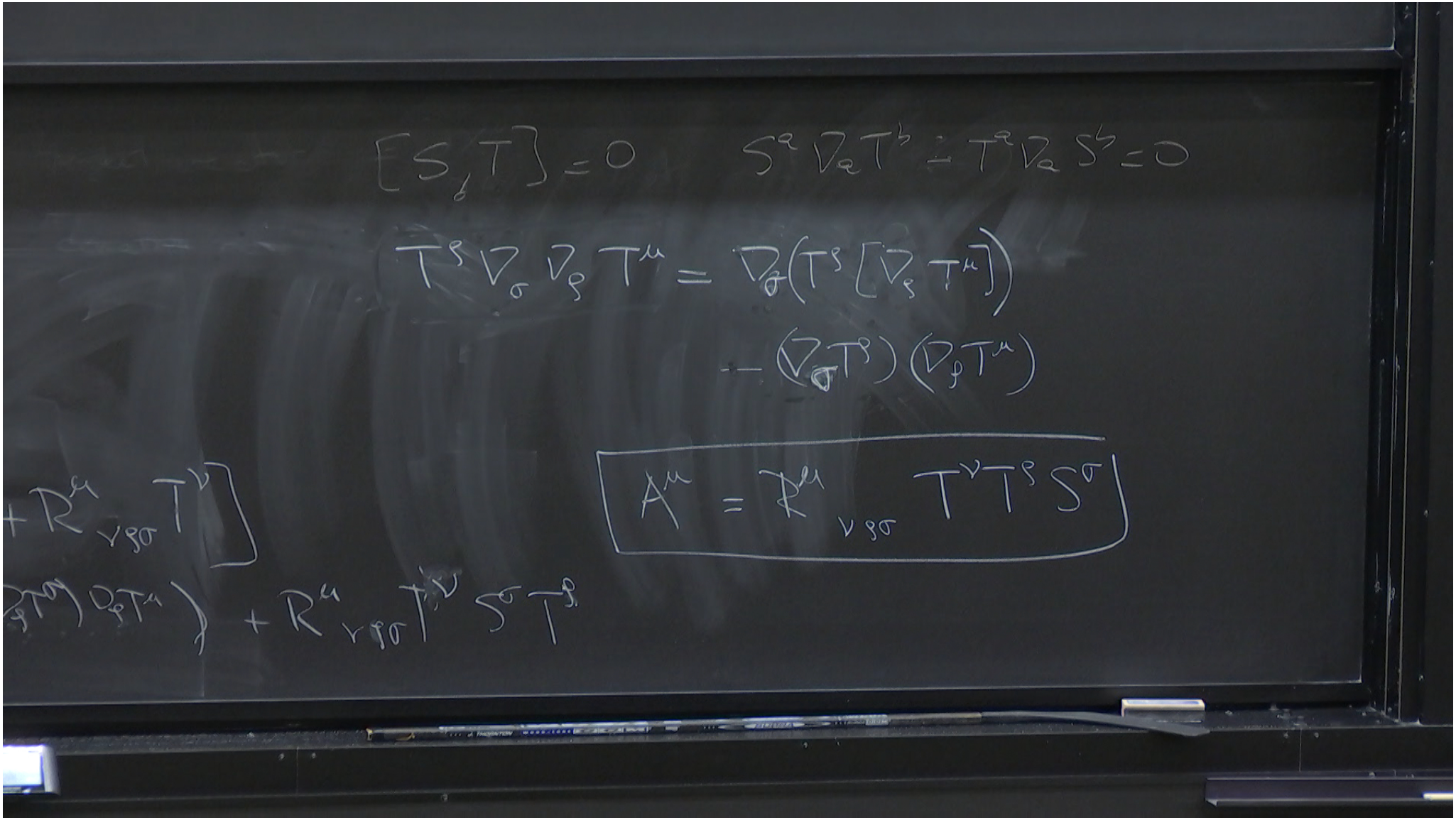
$A^\mu = T^\beta \nabla_\beta \left(S^\alpha \nabla_\alpha T^\mu \right)$ → relative velocity

