Title: Avoiding the Corners: Partition Functions of Abelian Chern-Simons Theories on Handlebodies

Speakers: Massimo Porrati

Collection: Quantum Gravity Around the Corner

Date: October 04, 2022 - 2:45 PM

URL: https://pirsa.org/22100040

Abstract: This talk reviews the use of radial quantization to compute Chern-Simons partition functions on handlebodies of arbitrary genus. The partition function is given by a particular transition amplitude between two states which are defined on the Riemann surfaces that define the (singular) foliation of the handlebody. By requiring that the only singularities of the gauge field inside the handlebody must be compatible with Wilson loop insertions, we find that the Wilson loop shifts the holonomy of the initial state. Together with an appropriate choice of normalization, this procedure selects a unique state in the Hilbert space obtained from a Kähler quantization of the theory on the constant-radius Riemann surfaces. Radial quantization allows us to find the partition functions of Abelian Chern-Simons theories for handlebodies of arbitrary genus. For non-Abelian compact gauge groups, we show that our method reproduces the known partition function and Wilson loop VEVs at genus one.

PARTITION FUNCTION OF ABELIAN CHERN-SIMONS THEORIES ON HANDLEBODIES

BASED ON

MP and Cedric Yu

JHEP 07 (2021) 194, arXiv:2104.12799, MP

- CHERN-SIMONS THEORIES ON OPEN MANIFOLDS: METRIC DEPENDENCE FROM BOUNDARY CONDITIONS
- PARTITION FUNCTIONS OF C-S ON OPEN MANIFOLDS AS WAVE FUNCTIONS
- AN EXPLICIT BASIS OF WAVE FUNCTION FOR ABELIAN C-S
- AN IMPLICIT BASIS OF PARTITION FUNCTIONS FOR ABELIAN AND NONABELIAN C-S
- RELATING THE BASES FOR HANDLEBODIES: 1) THE INITIAL CONDITION
- RELATING THE BASES: 2) RADIAL QUANTIZATION OF C-S AS PROJECTION OVER GAUGE INVARIANT WAVE FUNCTIONS

- WILSON LOOPS (AND THE FRAMING ANOMALY)
- THE PARTITION FUNCTION OF NON-ABELIAN C-S ON A TORUS HANDLEBODY
- THE PARTITION FUNCTION OF NON-ABELIAN C-S WITH WILSON LOOPS ON A TORUS HANDLEBODY (AND A CONJECTURED MATHEMATICAL IDENTITY)

THE ABELIAN CHERN-SIMONS BULK LAGRANGIAN

$$I = \int_M AdA$$

$$A = \text{Abelian connection on } M$$

THE ACTION IS TOPOLOGICAL BUT A DEPENDENCE ON NON-TOPOLOGICAL DATA IS INTRODUCED BY THE BOUNDARY TERM

$$I_B = \int_\Sigma d^2x A_z A_{\bar{z}} \ , \qquad \Sigma = \partial M$$

TO DEFINE THE BOUNDARY TERM WE MUST INTRODUCE A COMPLEX STRUCTURE (z AND z^*)

THE FUNCTIONAL INTEGRAL

$$Z[\bar{A}] = \int_{A_{\bar{z}}|_{\Sigma}=\bar{A}} [dA] \exp\left(-i\frac{k}{4\pi} I + \frac{k}{2\pi} I_B\right)$$

DEFINES A WAVE FUNCTION IN HOLOMORPHIC QUANTIZATION

GAUGE-INVARIANT SCALAR PRODUCT

$$(\Phi, \Psi) = \int [d\bar{A}] \Phi^*(\bar{A}) \Psi(\bar{A}) \exp\left(-\frac{k}{\pi} I_B\right)$$

GAUGE TRANSFORMATION

$$U(\lambda)\Psi(\bar{A}) = \exp\left(-\frac{k}{2\pi} \int_{\Sigma} d^2x (\partial\lambda\bar{\partial}\lambda - 2\partial\lambda\bar{A})\right) \Psi(\bar{A} - \bar{\partial}\lambda)$$

