Title: Quantum Field Theory I - Lecture 221014

Speakers: Gang Xu

Collection: Quantum Field Theory I (2022/2023)

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URL: https://pirsa.org/22100050

Goal: find all reps

see symmetry → group

$SO(N)$, $SU(N)$

Goal: find all reps

See symmetry → group

$SO(N)$, $SU(N)$

→ Rep

Goal → $((D(g))_{ij} = \delta_{ij} + i\epsilon_a (T_a)_{ij} + \ldots$

$1 \ldots n$

finite

$\xrightarrow[N\to\infty]{} \equiv \infty a l$

$A \equiv i\varphi_a T_a$

$\lim_{N\to\infty}\left(1+\frac{A}{N}\right)^N \equiv e^A$

$= 1 + A + \frac{1}{2}A^2 + \ldots$

① first order

$D_{(g)} \to$ Unitary $\quad T_{\text{general}} \to$ Hermitian

② second order (quest for $\tilde{f}(\tilde{\varphi}^{q}, \tilde{\rho}^{o})$)

$$[T_a, T_b] = i f_{abc} T_c$$

① first order

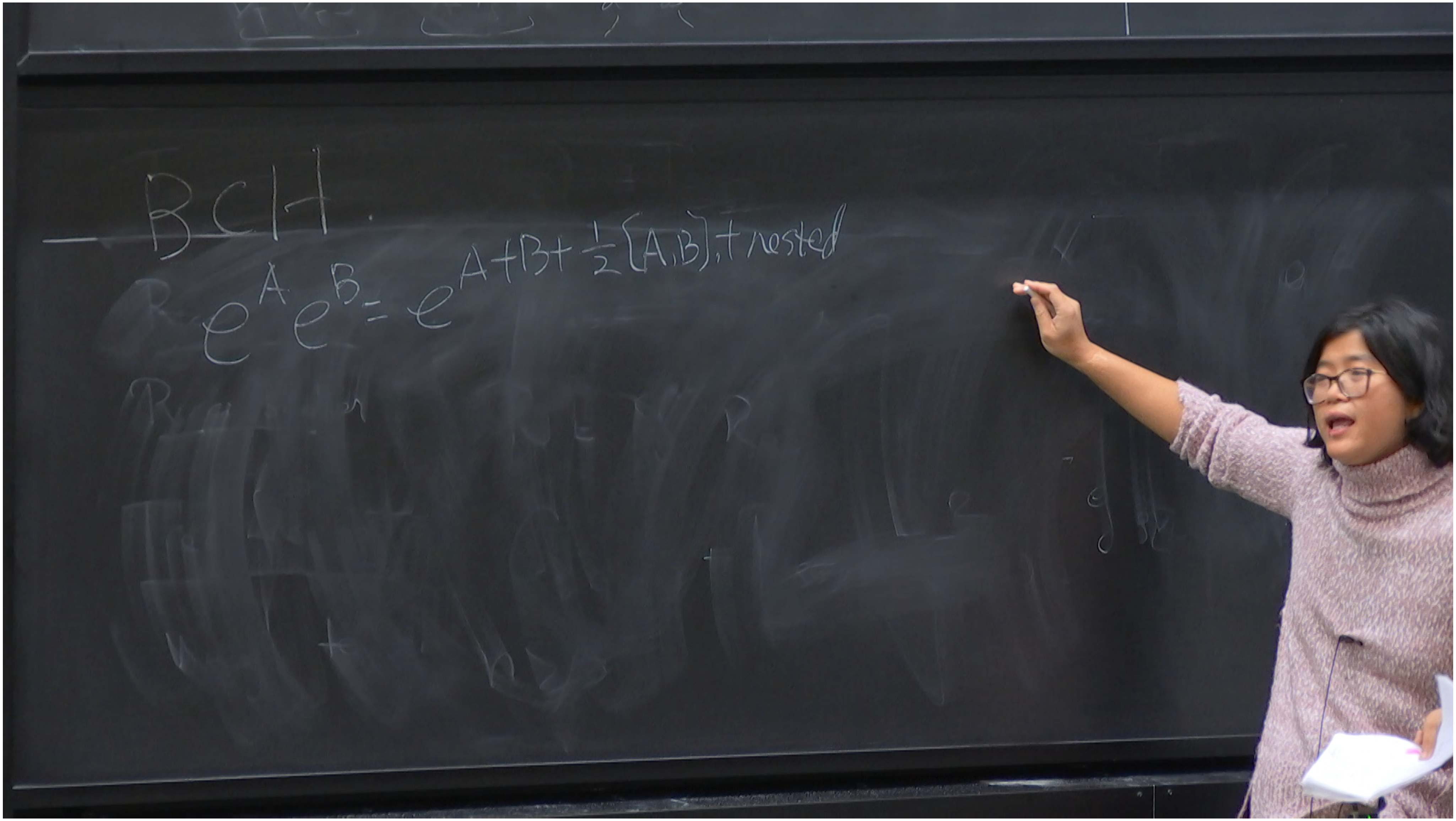
$D_{(g)} \to$ Unitary $\quad T_{\text{general}} \to$ Hermitian

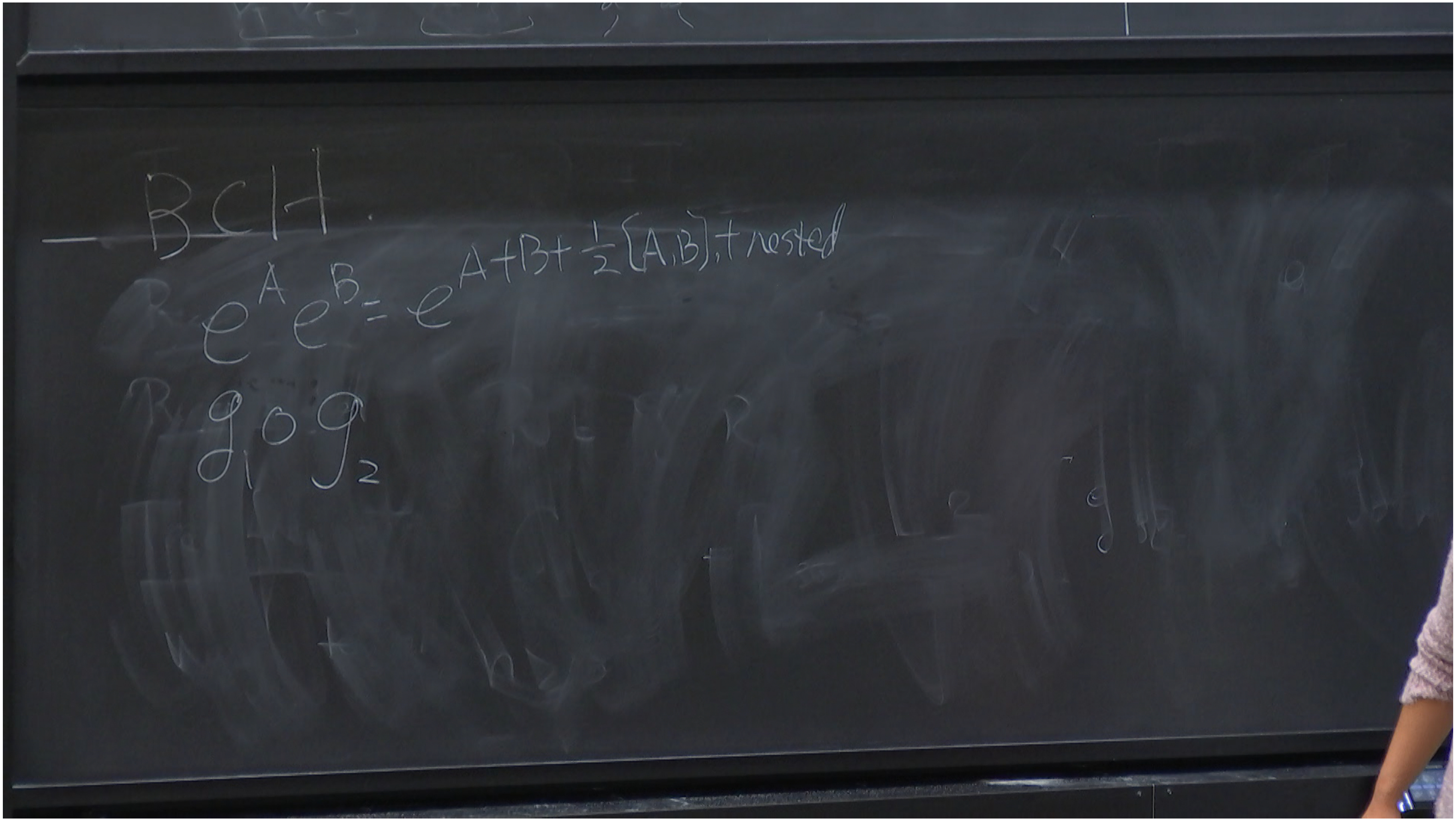
② second order (quest for $\tilde{f}(\tilde{\varphi}^a, \tilde{\rho}^a)$)

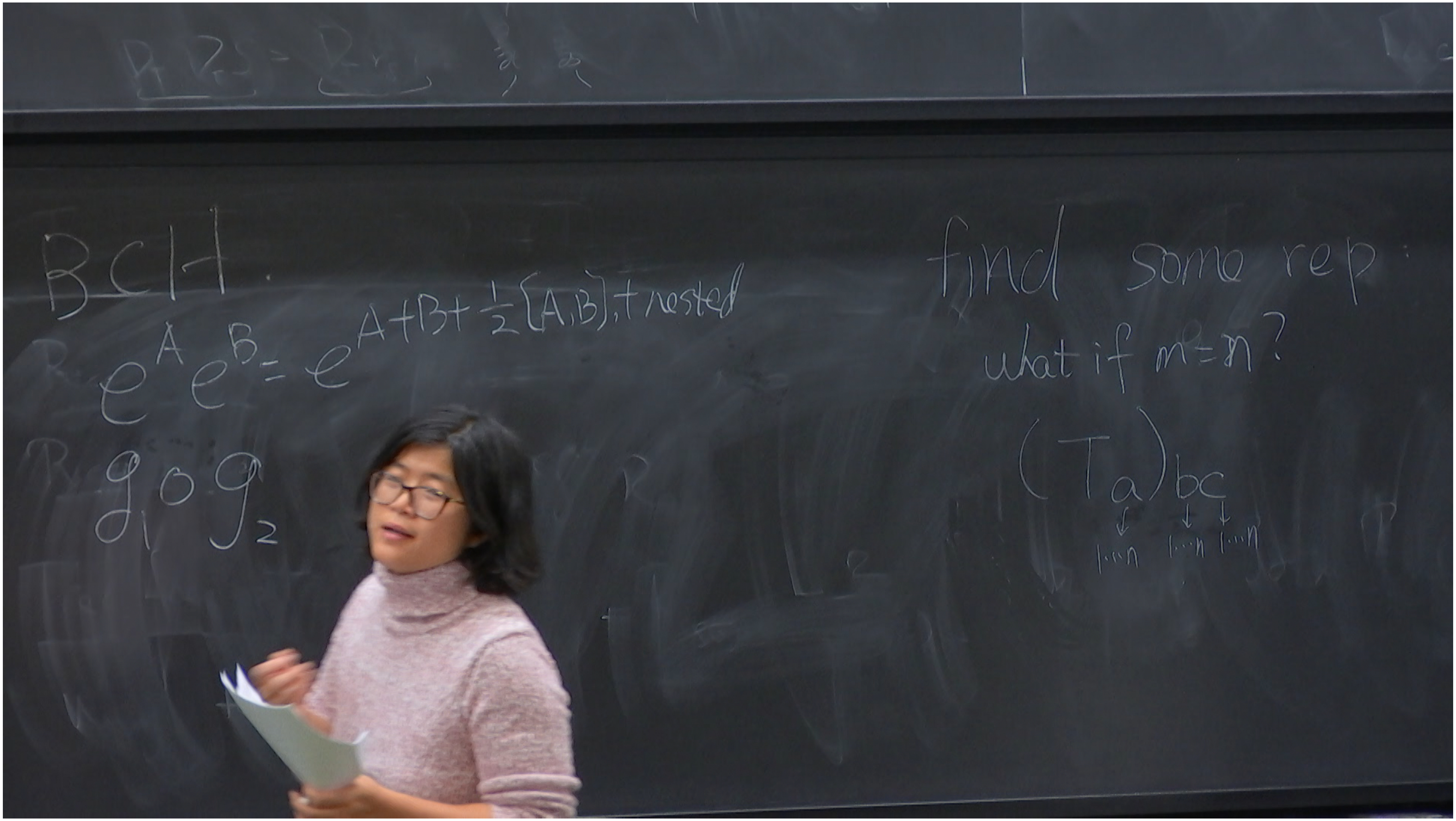
$$[T_a, T_b] = i f_{abc} T_c$$

close

③


BCH.
$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\text{nested}}$


BCH.
$e^{A}e^{B}=e^{A+B+\frac{1}{2}[A,B]+\text{nested}}$
$g_1 \circ g_2$


BCH
e^A e^B = e^{A+B+1/2[A,B]+nested}
g_1 o g_2
find some rep:
what if m = n?
(T_a)_bc
1…n 1…n 1…n

find some rep:

what if $m = n$?

$$(T^{ad}_{a})_{bc} = -i f_{abc}$$

$a \to 1 \cdots n$, $b \to 1 \cdots n$, $c \to 1 \cdots n$

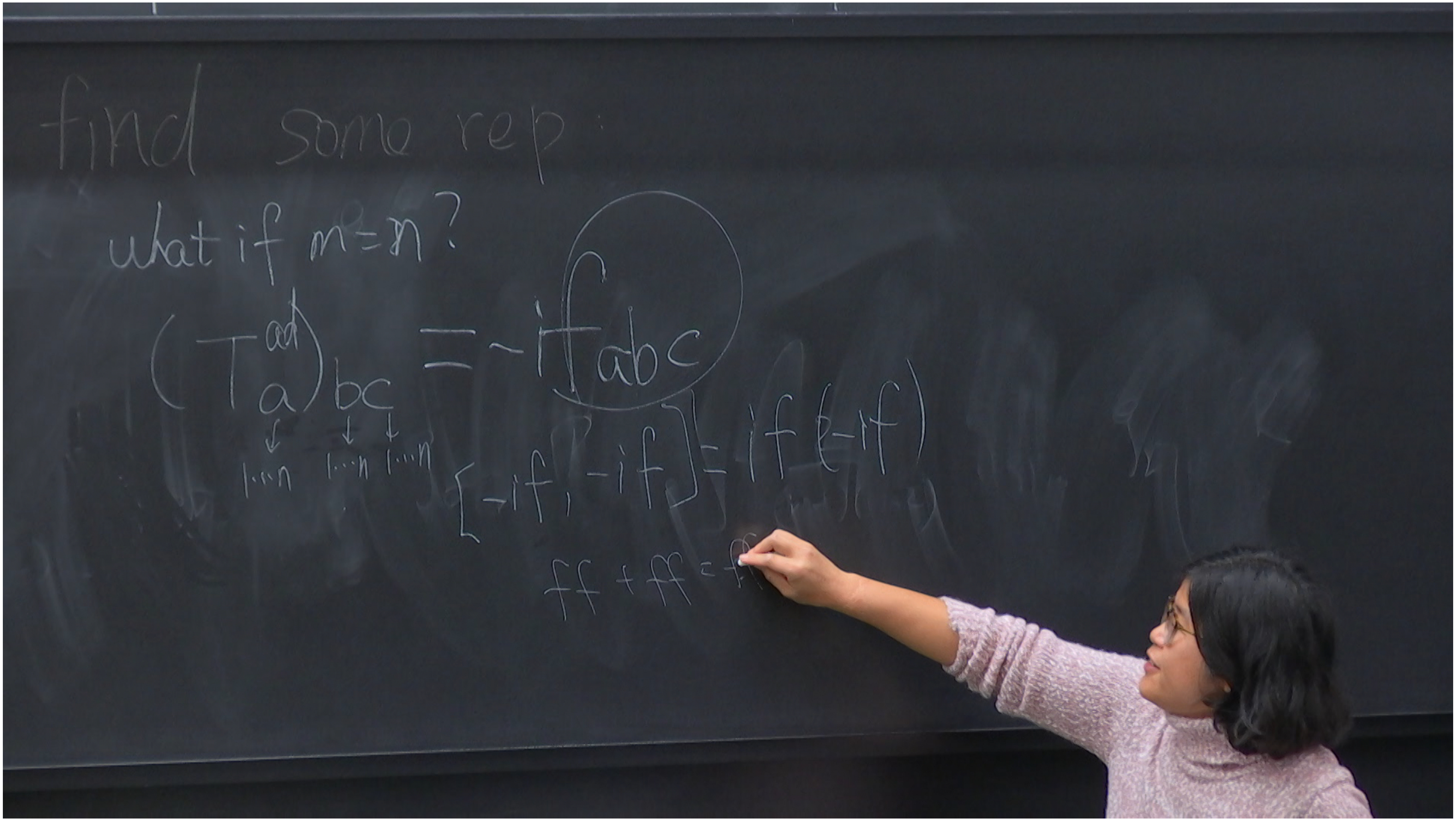
find some rep:

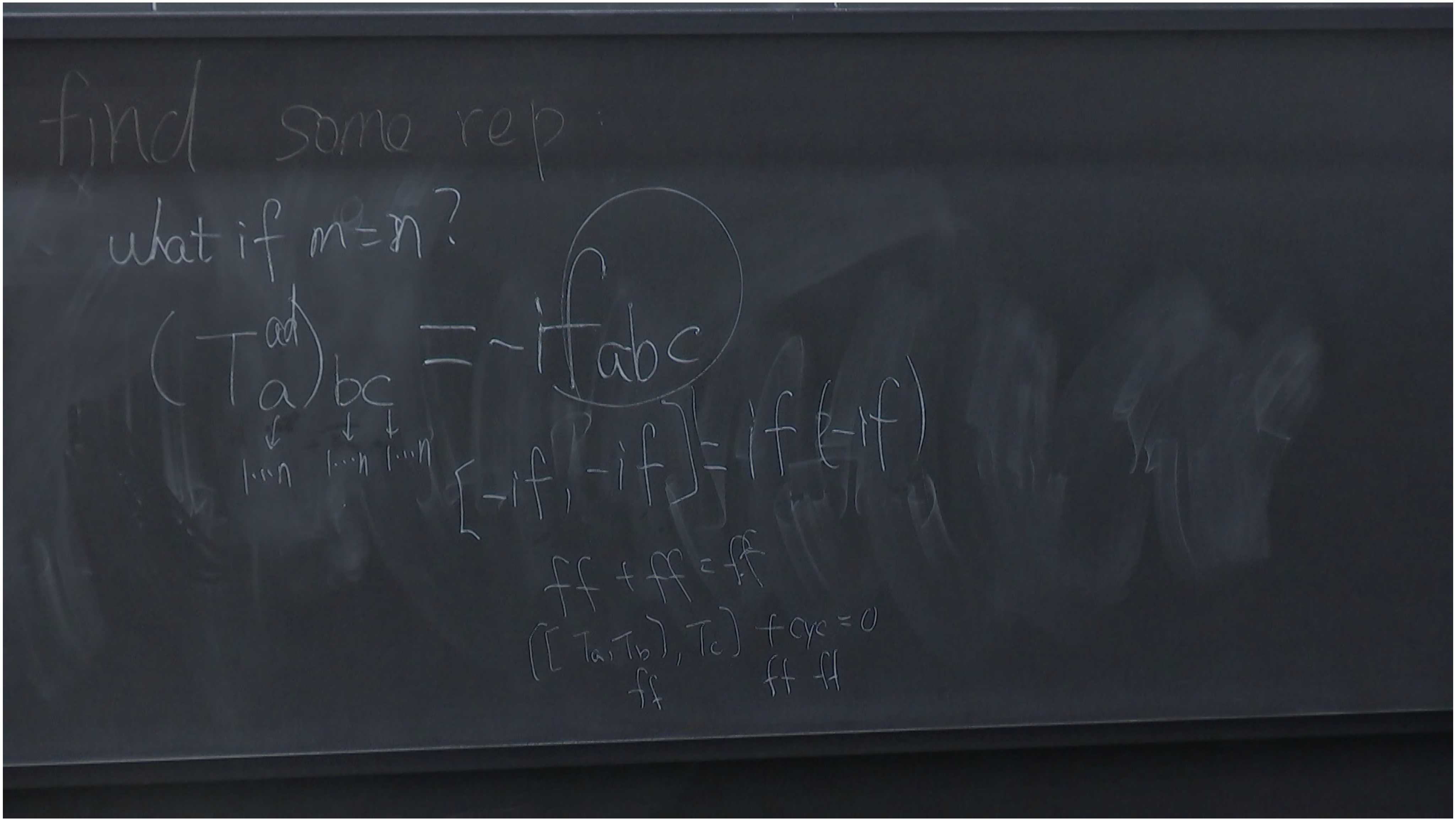
what if m=n?

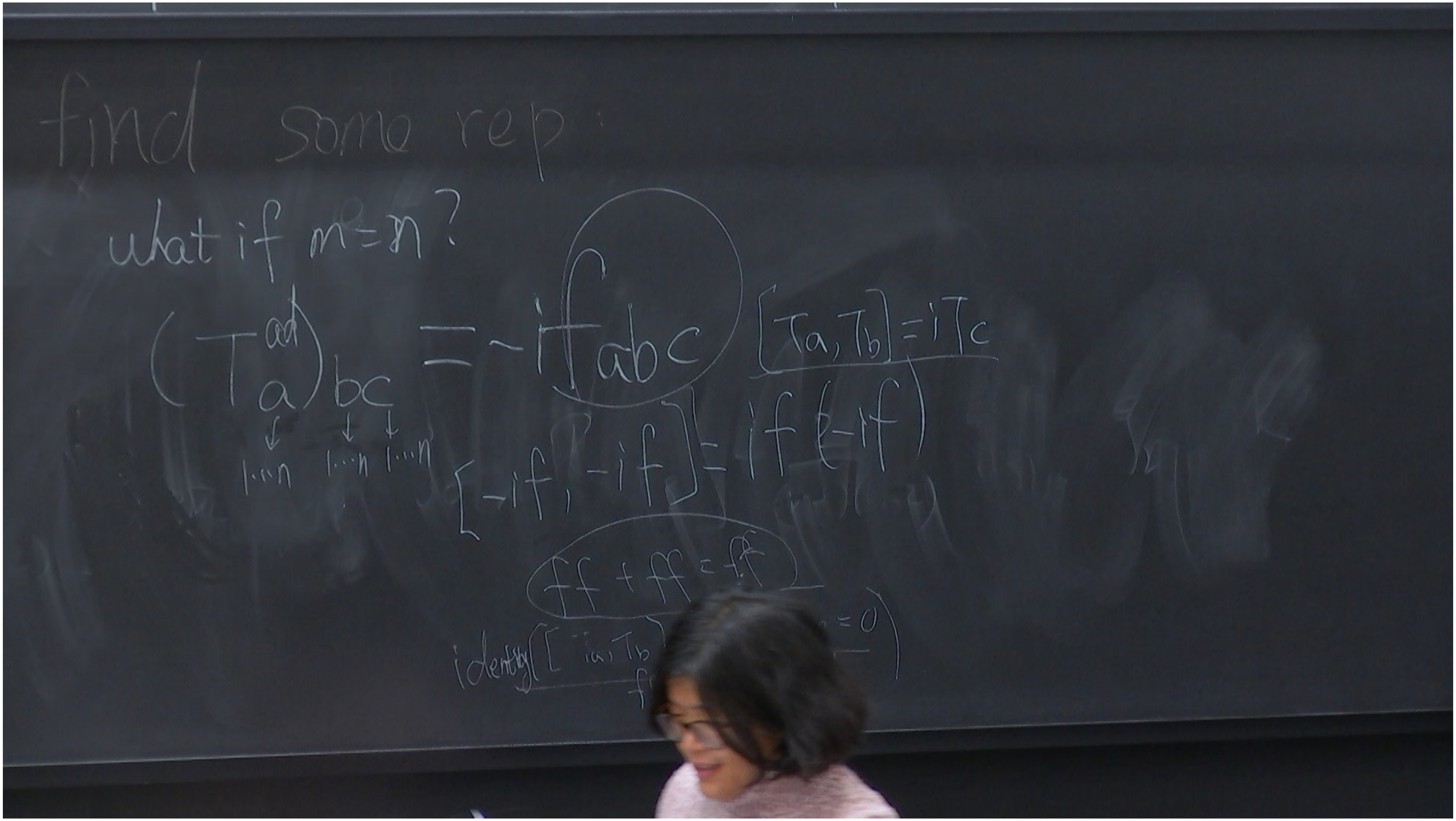
$$(T^{adj}_a)_{bc} = -i f_{abc}$$

$$a \to 1\cdots n \quad b \to 1\cdots n \quad c \to 1\cdots n$$

$$[-if, -if] = if(-if)$$


find some rep:
what if m=n?
$(T^{ad}_a)_{bc} = -i f_{abc}$
$1 \cdots n$ $1 \cdots n$ $1 \cdots n$
$[-if, -if] = if(-if)$
$ff + ff =$


find some rep:
what if m=n?
$(T^{ad}_a)_{bc} = -i f_{abc}$
1…n 1…n 1…n
$[-if, -if] = if(-if)$
ff + ff = ff
$([T_a, T_b], T_c)$ + cyc = 0
ff ff ff


find some rep:
what if m=n?
$(T^{ad}_a)_{bc} = -i f_{abc}$
$[T_a, T_b] = i T_c$
$[-if, -if] = if(-if)$
ff + ff = ff
identity

find some rep:

what if $m=n$?

$$(T^{ad}_a)_{bc} = -i f_{abc} \qquad [T_a, T_b] = i T_c$$

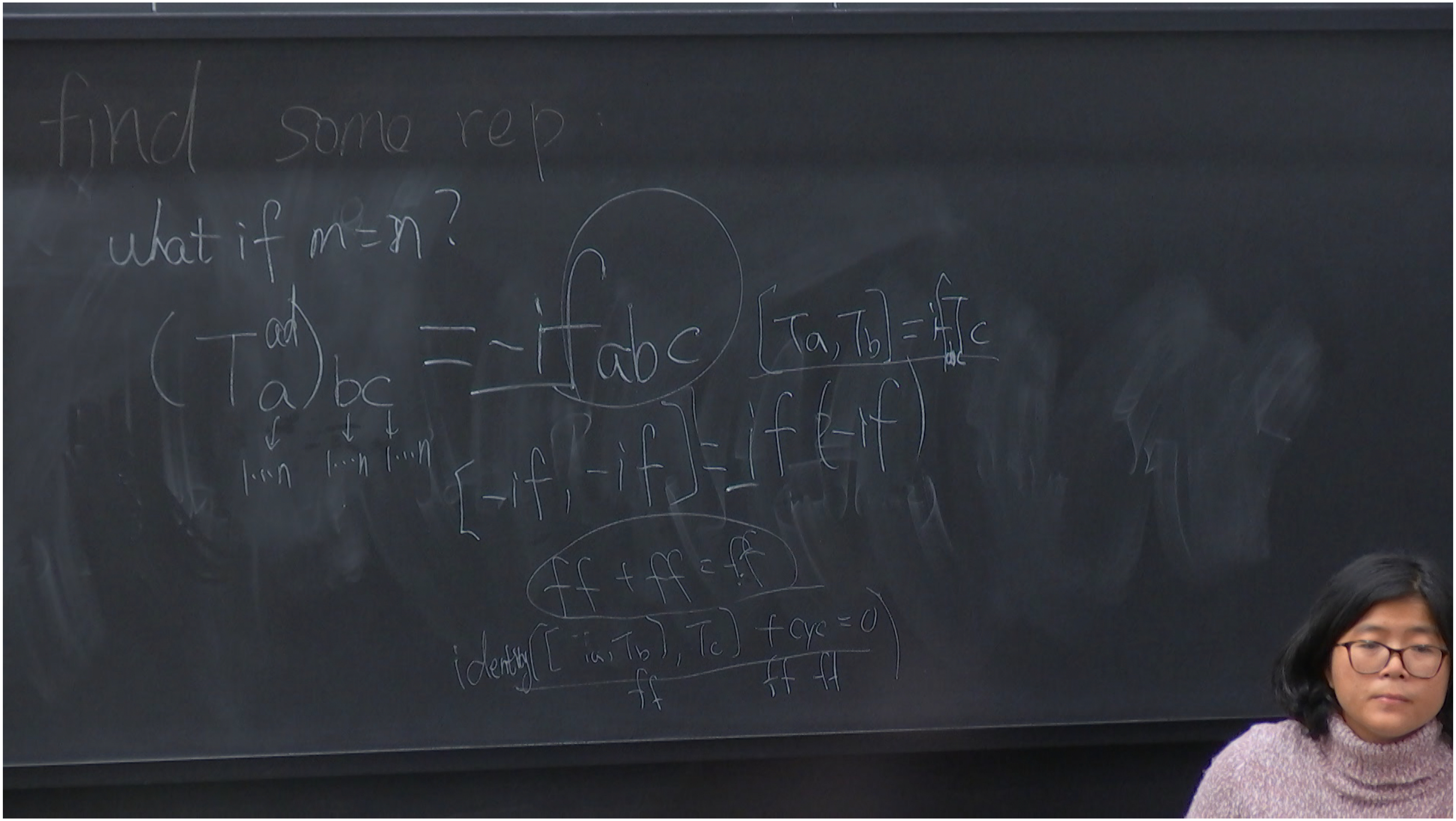
$a,\ b,\ c \to 1\cdots n$

$$[-if, -if] = if(-if)$$

$$ff + ff = ff$$

identity $([T_a, T_b], T_c) + \text{cyc} = 0$

$ff \quad ff \quad ff$


find some rep:
what if m=n?
(T^ad_a)_bc = -i f_abc
[T_a, T_b] = i f_abc T_c
1…n 1…n 1…n
[-if, -if] = if (-if)
ff + ff = ff
identity ([T_a, T_b], T_c] + cyc = 0

find some rep:

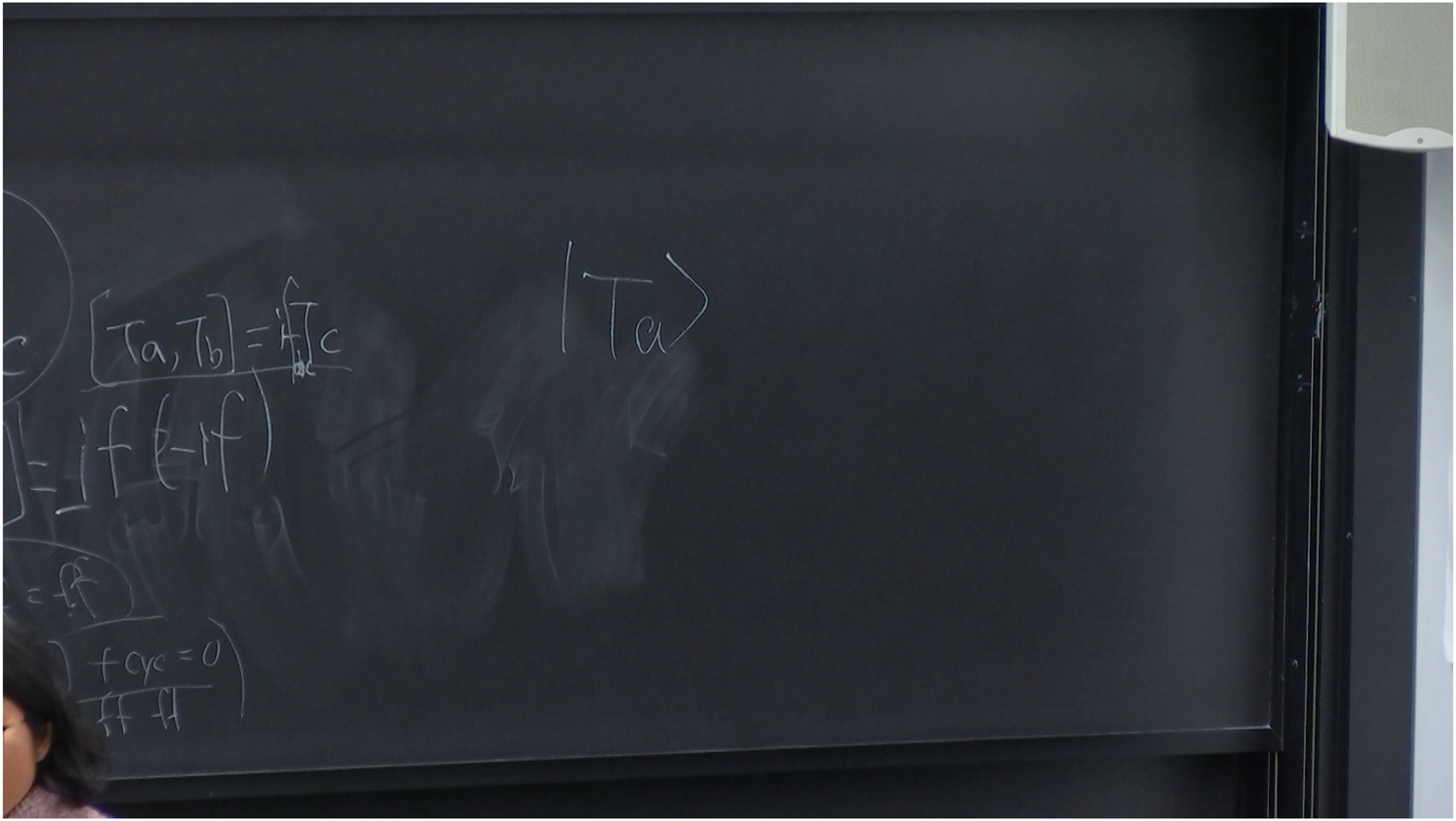
what if m=n?