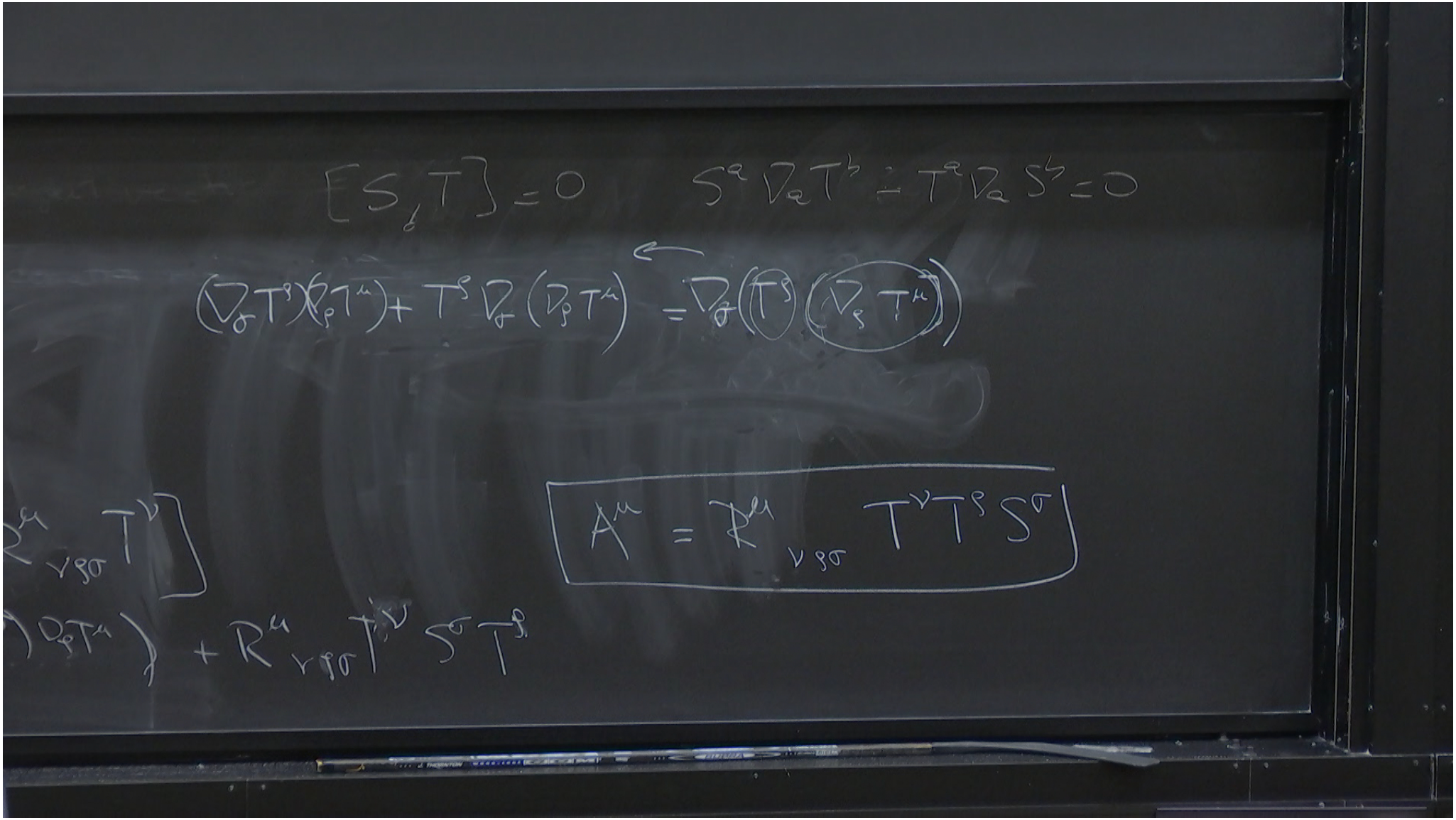
$= (T^\beta \nabla_\beta S^\alpha) \nabla_\alpha T^\mu + S^\alpha T^\beta (\nabla_\beta \nabla_\alpha T^\mu)$