Page 7 sur 44

FOR ABELIAN C-S AN EXPLICIT BASIS OF WAVE FUNCTIONS IS KNOWN (BOS AND NAIR 1990)

$$\Psi(\bar{A}, \mu, \Omega) = F(\Omega)^{-1/2} \exp\left[\frac{k\pi}{2} u(\mathrm{Im}\,\Omega)^{-1} u + \frac{k}{2\pi}\int_\Sigma d^2x \partial\chi\bar{\partial}\chi\right] \theta\left[\begin{array}{c}\mu/k \\ 0\end{array}\right](ku, k\Omega)$$

$$\mu \in Z_k^g, \qquad k \in 2\mathbb{Z}, \qquad \bar{A} = \sum_I u^I \bar{\omega}^I + \bar{\partial}\chi$$

$$\frac{\det' \Delta}{\mathrm{Im}\ \det \Omega} = |F(\Omega)|^2 \exp(-S)$$

Zograf-Takhtajan 1988, Quillen 1985

FOR ABELIAN C-S AN EXPLICIT BASIS OF WAVE FUNCTIONS IS KNOWN (BOS AND NAIR 1990)

$$\Psi(\bar{A}, \mu, \Omega) = F(\Omega)^{-1/2} \exp\left[\frac{k\pi}{2} u (\mathrm{Im}\,\Omega)^{-1} u + \frac{k}{2\pi} \int_\Sigma d^2x \partial\chi\bar{\partial}\chi\right] \theta \left[\begin{array}{c} \mu/k \\ 0 \end{array}\right] (ku, k\Omega)$$

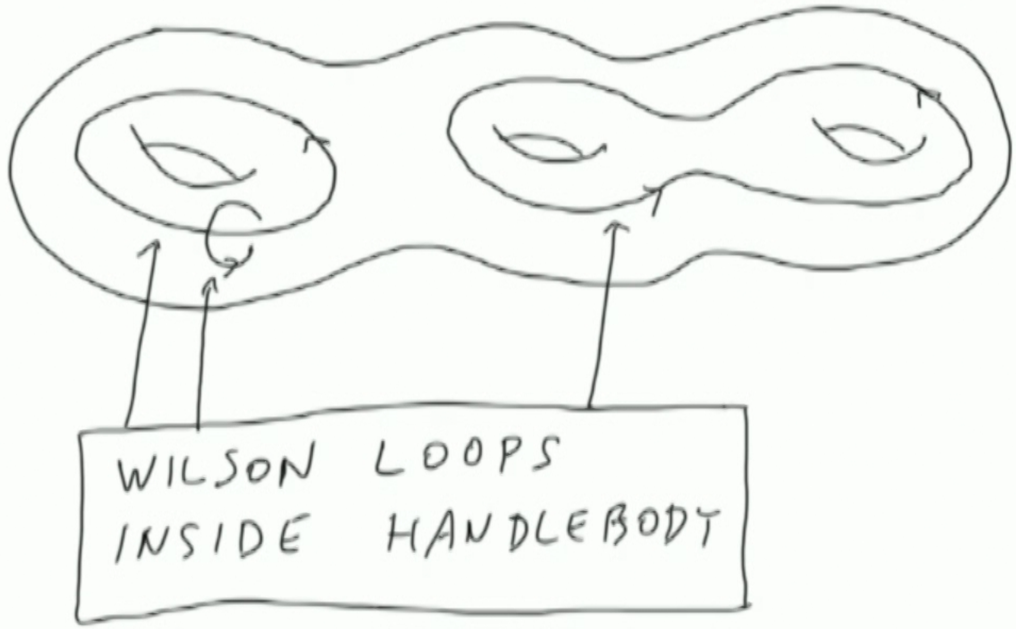
$$\mu \in Z_k^g, \qquad k \in 2\mathbb{Z}, \qquad \bar{A} = \sum_I u^I \bar{\omega}^I + \bar{\partial}\chi$$

$$\frac{\det' \Delta}{\mathrm{Im}\,\det\Omega} = |F(\Omega)|^2 \exp(-S)$$

Zograf-Takhtajan 1988, Quillen 1985

WHICH LINEAR COMBINATION OF THESE BASIS VECTORS IS THE PARTITION FUNCTION?

