Title: Review Talk: An introduction to coadjoint orbits

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Collection: Quantum Gravity Around the Corner

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Abstract: We begin by reviewing the role of coadjoint orbits in the representation theory of nilpotents groups and then, to connect with the recent applications in physics of coadjoint orbits "around the corner" of the mathematical framework developed by Kirillov, we review the classification of coadjoint orbits of the Virasoro group. This will allow us to connect with more recent developments, including e.g. the study of coadjoint orbits of BMS algebras.

INTRODUCTION TO COADJOINT ORBITS

Why studying them?

- quantise dynamics on classical phase spaces with SYMMETRIES;
- build IRREPS. of non-compact & (especially) ∞-dim groups

ADJ. & COADJ. REPR.

$$Ad : G \longrightarrow GL(\mathrm{Lie}(G)) \qquad Ad^* : G \longrightarrow GL(\mathfrak{g}^*)$$

$$g \longmapsto Ad_g \qquad p \longmapsto Ad^*_p$$

$$Ad_g(X) = \frac{d}{dt}\left[g e^{tX} g^{-1} \right]\Big|_{t=0} \qquad X \in \mathrm{Lie}(G)$$

(for matrix groups $Ad_g(X) = gXg^{-1}$)

$g \longmapsto Ad_g$

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$$ad_x(y) = \frac{d}{dt}\left[Ad_{e^{tx}}(y) \right]\Big|_{t=0} = [x, y]$$

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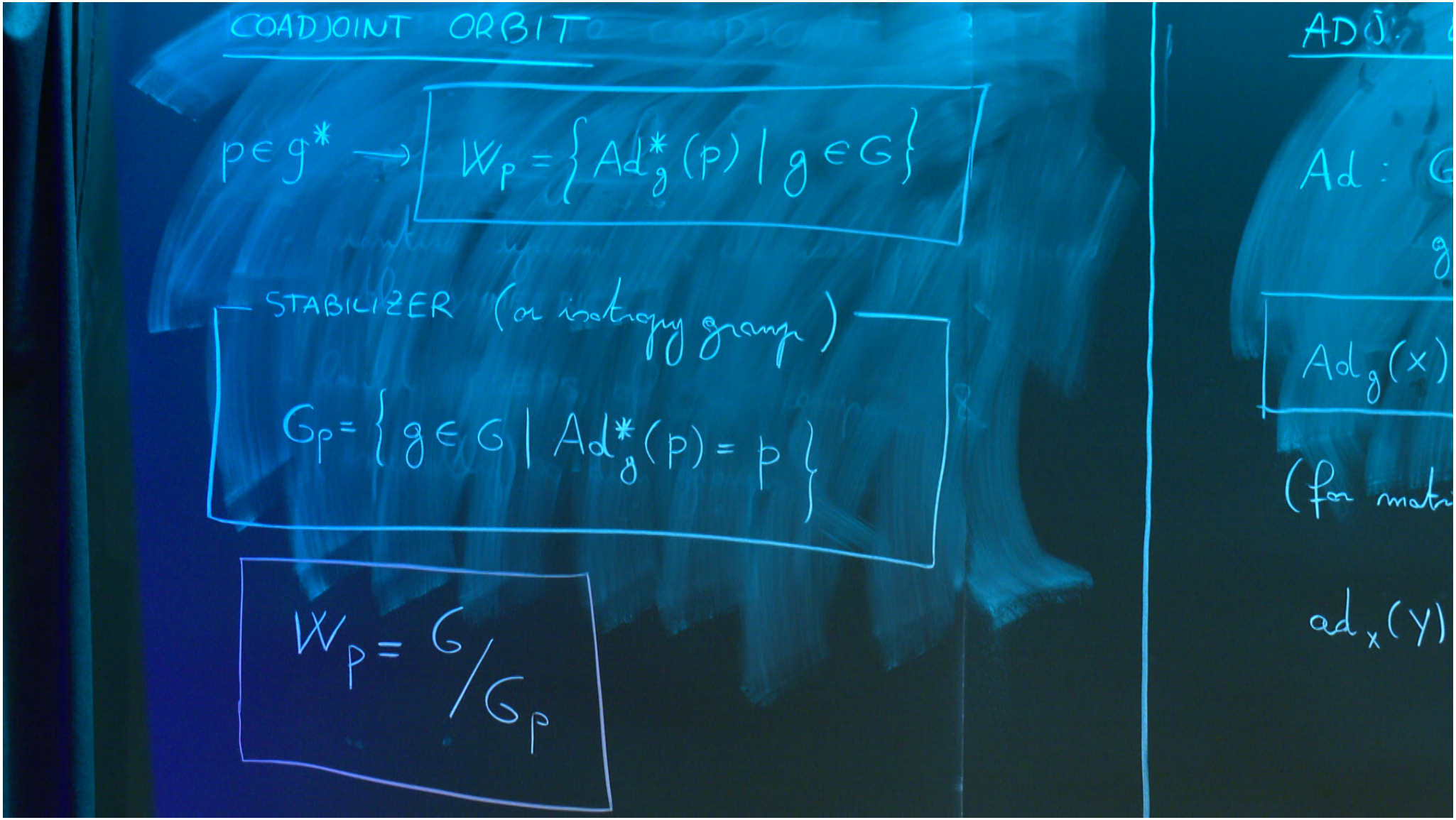
$$ad_X(Y) = \frac{d}{dt}\left[Ad_{e^{tX}}(Y) \right]\Big|_{t=0} = [X, Y]$$

