Title: Classical Physics - Lecture 221003

Speakers:

Collection: Classical Physics (2022/2023)

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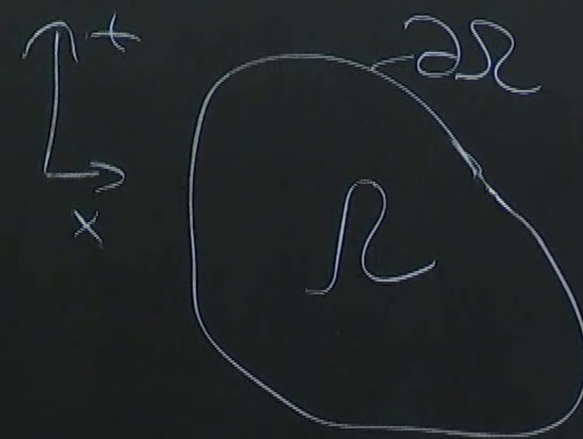
5) RELATIVISTIC FIELD THEORIES

a) ACTION PRINCIPLE

RELATIVISTIC FIELDS $\varphi(t,\vec{x})$ LIVE IN SPACETIME & BEHAVE NICELY UNDER LORENTZ (POINCARE) TRANSFORMATIONS

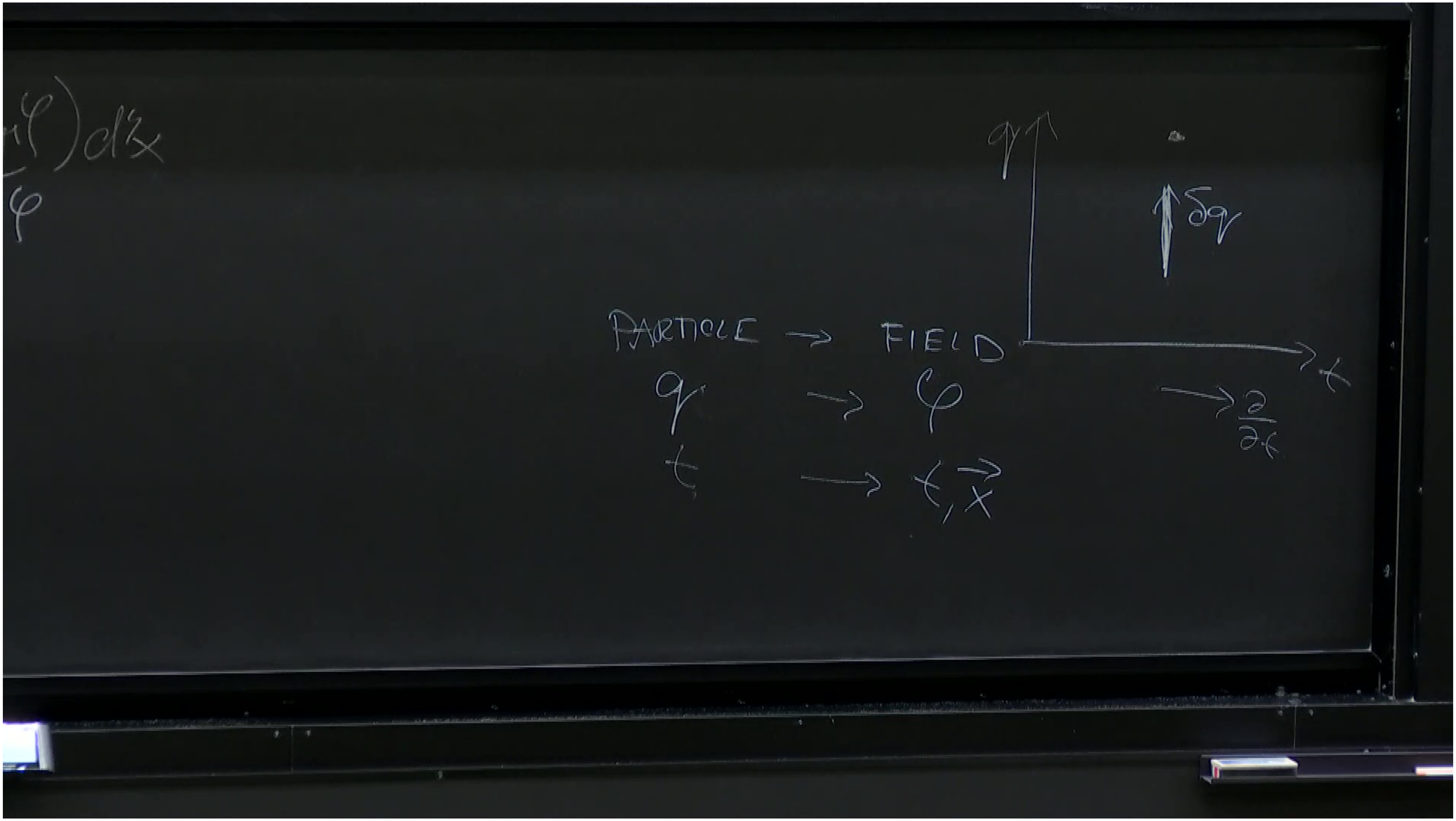
$$S[\varphi] = \int_{\Omega} \underbrace{dt\, d^3x}_{d^4x}\, \mathcal{L}(\varphi, \partial_\mu \varphi)$$

SPACETIME DOMAIN



$$\delta S = 0 = \int_{\Omega} \delta \mathcal{L} \, d^4x = \int_{\Omega} \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underbrace{\delta \partial_\mu \varphi}_{\partial_\mu \delta \varphi} \right) d$$

$$= \int_{\Omega} \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta \varphi$$


PARTICLE → FIELD
q → φ
t → t, x⃗
q
t
δq
→ ∂/∂t

$$= \int_{\Omega} \delta \mathcal{L}\, d^4x = \int_{\Omega} \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underbrace{\delta \partial_\mu \varphi}_{\partial_\mu \delta\varphi} \right) d^4x$$

$$\int_{\Omega} \left[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \right] \delta\varphi + \int_{\partial\Omega} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi\, d^3x$$

$$\mathcal{L}\, d^4x = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\underbrace{\delta\partial_\mu\varphi}_{\partial_\mu\delta\varphi}\right) d^4x$$

$$S[\varphi+\delta$$

$$-\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\Big]\delta\varphi + \int_{\partial\Omega}\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\, \underbrace{d^3x\, n^\mu}_{d\Sigma^\mu}$$

$$\boxed{\delta\varphi\big/_{\partial\Omega}=0}$$ "FIXED ENDPOINTS"

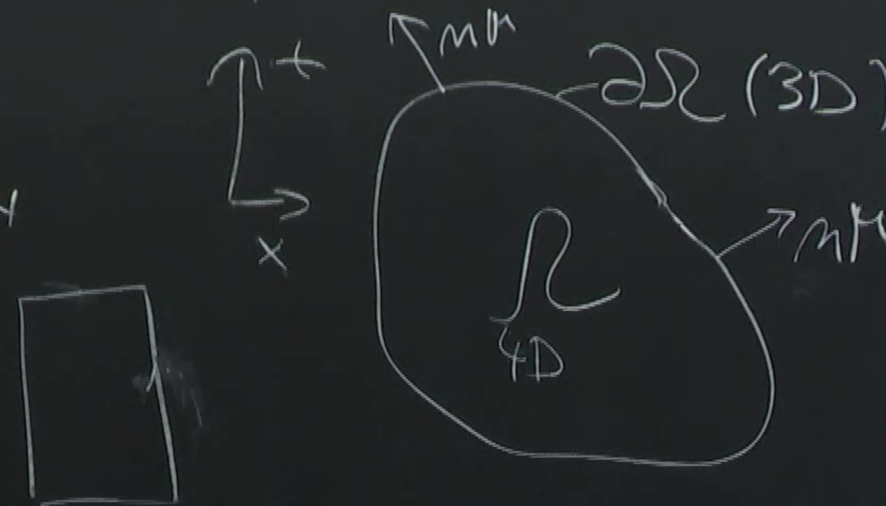
$$= \int_\Omega \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial\varphi} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Big)}_{0}\Big]\delta\varphi + \int_{\partial\Omega} \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi$$

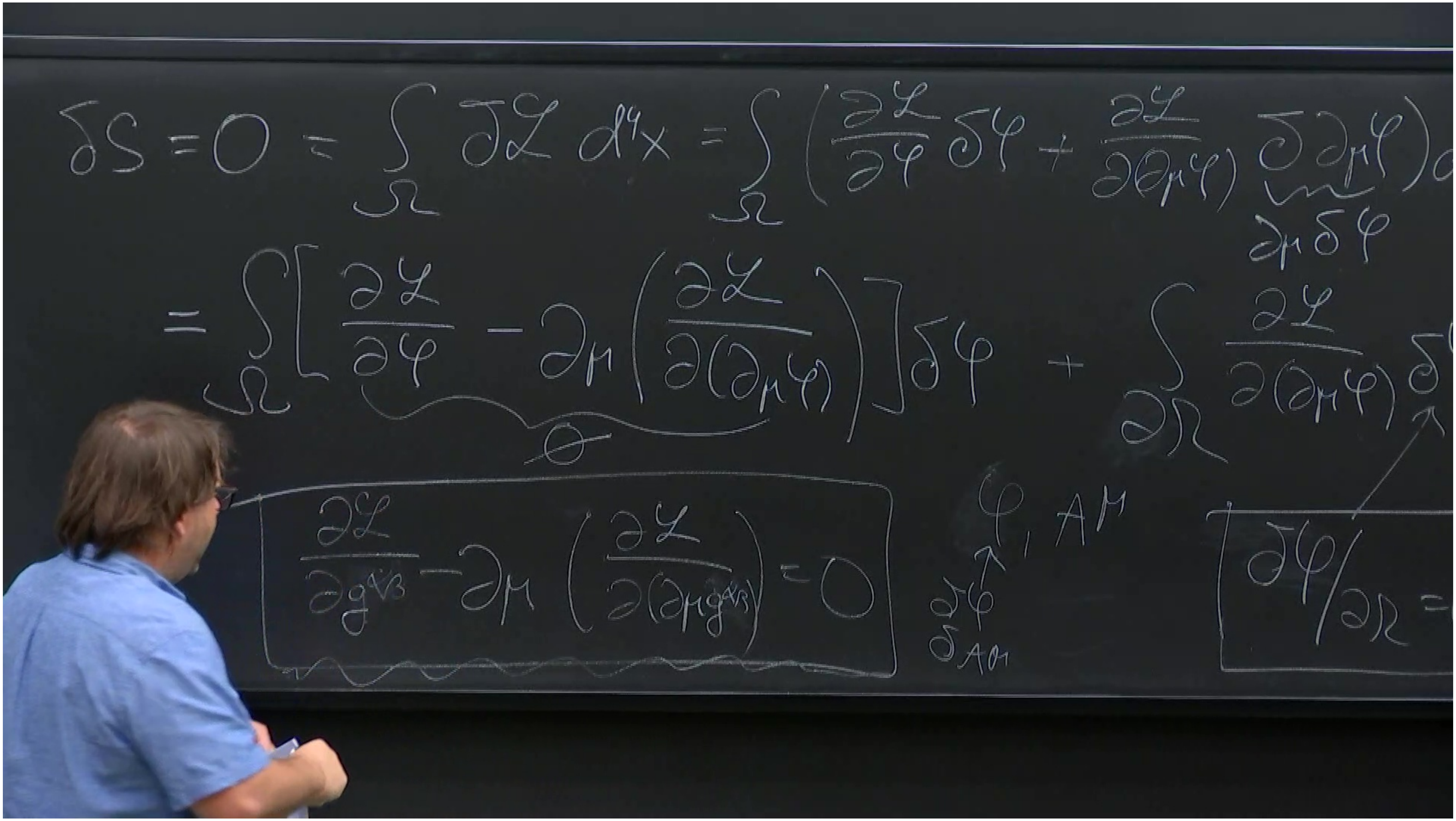
$$\boxed{\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right) = 0}$$