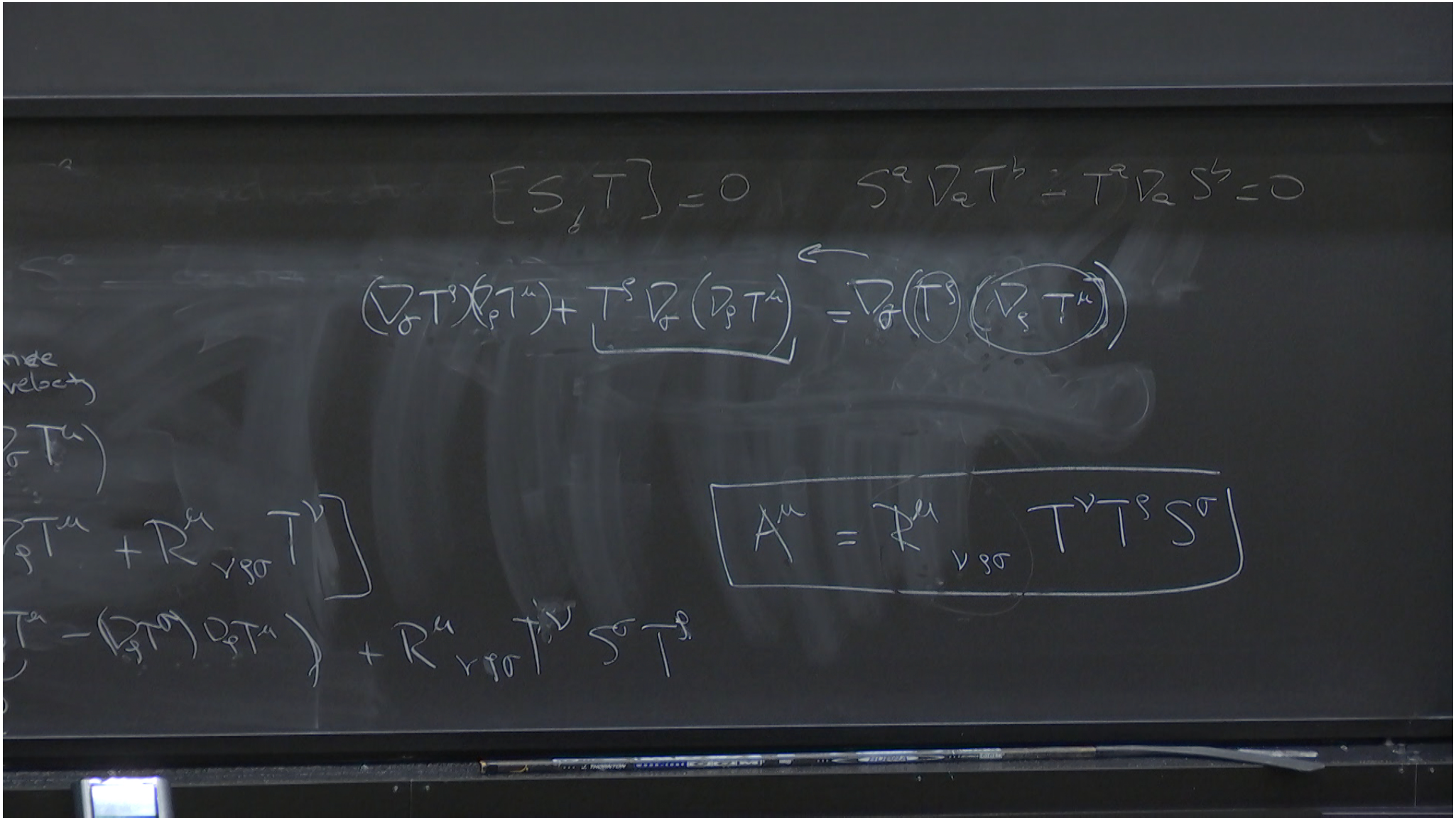
$+ S^\alpha T^\beta \left[\nabla_\alpha \nabla_\beta T^\mu + R^\mu{}_{\nu\beta\alpha} T^\nu \right]$

$+ S^\alpha \left(\nabla_\alpha T^\beta \nabla_\beta T^\mu - (\nabla_\beta T^\alpha) \nabla_\beta T^\mu \right) + R^\mu{}_{\nu\beta\alpha} T^\nu$