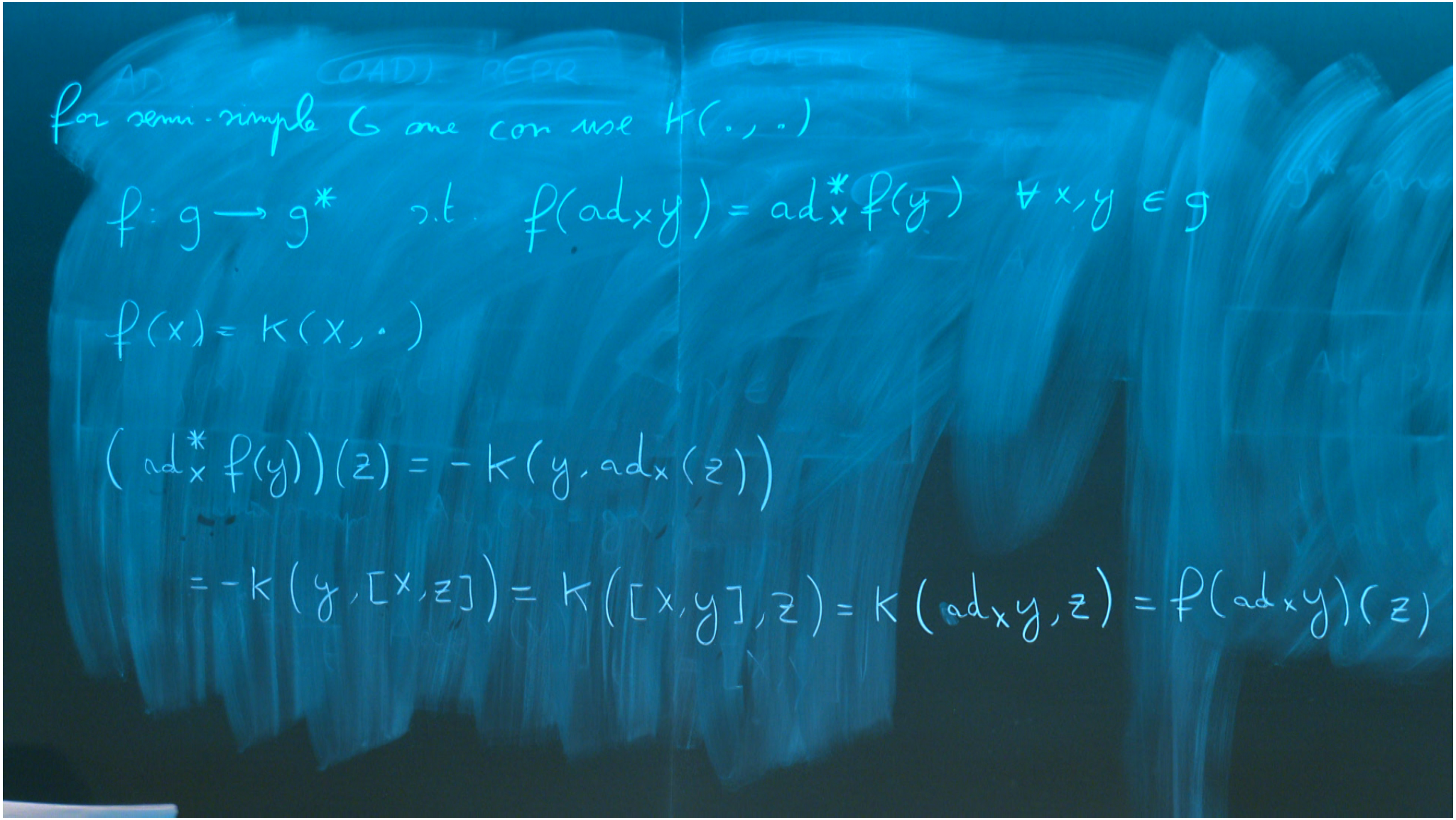
$\mathfrak{g}^*$ given by $p : \mathfrak{g} \longrightarrow \mathbb{R}$