$\delta\varphi($

$$S[\varphi] = \int_{\Omega} \underbrace{dt\, d^3x}_{d^4x} \; \mathcal{L}(\varphi, \partial_\mu \varphi)$$

SPACETIME DOMAIN





$\delta S = 0 = \int_\Omega \delta \mathcal{L}\, d^4x = \int_\Omega \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \underbrace{\delta \partial_\mu \varphi}_{\partial_\mu \delta\varphi} \right)$
$= \int_\Omega \Big[\underbrace{\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right)}_{\cancel{0}} \Big] \delta\varphi + \int_{\partial\Omega} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta$
$\boxed{\frac{\partial \mathcal{L}}{\partial g^{\alpha\beta}} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu g^{\alpha\beta})} \right) = 0}$
$\varphi,\ A^\mu$
$\delta\varphi$
δA^μ
$\delta\varphi / \partial\Omega =$

5) CONSERVED CURRENTS

NOETHER'S THEOREM (EXPLICIT)

LET $\tilde{\delta}$ BE A GLOBAL CONTINUOUS SYMMETRY (THAT IS OFF-SHELL WE FIND $\tilde{\delta}x^\mu, \tilde{\delta}\varphi$ SUCH THAT $\tilde{\delta}S = 0$) THEN

WE HAVE THE FOLLOWING CONSERVED CURRENT

ETRY

URRENT

$$\boxed{J^\mu = T_c{}^\mu{}_\nu\, \delta x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\,\delta\varphi}$$

$\uparrow$ CANONICAL ENERGY-MOMENTUM TENSOR

$$\boxed{\partial_\mu J^\mu = 0}$$

WHERE

$$\boxed{T_c{}^\mu{}_\nu = \eta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\,\partial_\nu \varphi}$$

$$\boxed{J^\mu = T_c{}^\mu{}_\nu \,\tilde{\delta} x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \tilde{\delta}\varphi}$$

CANONICAL ENERGY-MOMENTUM TENSOR

MMETRY

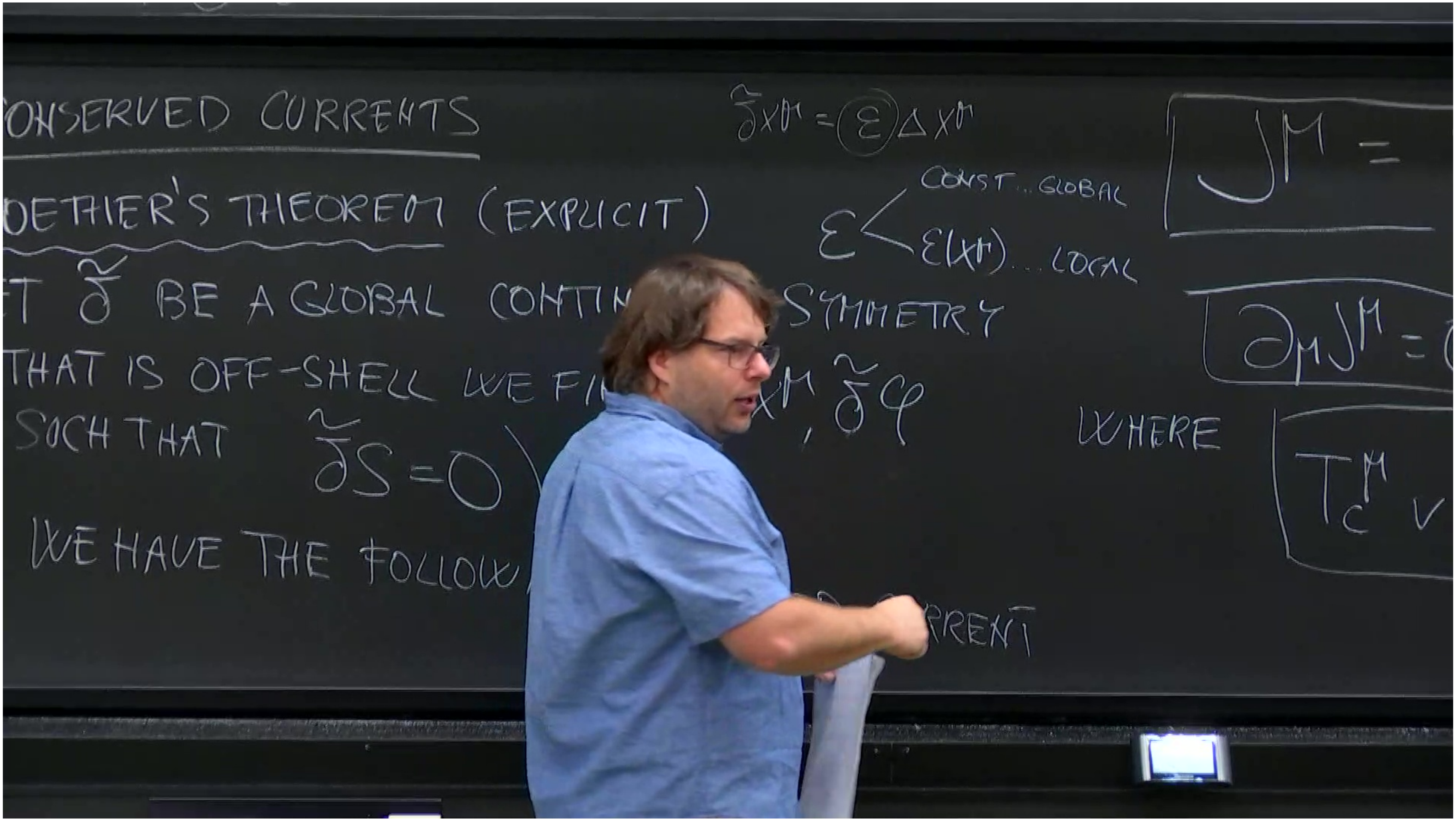
$\tilde{\delta}\varphi$

$$\boxed{\partial_\mu J^\mu = 0}$$

CURRENT

WHERE

$$\boxed{T_c{}^\mu{}_\nu = \eta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi}$$


ONSERVED CURRENTS
OETHER'S THEOREM (EXPLICIT)
T δ̃ BE A GLOBAL CONTIN
SYMMETRY
THAT IS OFF-SHELL WE FI
SUCH THAT δ̃S = 0
WE HAVE THE FOLLOW
δ̃xμ = ε Δxμ
CONST ... GLOBAL
ε(xμ) ... LOCAL
WHERE
RRENT

CONST ... GLOBAL

$\varepsilon <$ $\varepsilon(x^\mu)$ ... LOCAL

S SYMMETRY

$\tilde{\delta} x^\mu, \tilde{\delta}\varphi$

ERVED CURRENT

$$\boxed{J^\mu = T_c{}^\mu{}_\nu \, \tilde{\delta} x^\nu + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \tilde{\delta}\varphi}$$

CANONICAL ENERGY-MOMENTUM TE

$$\boxed{\partial_\mu J^\mu = 0}$$ ON-SHELL

WHERE

$$\boxed{T_c{}^\mu{}_\nu = \eta^\mu{}_\nu \mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi}$$

... GLOBAL

... LOCAL

$$\boxed{J^{\mu} = T_{c}{}^{\mu}{}_{\nu}\, \delta x^{\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}\, \delta\varphi}$$

CANONICAL ENERGY-MOMENTUM TENSOR

$$\boxed{\partial_{\mu} J^{\mu} = 0}$$ ON-SHELL

WHERE

$$\boxed{T_{c}{}^{\mu}{}_{\nu} = \eta^{\mu}{}_{\nu}\mathcal{L} - \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi)}\partial_{\nu}\varphi}$$

$$I = \underbrace{\frac{\partial L}{\partial \dot{q}}}_{\text{"p"}} \tilde{\partial} q + \Big(\underbrace{L - \dot{q}\frac{\partial L}{\partial \dot{q}}}_{\text{"E"}}\Big) \tilde{\partial} t$$