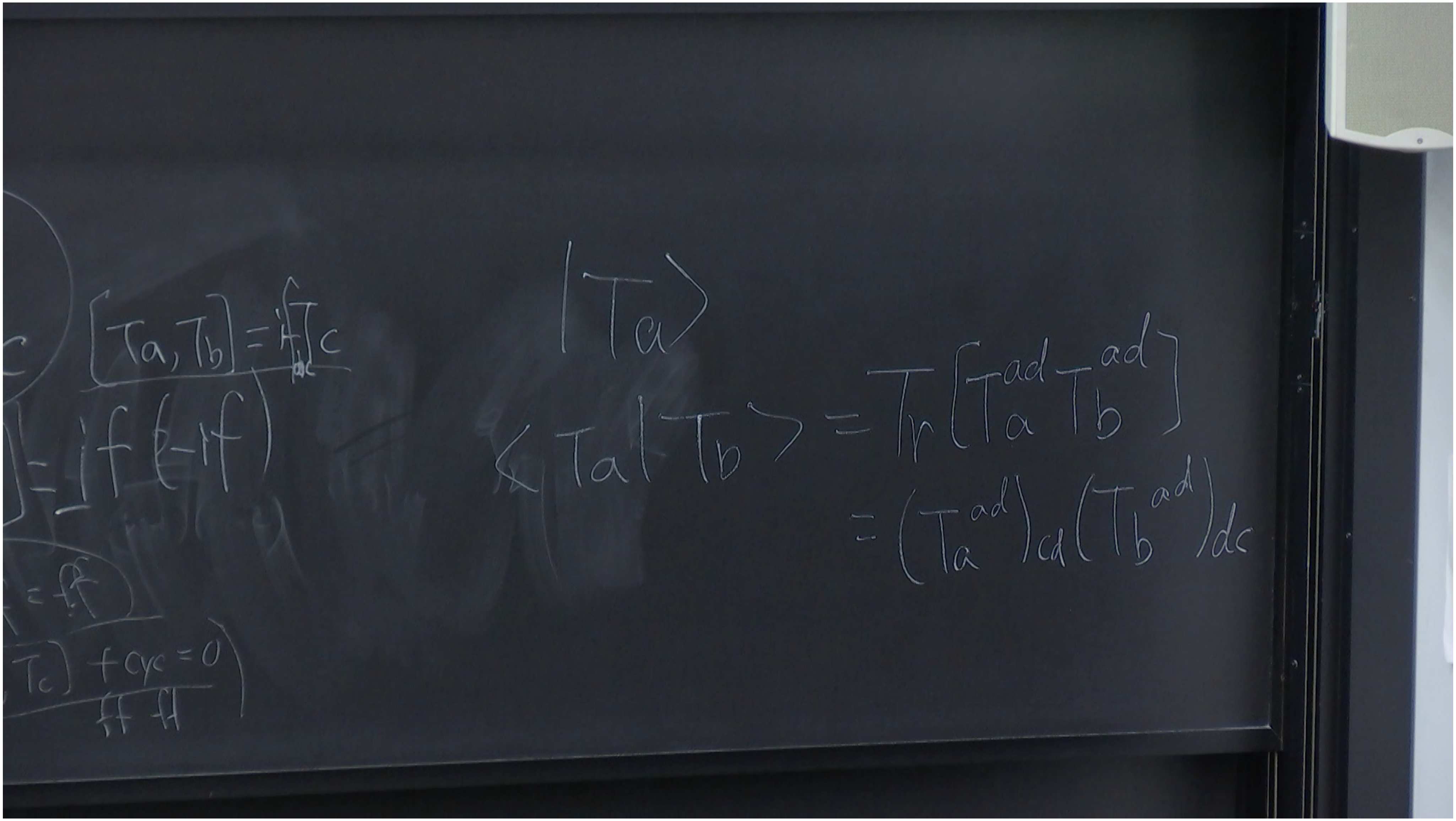
$$(T^{ad}_a)_{bc} = -i f_{abc} \qquad [T_a, T_b] = i f_{abc} T_c$$

$$[-i f_{abc}, -i f] = i f (-i f)$$

$$ff + ff = ff$$

identity $\underbrace{[[T_a, T_b], T_c]}_{ff} + \underbrace{cyc}_{ff \; ff} = 0$


[Ta, Tb]
|Ta>
f cyc = 0


$|T_a\rangle$
$\langle T_a | T_b \rangle = \mathrm{Tr}\left[T_a^{ad} T_b^{ad}\right]$
$= (T_a^{ad})_{cd} (T_b^{ad})_{dc}$
$[T_a, T_b] = i f_{abc} T_c$

$[T_a, T_b] = i f_{abc} T_c$

$= if(-if)$

$= ff$

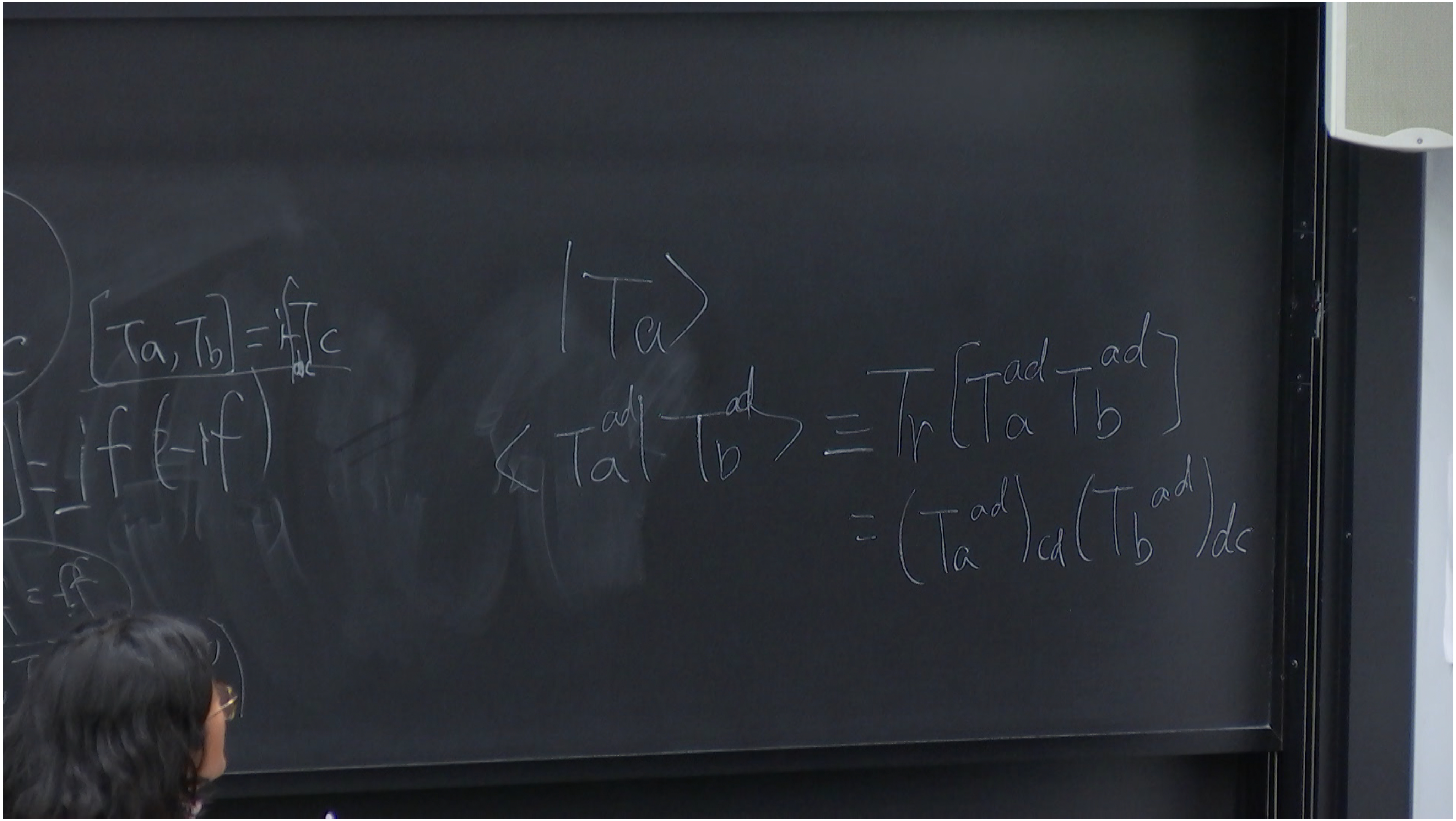
$[\ldots, T_c]\ f\ \text{cyc} = 0$

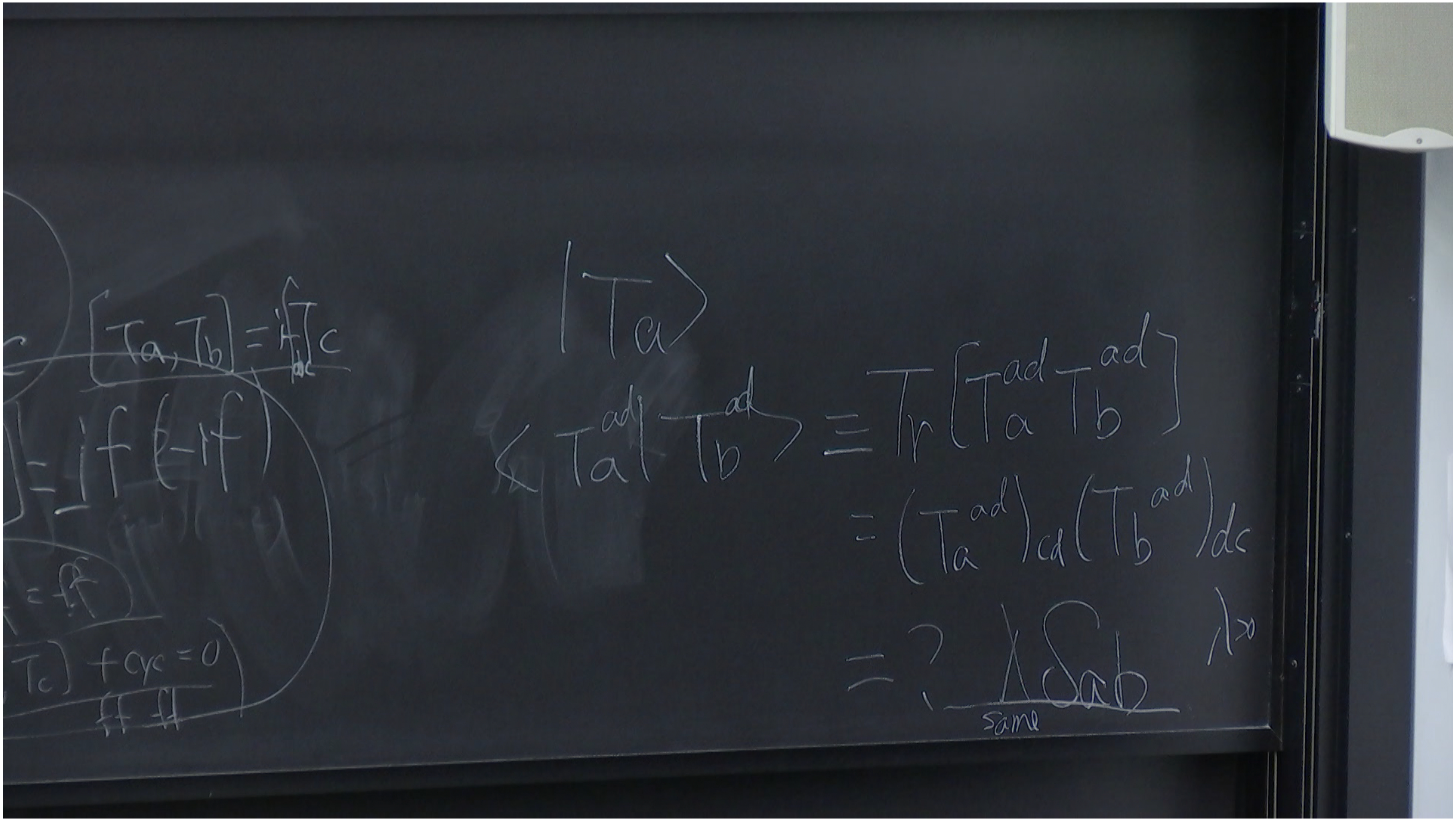
$ff\ ff$

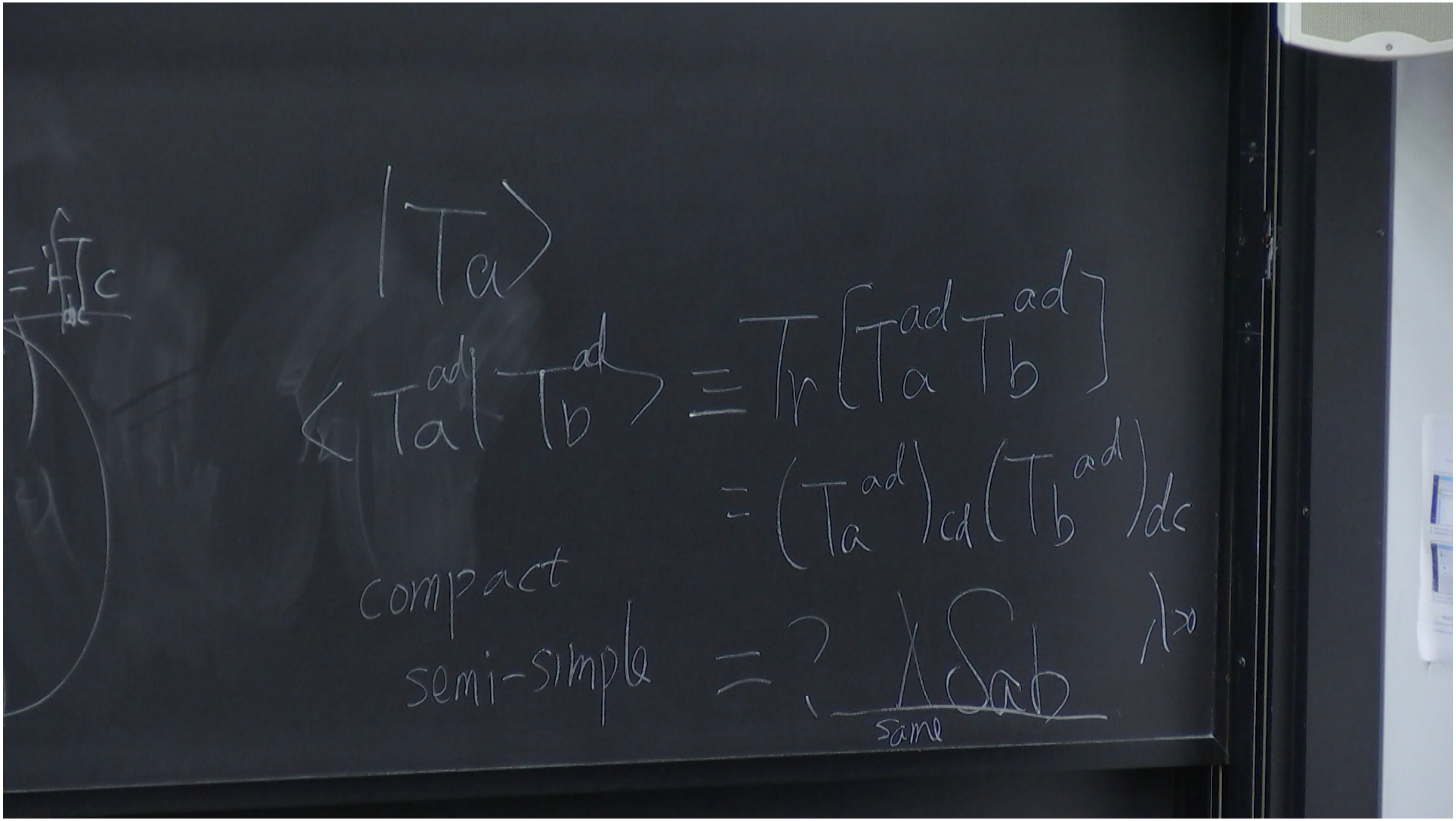
$|T_a\rangle$

$$\langle T_a | T_b \rangle \equiv \mathrm{Tr}\left[T_a^{ad} T_b^{ad}\right]$$

$$= \left(T_a^{ad}\right)_{cd} \left(T_b^{ad}\right)_{dc}$$


$|T_a\rangle$
$\langle T_a^{ad} | T_b^{ad} \rangle \equiv \mathrm{Tr}[T_a^{ad} T_b^{ad}]$
$= (T_a^{ad})_{cd} (T_b^{ad})_{dc}$


$[T_a, T_b] = i f_{abc} T_c$
$f_{cyc} = 0$
$|T_a\rangle$
$\langle T_a^{ad} | T_b^{ad}\rangle \equiv \mathrm{Tr}[T_a^{ad} T_b^{ad}]$
$= (T_a^{ad})_{cd} (T_b^{ad})_{dc}$
$=? \lambda \delta_{ab}$
$\lambda > 0$
same


$|T_a\rangle$
$\langle T_a^{ad} | T_b^{ad} \rangle \equiv \mathrm{Tr}[T_a^{ad} T_b^{ad}]$
$= (T_a^{ad})_{cd} (T_b^{ad})_{dc}$
compact
semi-simple
$\stackrel{?}{=} \lambda \delta_{ab}$
same
$\lambda > 0$

Math Interlude

direct product

$$(g_1, g_1') \cdot (g_2, g_2')$$

$$= (g_1 \circ g_2, g_1' \circ' g_2')$$

Math Interlude

direct product

$$G \times G' \quad (\overset{G}{g_1}, \overset{G'}{g_1'}) \cdot (g_2, g_2')$$

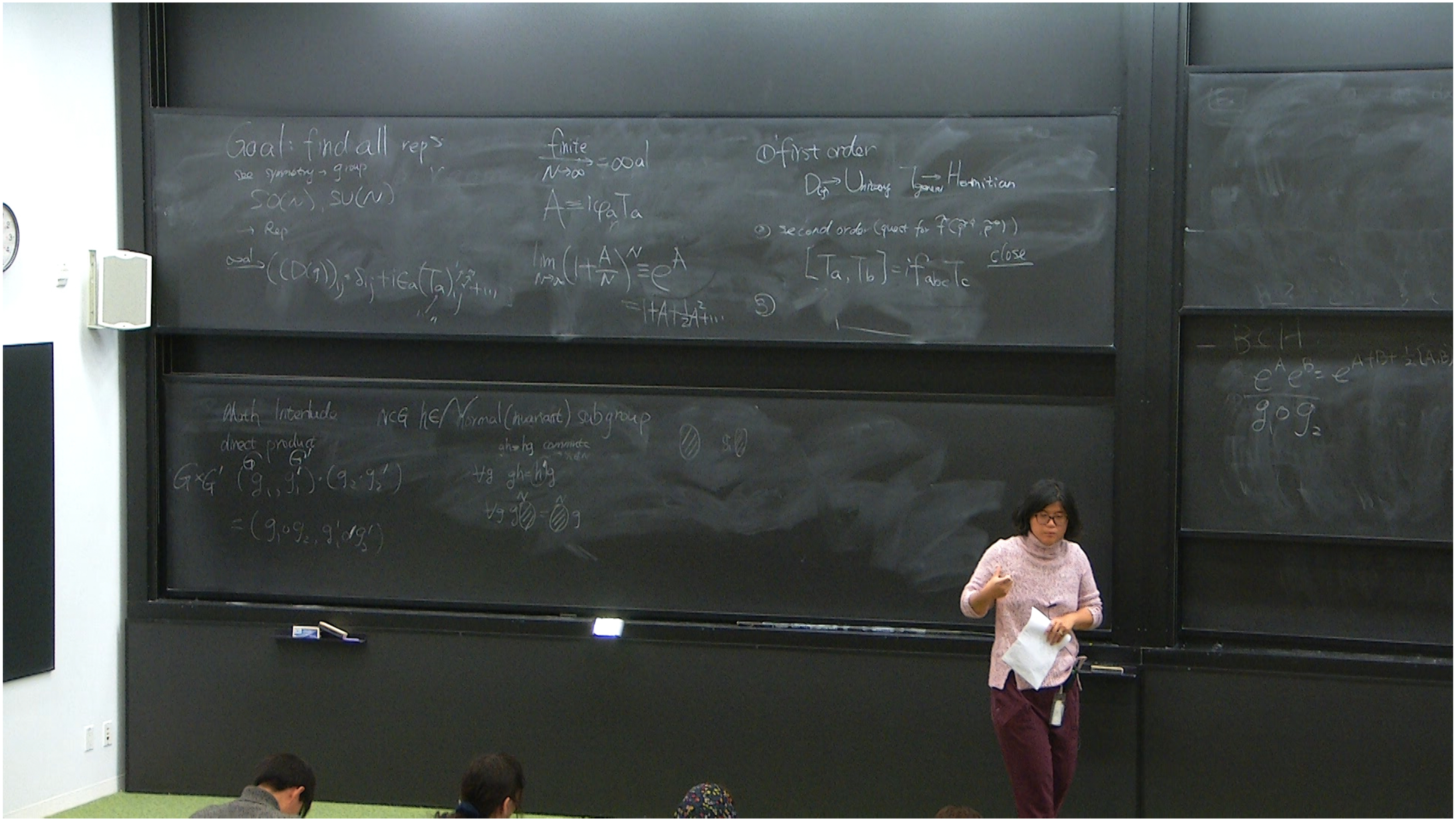
$$= (g_1 \circ g_2, \; g_1' \circ' g_2')$$

