

Title: Residual gauge symmetries in the front form

Speakers: Sucheta Majumdar

Collection: Quantum Gravity Around the Corner

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Abstract: The goal of this talk is to discuss residual gauge symmetries in electromagnetism and gravity in Dirac's front form. Working in the light-cone gauge, I will demonstrate how the large gauge transformations and BMS supertranslations may be obtained from residual gauge invariance of the Hamiltonian action. The residual gauge symmetries in this (2+2) formulation share some striking similarities with the asymptotic symmetries in the conventional (3+1) Hamiltonian formulation. I will illustrate this fact using the example of electromagnetism and show how the the zero modes play a crucial role akin to boundary degrees of freedom in the asymptotic analysis at spatial infinity à la Henneaux-Troessaert.

Residual gauge symmetries in the front form

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Dirac's Front Form

REVIEWS OF MODERN PHYSICS

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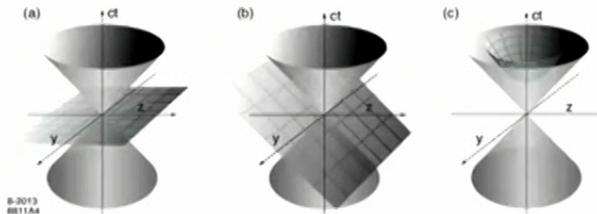
JULY, 1949

Forms of Relativistic Dynamics

P. A. M. DIRAC
St. John's College, Cambridge, England

For the purposes of atomic theory it is necessary to combine the restricted principle of relativity with the Hamiltonian formulation of dynamics. This combination leads to the appearance of ten fundamental quantities for each dynamical system, namely the total energy, the total momentum and the 6-vector which has three components equal to the total angular momentum. The usual form of dynamics expresses everything in terms of dynamical variables at one instant of time, which results in specially simple expressions for six of these ten, namely the components of momentum and of angular momentum. There are other forms for relativistic dynamics in which others of the ten are specially simple, corresponding to various sub-groups of the inhomogeneous Lorentz group. These forms are investigated and applied to a system of particles in interaction and to the electromagnetic field.

- Dirac introduced three forms of relativistic dynamics



DOI:10.1016/j.physrep.2015.05.001

- (a) Instant form: time x^0
Initial data on a spatial hypersurface
- (b) Front form: time $x^+ = \frac{x^0 + x^3}{\sqrt{2}}$
Initial data on a null hypersurface
- (c) Instant form: Proper time τ

- Front form → Use a null time parameter to study the dynamics

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Dirac's front form

- Light-cone coordinates

$$x^+ = \frac{x^0 + x^3}{\sqrt{2}}, \quad x^- = \frac{x^0 - x^3}{\sqrt{2}}, \quad x^i \quad (i = 1, 2)$$

$$x^+ \quad \text{Light-cone time} \quad \Rightarrow \quad P_+ = i\partial_+ = -P^- \quad \text{Hamiltonian}$$

- The three "Hamiltonians" in the front form

Poincaré generators in the instant form: $(P_\mu, M_{\mu\nu})$

$$[P, P] \sim 0, \quad [P, M] \sim P, \quad [M, M] \sim M$$

$(P_0, M_{0i}) \rightarrow$ four dynamical generators or "Hamiltonians"

Poincaré generators in front form

Kinematical $K = \{P_i, P_-, M_{ij}, M_{+-}\},$

Dynamical $D = \{P_+, M_{i+}\} \rightarrow$ three "Hamiltonians" pick up corrections

- Constraint equations in the front form are often soluble: [Unconstrained Hamiltonian systems](#)



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Outline

- Electromagnetism in the front form
- Residual gauge symmetries in light-cone EM
- Links to asymptotic analysis at spatial infinity
- Light-cone gravity in the front form
- BMS symmetries in light-cone gravity



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- Electromagnetism in the front form
- Residual gauge symmetries in light-cone EM
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One-line summary

A particular example of null-front Hamiltonian analysis and its connections with the corner

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Electromagnetism in the front form

- Light-cone gauge

$$A_- = -A^+ = -\frac{A^0 + A^3}{\sqrt{2}} = 0$$

- Maxwell equations: $\partial_\mu F^{\mu\nu} = 0$

1) Constraints

$$(\nu = +): \quad \partial_-^2 A^- + \partial_i \partial_- A^i = 0 \quad \Rightarrow \quad A^- = -\frac{\partial_i A^i}{\partial_-} + \alpha(x^+, x^i) x^- + \beta(x^+, x^i)$$

$$(\nu = -): \quad \text{constraint relating } \alpha \text{ and } \beta \quad \Rightarrow \quad \text{only one arbitrary constant}$$