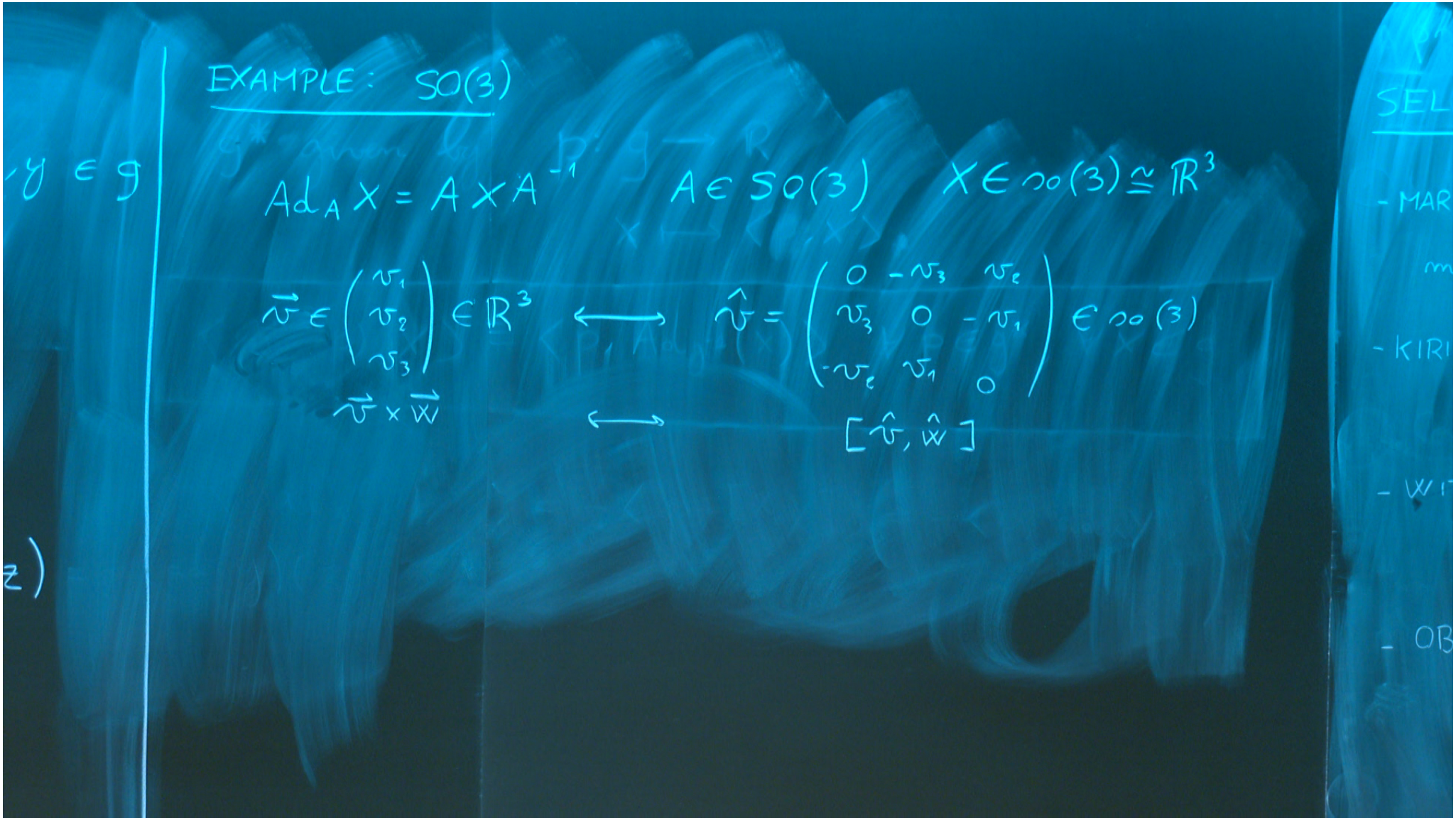
$X \longmapsto \langle p, X \rangle$.