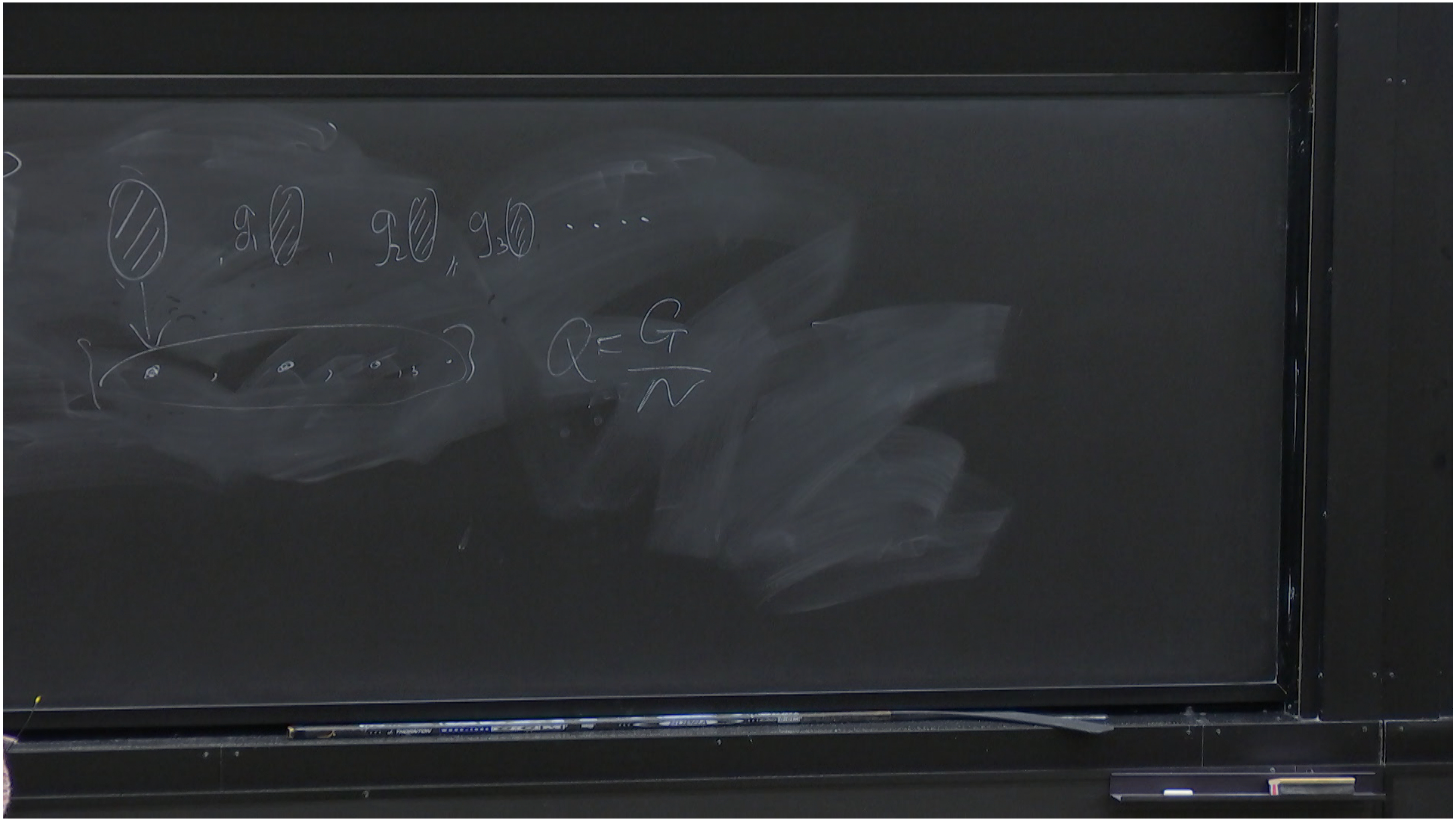
$h \in$ Normal (invariant) subgroup

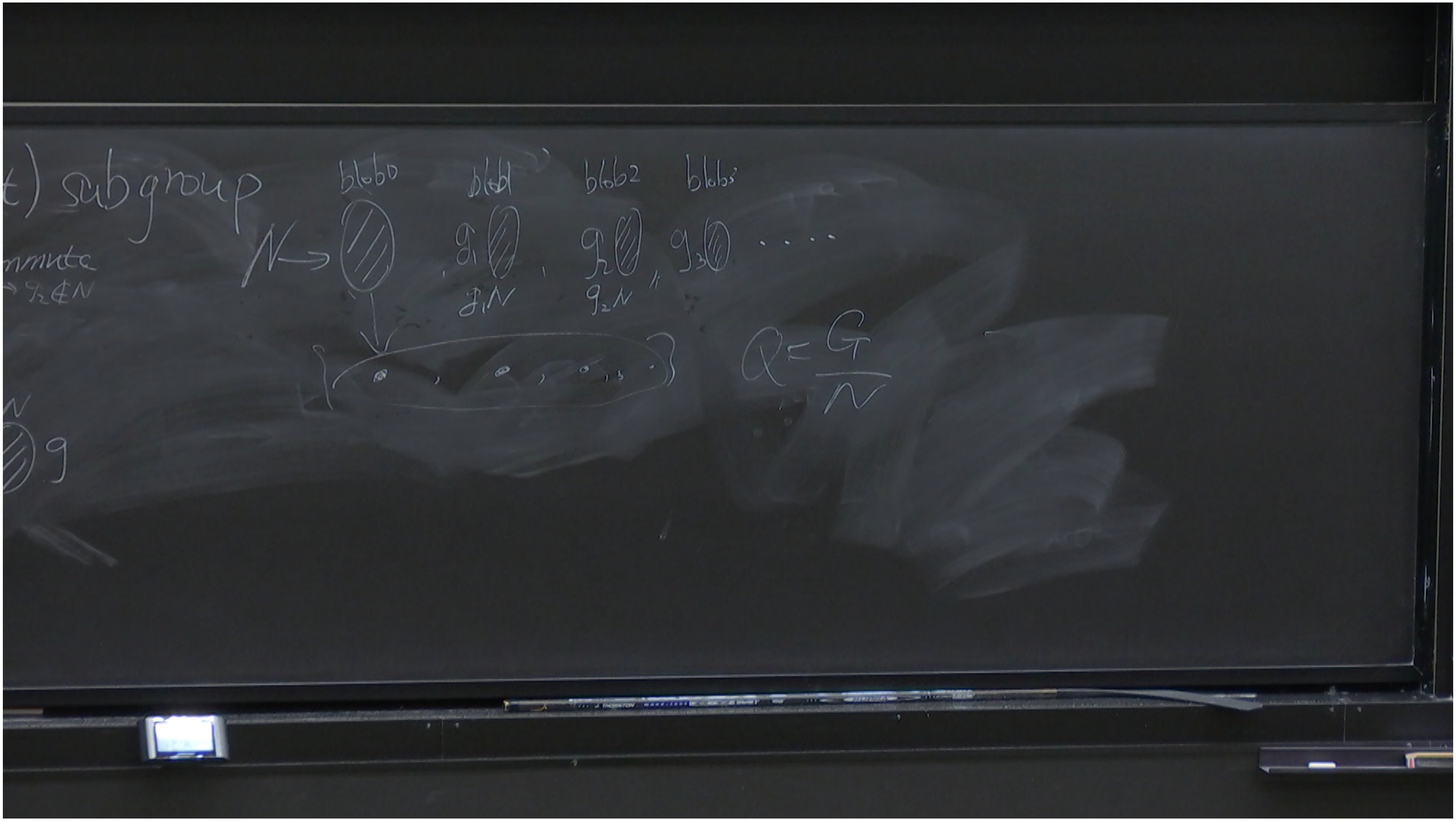
$gh = hg$ commute

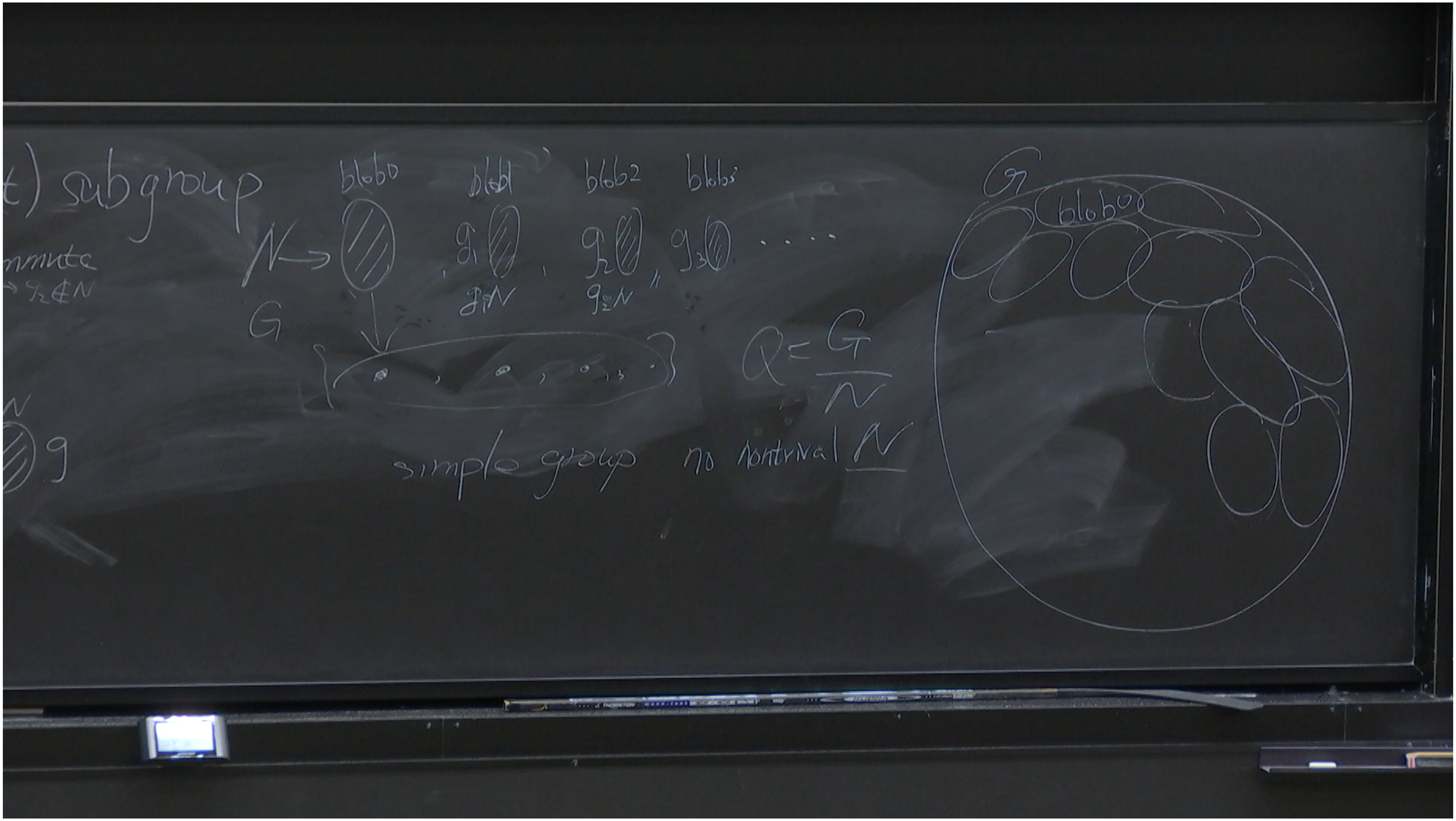
$\forall g \quad gh = h'' g \qquad h'' \in N$

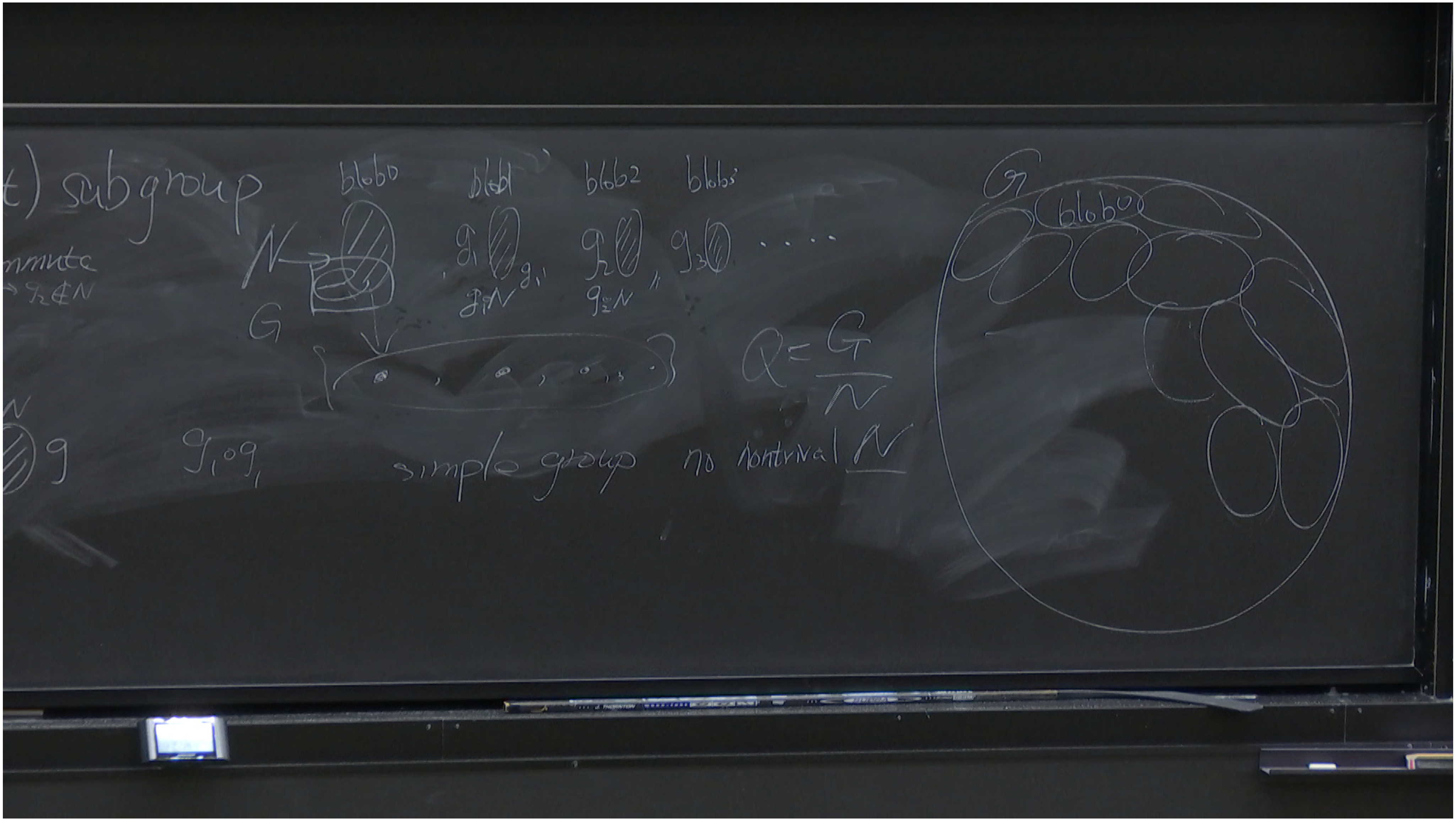
$\forall g \quad g\,N = N\,g$


Goal: find all reps
finite
first order
Hermitian
second order
close
BCH
Math Interlude
direct product
Normal (invariant) subgroup
commute


g1, g2, g3 . . . . .
Q = G/N


t) subgroup
blob0
blob1
blob2
blob3
N →
g1 , g2 , g3 ....
g1N
g2N
Q = G/N
mmute
→ g2 ∉ N
N
g


t) subgroup
mmute
$g_2 \notin N$
blob0
blob1
blob2
blob3
$N \rightarrow$
g_1 , g_2 , g_3 . . . . .
$g_1 N$
$g_2 N$
G
$Q = \frac{G}{N}$
simple group no nontrivial N
G
blob0