$T^a \nabla_a T^b = 0$
$[S, T] = 0$
$A^\mu = T^\beta \nabla_\beta (S^\alpha \nabla_\alpha T^\mu)$
relative velocity
$= (T^\beta \nabla_\beta S^\alpha) \nabla_\alpha T^\mu + S^\alpha T^\beta (\nabla_\beta \nabla_\alpha T^\mu)$
$+ S^\alpha T^\beta [\nabla_\alpha \nabla_\beta T^\mu + R^\mu{}_{\nu\beta\alpha} T^\nu]$
$+ S^\alpha (\nabla_\alpha T^\beta \nabla_\beta T^\mu - (\nabla_\beta T^\alpha) \nabla_\beta T^\mu) + R^\mu{}_{\nu\beta\alpha} T^\nu$


$[S, T] = 0$
$S^a \nabla_a T^b - T^a \nabla_a S^b = 0$
$T^\sigma \nabla_\sigma \nabla_\rho T^\mu = \nabla_\sigma (T^\sigma [\nabla_\rho T^\mu])$
$- (\nabla_\sigma T^\rho)(\nabla_\rho T^\mu)$
$A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$
$+ R^\mu{}_{\nu\rho\sigma} T^\nu]$
$\nabla_\rho T^\mu) \nabla_\rho T^\mu)$
$+ R^\mu{}_{\nu\rho\sigma} T^\nu S^\sigma T^\rho$


$[S, T] = 0$
$S^a \nabla_a T^b - T^a \nabla_a S^b = 0$
$A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$