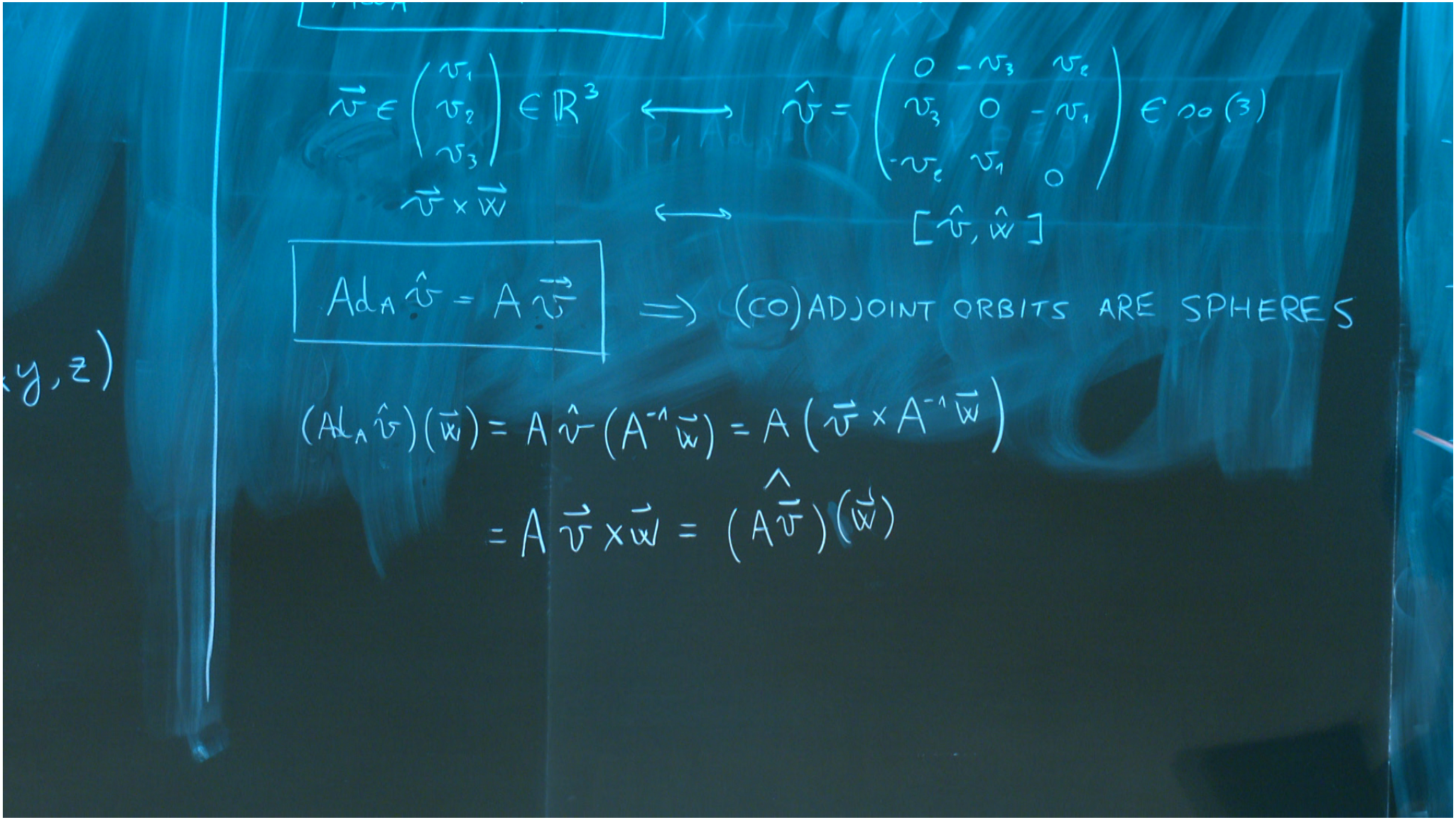
$$\langle Ad^*_g(p), X \rangle \equiv \langle p, Ad_{g^{-1}}(X) \rangle \quad \forall p \in \mathfrak{g}^*, \forall X \in \mathfrak{g}$$

$$\langle ad^*_X(p), Y \rangle = -\langle p, [X,Y] \rangle \left(= -\langle p, ad_X Y \rangle \right)$$


COADJOINT ORBIT
$p \in \mathfrak{g}^* \longrightarrow W_p = \{ Ad^*_g(p) \mid g \in G \}$
STABILIZER (or isotropy group)
$G_p = \{ g \in G \mid Ad^*_g(p) = p \}$
$W_p = G/G_p$
ADJ
Ad : G
g
$Ad_g(x)$
(for matr
$ad_x(y)$


for semi-simple G one can use K(.,.)
$f: \mathfrak{g} \to \mathfrak{g}^*$ s.t. $f(ad_x y) = ad_x^* f(y) \quad \forall x, y \in \mathfrak{g}$
$f(x) = K(x, \cdot)$
$(ad_x^* f(y))(z) = -K(y, ad_x(z))$
$= -K(y, [x, z]) = K([x, y], z) = K(ad_x y, z) = f(ad_x y)(z)$


EXAMPLE: SO(3)
$Ad_A X = A X A^{-1}$
$A \in SO(3)$
$X \in \mathfrak{so}(3) \cong \mathbb{R}^3$
$\vec{v} \in \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \in \mathbb{R}^3 \longleftrightarrow \hat{v} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix} \in \mathfrak{so}(3)$
$\vec{v} \times \vec{w} \longleftrightarrow [\hat{v}, \hat{w}]$


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$\vec{v} \times \vec{w} \longleftrightarrow [\hat{v}, \hat{w}]$
$Ad_A \hat{v} = A\hat{v}$
$\Rightarrow$ (CO)ADJOINT ORBITS ARE SPHERES
$(Ad_A \hat{v})(\vec{w}) = A\hat{v}(A^{-1}\vec{w}) = A(\vec{v} \times A^{-1}\vec{w})$
$= A\vec{v} \times \vec{w} = (\widehat{A\vec{v}})(\vec{w})$

$\vec{v} \times \vec{w} \longleftrightarrow [\hat{v}, \hat{w}]$

$\boxed{Ad_A \hat{v} = A\vec{v}} \implies$ (CO)ADJOINT ORBITS ARE SPHERES

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$\mathfrak{so}(3)^* \simeq \mathbb{R}^3$ via $\langle \vec{p}, \vec{v} \rangle = \vec{p} \cdot \vec{v}$

$g(t)$ = curve in G tangent to $X \in \mathfrak{g}$ at $t=0$

$\| \; e^{tX}$

$p(t) = Ad^*_{g(t)} p \qquad p \in W_p$

$\langle p(t), Y \rangle = \langle p, Ad_{g^{-1}(t)} Y \rangle$

$\langle \dot{p}(0), Y \rangle = - \langle p, ad_X Y \rangle = \langle ad^*_X p, Y \rangle$

$\uparrow$

$T_q g^* \cong g^*$

W_p ALWAYS HAS A NATURAL SYMPLECTIC STRUCTURE

$\omega(\vec{v}, \vec{w})$ $\quad \vec{v} \in T_p W_p$ $\quad$ & $\quad d\omega = 0$

HOW TO IDENTIFY $T_q W_p$?