REMARK: FOR PARTICLE

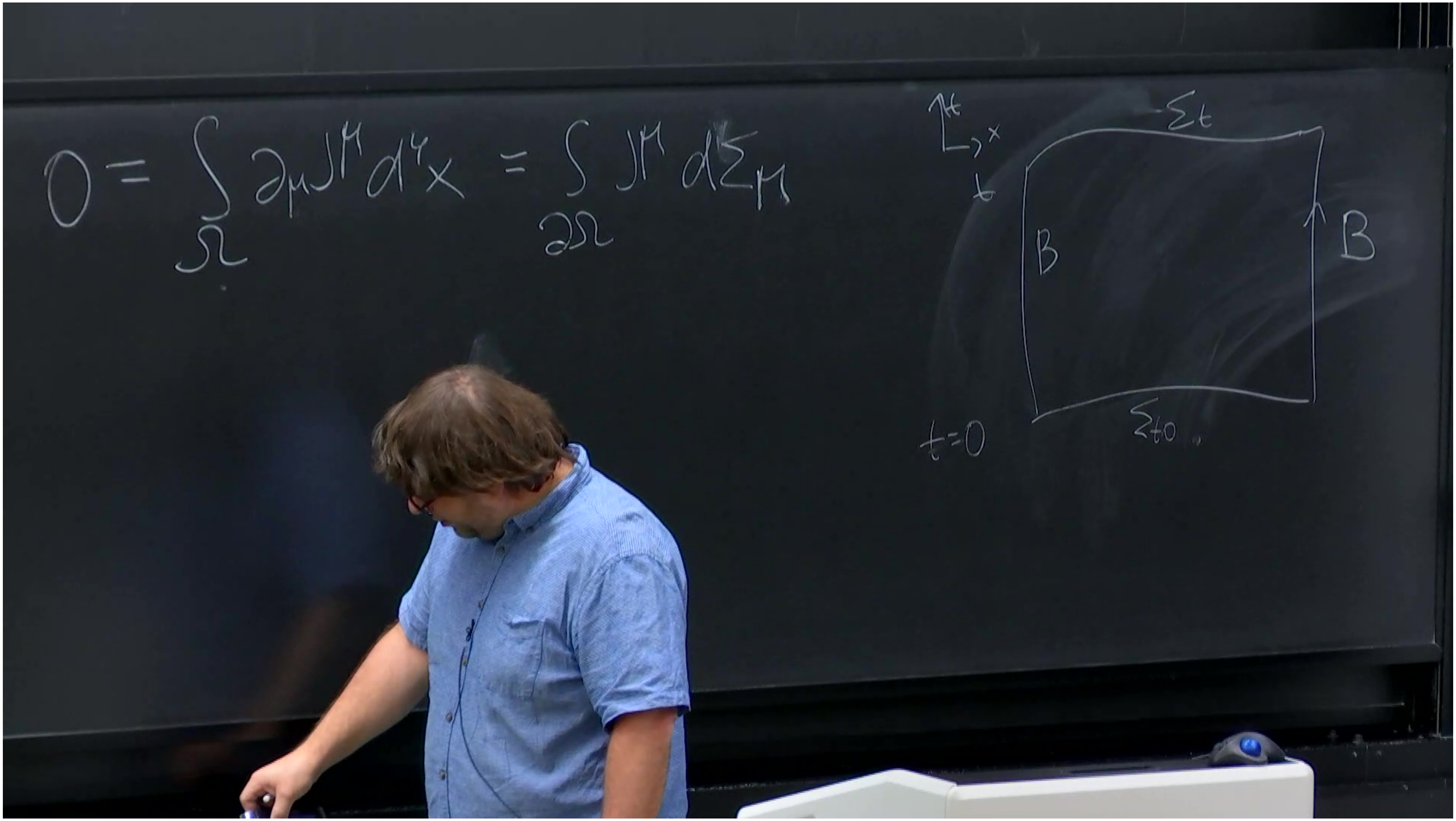
$$I = \underbrace{\frac{\partial L}{\partial \dot q}}_{\text{"p"}} \frac{\partial}{\partial q} + \underbrace{\left(L - \dot q \frac{\partial L}{\partial \dot q}\right)}_{\text{"E"}} \frac{\partial}{\partial t}$$

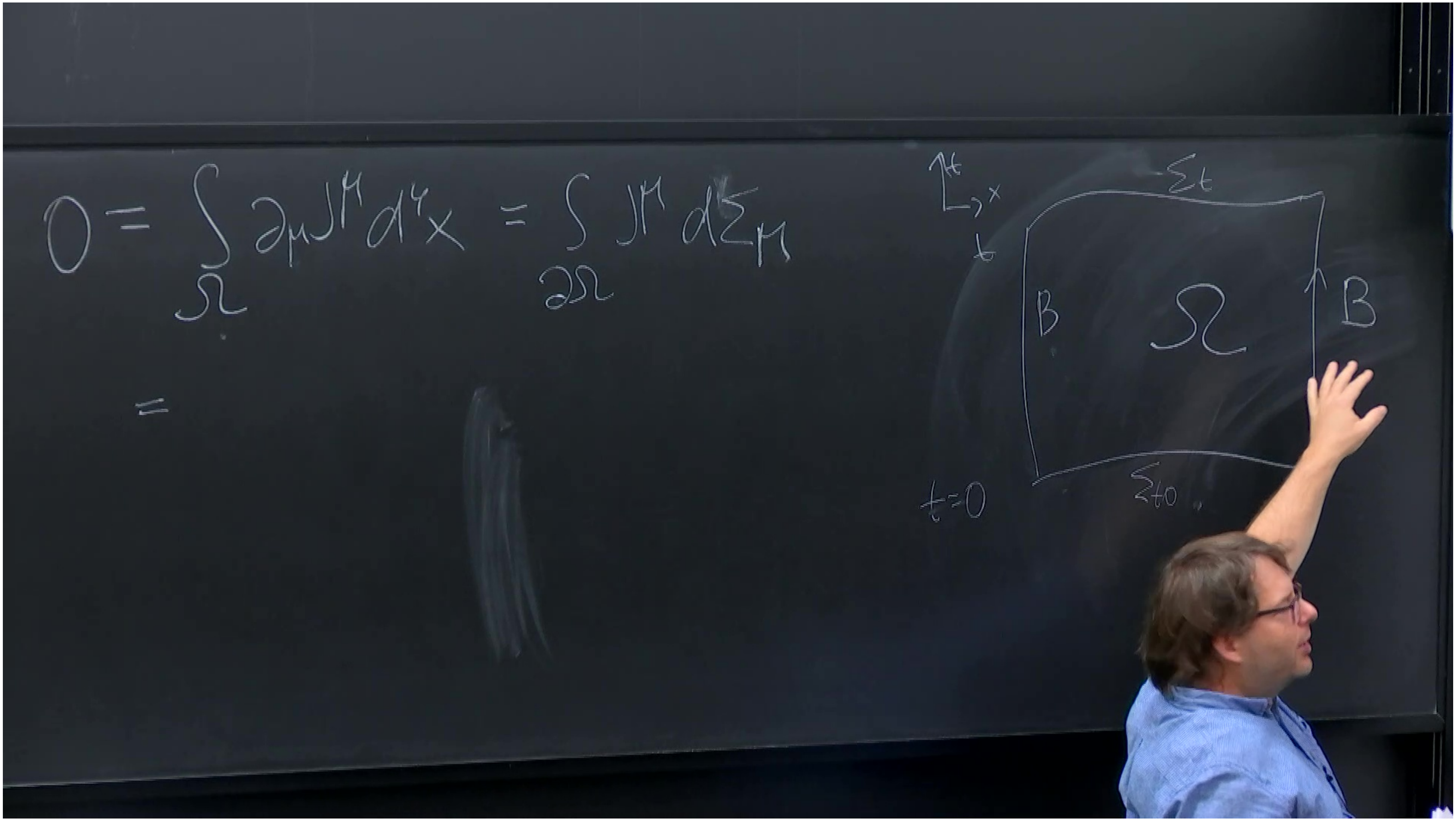
- **CHARGE CONSERVATION**

"COVARIANT VERSION" OF $Q = \int \rho \, d^3V$

$$J^\mu = (\rho, \vec{j})$$

$$\frac{dQ}{dt} = 0$$ ... NO FLUX THROUGH ∂V

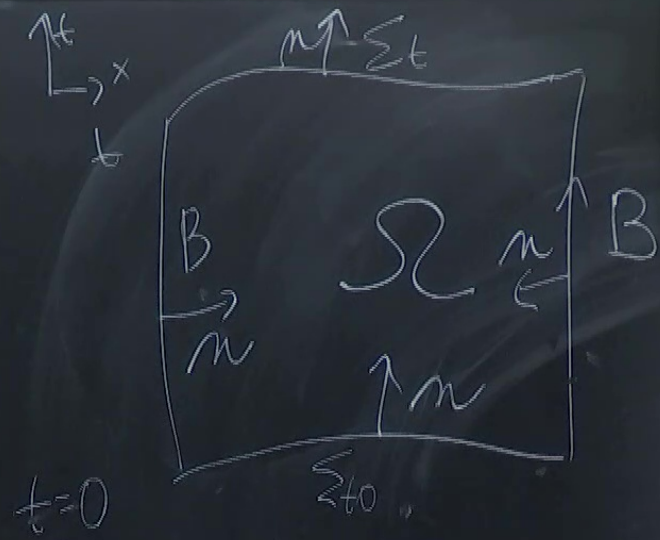

$0 = \int_{\Omega} \partial_\mu J^\mu d^4x = \int_{\partial\Omega} J^\mu d\Sigma_\mu$
t
x
t
Σ_t
B
B
t=0
Σ_{t0}