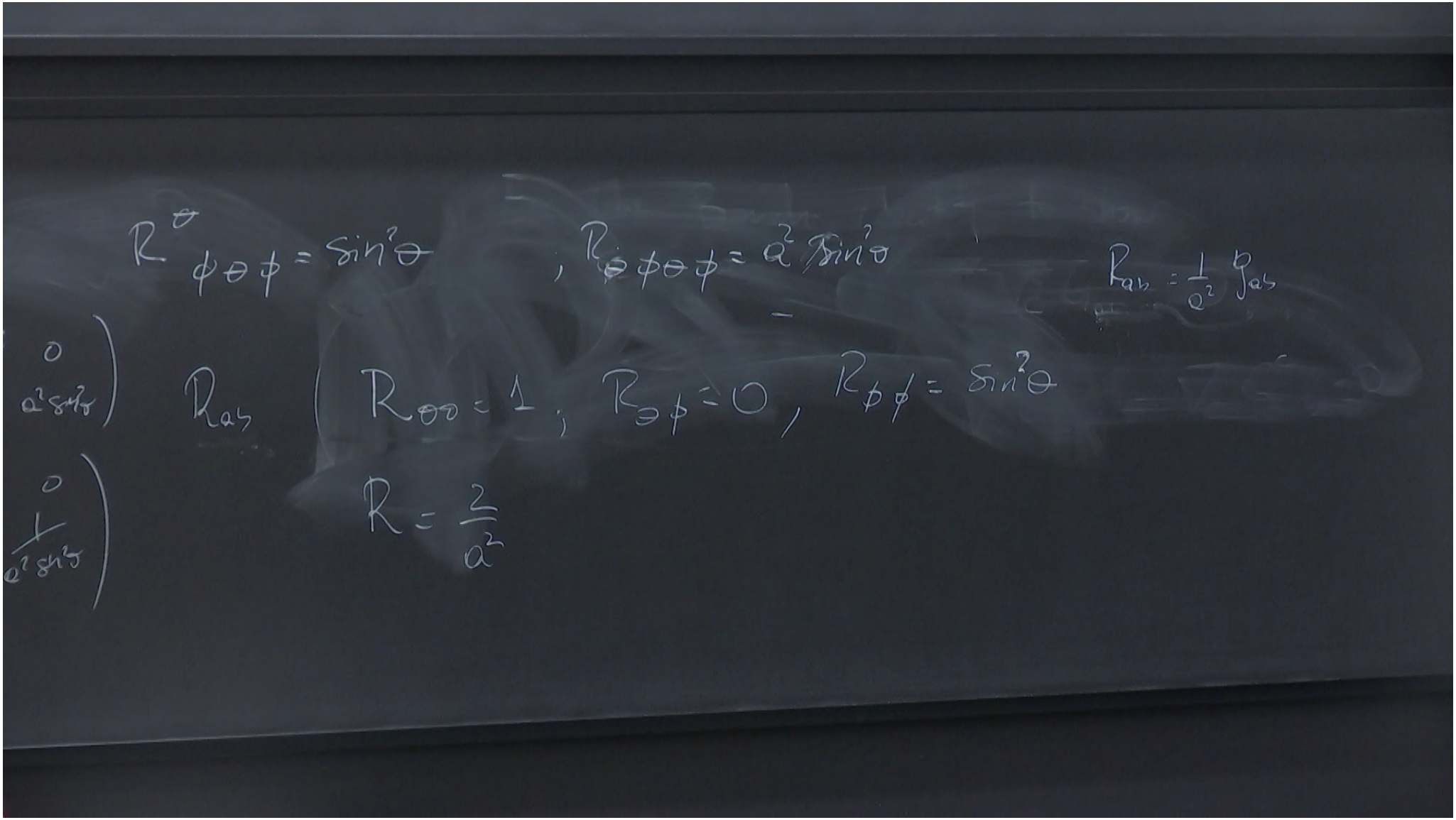

$[S, T] = 0$
$S^a \nabla_a T^b - T^a \nabla_a S^b = 0$
$(\nabla_\sigma T^\nu)(\nabla_\rho T^\mu) + T^\rho \nabla_\sigma (\nabla_\rho T^\mu) = \nabla_\sigma (T^\rho (\nabla_\rho T^\mu))$
$A^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma$

$$ds^2 = g_{ab}\, dx^a dx^b = a^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2 \right)$$

$n = 2$

$$\left.\begin{aligned} \Gamma^{\theta}_{\phi\phi} &= -\sin\theta\cos\theta \\ \Gamma^{\phi}_{\theta\phi} &= \cot\theta = \Gamma^{\phi}_{\phi\theta} \end{aligned}\right\}$$

$$\Gamma^{\phi}_{\theta\theta} = 0 = \Gamma^{\phi}_{\phi\phi}$$


$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta$, $R_{\theta\phi\theta\phi} = a^2 \sin^2\theta$
$R_{ab} = \frac{1}{a^2} g_{ab}$
R_{ab} ($R_{\theta\theta} = 1$; $R_{\theta\phi} = 0$, $R_{\phi\phi} = \sin^2\theta$
$R = \frac{2}{a^2}$

$ds^2 = g_{\alpha\beta}\, dx^\alpha dx^\beta = a^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$

$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta$

$n = 2$

$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\cos\theta$

$\Gamma^{\phi}_{\theta\phi} = \cot\theta = \Gamma^{\phi}_{\phi\theta}$

$g_{\alpha\beta} = \begin{pmatrix} a^2 & 0 \\ 0 & a^2\sin^2\theta \end{pmatrix}$

$g^{\alpha\beta} = \begin{pmatrix} 1/a^2 & 0 \\ 0 & 1/a^2\sin^2\theta \end{pmatrix}$