$g(t)$ = curve in G tangent to $X \in \mathfrak{g}$ at $t = 0$

$\| \quad e^{tX}$

$p(t) = Ad^*_{g(t)} P \qquad P \in W_p$

$\langle p(t), Y \rangle = \langle P, Ad_{g^{-1}(t)} Y \rangle$

$\dot{p}(0) = ad^*_X p \Rightarrow T_q W_p = \{ ad^*_X p \mid X \in \mathfrak{g} \}$

TANGENT SPACE SPAN THE COADJOINT ACTION

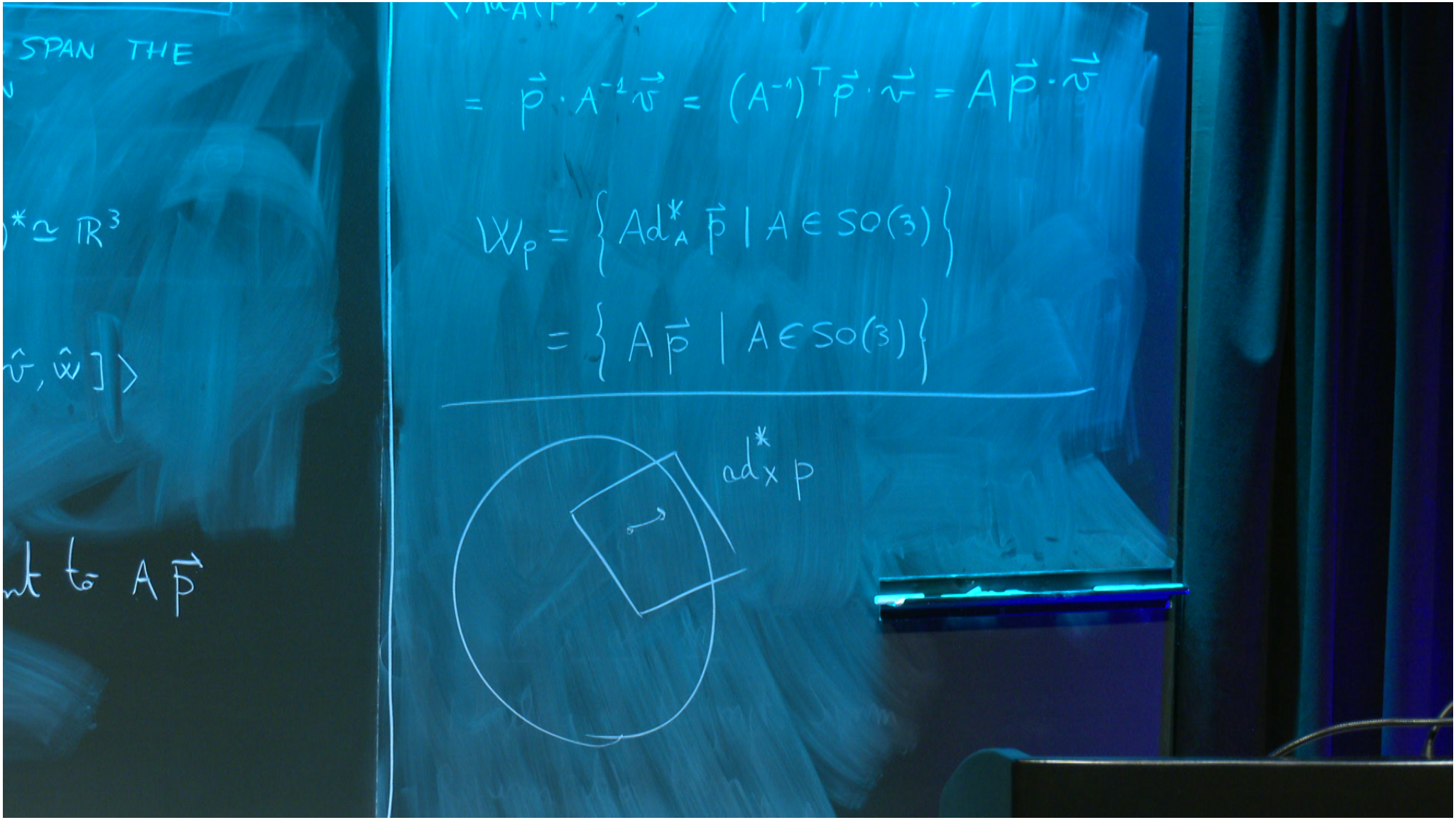
EXAMPLE: $SO(3)$

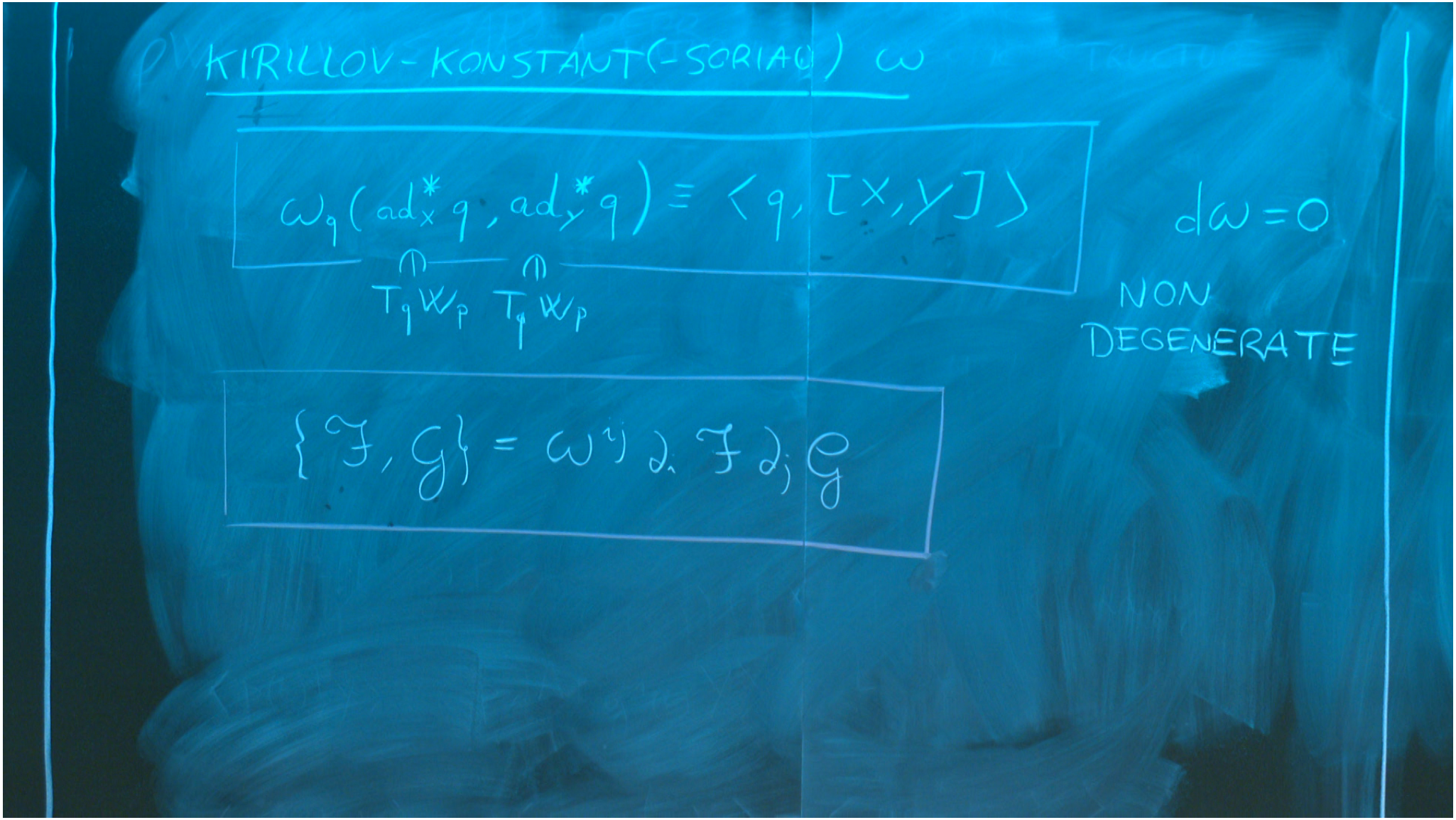
$(\mathfrak{so}(3), [\cdot,\cdot]) \simeq (\mathbb{R}^3, \times)$ & $\mathfrak{so}(3)^* \simeq \mathbb{R}^3$

$\langle \dot{p}(0), \hat{w} \rangle = \langle ad^*_{\hat{v}} p, \hat{w} \rangle = \langle p, -[\hat{v}, \hat{w}] \rangle$

$= -\vec{p} \cdot (\vec{v} \times \vec{w}) = -(\vec{p} \times \vec{v}) \cdot \vec{w}$

$\Rightarrow \dot{\vec{p}}(0) = -\vec{p} \times \vec{v} \Rightarrow$ it is tangent to $A\vec{p}$


SPAN THE
$= \vec{p} \cdot A^{-1}\vec{v} = (A^{-1})^T \vec{p} \cdot \vec{v} = A\vec{p} \cdot \vec{v}$
$W_p = \{ Ad^*_A \vec{p} \mid A \in SO(3) \}$
$= \{ A\vec{p} \mid A \in SO(3) \}$
$ad^*_x p$


KIRILLOV-KONSTANT(-SORIAU) ω
$\omega_q(\mathrm{ad}^*_x q, \mathrm{ad}^*_y q) = \langle q, [x,y] \rangle$
$T_q W_p$ $T_q W_p$
$d\omega = 0$
NON DEGENERATE
$\{f, g\} = \omega^{ij} \partial_i f \partial_j g$