t) subgroup
blob0
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G
g1
g2
g3
g1N
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Q = G/N
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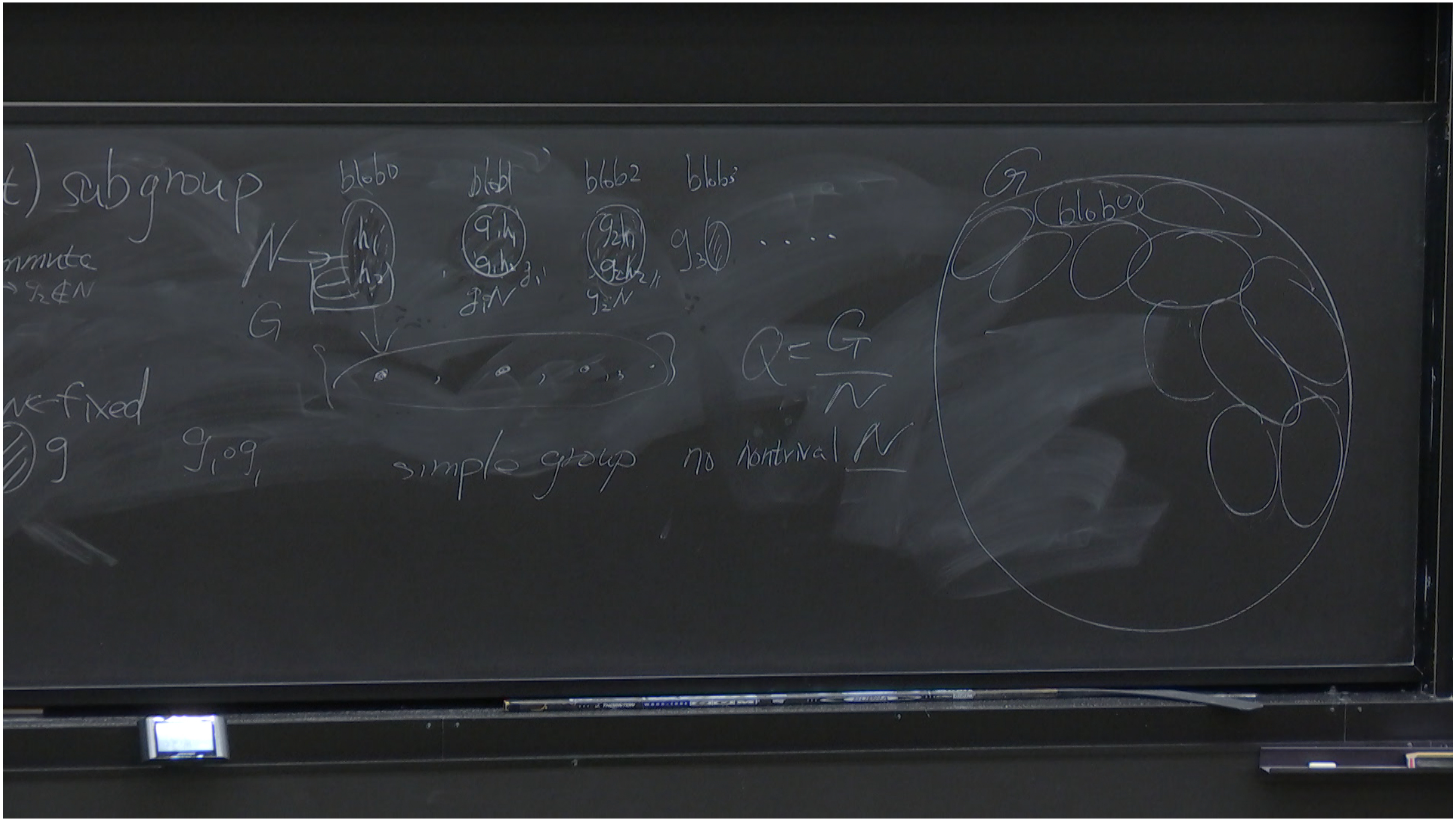
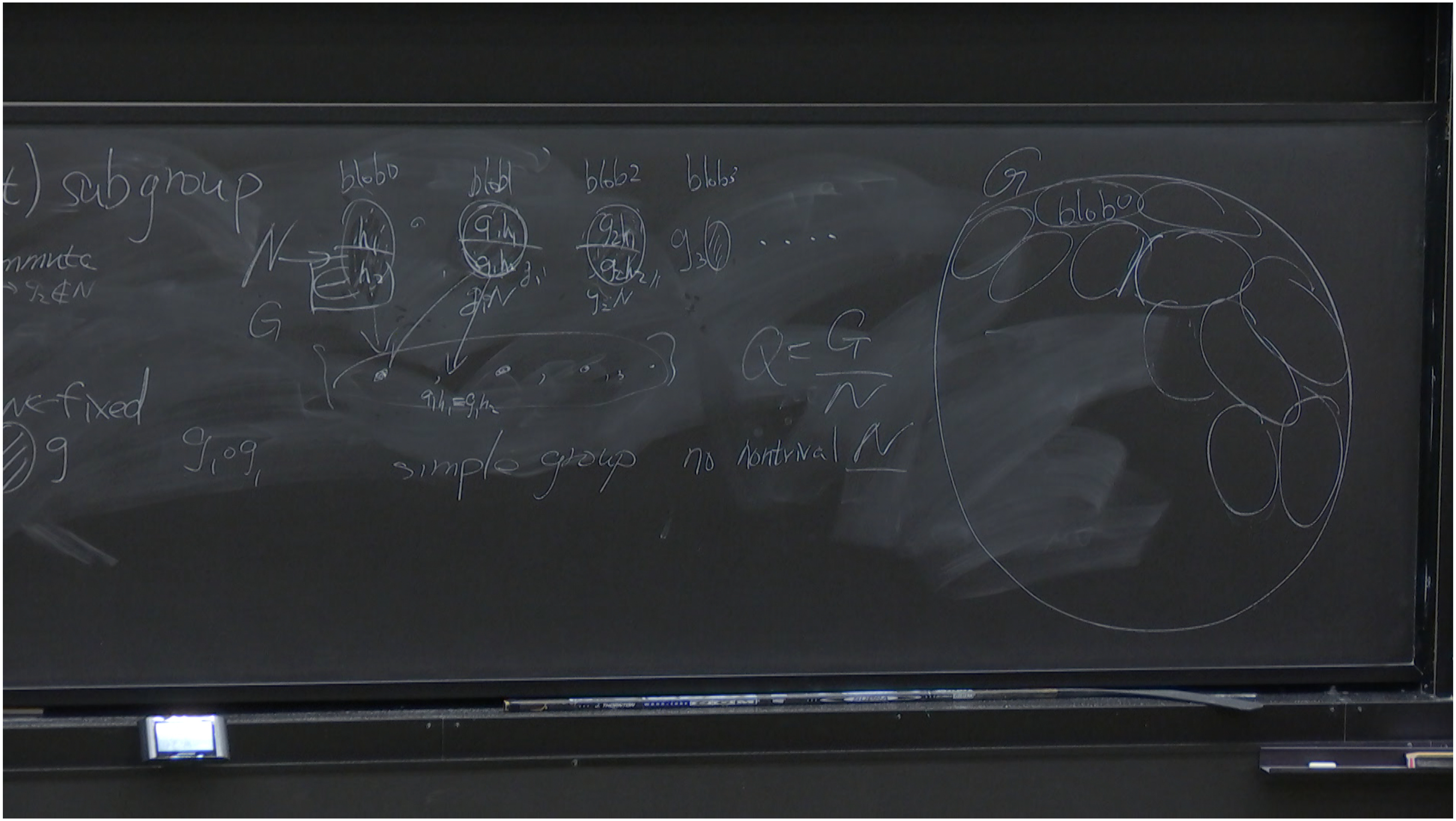
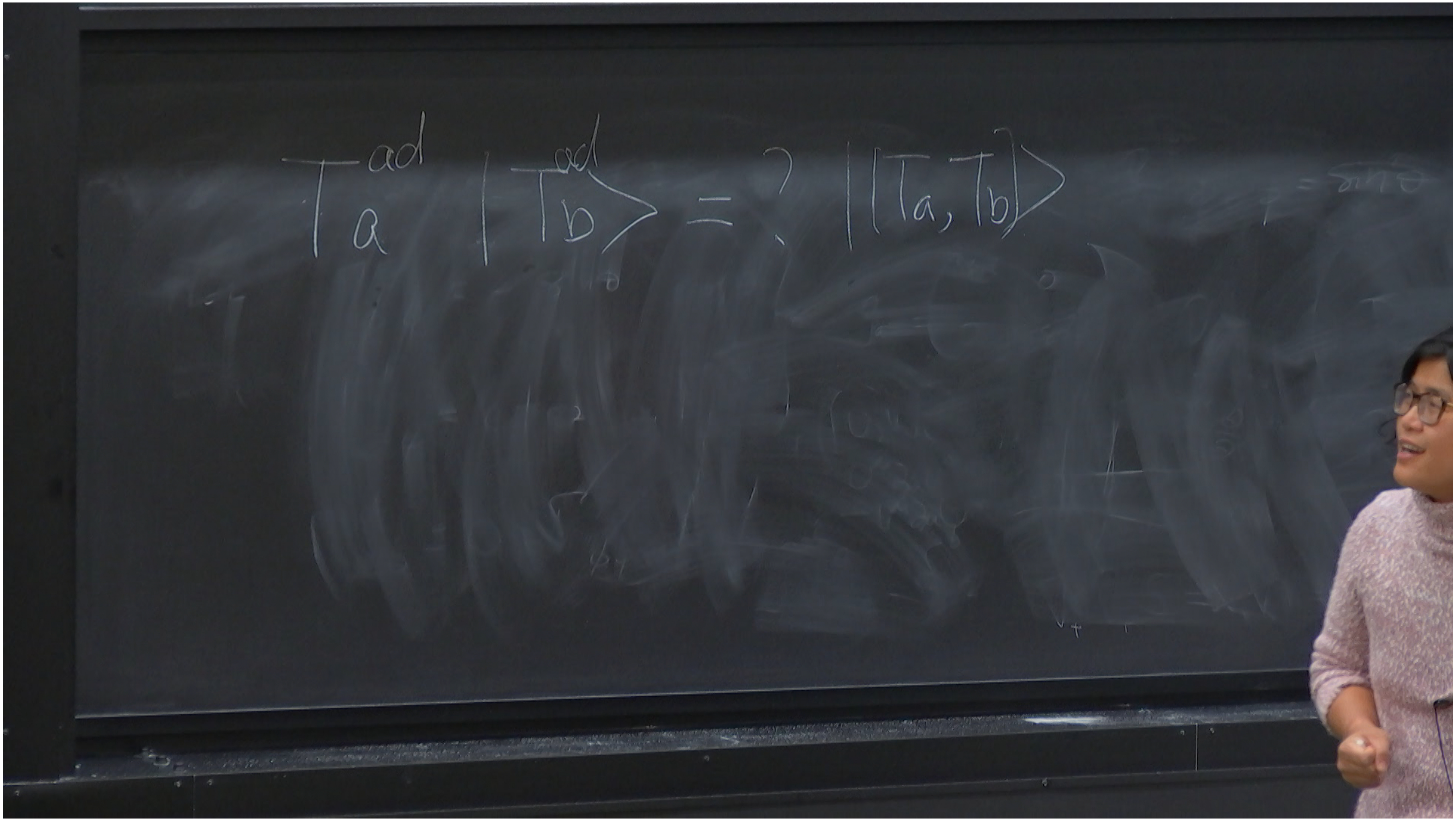
blob0


t) subgroup
blob0
blob1
blob2
blob3
mmute
fixed
N
G
Q = G/N
simple group no nontrival N
G
blob0

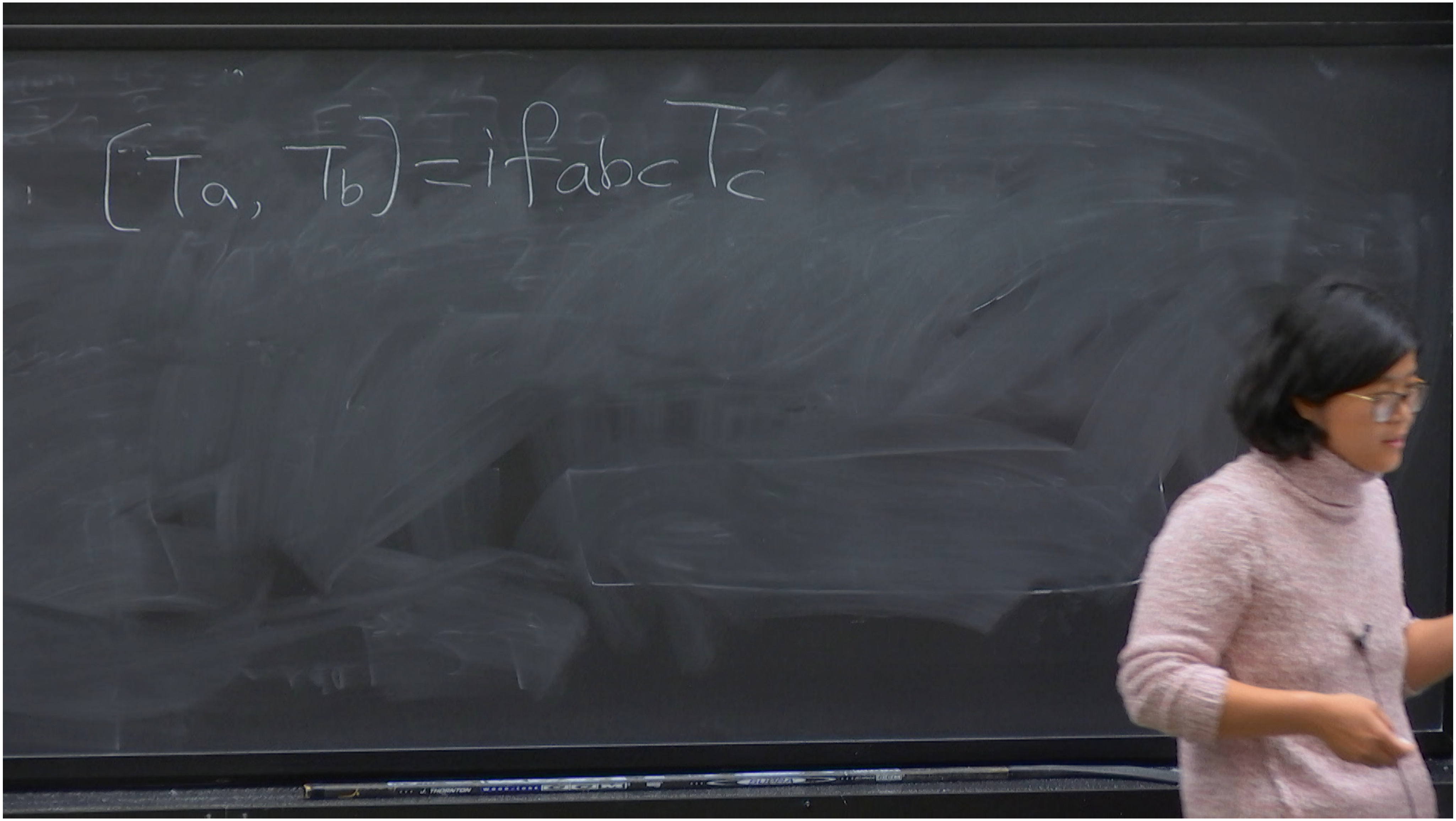

t) subgroup
blob0
blob1
blob2
blob3
mmute
N
G
$g_1 N$
$g_2 N$
$g_1 h_1 = g_1 h_2$
$Q = \frac{G}{N}$
fixed
simple group no nontrivial N
G
blob0


$T_a^{ad} \, | T_b^{ad} \rangle \overset{?}{=} | [T_a, T_b] \rangle$

$O(B_1)$ is our only hope $\times$

$R^2 \times D_2$ is " " " Lie

$SU(2)$ is . . . . .