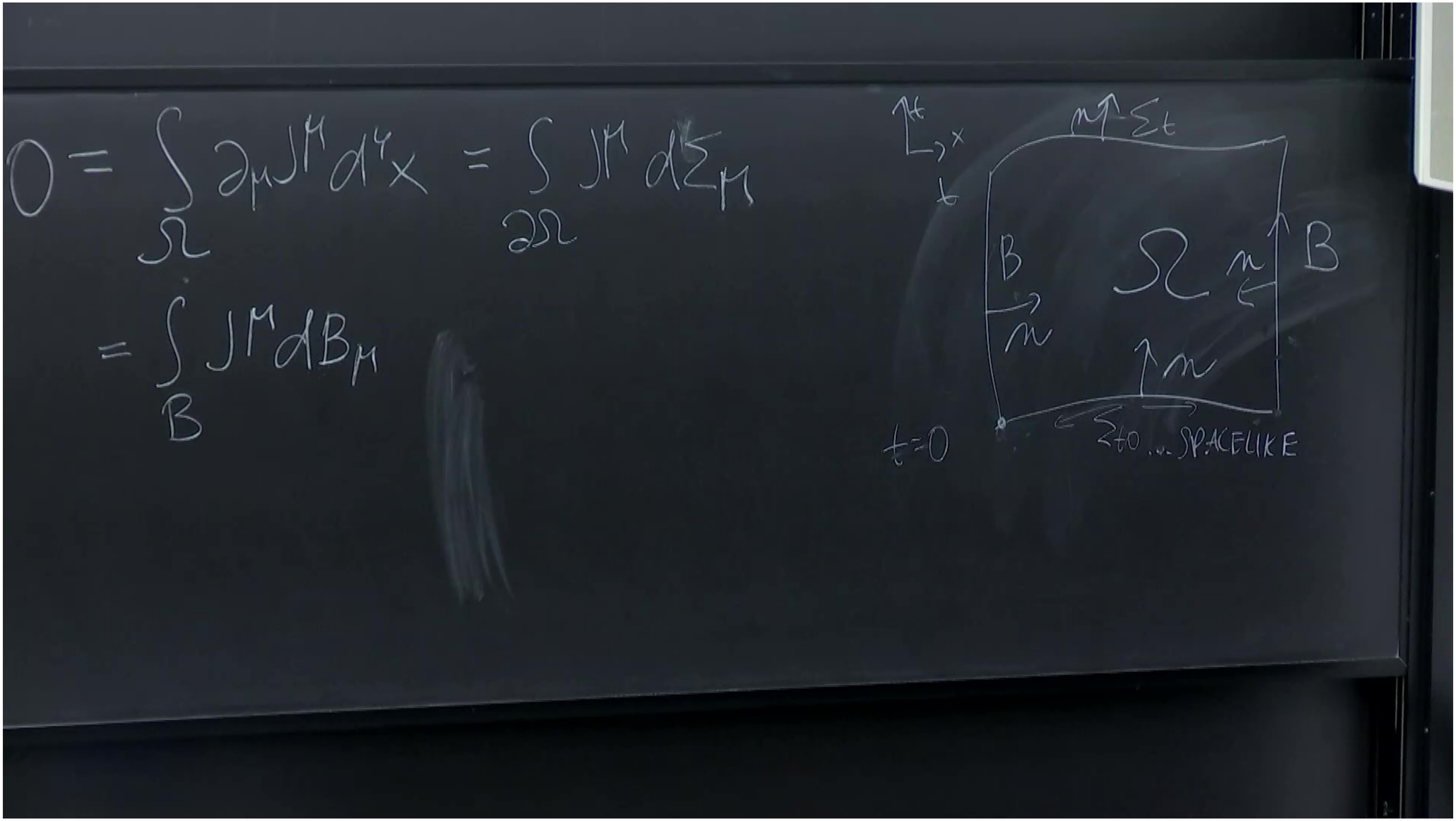


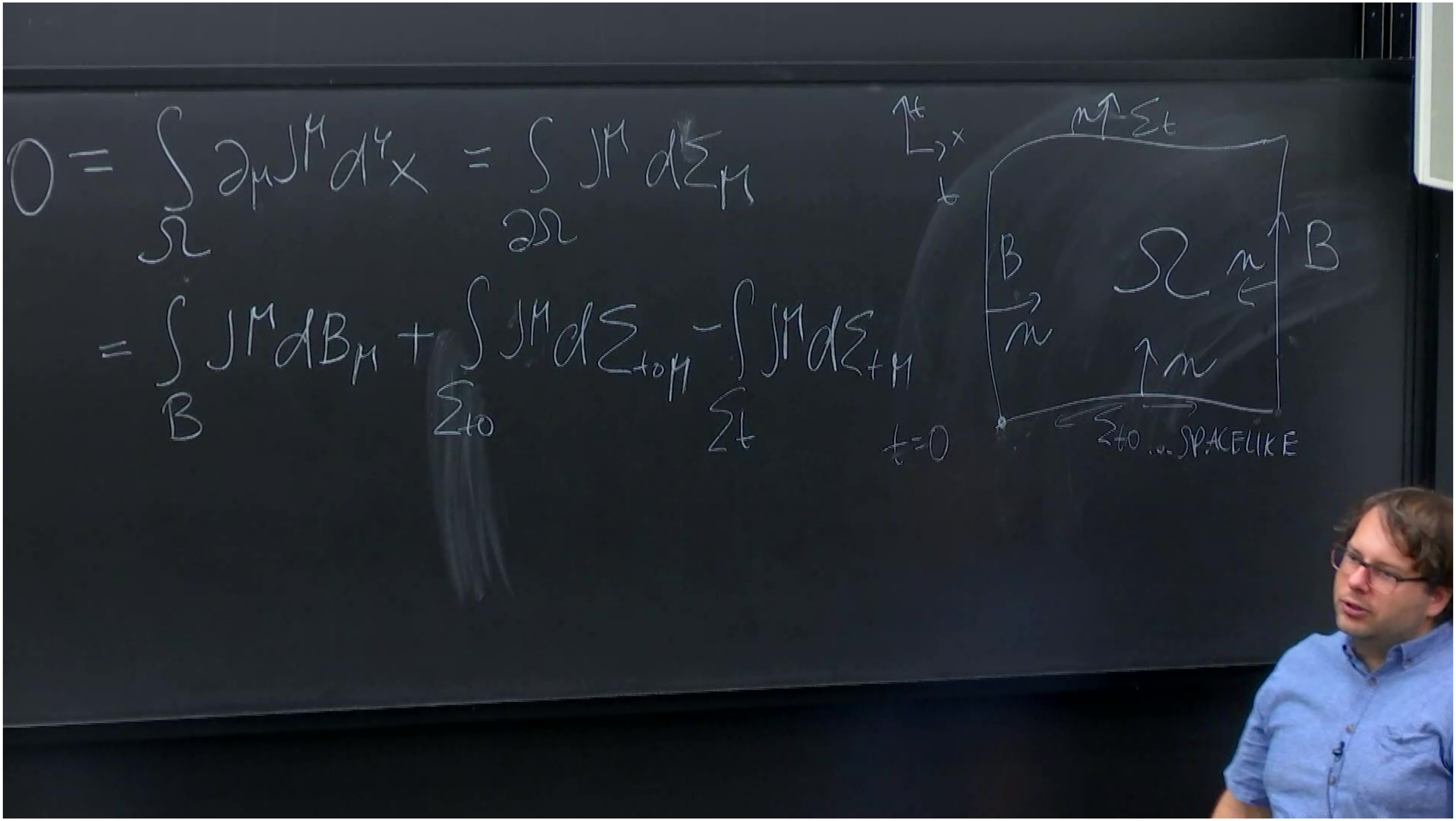
$0 = \int_{\Omega} \partial_\mu J^\mu d^4x = \int_{\partial\Omega} J^\mu d\Sigma_\mu$
$=$
t
x
Σ_t
t
B
Ω
B
$t=0$
Σ_{t0}

$$0 = \int_{\Omega} \partial_\mu J^\mu d^4x = \int_{\partial\Omega} J^\mu d\Sigma_\mu$$

$$= \int_{B} J^\mu dB_\mu$$



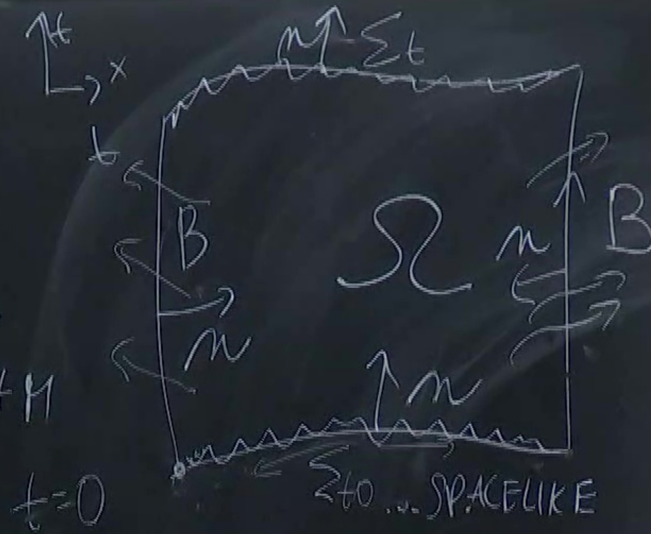

$0 = \int_{\Omega} \partial_\mu J^\mu d^4x = \int_{\partial\Omega} J^\mu d\Sigma_\mu$
$= \int_B J^\mu dB_\mu$
t
x
n
Σ_t
B
Ω
n
B
n
n
$t=0$
Σ_{t0} ... SPACELIKE


$0 = \int_\Omega \partial_\mu J^\mu d^4x = \int_{\partial\Omega} J^\mu d\Sigma_\mu$
$= \int_B J^\mu dB_\mu + \int_{\Sigma_{t0}} J^\mu d\Sigma_{t0\mu} - \int_{\Sigma_t} J^\mu d\Sigma_{t\mu}$
t
x
t
n
Σ_t
B
Ω
n
B
$\vec{n}$
n
$t=0$
Σ_{t0} ... SPACELIKE

$$0 = \int_{\Omega} \partial_\mu J^\mu d^4x = \int_{\partial\Omega} J^\mu d\Sigma_\mu$$

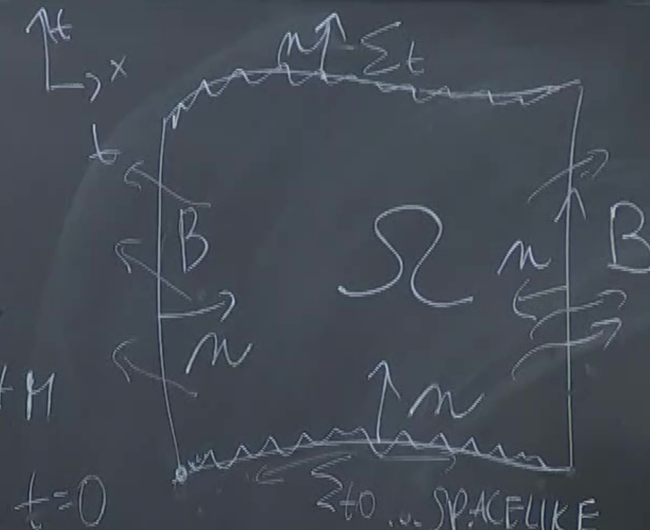
$$= \underbrace{\int_{B} J^\mu dB_\mu}_{\emptyset} + \int_{\Sigma_{t0}} J^\mu d\Sigma_{t0\,\mu} - \int_{\Sigma_t} J^\mu d\Sigma_{t\,\mu}$$

$$\boxed{Q = \int_{\Sigma_{t0}} J^\mu d\Sigma_{t0\,\mu} = \int_{\Sigma_t} J^\mu d\Sigma_{t\,\mu}}$$