HAMILTONIAN FIELD & MOMENTUM MAP

$$H \in C^\infty(M) \longrightarrow \xi_H = -\{H, \cdot\}$$

$$\dot{x}(t) = \xi_H(x(t)) = \{x^i, H\}$$

$$\xi_H^i = \omega^{ij}\partial_j H \iff \partial_j H = \omega_{ij}\xi_H^j$$

$$[\xi_f, \xi_g] = -\xi_{\{f,g\}}$$

$$\xi_X = \frac{d}{dt}\left(e^{tX}\cdot q\right)\Big|_{t=0}$$

GOAL: J_X s.t. $\xi_X = \xi_{J_X} = -\{J_X, \cdot\}$

$$\Rightarrow \{\{J_X, J_Y\}, \mathcal{F}\} = \{J_{[X,Y]}, \mathcal{F}\}$$

$$\Rightarrow \{J_X, J_Y\} = J_{[X,Y]} + \cancel{c(X,Y)} \quad X, Y \in \mathfrak{g}$$

FOR COADJOINT ORBITS

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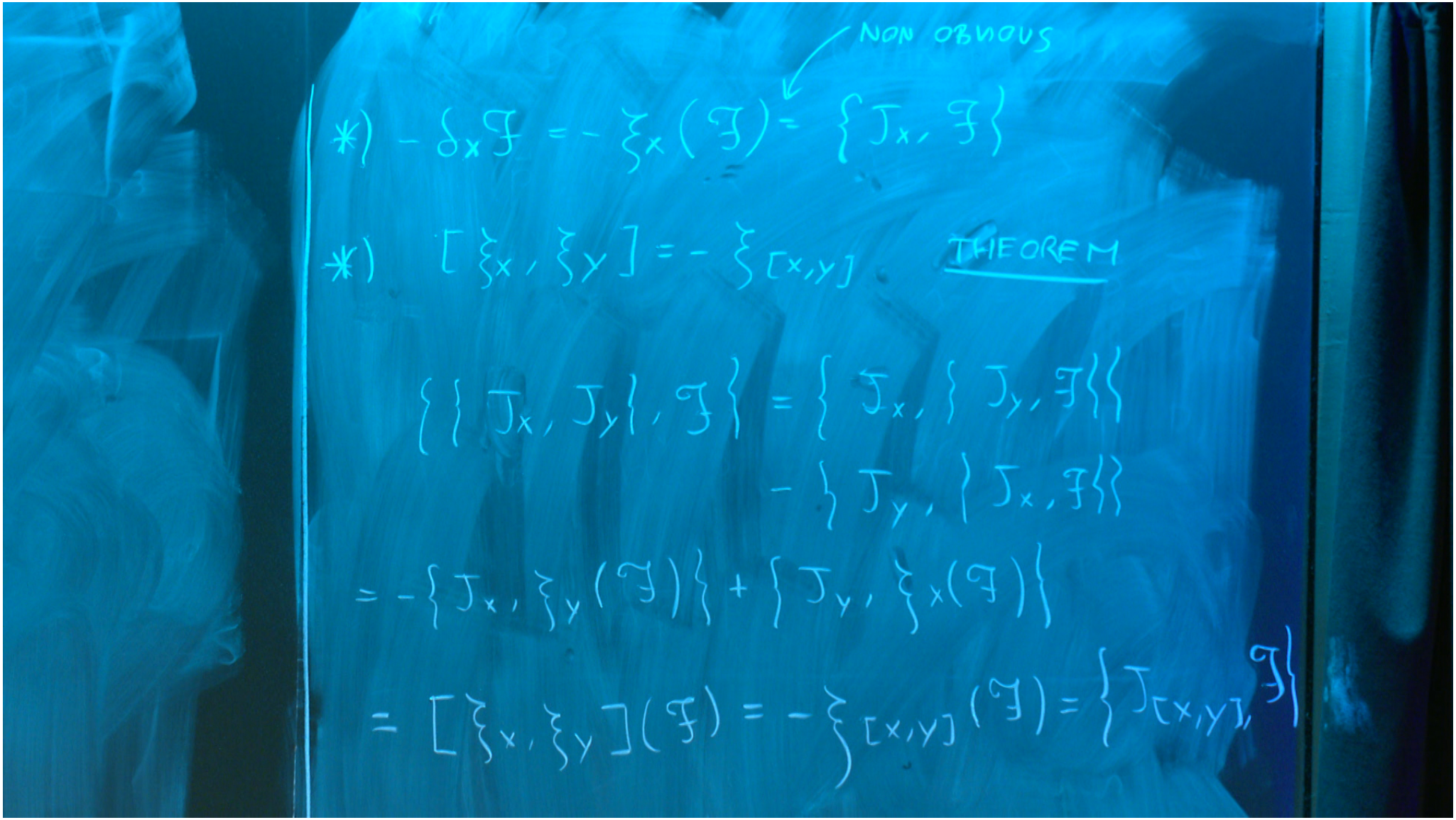
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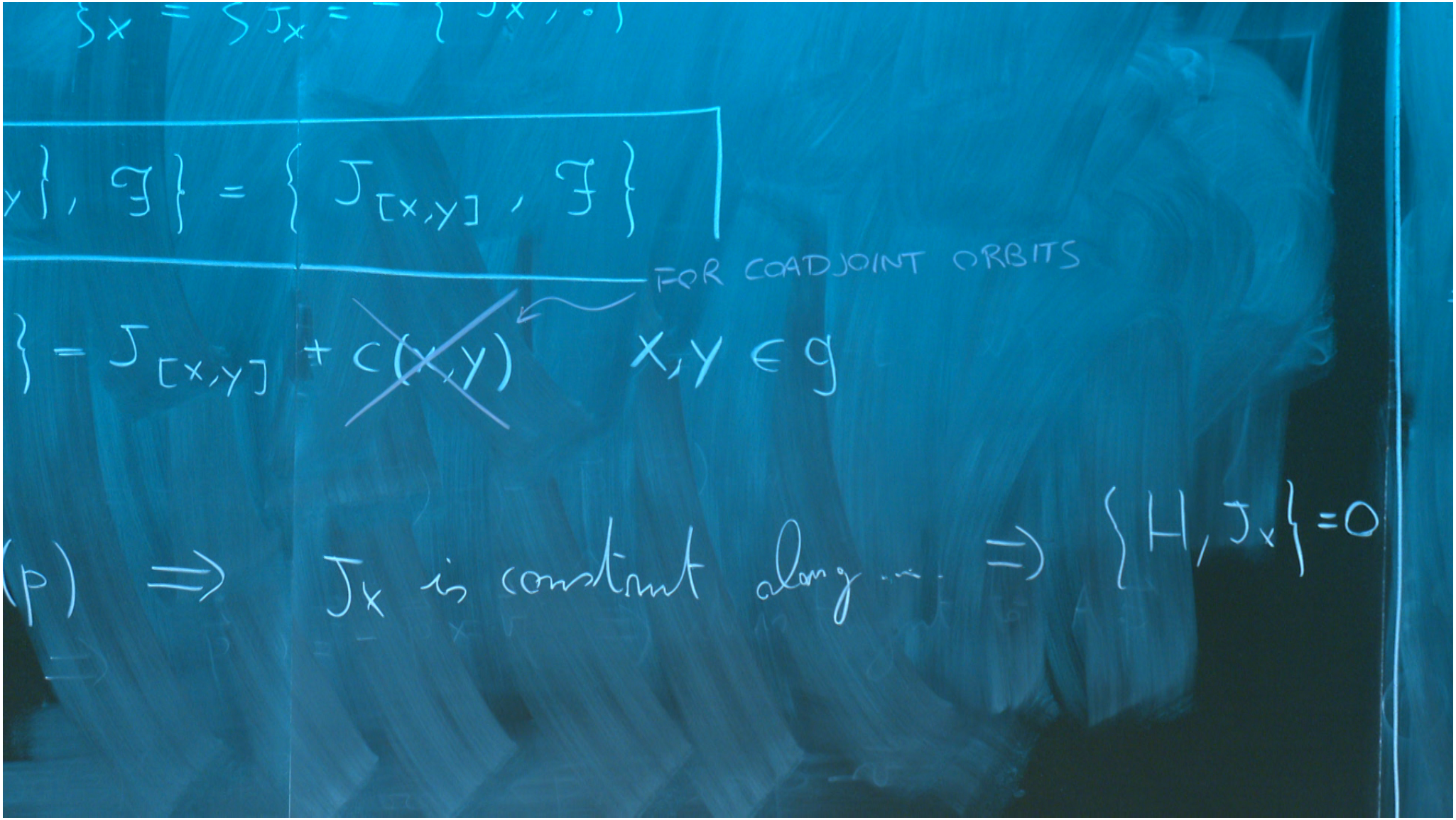
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FOR COADJOINT ORBITS