WE KNOW THE ANSWER IMPLICITLY IN ANOTHER BASIS (WITTEN 1989)



WILSON LOOP GENERATE A COMPLETE BASIS OF WAVE FUNCTIONS FOR COMPACT ABELIAN CS BY THE STATE OPERATOR CORRESPONDENCE

WILSON LOOPS GENERATE A COMPLETE SET OF WAVE FUNCTIONS ON THE RIEMANN SURFACE BY THE STATE-OPERATOR CORRESPONDENCE

$$Z[\bar{A}|\mu] = \int_{A_{\bar{z}}|_{\Sigma}=\bar{A}} [dA] \prod_{I=1}^{g} \exp\left(i\mu^I \oint_{C^I} A\right) \exp\left(-i\frac{k}{4\pi} I + \frac{k}{2\pi} I_B\right), \qquad \mu \in \mathbb{Z}_k^g$$

HOW ARE THESE WAVE FUNCTIONS RELATED TO THE EXPLICIT BOS-NAIR BASIS?

BOS-NAIR IS HOLOMORPHIC IN THE GAUGE CONNECTION

IT IS ALSO HOLOMORPHIC IN THE COMPLEX STRUCTURE

WE COULD TRY TO DECOMPOSE THE HANDLEBODY INTO AN N-HOLED SPHERE AND N SOLID CYLINDERS (HANDLES)

BUT THE CORNERS IN THE CYLINDER AND IN THE HOLED SPHERE INTRODUCE EXTRA DEGREES OF FREEDOM THAT WE WERE NOT ABLE TO ACCOUNT CORRECTLY

THE CORRECT ANSWER IS SUGGESTED BY THE STATE-OPERATOR CORRESPONDENCE CONSTRUCTION

FOLIATE THE HANDLEBODY AS

$$M = \Sigma \times [0, R]$$

USE RADIAL QUANTIZATION

$$Z = (A_{\bar{z}} = \bar{A}| \exp(-RH)|\Psi_0\rangle$$

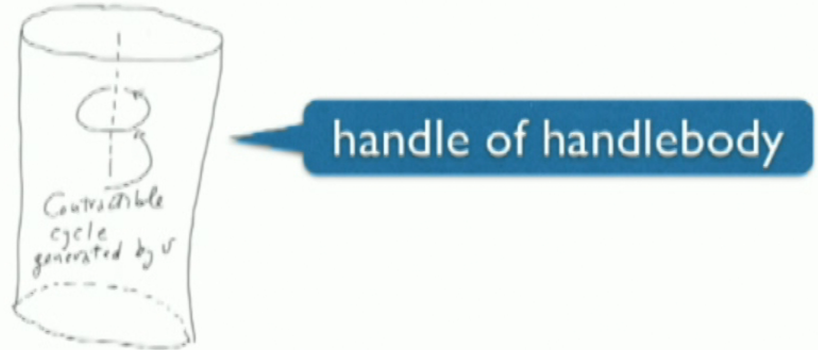
TO WIT: WRITE THE FUNCTIONAL INTEGRAL AS A KERNEL BETWEEN AN APPROPRIATE INITIAL STATE AND A COHERENT FINAL STATE OF THE EVOLUTION UNDER THE RADIAL HAMILTONIAN H

WE NEED TO FIND THE APPROPRIATE INITIAL STATE AND H

CASE A): NO WILSON LOOP

THE FOLIATION DEGENERATES AT R=0 BUT THE MANIFOLD M IS REGULAR EVERYWHERE

THIS IMPOSES A STRONG CONDITION ON THE INITIAL STATE



STREBEL DIFFERENTIAL h: MEROMORPHIC, CLOSED.
ITS HORIZONTAL TRAJECTORIES COVER A SET OF CODIMENSION ZERO

HORIZONTAL TRAJECTORY IS GIVEN BY

$$\text{for trajectory } z(t), \quad h\left(\frac{dz}{dt}\right)^2 \text{ is real and positive}$$