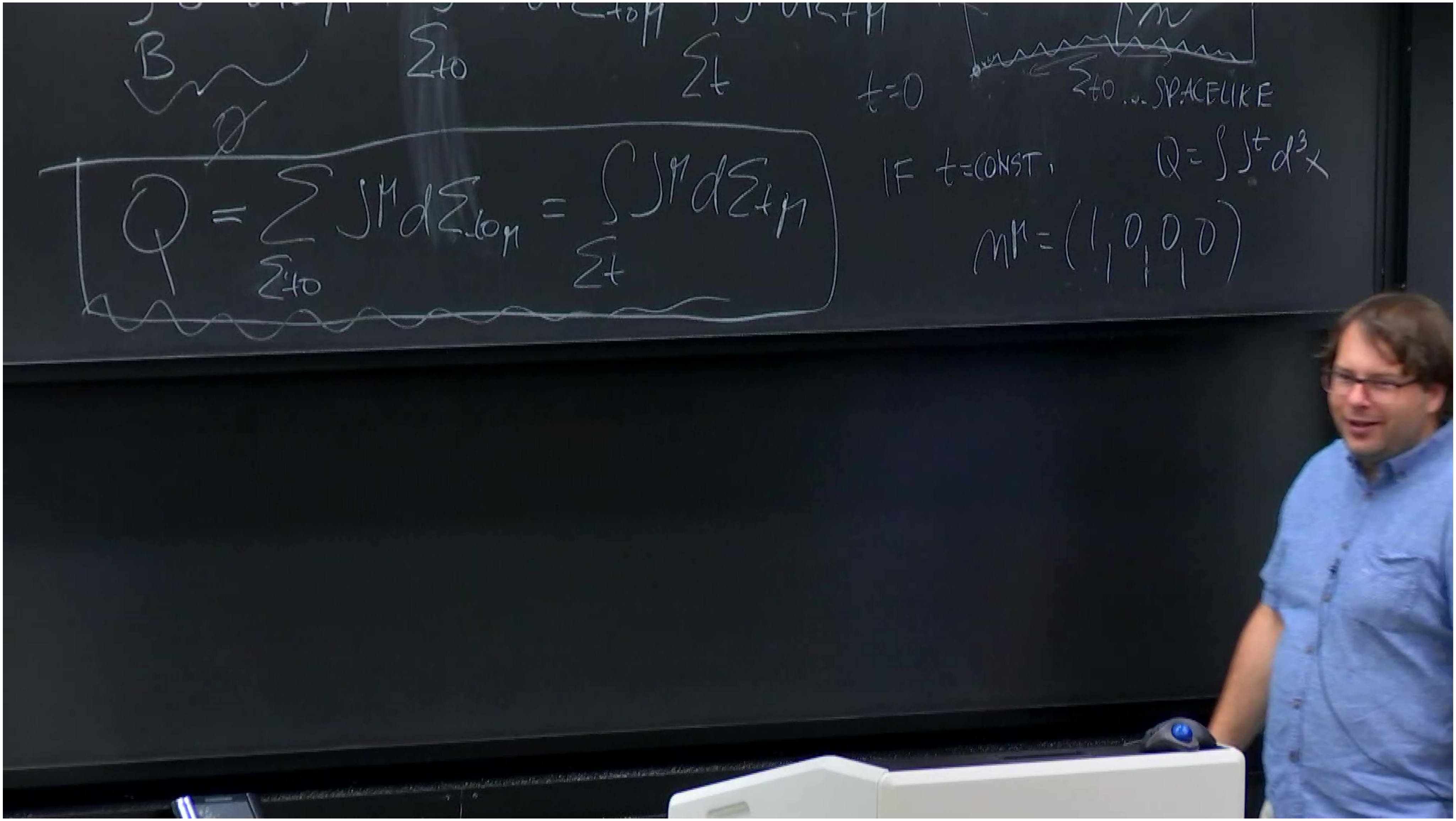


$$0 = \int_{\Omega} \partial_\mu J^\mu d^4x = \int_{\partial\Omega} J^\mu d\Sigma_\mu$$

$$= \underbrace{\int_{B} J^\mu dB_\mu}_{\emptyset} + \int_{\Sigma_{t0}} J^\mu d\Sigma_{t0\mu} - \int_{\Sigma_t} J^\mu d\Sigma_{t\mu}$$



$$\boxed{Q = \sum_{\Sigma_{t0}} J^\mu d\Sigma_{t0\mu} = \int_{\Sigma_t} J^\mu d\Sigma_{t\mu}}$$


t=0
Σt0 ... SPACELIKE
IF t=CONST.
Q = ∫ J^t d^3x
n^μ = (1,0,0,0)
Q = ∫ J^μ dΣ_{t0 μ} = ∫ J^μ dΣ_{t μ}

K: FOR PARTICLE

$$= \underbrace{\frac{\partial L}{\partial \dot q}}_{\text{"}p\text{"}}\tilde\delta q + \underbrace{\left(L - \dot q \frac{\partial L}{\partial \dot q}\right)}_{\text{"}E\text{"}}\tilde\delta t$$

RGE CONSERVATION

ARIANT VERSION" OF $Q = \int \rho \, d^3V$

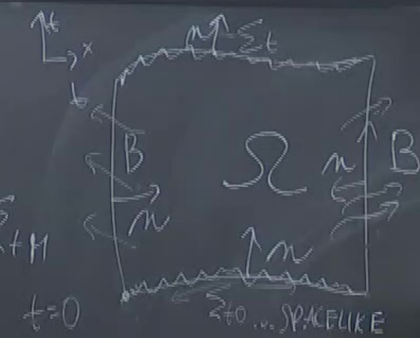
$$J^\mu = (\rho, \vec{j})$$

$$\frac{dQ}{dt} = 0$$ ... NO FLUX THROUGH ∂V

$$0 = \int_\Omega \partial_\mu J^\mu d^4x = \int_{\partial\Omega} J^\mu d\Sigma_\mu$$

$$= \underbrace{\int_B J^\mu dB_\mu}_{0} + \int_{\Sigma_{t0}} J^\mu d\Sigma_{t_0\mu} - \int_{\Sigma_t} J^\mu d\Sigma_{t\mu}$$

$$\boxed{Q = \int_{\Sigma_{t0}} J^\mu d\Sigma_{t_0\mu} = \int_{\Sigma_t} J^\mu d\Sigma_{t\mu}}$$



IF t=CONST, $Q = \int J^t d^3x$

$$n^\mu = (1,0,0,0)$$

- **REMARK**: CURRENT IS NOT UNIQUE.

$$J^\mu \to J^\mu + \partial_\nu \Psi^{\nu\mu} = \tilde{J}^\mu$$ ALSO CONSERVED.

$$\partial_\mu J^\mu = 0 \to \underbrace{\partial_\mu J^\mu}_{0} + \underbrace{\partial_\mu \partial_\nu \Psi^{\nu\mu}}_{0}$$

- REMARK: CURRENT IS NOT UNIQUE.

$$J^\mu \to J^\mu + \partial_\nu \psi^{\nu\mu} = \tilde{J}^\mu \quad \text{ALSO CONSERVED,}$$

$$\partial_\mu J^\mu = 0 \to \underbrace{\partial_\mu J^\mu}_{0} + \underbrace{\partial_\mu \partial_\nu \psi^{\nu\mu}}_{0}$$

RK : CURRENT IS NOT UNIQUE

$$J^\mu \to J^\mu + \partial_\nu \Psi^{\nu\mu} = \tilde{J}^\mu$$

ALSO CONSERVED

SUPERPOTENTIAL

$$\partial_\mu J^\mu = 0 \to \underbrace{\partial_\mu J^\mu}_{0} + \underbrace{\partial_\mu \partial_\nu \Psi^{\nu\mu}}_{0}$$

- REMARK: CURRENT IS NOT UNIQUE.

$$J^\mu \to J^\mu + \partial_\nu \psi^{\nu\mu} = \tilde{J}^\mu$$

ALSO CONSERVED,

$$\partial_\mu J^\mu = 0 \to \underbrace{\partial_\mu J^\mu}_{0} + \underbrace{\partial_\mu \partial_\nu \psi^{\nu\mu}}_{0}$$

SUPERPOTENTIAL

5) CONSERVED CURRENTS

NOETHER'S THEOREM (EXPLICIT)

$\varepsilon <$ CONST ... GLOBAL; $\varepsilon(x^\mu)$ ... LOCAL

LET $\tilde{\delta}$ BE A GLOBAL CONTINUOUS SYMMETRY (THAT IS OFF-SHELL WE FIND $\tilde{\delta}x^\mu, \tilde{\delta}\varphi$ SUCH THAT $\tilde{\delta}S = 0$) THEN