$[T_a, T_b] = i f_{abc} T_c$

$$[T_a, T_b] = i f_{abc} T_c$$

$$[J_a, J_b] = i \epsilon_{abc} J_c$$

$n=2$

$$[T_a, T_b] = i f_{abc} T_c$$

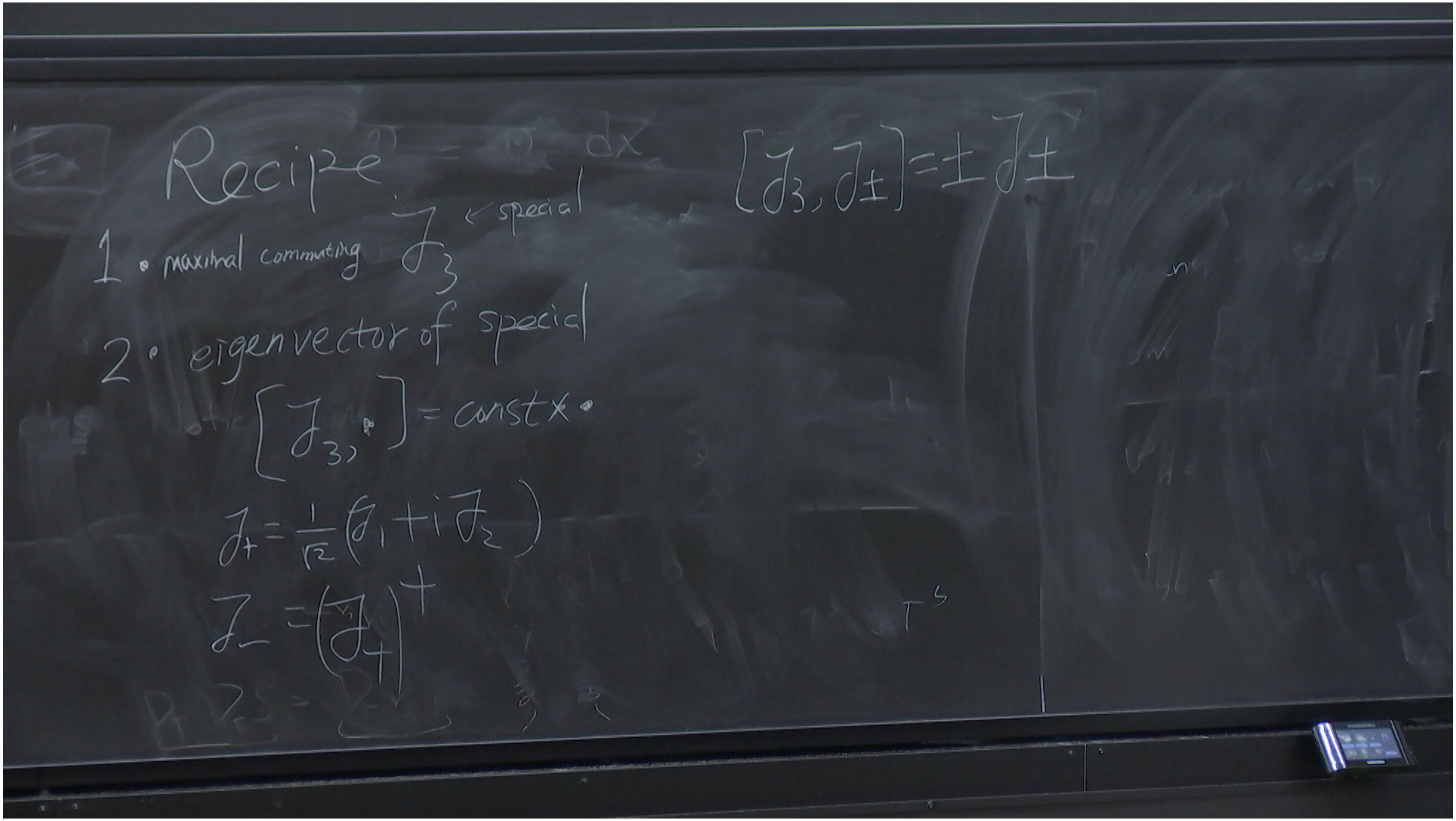
$$[J_a, J_b] = i \epsilon_{abc} J_c$$

$n=3$

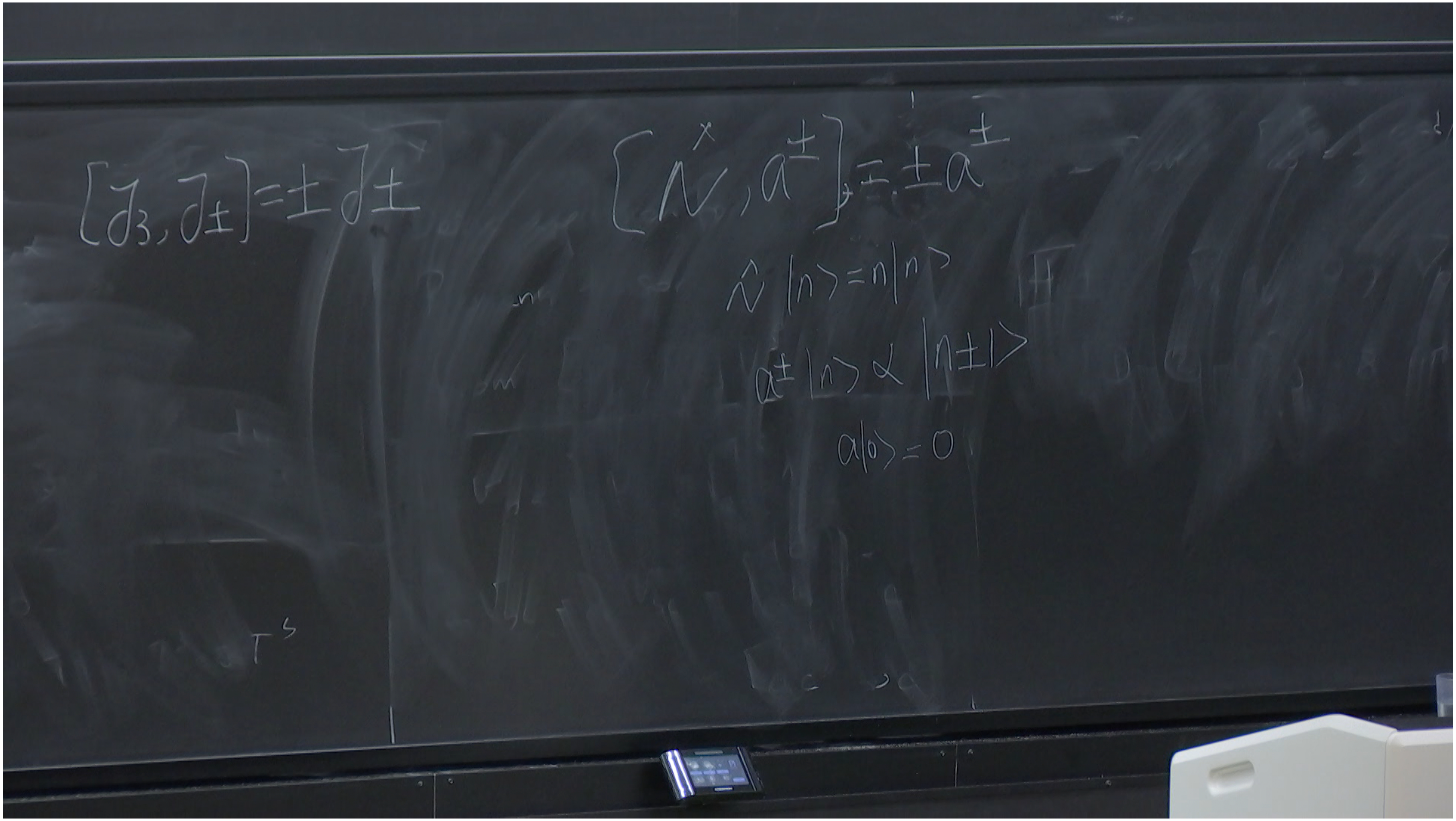
$n=2$ Pauli

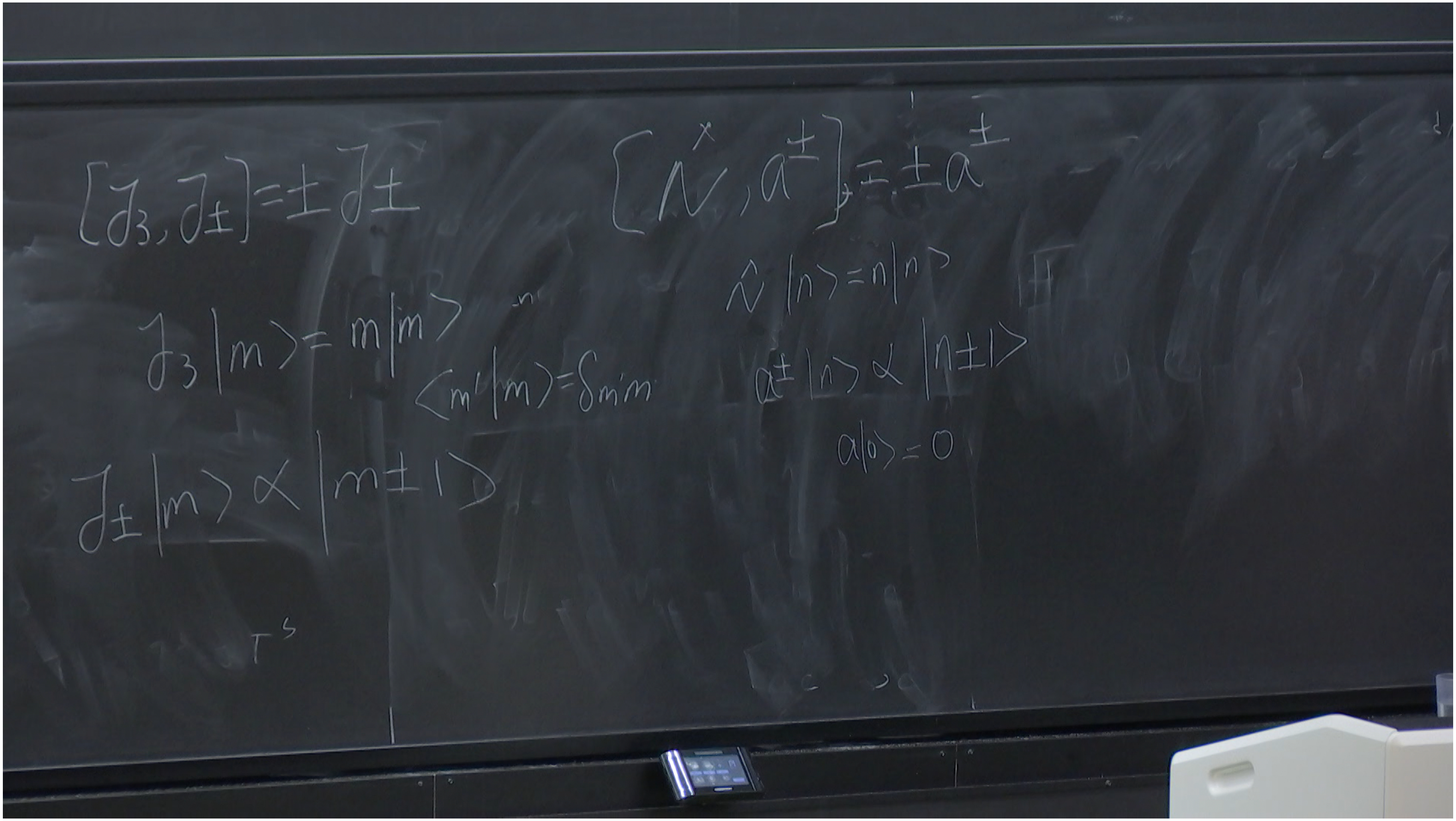
$n=3$

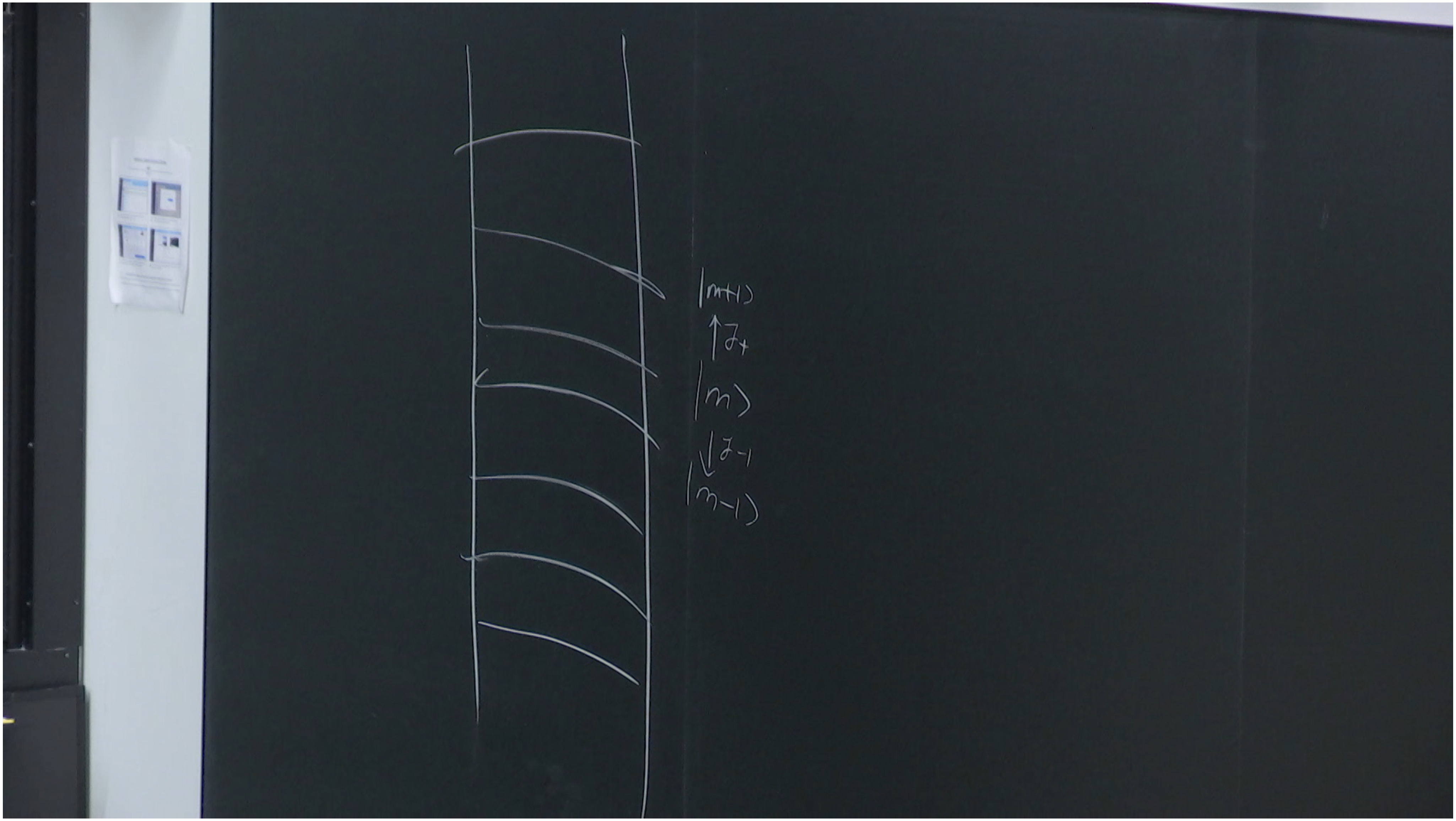
$$\begin{pmatrix} & & \\ & & 1 \\ & -1 & \end{pmatrix} T_1$$



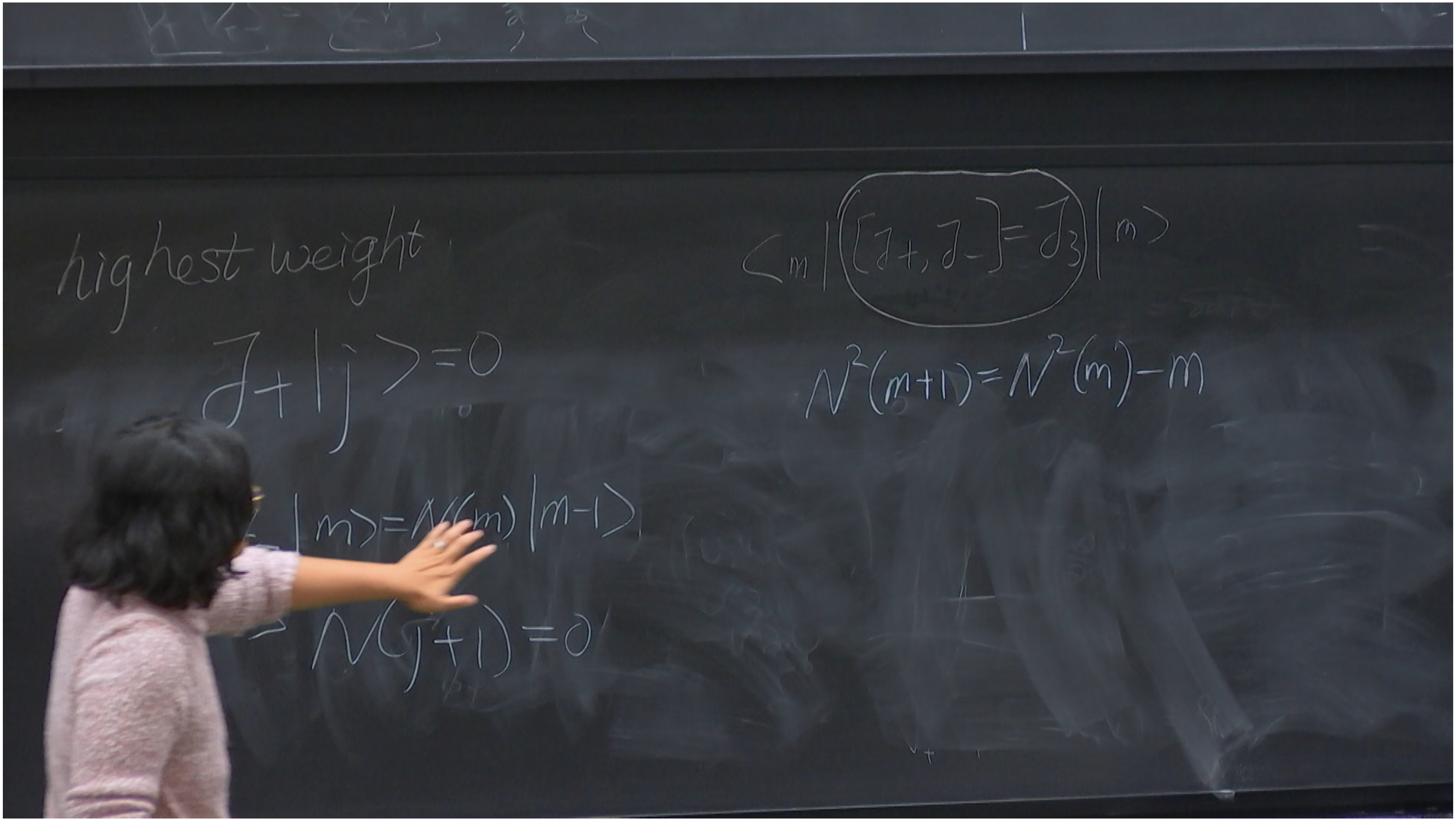
Recipe
$[J_3, J_\pm] = \pm J_\pm$
J_3 ← special
1. maximal commuting J_3
2. eigenvector of special
$[J_3, \cdot] = \text{const} \times \cdot$
$J_+ = \frac{1}{\sqrt{2}}(J_1 + i J_2)$
$J_- = (J_+)^\dagger$