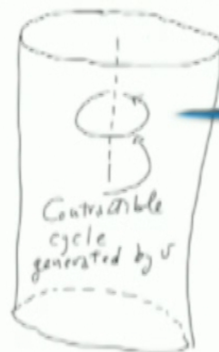
THE STREBEL DIFFERENTIAL DEFINES A FLAT METRIC g AND A 2-VALUED VECTOR FIELD v

$$g = \sqrt{h\bar{h}}, \quad v = \frac{1}{\sqrt{h}}\partial_z + \frac{1}{\sqrt{\bar{h}}}\partial_{\bar{z}}$$

$$dz/dt = fv, \quad f = \text{real function}$$

THE COMPONENTS OF THE VECTOR FIELD v MAY HAVE BRANCH CUTS BUT THEY CANCEL IN THE RATIO v*/v

THE HORIZONTAL TRAJECTORIES OF THE STREBEL DIFFERENTIAL CAN BE CHOSEN TO BE HOMOLOGOUS TO HOMOLOGY CYCLES OF THE RIEMANN SURFACE THAT FOLIATES THE MANIFOLD M WHICH ARE CONTRACTIBLE IN M



homologous to a horizontal trajectory of the Strebel differential

SINCE THE HANDLEBODY IS NOT SINGULAR AT r=0 WHERE THE FOLIATION DEGENERATES, THE INITIAL STATE MUST BE SUCH THAT THE GAUGE FIELD VANISHES ALONG THE VECTOR v THAT GENERATES THE CONTRACTIBLE CYCLE.

$$(\bar{v}A_{\bar{z}} + vA_z)|\Psi_0\rangle = 0, \qquad A_z = \frac{\pi}{k}\frac{\delta}{\delta A_{\bar{z}}}$$

THE INITIAL STATE OBEYING THIS CONDITION IS A SQUEEZED STATE

$$(A_{\bar{z}}|\Psi_0\rangle = \exp\left(-\frac{k}{2\pi}\int_\Sigma d^2x \frac{\bar{v}}{v} A_{\bar{z}}^2\right)$$

THIS STATE DOES NOT OBEY THE GAUSS LAW, BUT ITS RADIAL EVOLUTION DOES THANKS TO THE R INDEPENDENT IDENTITY

$$\sum\!\!\!\!\!\!\int [d\lambda](A_{\bar{z}}|U(\lambda)|\Psi_0\rangle = (A_{\bar{z}}|\exp(-RH)|\Psi_0\rangle$$

SO RADIAL EVOLUTION PROJECTS OVER GAUGE INVARIANT STATES. WE CAN THEREFORE USE OUR NON GAUGE INVARIANT INITIAL STATE AS A "SEED" THAT GENERATES THE WAVE FUNCTION CORRESPONDING TO THE VACUUM PARTITION FUNCTION.

THE INTEGRAL OVER GAUGE ORBITS CAN BE DONE EXPLICITLY BECAUSE IS QUADRATIC

$$\sum\!\!\!\!\int [d\lambda] \langle A_{\bar{z}}|U(\lambda)|\Psi_0\rangle = \sum\!\!\!\!\int [d\lambda] \exp\left(-\frac{k}{2\pi}\int d^2x\, 2(\partial\lambda+\theta)\bar{A} + (\partial\lambda+\theta)(\bar{\partial}\lambda+\bar{\theta}) + \frac{\bar{v}}{v}(\bar{A}+\bar{\partial}\lambda+\bar{\theta})^2\right)$$

large gauge transformation

THE NONTRIVIAL PART OF THE GAUSSIAN INTEGRAL IS THE EQUATION FOR THE SADDLE POINT BECAUSE IT APPEARS TO DEPEND ON THE METRIC AND ALSO ON THE CHOICE OF THE VECTOR v