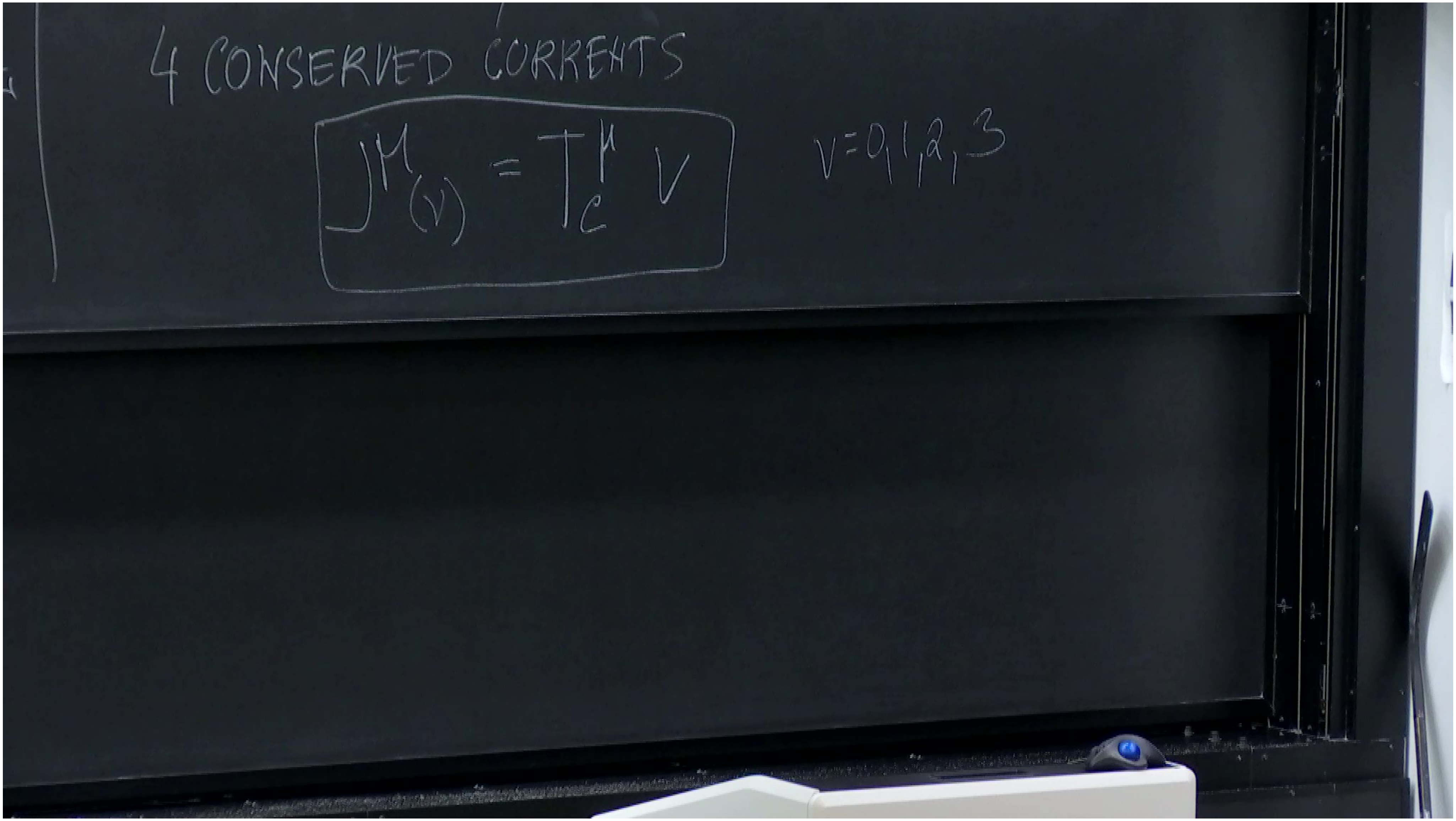
WHERE

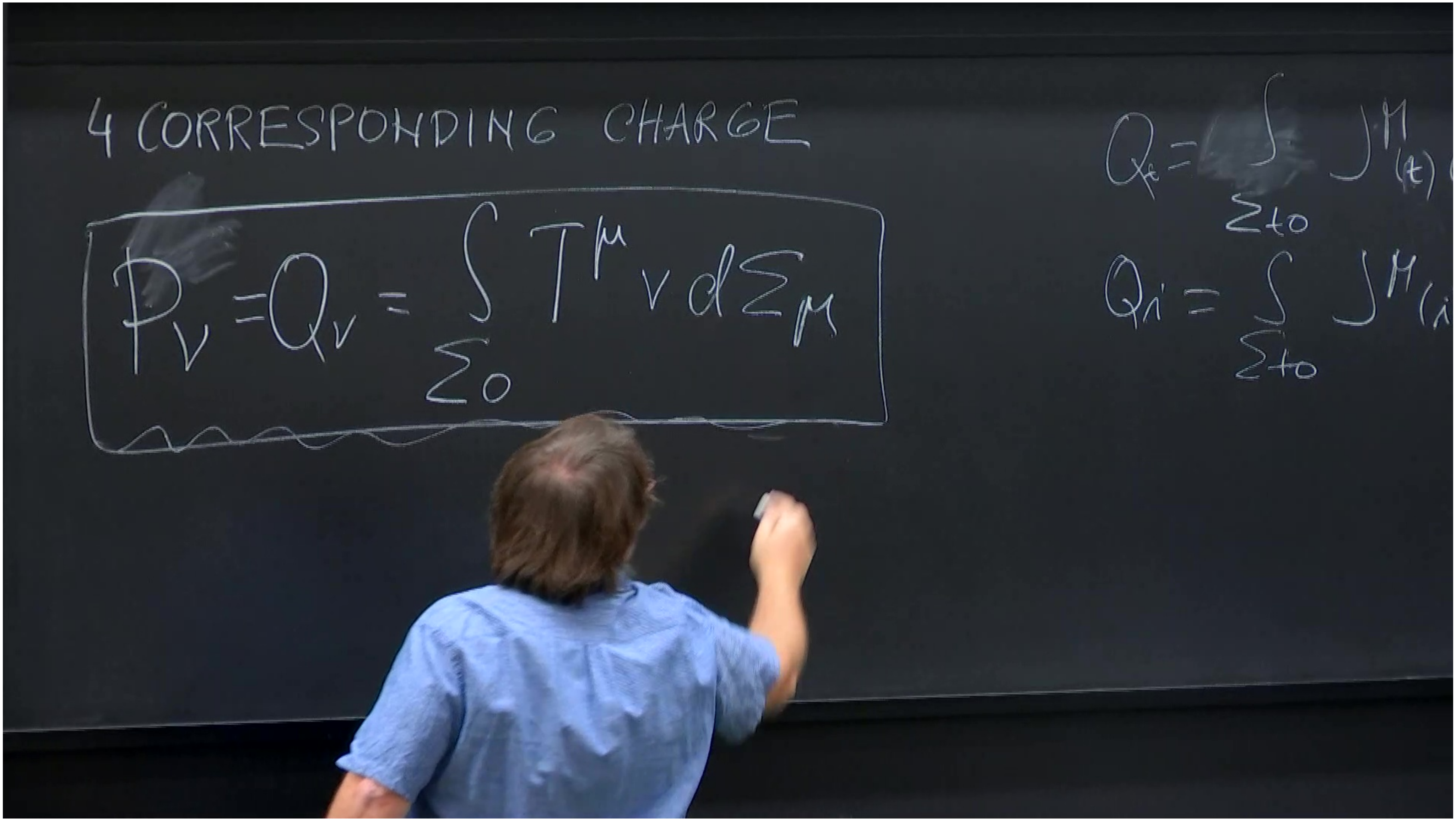
WE HAVE THE FOLLOWING CONSERVED CURRENT

c) CANONICAL ENERGY-MOMENTUM TENSOR

• CONSIDER SPACETIME TRANSLATIONS

$$\tilde{\delta} x^\mu = \epsilon^\mu \,, \quad \tilde{\delta} \varphi = 0$$


4 CONSERVED CURRENTS
$J^{\mu}_{(\nu)} = T^{\mu}_{c}\ \nu$
$\nu = 0, 1, 2, 3$


4 CORRESPONDING CHARGE
$P_\nu = Q_\nu = \int_{\Sigma_0} T^\mu{}_\nu \, d\Sigma_\mu$
$Q_t = \int_{\Sigma_{t_0}} J^\mu_{(t)}$
$Q_i = \int_{\Sigma_{t_0}} J^\mu_{(i)}$