$$\Rightarrow \{J_X, J_Y\} = J_{[X,Y]} + \cancel{c(X,Y)} \qquad X, Y \in \mathfrak{g}$$


NON OBVIOUS
*) $-\delta_x \mathcal{F} = -\xi_x(\mathcal{F}) = \{J_x, \mathcal{F}\}$
*) $[\xi_x, \xi_y] = -\xi_{[x,y]}$
THEOREM
$\{\{J_x, J_y\}, \mathcal{F}\} = \{J_x, \{J_y, \mathcal{F}\}\}$
$- \{J_y, \{J_x, \mathcal{F}\}\}$
$= -\{J_x, \xi_y(\mathcal{F})\} + \{J_y, \xi_x(\mathcal{F})\}$
$= [\xi_x, \xi_y](\mathcal{F}) = -\xi_{[x,y]}(\mathcal{F}) = \{J_{[x,y]}, \mathcal{F}\}$


FOR COADJOINT ORBITS
$X, Y \in \mathfrak{g}$
$\Rightarrow$
$\{H, J_x\} = 0$

GOAL: J_x s.t. $\{x = \{J_x = \{J_x, \cdot\}$

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FOR COADJOINT ORB

$$\Rightarrow \{J_x, J_y\} = J_{[x,y]} + \cancel{c(x,y)} \qquad x, y \in \mathfrak{g}$$

$H(g \cdot p) = H(p) \Rightarrow J_x$ is constant along ... $\Rightarrow$