$\Gamma^{\phi}_{\theta\theta} = 0 = \Gamma^{\phi}_{\phi\phi}$

$g^{\alpha\beta} g_{\beta c} = \delta^{\alpha}_{c}$

$R_{\alpha\beta}$ ($R_{\theta\theta} = 1$

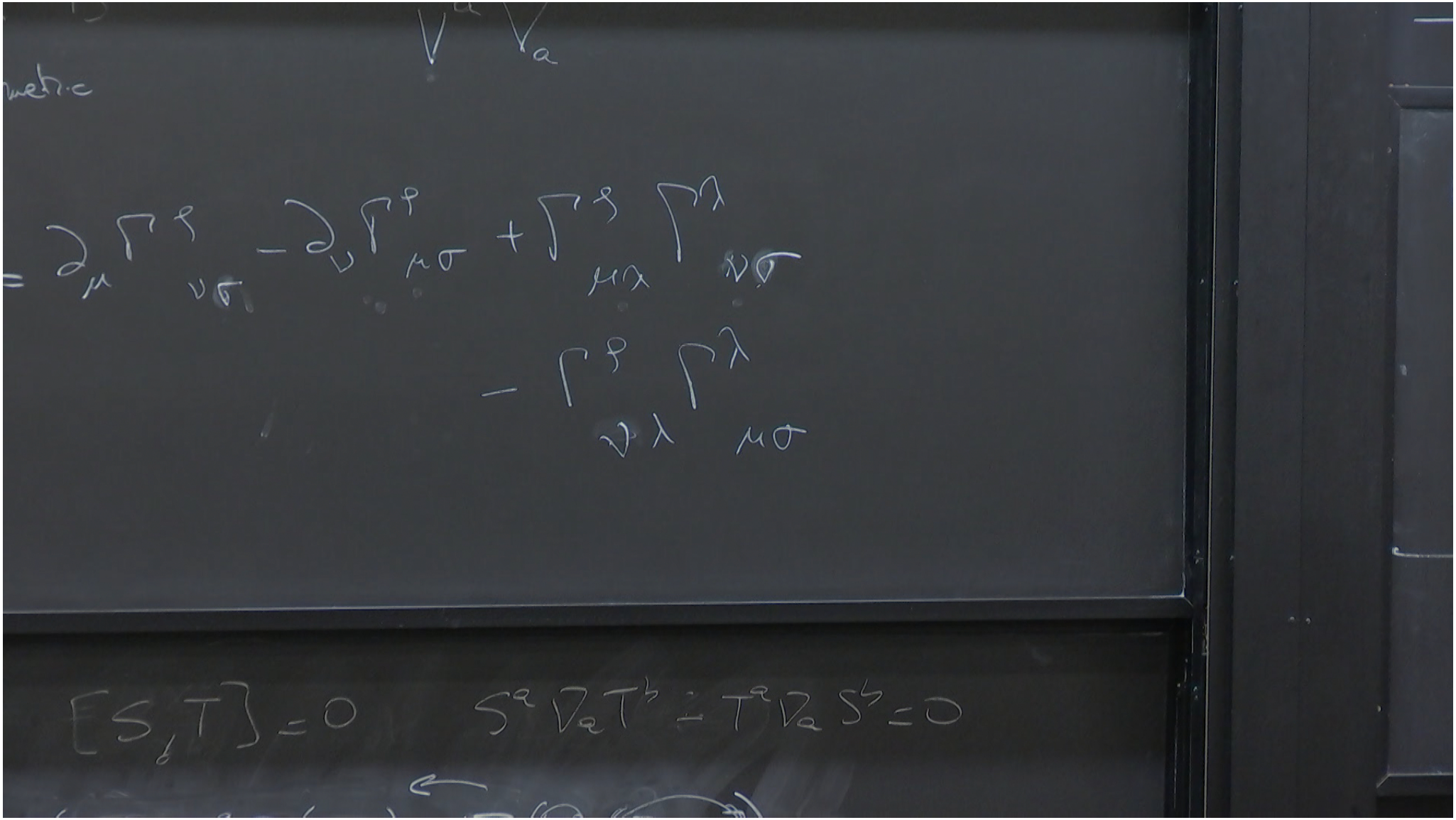
$R = \frac{2}{a^2}$

$$R^{\theta}{}_{\phi\theta\phi} = \sin^2\theta \quad ; \quad R_{\theta\phi\theta\phi} = a^2 \sin^2\theta \qquad R_{ab} = \frac{1}{a^2} g_{ab}$$

$$\begin{pmatrix} a^2 & 0 \\ 0 & a^2 \sin^2\theta \end{pmatrix} \qquad R_{ab} \left(R_{\theta\theta} = 1 \; ; \; R_{\theta\phi} = 0 \; , \; R_{\phi\phi} = \sin^2\theta \right.$$

$$\begin{pmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{a^2 \sin^2\theta} \end{pmatrix}$$

$$\boxed{R = \frac{2}{a^2}}$$


$V^a V_a$
metric
$= \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma}$
$- \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$
$[S, T] = 0$
$S^a \nabla_a T^b - T^a \nabla_a S^b = 0$