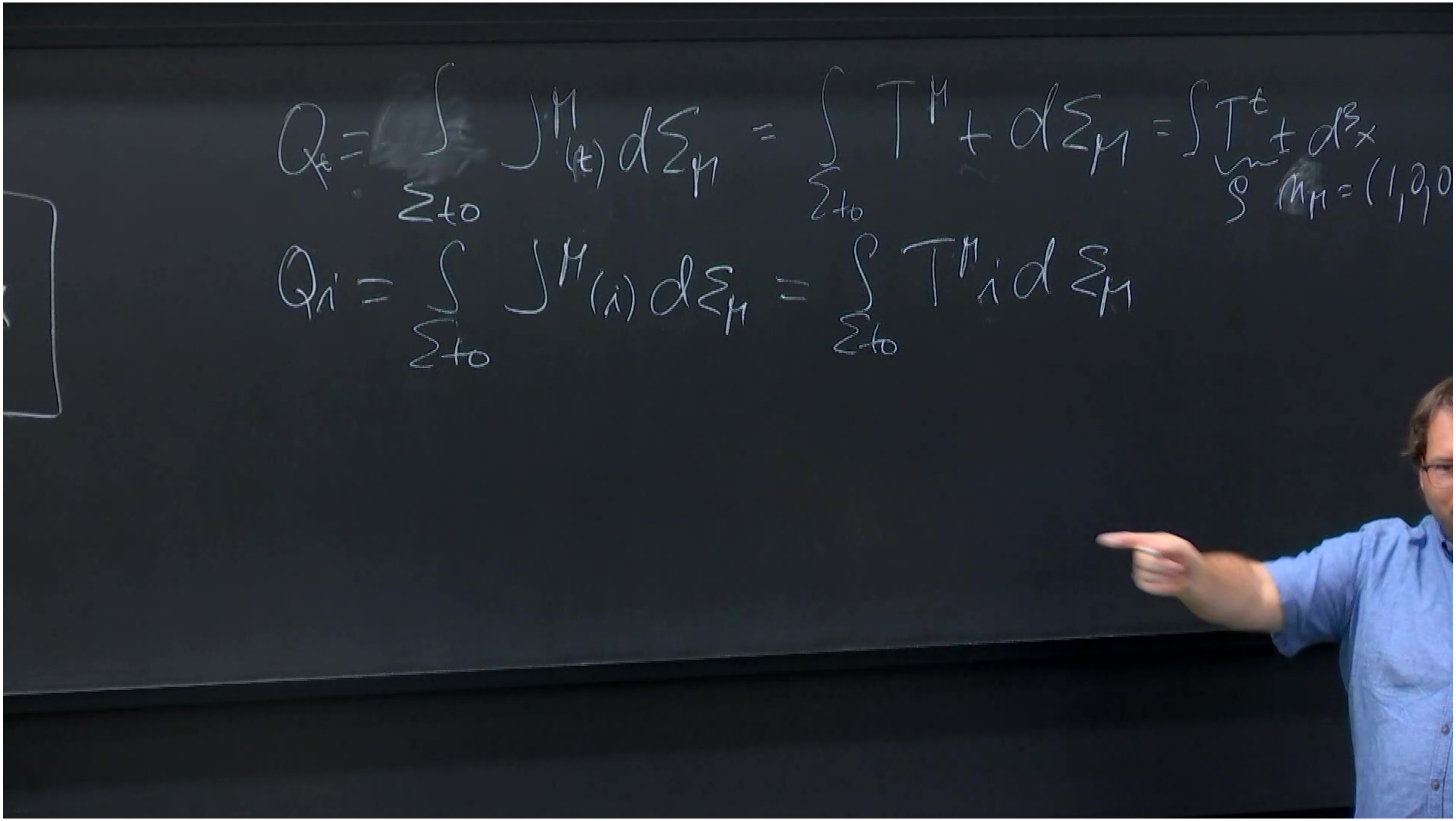
4 CORRESPONDING CHARGE

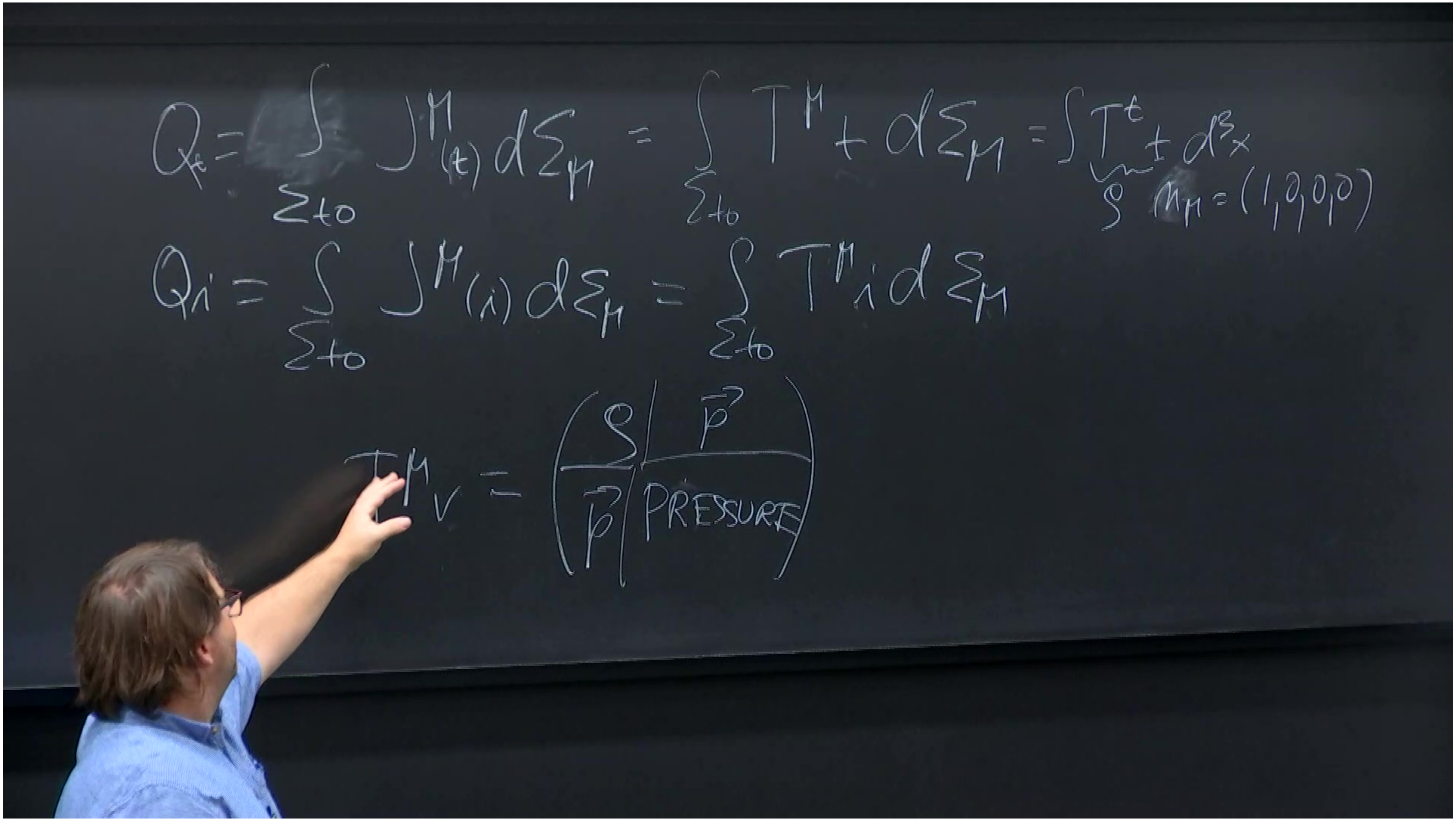
$$\boxed{P_\nu = Q_\nu = \int_{\Sigma_0} T^\mu{}_\nu \, d\Sigma_\mu}$$

4-MOMENTUM $(E, \vec{P})$

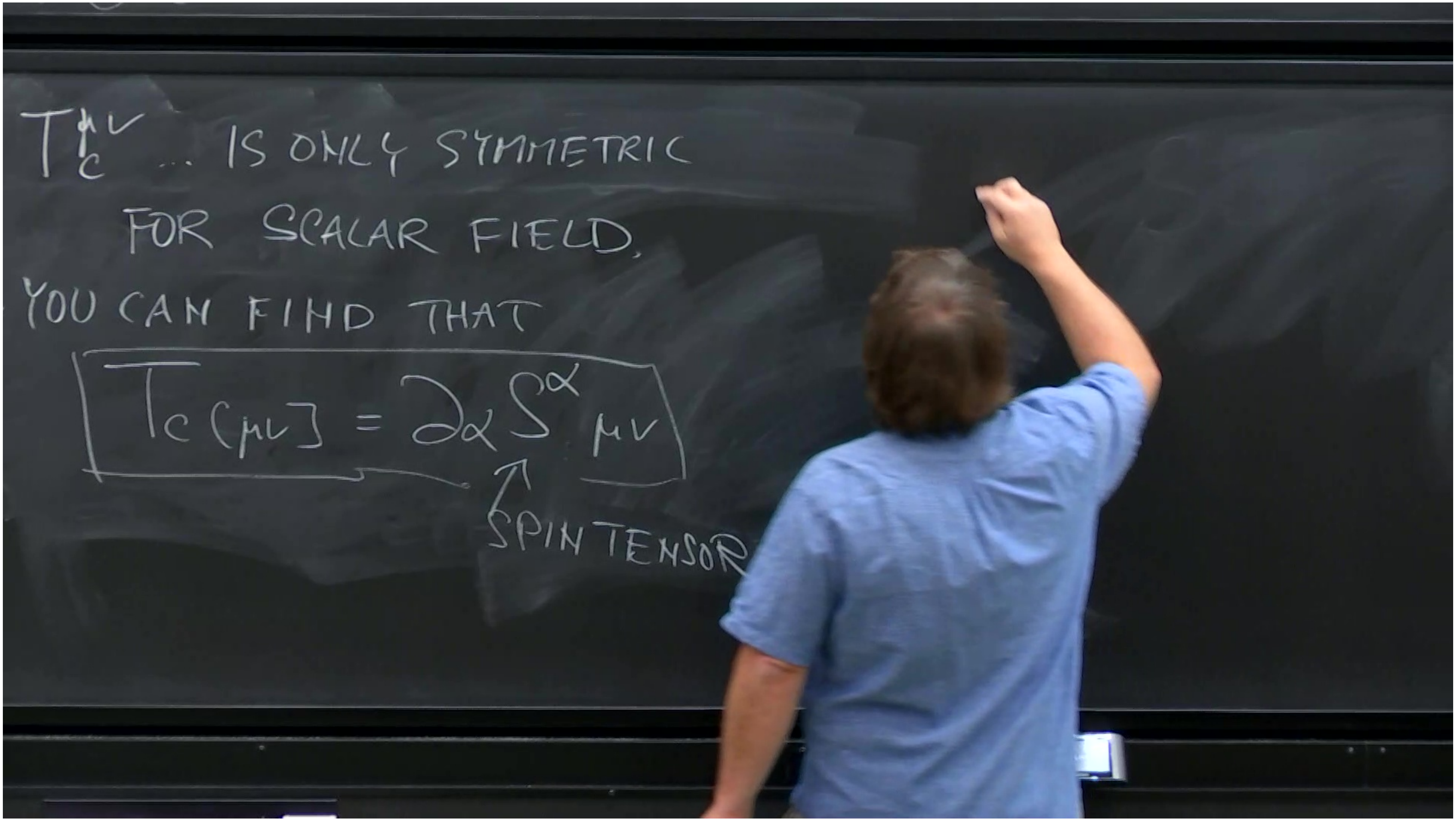
$$Q_t = \int_{\Sigma_{t_0}} J^\mu_{(t)}$$

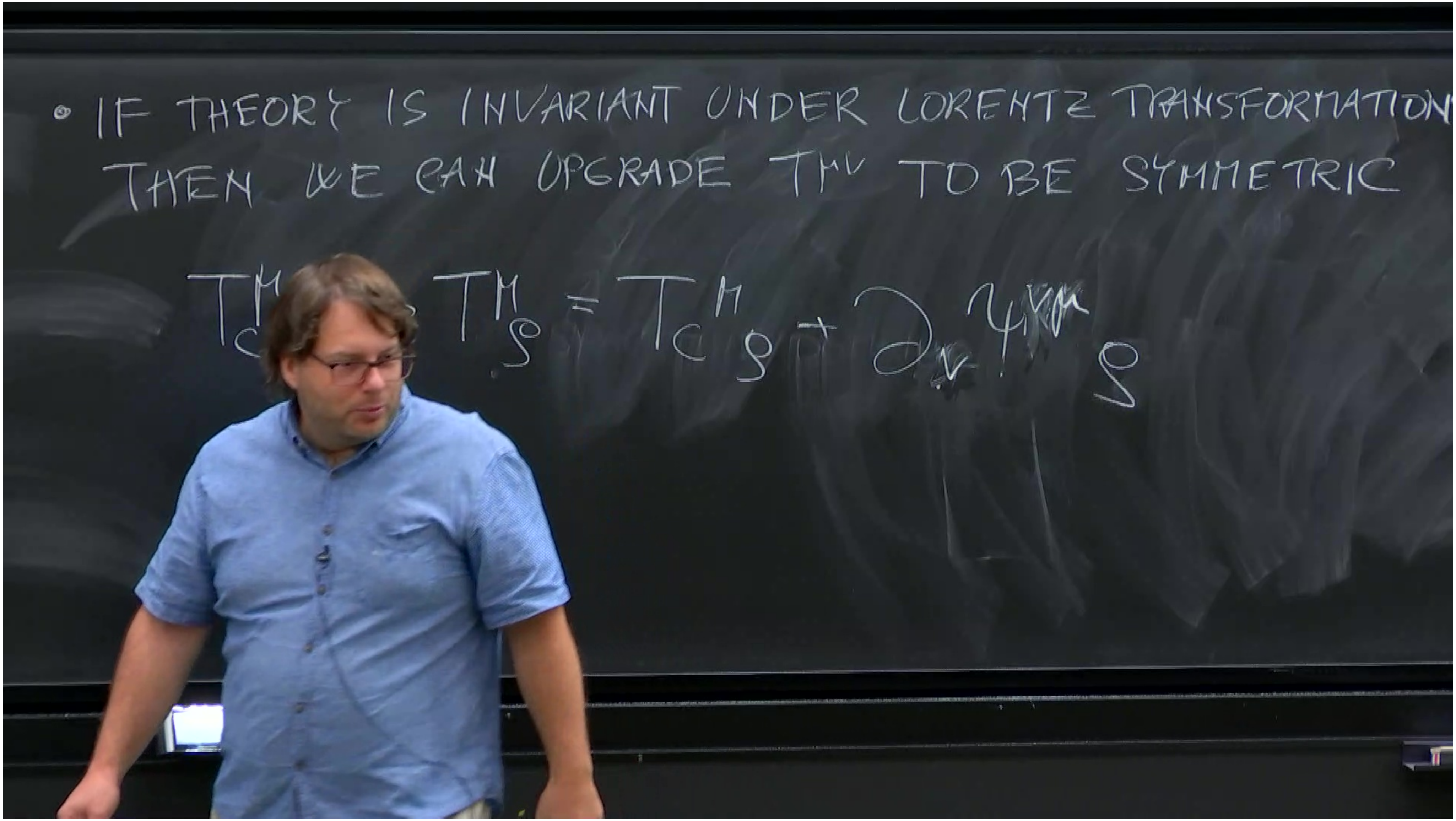
$$Q_i = \int_{\Sigma_{t_0}} J^\mu_{(i)}$$