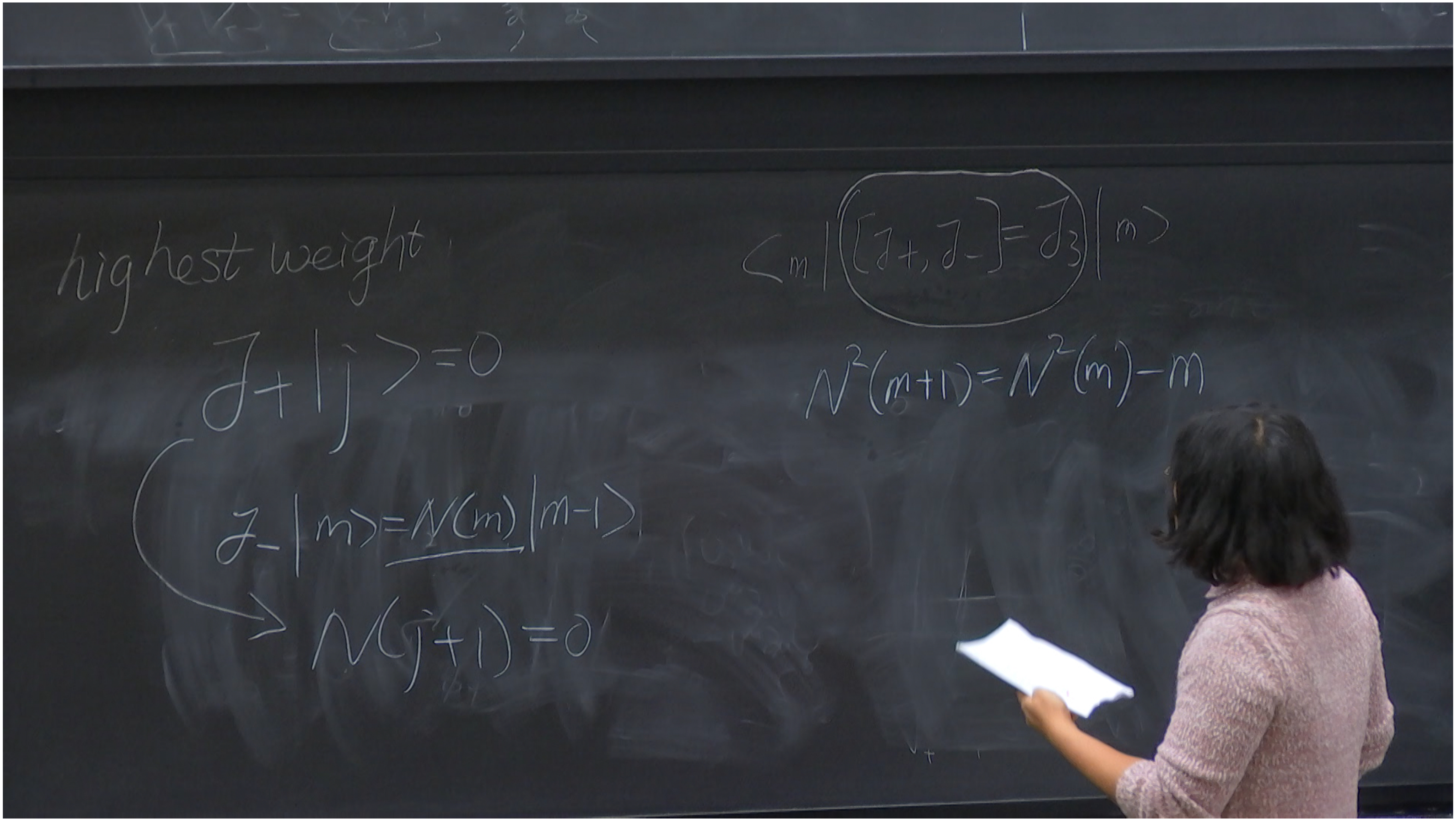

$[J_3, J_\pm] = \pm J_\pm$
$[\hat{N}, a^\pm] = \pm a^\pm$
$\hat{N}|n\rangle = n|n\rangle$
$a^\pm|n\rangle \propto |n\pm 1\rangle$
$a|0\rangle = 0$

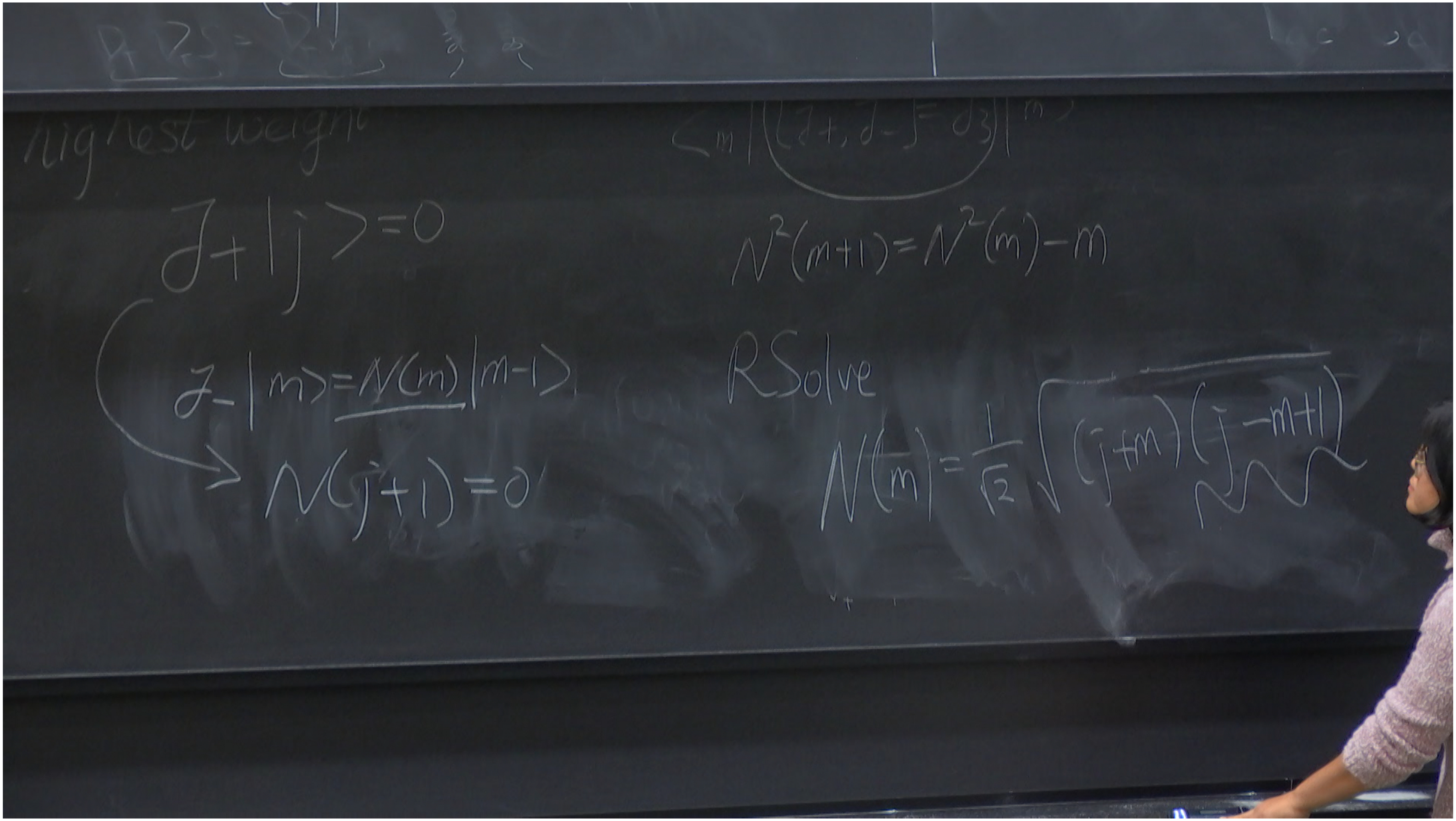

$[J_3, J_\pm] = \pm J_\pm$
$[\hat{N}, a^\pm] = \pm a^\pm$
$\hat{N}|n\rangle = n|n\rangle$
$J_3|m\rangle = m|m\rangle$
$\langle m'|m\rangle = \delta_{m'm}$
$a^\pm|n\rangle \propto |n\pm 1\rangle$
$a|0\rangle = 0$
$J_\pm|m\rangle \propto |m\pm 1\rangle$



|m+1⟩
J+
|m⟩
J-
|m-1⟩


highest weight
J+|j>=0
|m>=N(m)|m-1>
N(j+1)=0
<m|[J+,J-]=J3|m>
N2(m+1)=N2(m)-m


highest weight
$J_+|j\rangle = 0$
$J_-|m\rangle = N(m)|m-1\rangle$
$N(j+1) = 0$
$\langle m|([J_+, J_-] = J_3)|m\rangle$
$N^2(m+1) = N^2(m) - m$


highest weight
$J_+|j\rangle = 0$
$\langle m|[J_+, J_-] = J_3|m\rangle$
$N^2(m+1) = N^2(m) - m$
$J_-|m\rangle = \underline{N(m)}|m-1\rangle$
RSolve
$N(j+1) = 0$
$N(m) = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$

highest weight

$$J_+|j\rangle = 0$$

$$J_-|m\rangle = N(m)|m-1\rangle$$

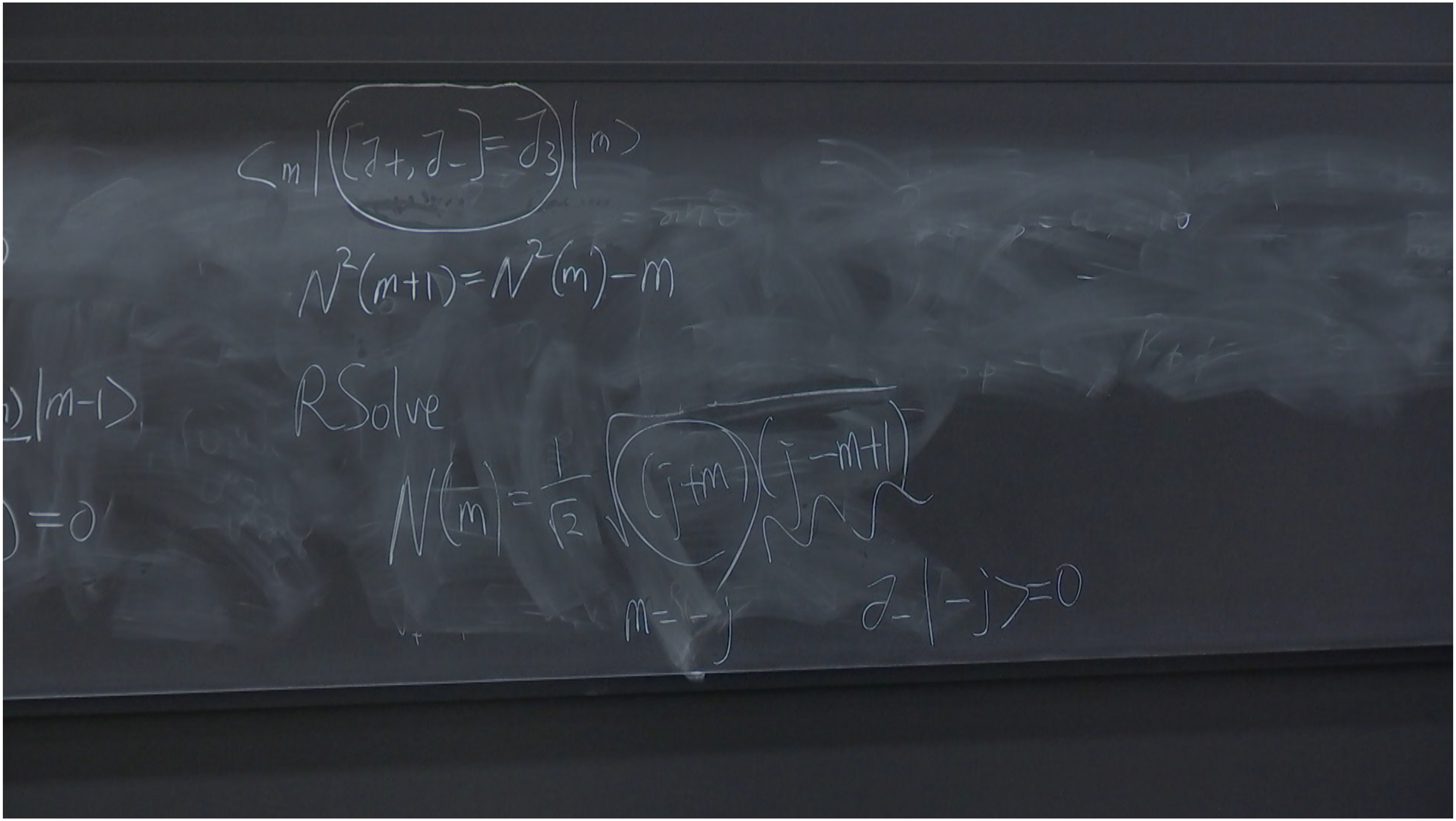
$$N(j+1) = 0$$

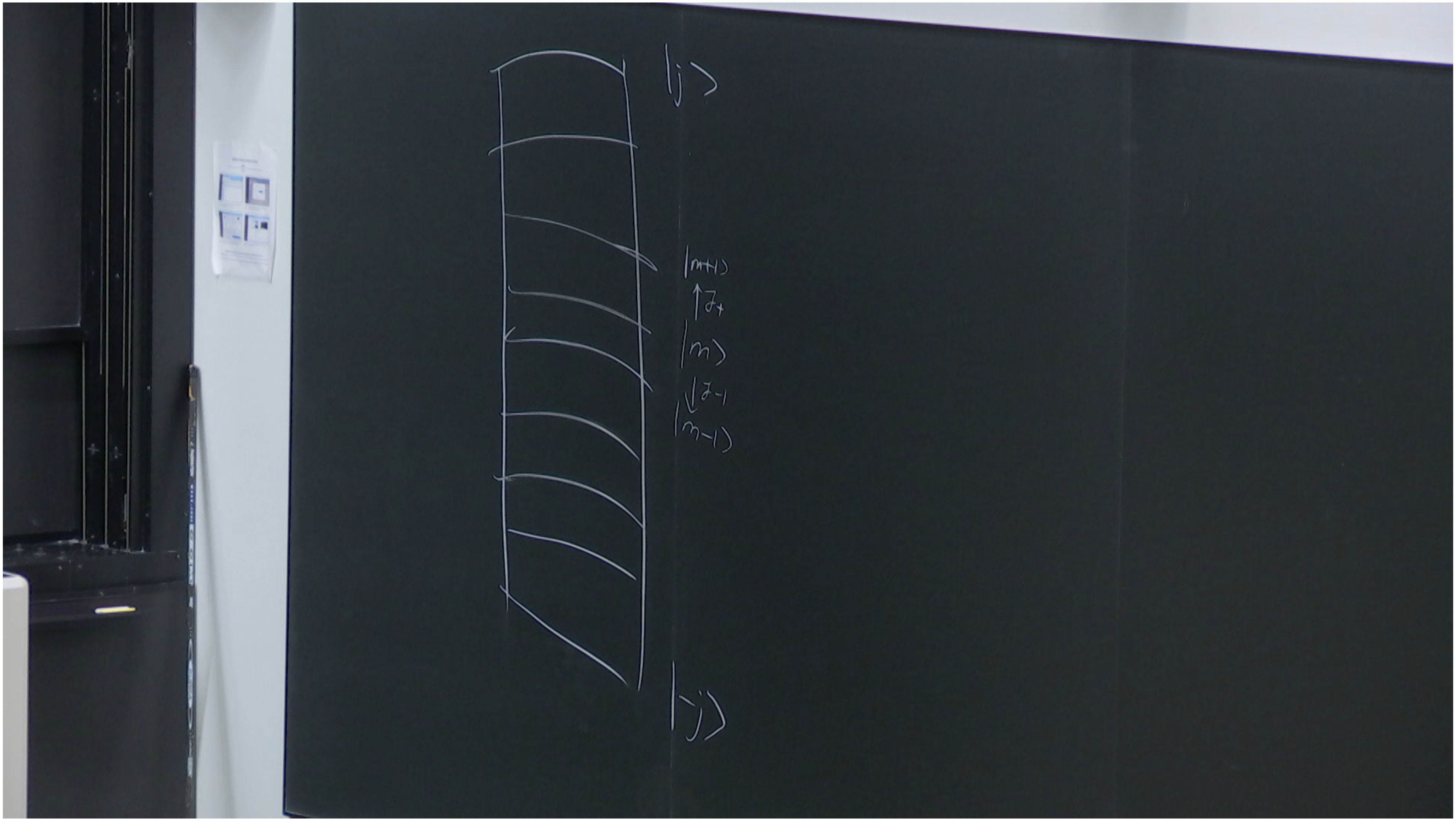
$$\langle m|\left([J_+, J_-] = J_3\right)|m\rangle$$