THE SADDLE POINT EQUATION IS

$$\partial\bar{\partial}\lambda + \partial\bar{A} + \bar{\partial}\frac{\bar{v}}{v}(\bar{\partial}\lambda + \bar{\theta} + \bar{A}) = 0$$

DECOMPOSE GAUGE FIELD IN EXACT + HARMONIC PART

$$\bar{A} = \bar{\partial}\chi + \bar{\Theta}$$

SIMPLE MANIPULATIONS TRANSFORM THE SADDLE POINT EQUATION INTO

$$\partial_h(\lambda + \chi) + v(\theta + \Theta) + \bar{v}(\bar{\theta} + \bar{\Theta}) = \text{ holomorphic function}, \quad \partial_h \equiv v\partial + \bar{v}\bar{\partial}$$

INTEGRATING THIS EQUATION OVER A CONTRACTIBLE CYCLE AND NOTICING THAT THE INTEGRAL OF THE EXACT PART VANISHES WE GET

$$\partial_h(\lambda + \chi) = v(\tilde{\theta} + \tilde{\Theta}) - \bar{v}(\bar{\theta} + \bar{\Theta})$$

$$\bar{\theta} + \bar{\Theta} = U_i\bar{\omega}^i \qquad \tilde{\theta} + \tilde{\Theta} = U_i\omega^i$$

THE SADDLE POINT EQUATION IS

$$\partial\bar\partial\lambda + \partial\bar A + \bar\partial\frac{\bar v}{v}(\bar\partial\lambda + \bar\theta + \bar A) = 0$$

DECOMPOSE GAUGE FIELD IN EXACT + HARMONIC PART

$$\bar A = \bar\partial\chi + \bar\Theta$$

SIMPLE MANIPULATIONS TRANSFORM THE SADDLE POINT EQUATION INTO

$$\partial_h(\lambda + \chi) + v(\theta + \Theta) + \bar v(\bar\theta + \bar\Theta) = \text{ holomorphic function}, \quad \partial_h \equiv v\partial + \bar v\bar\partial$$

INTEGRATING THIS EQUATION OVER A CONTRACTIBLE CYCLE AND NOTICING THAT THE INTEGRAL OF THE EXACT PART VANISHES WE GET

$$\partial_h(\lambda + \chi) = v(\tilde{\bar\theta} + \tilde{\bar\Theta}) - \bar v(\bar\theta + \bar\Theta)$$

$$\bar\theta + \bar\Theta = U_i\bar\omega^i \qquad \tilde{\bar\theta} + \tilde{\bar\Theta} = U_i\omega^i$$

SUBSTITUTE INTO ACTION AND FIND A PLEASANT SURPRISE

ALL TERMS CONTAINING THE HARMONIC PART OF THE GAUGE CONNECTION DEPEND NEITHER ON THE METRIC g NOR ON v

THE QUADRATIC FLUCTUATIONS AROUND THE SADDLE POINT ARE EASILY EVALUATED AND THE FINAL RESULT IS (AFTER DISCARDING AN INFINITE SUM OVER IDENTICAL GAUGE COPIES)

$$(\bar{A}|\exp(-RH)|\Psi_0\rangle = F(\Omega)^{-1/2}\exp\left[\frac{k\pi}{2}u(\mathrm{Im}\,\Omega)^{-1}u + \frac{k}{2\pi}\int_\Sigma d^2x\partial\chi\bar{\partial}\chi\right]\theta\left[\begin{array}{c}0\\0\end{array}\right](ku, k\Omega)$$

THIS IS THE BOS-NAIR WAVE FUNCTION FOR $\mu = 0$

CASE B): WILSON LOOP INSERTION

WE START BY DEFINING A WILSON LOOP "SPREAD OUT" OVER THE RIEMANN SURFACE

$$\exp\left(i\mu\oint_C A\right) \Rightarrow \exp\left(\int_\Sigma d^2x(wA_{\bar{z}} + \bar{w}A_z)\right) \equiv W[\Sigma, w]$$