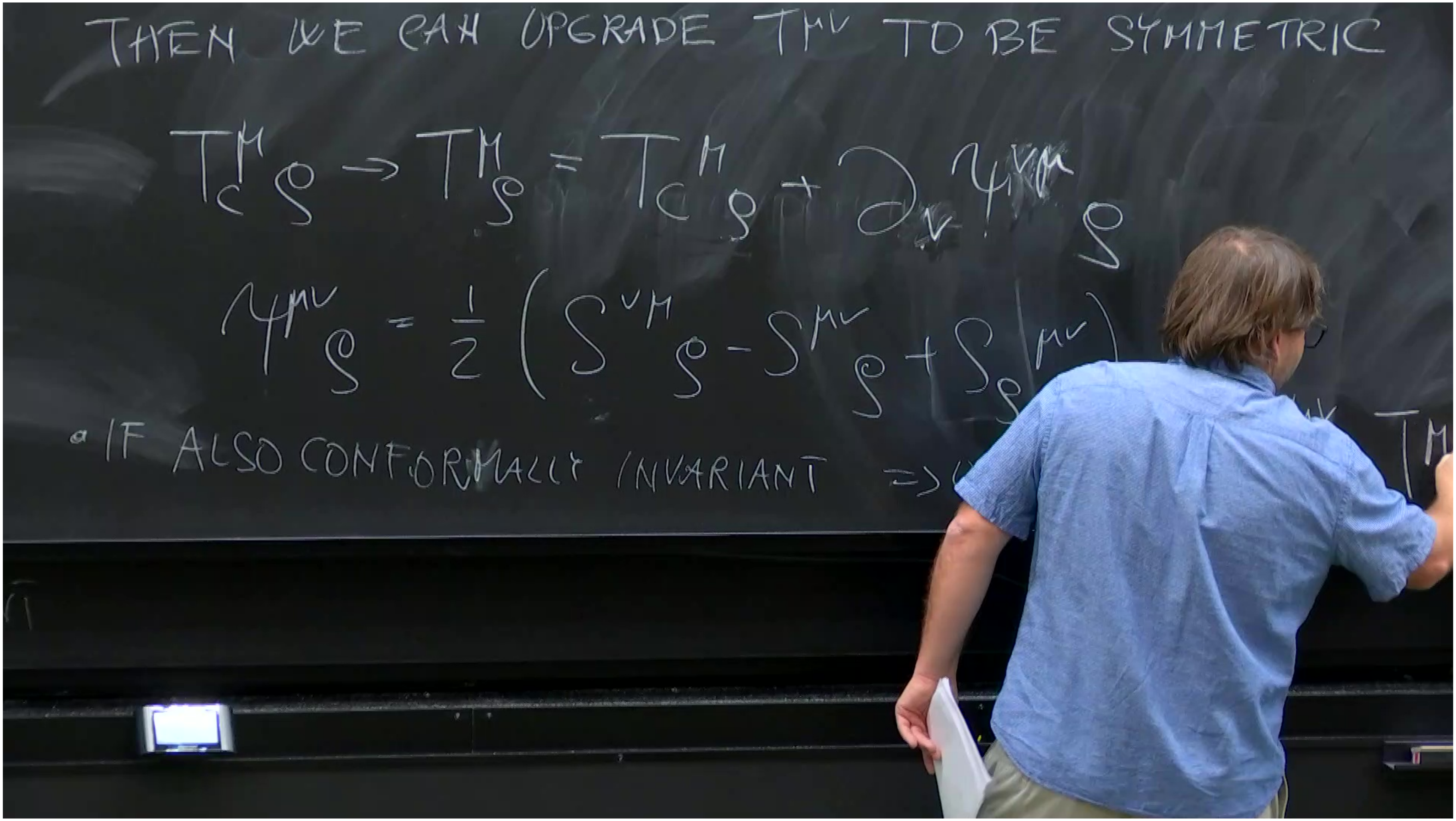

$Q_t = \int_{\Sigma_{t_0}} J^\mu{}_{(t)}\, d\Sigma_\mu = \int_{\Sigma_{t_0}} T^\mu{}_t\, d\Sigma_\mu = \int \underbrace{T^t{}_t}_{\rho}\, d^3x$
$n_\mu = (1,0,0$
$Q_i = \int_{\Sigma_{t_0}} J^\mu{}_{(i)}\, d\Sigma_\mu = \int_{\Sigma_{t_0}} T^\mu{}_i\, d\Sigma_\mu$

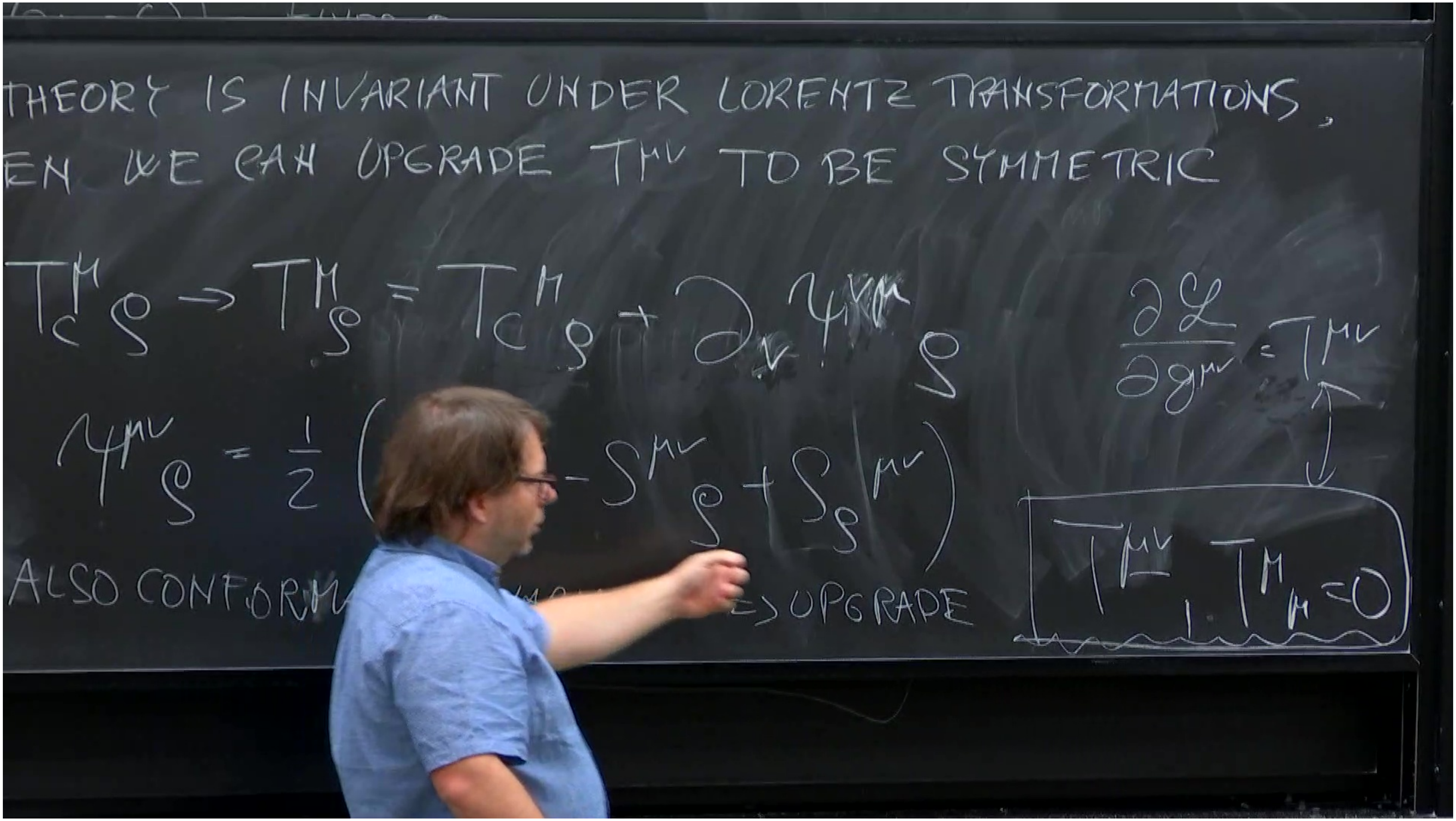


$Q_t = \int_{\Sigma_{t_0}} J^{\mu}{}_{(t)}\, d\Sigma_{\mu} = \int_{\Sigma_{t_0}} T^{\mu}{}_{t}\, d\Sigma_{\mu} = \int \underbrace{T^{t}{}_{t}}_{\rho}\, d^3x \quad n_{\mu} = (1,0,0,0)$
$Q_i = \int_{\Sigma_{t_0}} J^{\mu}{}_{(i)}\, d\Sigma_{\mu} = \int_{\Sigma_{t_0}} T^{\mu}{}_{i}\, d\Sigma_{\mu}$
$T^{\mu}{}_{\nu} = \left(\begin{array}{c|c} \rho & \vec{p} \\ \hline \vec{p} & \text{PRESSURE} \end{array}\right)$


$T_c^{\mu\nu}$ ... IS ONLY SYMMETRIC
FOR SCALAR FIELD.
YOU CAN FIND THAT
$\boxed{T_{c[\mu\nu]} = \partial_\alpha S^\alpha{}_{\mu\nu}}$
SPIN TENSOR


• IF THEORY IS INVARIANT UNDER LORENTZ TRANSFORMATION
THEN WE CAN UPGRADE $T^{\mu\nu}$ TO BE SYMMETRIC
$T^{\mu}{}_{\rho} = T_C{}^{\mu}{}_{\rho} + \partial_\nu Y^{\nu\mu}{}_{\rho}$


THEN WE CAN UPGRADE $T^{\mu\nu}$ TO BE SYMMETRIC
$T_C{}^{\mu}{}_{\rho} \to T^{\mu}{}_{\rho} = T_C{}^{\mu}{}_{\rho} + \partial_\nu Y^{\nu\mu}{}_{\rho}$
$Y^{\mu\nu}{}_{\rho} = \frac{1}{2}\left(S^{\nu\mu}{}_{\rho} - S^{\mu\nu}{}_{\rho} + S_{\rho}{}^{\mu\nu}\right)$
• IF ALSO CONFORMALLY INVARIANT $\Rightarrow$


THEORY IS INVARIANT UNDER LORENTZ TRANSFORMATIONS
EN WE CAN UPGRADE T^μν TO BE SYMMETRIC
ALSO CONFORM
UPGRADE