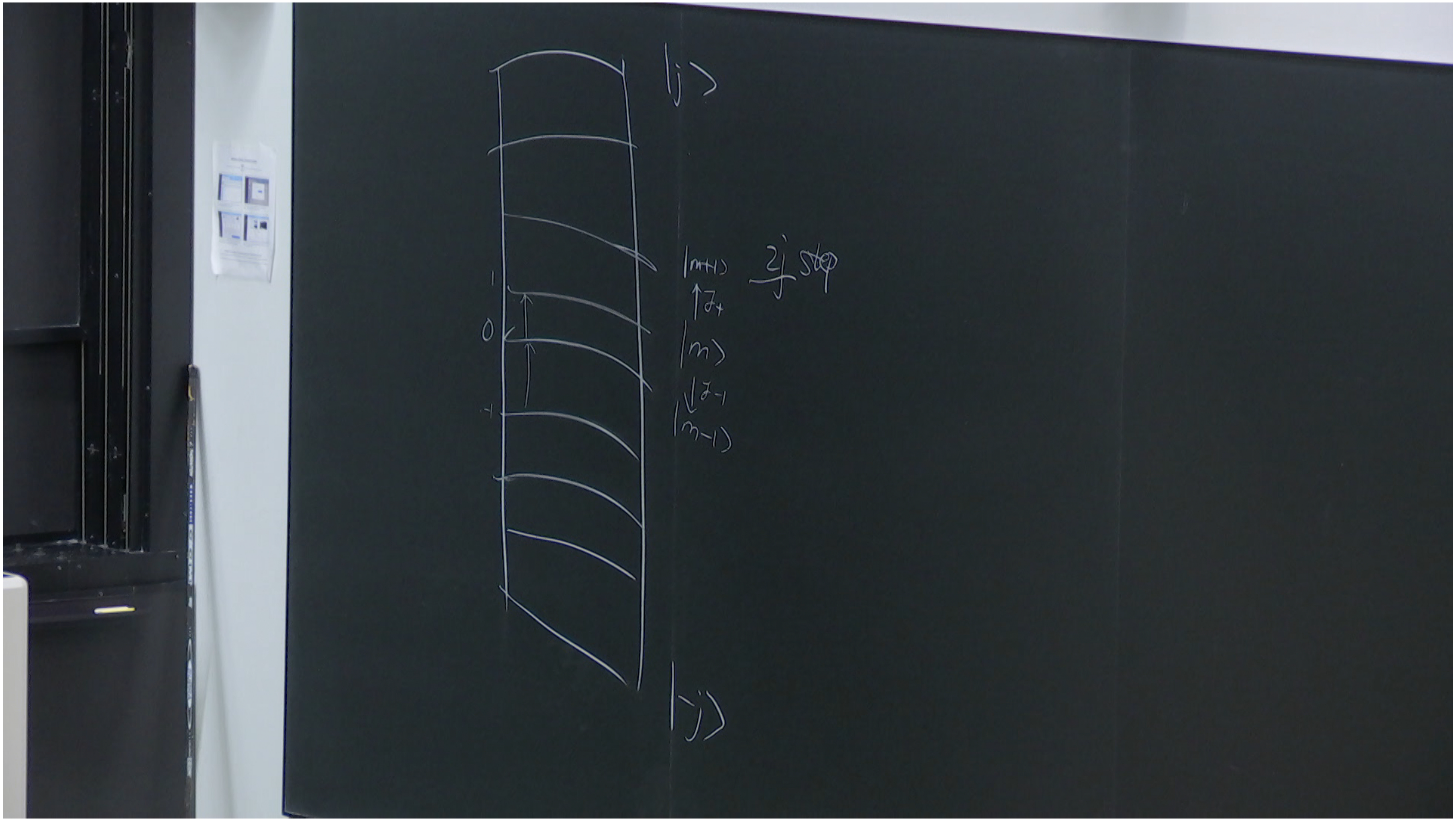
$$N^2(m+1) = N^2(m) - m$$

RSolve

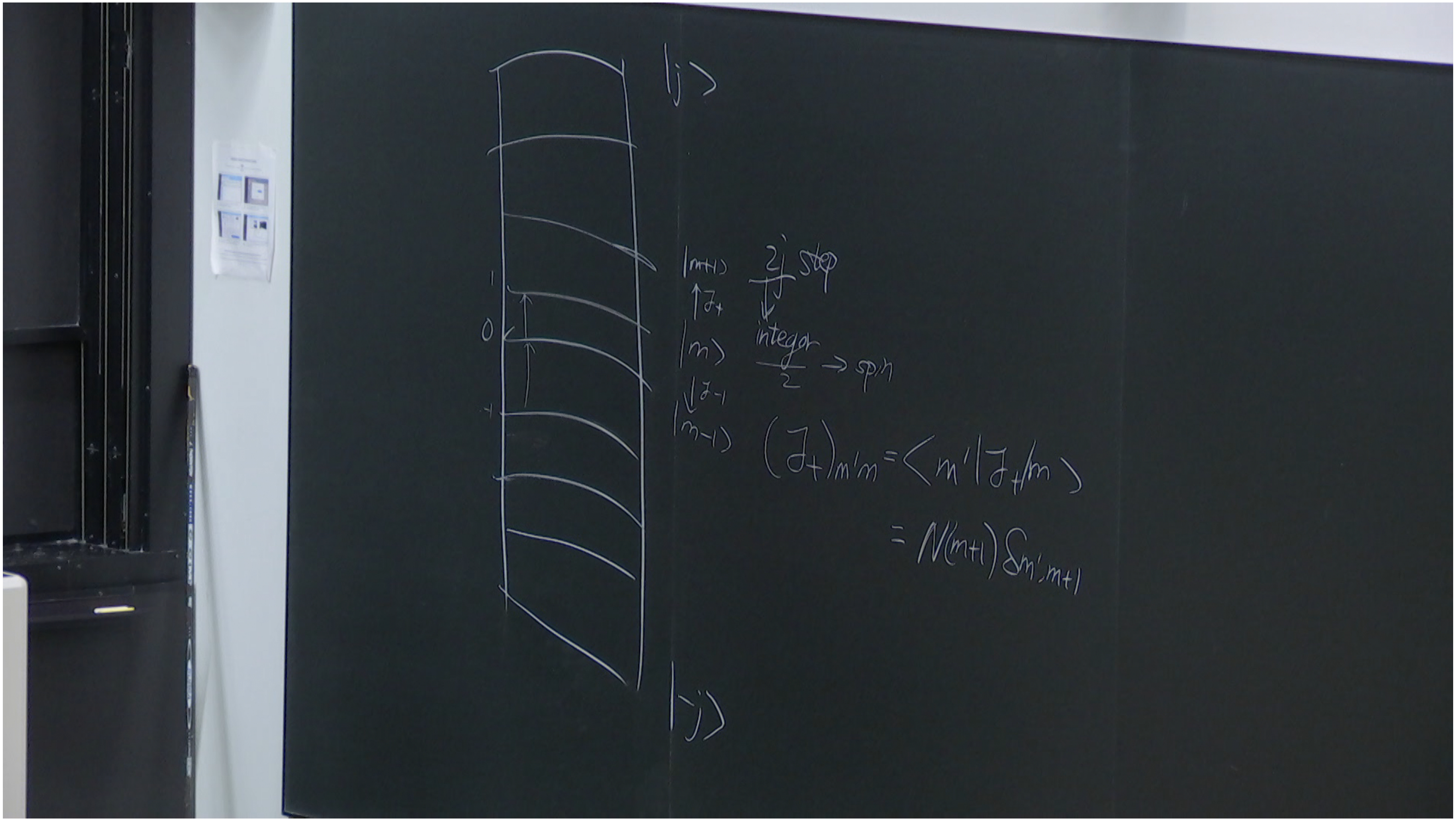
$$N(m) = \frac{1}{\sqrt{2}}\sqrt{(j+m)(j-m+1)}$$

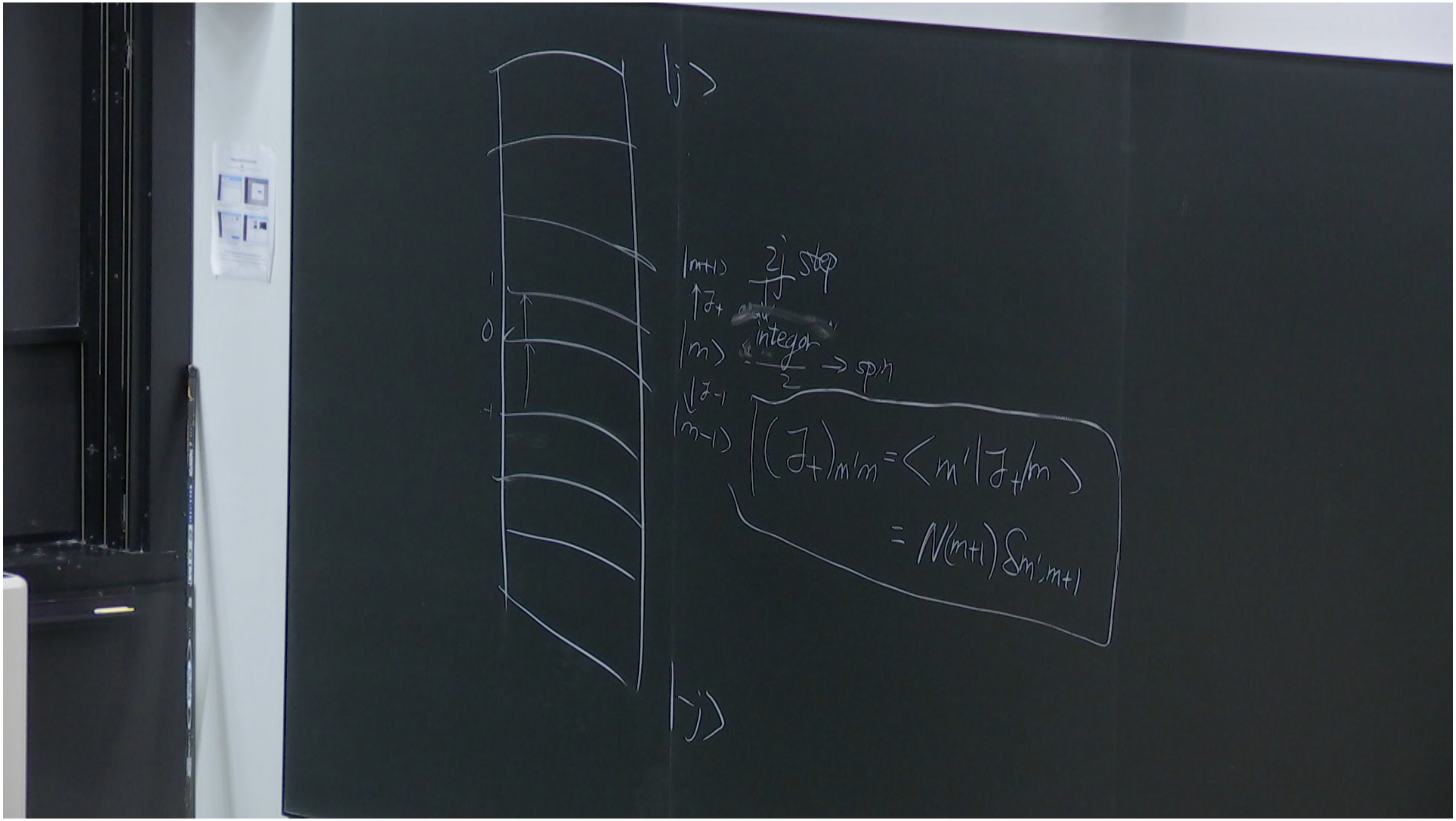

⟨m|([J+, J−] = J3)|m⟩
N²(m+1) = N²(m) − m
RSolve
|m−1⟩
= 0
N(m) = 1/√2 √((j+m)(j−m+1))
m = −j
J−|−j⟩ = 0


$|j\rangle$
$|m+1\rangle$
J_+
$|m\rangle$
J_-
$|m-1\rangle$
$|-j\rangle$

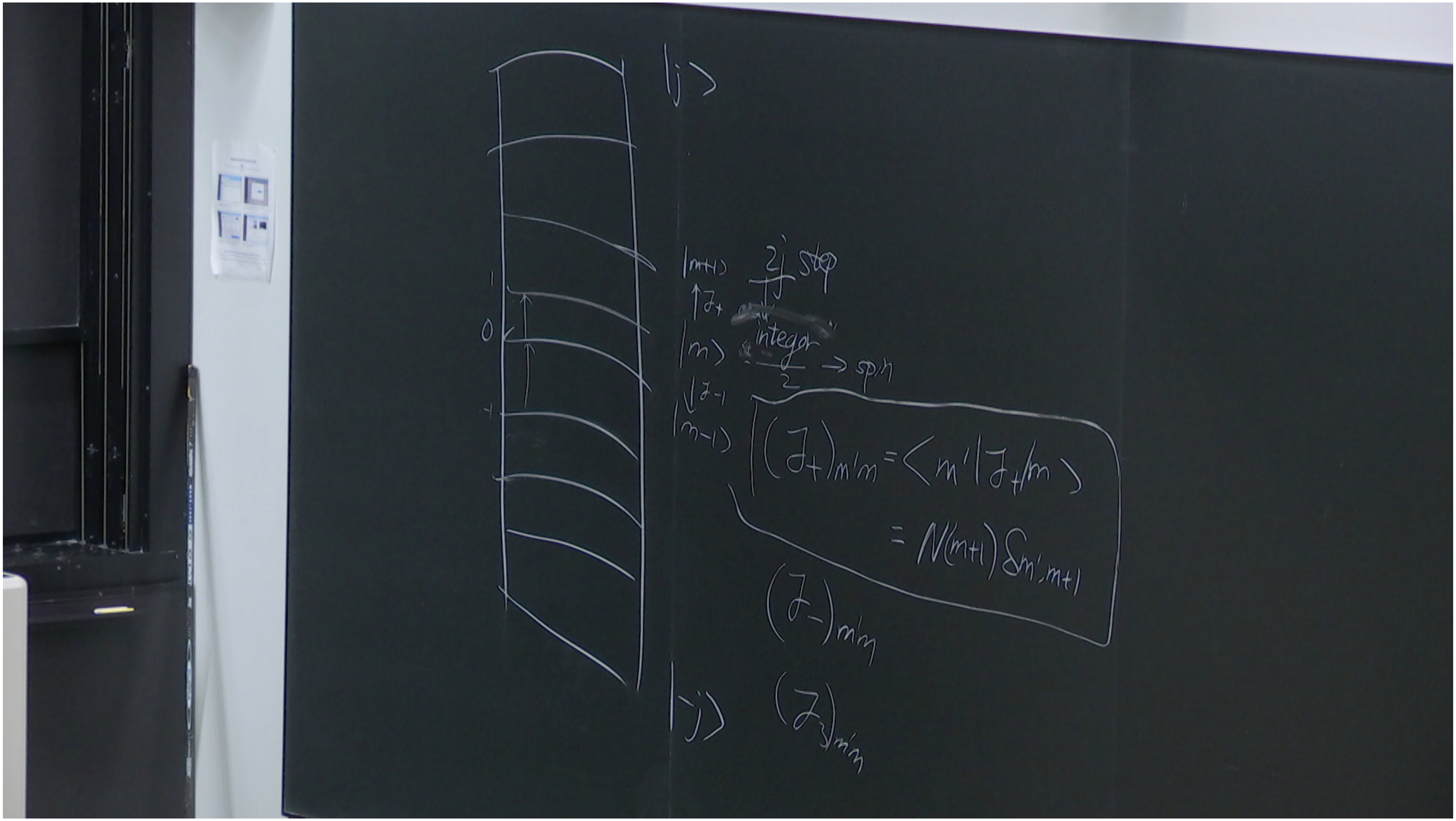


|j⟩
|m+1⟩
2j step
J+
|m⟩
J−
|m−1⟩
|−j⟩
1
0
−1


|j⟩
|m+1⟩
2j step
J+
integer
2
→ spin
|m⟩
J-
|m-1⟩
(J+)m'm = ⟨m'|J+|m⟩
= N(m+1) δm',m+1
|-j⟩



|j>
1
0
-1
|m+1>
J+
|m>
J-
|m-1>
2j step
integer
2
→ spin
(J+)m'm = <m'|J+|m>
= N(m+1) δm',m+1
|-j>



|j>
|m+1>
2j step
integer
|m>
→ spin
|m−1>
(J+)m'm = <m'|J+|m>
= N(m+1) δm',m+1
(J−)m'm
(Jz)m'm
|−j>

$|m+1\rangle$ $\quad 2j$ step

$\uparrow J_+$ $\quad 2j$ integer

$|m\rangle$ $\quad \frac{\text{integer}}{2} \to$ spin

$\downarrow J_-$

$|m-1\rangle$

$$\boxed{\begin{aligned}(J_+)_{m'm} &= \langle m'|J_+|m\rangle \\ &= N(m+1)\,\delta_{m',m+1}\end{aligned}}$$

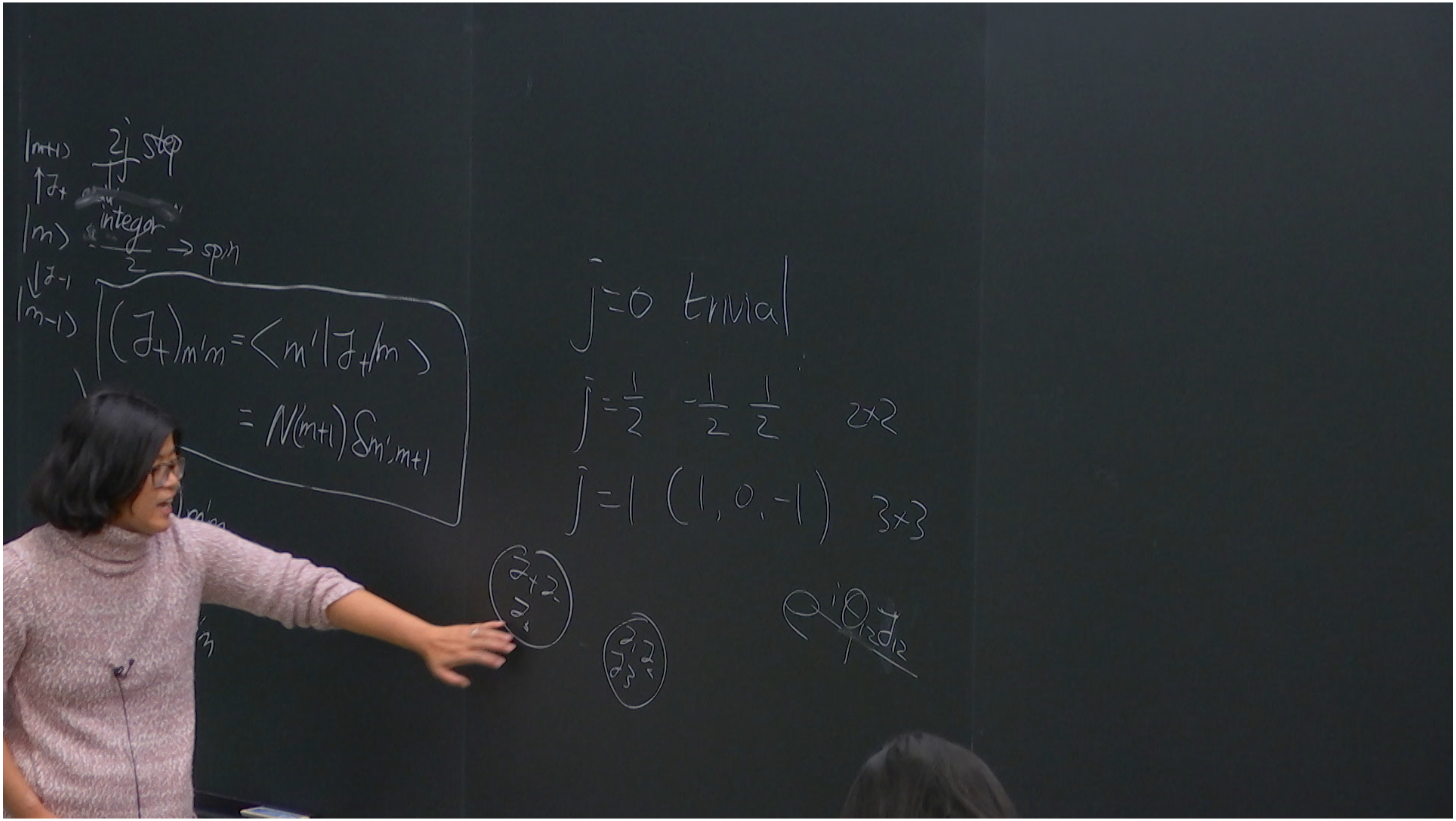
$$(J_-)_{m'm}$$

$|-j\rangle$ $\quad (J_z)_{m'm}$

$j=0$ trivial

$j=\frac{1}{2}$ $\quad -\frac{1}{2}$ $\quad \frac{1}{2}$ $\quad 2\times2$

$j=1$ $\quad (1,0,-1)$ $\quad 3\times3$


2j step
integer
spin
j=0 trivial
2×2
3×3