A further choice: **set the constant to zero**

2) Dynamical equation

$$(\nu = i): \quad (2\partial_- \partial_+ - \partial_i \partial^i) A^i = \square_{lc} A^i = 0 \quad \Rightarrow \quad \text{two propagating modes of the photon}$$

The "inverse derivative" operator

$$\partial_- f(x^-) = g(x^-) \quad \Rightarrow \quad f(x^-) = \frac{1}{\partial_-} g(x^-) = -\int \epsilon(x^- - y^-) g(y^-) dy^- + \text{"constant"}$$

[Mandelstam '83, Leibbrandt '83]

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Electromagnetism in the front form

- Complexify the x^i

$$x = \frac{x^1 + ix^2}{\sqrt{2}}, \quad \bar{x} = \frac{x^1 - ix^2}{\sqrt{2}} \quad \partial_i \rightarrow (\partial, \bar{\partial})$$

$A^i \rightarrow (A, \bar{A})$: ± 1 helicity states of the photon

- Light-cone action for electromagnetism

$$S = \frac{1}{2} \int d^4x \bar{A} \square_{lc} A = \int d^4x \bar{A} (\partial_+ \partial_- - \partial \bar{\partial}) A$$

→ lc_2 formalism of electromagnetism

- Hamiltonian and Poisson brackets (recall: x^+ is time)

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_+ A)} = -\partial_- \bar{A}, \quad \bar{\pi} = \frac{\delta \mathcal{L}}{\delta(\partial_+ \bar{A})} = -\partial_- A$$

$(\pi, \bar{\pi})$ not independent variables \Rightarrow Half the d.o.f than in the 3+1 formalism

→ a feature of *all* null-front Hamiltonian systems

Poisson brackets

$$[A(x), \bar{A}(y)] = \epsilon(x^- - y^-) \delta^{(2)}(x - y), \quad [A(x), A(y)] = [\bar{A}(x), \bar{A}(y)] = 0.$$

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Residual gauge transformations

Symmetries in light-cone formulation

- Canonical transformation in the phase space: $(A, \bar{A}) \xrightarrow{\delta_X} (\tilde{A}, \tilde{\bar{A}})$
- *Strict* invariance of action: $\delta_X S[A, \bar{A}] = 0$
- Transformation = Poisson bracket with a generator $G_X[A, \bar{A}]$,

$$\delta_X A = [A, G_X]_{PB}$$

Is there any residual gauge freedom left?

- $A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \Lambda(x)$

All such $\Lambda(x)$ that respects the light-cone gauge choice: $A_- = 0$

$$\partial_- \Lambda(x) = 0 \quad \Rightarrow \quad \Lambda = \Lambda(x^+, x, \bar{x})$$



Residual gauge transformations

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- Check for invariance of the light-cone action

$$\text{Extra condition: } \partial \bar{\partial} \Lambda(x) = 0 \quad \Rightarrow \quad \Lambda(x) = \overset{I}{f(x, x^+)} + g(\bar{x}, x^+)$$

Not the most general function of (x^+, x, \bar{x})



Role of the zero modes

Resolution: Put back the integration constant (zero modes)

$$A^- = -\frac{\partial_i A^i}{\partial_-} + \Delta \alpha(x^+, x, \bar{x}) x^- + \beta(x^+, x, \bar{x});$$
$$\Delta \beta = \partial_+ \alpha; \quad \Delta = 2\partial \bar{\partial}$$

Modified light-cone action

$$S[A, \bar{A}, \alpha] = \int d^4x [\bar{A}(\partial_+ \partial_- - \partial \bar{\partial}) A - 2(\bar{A} \Delta \partial \alpha + A \Delta \bar{\partial} \alpha) - 2\partial \alpha \Delta \bar{\partial} \alpha];$$

Residual gauge symmetry: $\delta \alpha = -\Lambda \rightarrow$ **arbitrary function of** (x^+, x, \bar{x})