THE HARMONIC PART w' OF THE VECTOR FIELD w MUST OBEY THE FOLLOWING CONDITIONS TO ENSURE GAUGE INVARIANCE UNDER SMALL AND LARGE GAUGE TRANSFORMATIONS

$$\bar{w}' = (\mathrm{Im}\,\Omega)^{-1}(\Omega\mu + N)\bar{\omega} \qquad \mu, N \in \mathbb{Z}^g$$

THE INITIAL STATE WITH WILSON LOOP INSERTION IS

$$W[\Sigma, w]|\Psi_0\rangle = |\Psi_0\rangle \exp\left(-i\int_\Sigma d^2x\bar{w}\left[\bar{w}\frac{\bar{v}}{v} - w\right]A_{\bar{z}} + \frac{\pi}{2k}\int_\Sigma d^2x\bar{w}\left[\bar{w}\frac{\bar{v}}{v} - w\right]\right)$$

INSERT THIS INITIAL STATE INTO THE INTEGRAL OVER GAUGE ORBITS TO GET THE PARTITION FUNCTION WITH WILSON LOOP INSERTION

$$(\bar{A}|\exp(-RH)W[\Sigma,\mu]|\Psi_0\rangle = F(\Omega)^{-1/2}\exp\left[\frac{k\pi}{2}u(\mathrm{Im}\,\Omega)^{-1}u + \frac{k}{2\pi}\int_\Sigma d^2x\partial\chi\bar{\partial}\chi + i\frac{\pi}{k}\mu N\right]\theta\left[\begin{array}{c}\frac{\mu}{k}\\0\end{array}\right](ku, k\Omega)$$

this phase has an interesting interpretation

WRITE THE WILSON LOOP WITH VECTOR w AS THE PRODUCT

$$W[\Sigma, w] = W[\Sigma, w_\mu]W[\Sigma, w_N]\exp\left(+i\frac{\pi}{k}\mu N\right)$$

$$\bar{w}_\mu = \bar{\partial}f + (\mathrm{Im}\,\Omega)^{-1}\Omega\mu\bar{\omega} \qquad \bar{w}_N = \bar{\partial}g + (\mathrm{Im}\,\Omega)^{-1}N\bar{\omega}$$

THE FACTORS IN THE PRODUCT HAVE A NATURAL INTERPRETATION

$$W[\Sigma, w_\mu] = \text{blowup of Wilson loop along noncontractible cycle}$$

$$W[\Sigma, w_N] = \text{blowup of Wilson loop along contractible cycle}$$

THE WILSON LOOP ALONG THE CONTRACTIBLE CYCLE VANISHES ON THE INITIAL STATE

$$W[\Sigma, w]|\Psi_0\rangle = W[\Sigma, w_\mu]W[\Sigma, w_N]\exp\left(+i\frac{\pi}{k}\mu N\right)|\Psi_0\rangle = W[\Sigma, w_\mu]\exp\left(+i\frac{\pi}{k}\mu N\right)|\Psi_0\rangle$$

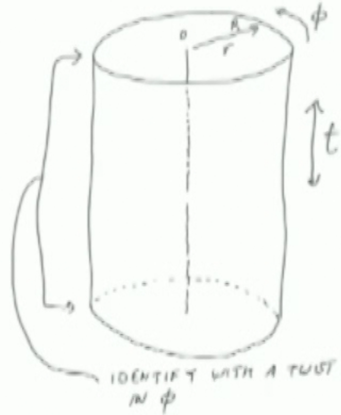
this phase is the only effect of inserting a Wilson loop winding N times along the contractible cycles of the handlebody

WHEN

$$N = \mu N', \quad N' \in \mathbb{Z}$$

THE PHASE IS THE WELL KNOWN FRAMING ANOMALY

NONABELIAN C-S ON THE TORUS



CHOOSE AS INITIAL BOUNDARY CONDITION AT r=0

$$A_\phi = 2\pi \frac{\mu}{k}, \quad \mu \in \frac{\Lambda_w}{W \ltimes k\Lambda_r} \subset \mathcal{C}, \quad \mathcal{C} = \text{Cartan subalgebra}$$

Weyl alcove (undilated)