Role of the zero modes

Resolution: Put back the integration constant (zero modes)

$$A^- = -\frac{\partial_i A^i}{\partial_-} + \Delta\alpha(x^+, x, \bar{x})x^- + \beta(x^+, x, \bar{x});$$

$$\Delta\beta = \partial_+\alpha; \quad \Delta = 2\partial\bar{\partial}$$

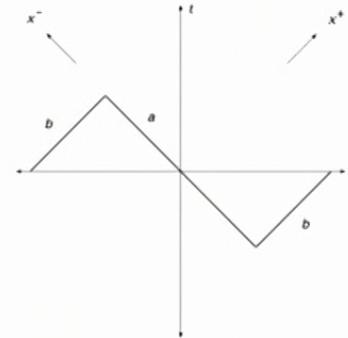
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Residual gauge symmetry: $\delta\alpha = -\Lambda \rightarrow$ **arbitrary function of (x^+, x, \bar{x})**

- Putting α, β to zero amounts to residual gauge fixing
- Zero modes \rightarrow a part of the initial data set
- Cauchy surface equivalent:
 $a(x^+ = \text{constant}) \cup b(x^- = \text{constant})$

[Image courtesy: <https://arxiv.org/abs/hep-th/9501107>]



[McCartor-Roberston' 95]

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Asymptotic analysis of EM à la Henneaux-Troessaert

- Hamiltonian action

$$S[A_i, \pi^i, A_0] = \int dt \left\{ \int d^3x \pi^i \dot{A}_i - \int d^3x \left(\frac{1}{2} \pi^i \pi_i + \frac{1}{4} F^{ij} F_{ij} + A_0 \mathcal{G} \right) + F_\infty \right\}$$

$$\text{Gauss constraint, } \mathcal{G} = \partial_i \pi^i \approx 0$$

- Fall-off conditions:

$$A_i = \frac{1}{r} \bar{A}_i + \mathcal{O}(r^{-2}), \quad \pi^i = \frac{1}{r^2} \bar{\pi}^i + \mathcal{O}(r^3)$$

(Gauge-twisted) parity conditions

$$\begin{aligned} \bar{A}_r &= (\bar{A}_r)^{odd}, & \bar{A}_B &= (\bar{A}_B)^{even} + \partial_B \Phi, & \Phi &= \text{even} \\ \bar{\pi}^r &= (\bar{\pi}^r)^{even}, & \bar{\pi}^A &= (\bar{\pi}^A)^{odd} \end{aligned}$$

Canonical generator for gauge symmetries

$$G[\epsilon] = \int d^3x \epsilon \mathcal{G} + \oint d^2S_i \bar{\epsilon} \bar{\pi}^i$$

- Gauge symmetry: surface charge = 0 → Proper gauge transformations
- True symmetry: surface charge ≠ 0 → Improper gauge transformations



Electromagnetism in the Hamiltonian formulation

- Symmetry \equiv *Strict* invariance of the symplectic form (not upto boundary terms)

$$\mathcal{L}_\xi \Omega = 0, \quad \Omega = \int d^3x d_V \pi^i d_V A_i, \quad d_V \equiv \text{exterior derivative in field space}$$

Canonical generator

$$d_V(\iota_\xi \Omega) = 0 \quad \Rightarrow \quad \iota_\xi \Omega = -d_V G_\xi$$

Invariance under boosts is violated with the gauge-twisted parity conditions

- Resolution: Add a surface dof to the symplectic form

$$- \oint d^2x \sqrt{\gamma} d_V \bar{A}_r d_V \bar{\Psi}$$

$\bar{\Psi}$, which is related to A^0 , has its own gauge symmetry

$$\delta_\mu \bar{\Psi} = \mu, \quad \mu = \text{odd}$$

- Full angle-dependent $U(1)$ gauge symmetry recovered

$$G_{\epsilon, \mu}[A_i, \Psi, \pi^i] = \int d^3x \epsilon \mathcal{G} + \oint d^2x (\bar{\epsilon} \bar{\pi}^r - \sqrt{\gamma} \bar{\mu} \bar{A}_r) \quad \mathbb{I}$$

$(\bar{\epsilon}_{\text{even}}, \bar{\mu}_{\text{odd}})$: Non-zero surface charge \rightarrow Improper gauge transformations

[Henneaux-Troessaert '18]

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Does (2+2) equal (3+1)?