CHOOSE AS FINAL BOUNDARY CONDITION AT r=R

$$A_{\bar{z}} = i \frac{u}{\text{Im}\,\tau}, \quad u \in \mathcal{C}, \quad \mathcal{C} = \text{Cartan subalgebra}$$

THE TRANSITION AMPLITUDE BETWEEN THE STATES

$$A_\phi|\mu\rangle = 2\pi\frac{\mu}{k}|\mu\rangle, \qquad A_{\bar{z}}|u) = i\frac{u}{\mathrm{Im}\,\tau}|u)$$

IS GIVEN BY THE FUNCTIONAL INTEGRAL

$$(u|\exp(-RH)|\mu\rangle = \int [dA]\delta[F_{z\bar{z}}]\exp(iI)$$

IT IS COMPUTED BY USING FOLLOWING DECOMPOSITION THE GAUGE FIELD ON THE 2-TORUS

$$A = g^{-1}(d+a)g \qquad g \in G \qquad a \in \mathcal{C}$$

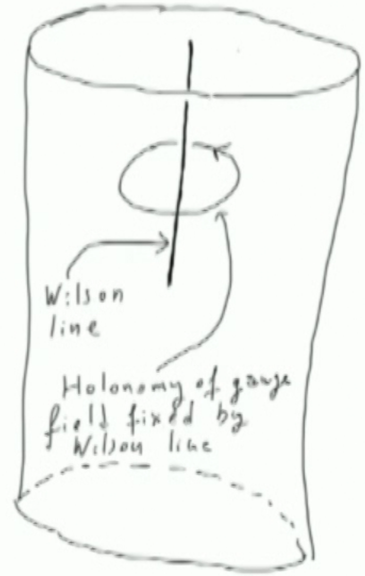
THE ACTION REDUCES TO THAT OF A CHIRAL WESS-ZUMINO MODEL AND THE INTEGRAL OVER g GIVES (PERROT 1991)

$$(u|\exp(-RH)|\mu\rangle = \exp\left(-\frac{\pi k}{2\mathrm{Im}\,\tau}\mathrm{Tr}\,u^2\right)\chi_{\mu,k}(u,\tau)$$

$$\chi_{\mu,k}(u,\tau) = \text{Weyl-Kac character}$$

ON THE OTHER HAND THE WILSON LOOP FIXES THE HOLONOMY ALONG THE CONTRACTIBLE CYCLE

$$\frac{k}{2\pi} F_{z\bar{z}} = \mu \delta^2(z, \bar{z}) \rightarrow A_\phi = 2\pi \frac{\mu}{k}$$



THE EQUALITY OF THE TWO FORMULAS IMPLIES AN IDENTITY AMONG WEYL CHARACTERS THAT WE COULD ONLY PROVE BY BRUTE FORCE FOR G=SU(2) ONLY

$$\sum_{\nu \in \Omega_\mu} \chi_\nu = \chi_\mu$$

SUMMARY

- THE PATH INTEGRAL OF ABELIAN C-S THEORY ON HANDLEBODIES CAN BE COMPUTED EXPLICITLY BY RADIAL QUANTIZATION
- RADIAL EVOLUTION FROM AN INITIAL STATE PROJECTS OVER GAUGE INVARIANT WAVE FUNCTIONS
- THE INITIAL WAVE FUNCTION IS FIXED BY REQUIRING THAT THE GAUGE FIELD ALONG CONTRACTIBLE HOLONOMY CYCLES IS DETERMINED BY THE WILSON LOOP
- PARTITION FUNCTIONS OF NONABELIAN C-S THEORIES ON TORUS HANDLEBODIES CAN BE COMPUTED BY METHODS SIMILAR TO THOSE USED IN THE ABELIAN CASE

SUMMARY

- FOR NONABELIAN C-S ON TORUS HANDLEBODIES, WE FOUND THAT THE VALIDITY OF THE RADIAL QUANTIZATION METHOD REQUIRES A CURIOUS IDENTITY AMONG WEYL CHARACTERS THAT WE COULD PROVE ONLY FOR SU(2)