(3+1): Asymptotic symmetries at spatial infinity

- Symmetry \equiv invariance of symplectic form or Hamiltonian action
- Boundary value problem on a Cauchy hypersurface
- **Spin 1:** Must include a surface dof $\bar{\Psi}$ to obtain full U(1) gauge symmetries
Setting $\bar{\Psi}$ to zero amounts to improper gauge fixing

(2+2): Residual gauge symmetries in light-cone formulation

- Symmetry \equiv invariance of light-cone action
- Characteristic initial value problem on a null hypersurface
- **Spin 1:** Must include the zero mode α to obtain all residual gauge symmetries
Setting α to zero amounts to residual gauge fixing



Gravity in the (2+2) formulation

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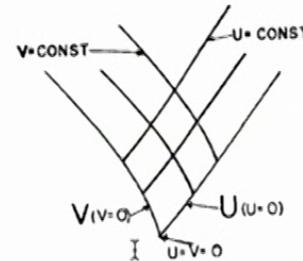
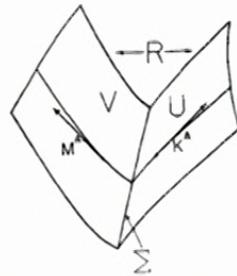
(2+2) or “double-null” formulation of gravity

“On the characteristic initial value problem in gravitational theory” [R. K. Sachs '62]

“Covariant 2+2 formulation of the initial-value problem in general relativity”

[d’Inverno and Smallwood '79]

[Gambini-Restuccia, Nagarajan-Goldberg, C. Torre, M. Kaku, S. Hayward...]



- Spacelike foliation of codim 2 (instead of 1)
- Unconstrained Hamiltonian systems
- Gravitational d.o.f. identified with the “conformal two-metric”

Our focus

A particular example of 2+2 formulation of gravity: lc_2 gravity

Light-cone gravity à la Scherk-Schwartz

- Light-cone gauge: Set the “minus” components to zero

$$g_{--} = g_{-i} = 0, \quad (i = 1, 2)$$

$$10 - 3 = 7$$

Parametrization

$$g_{+-} = -e^\phi, \quad g_{ij} = e^\psi \gamma_{ij}$$

ϕ, ψ, γ_{ij} are real and $\det \gamma_{ij} = 1$

Light-cone metric

$$dS_{LC}^2 = g_{\mu\nu} dx^\mu dx^\nu = -2e^\phi dx^+ dx^- + g_{++} (dx^+)^2 + g_{+i} dx^+ dx^i + e^\psi \gamma_{ij} dx^i dx^j$$

given in terms of 7 functions $\{\phi, \psi, \gamma_{ij}, g_{++}, g_{+i}\}$

- “2+2” split of the Einstein field equations $R_{\mu\nu} = 0$ [Sachs, d’Inverno-Smallwood, ...]

Dynamical equations: $R_{ij} = 0$

Constraint equations: $R_{--} = R_{-i} = 0$

Subsidiary equations: $R_{++} = R_{+i} = 0$

Trivial equations: $R_{+-} = 0$



Gravity in the light-cone gauge

- Constraint equation $R_{--} = 0$

$$2\partial_{-}\phi\partial_{-}\psi - (\partial_{-}\psi)^2 - 2\partial_{-}^2\psi + \frac{1}{2}\partial_{-}\gamma^{ij}\partial_{-}\gamma_{ij} = 0.$$

Fourth gauge choice

$$\phi = \frac{\psi}{2}$$

$$7 - 1 = 6$$

allows us to integrate[†] out ψ

$$\psi = \frac{1}{4} \frac{1}{\partial_{-}^2} (\partial_{-}\gamma^{ij}\partial_{-}\gamma_{ij})$$

$$6 - 1 = 5$$

- The constraint $R_{-i} = 0$ eliminates g_{+i}

$$5 - 2 = 3$$

- $R_{-+} = 0$ allows us to eliminate g_{++}

$$3 - 1 = 2$$

- Gravitational d.o.f. identified with the “conformal two-metric”: γ_{ij}

[†] All integration constants set to zero



Light-cone action for gravity

- Closed form expression

$$S[\gamma_{ij}] = \frac{1}{2\kappa^2} \int d^4x e^\psi \left(2\partial_+\partial_-\phi + \partial_+\partial_-\psi - \frac{1}{2}\partial_+\gamma^j{}_i\partial_-\gamma_{ij} \right) - \frac{1}{2}e^{\phi-2\psi}\gamma^{ij}\frac{1}{\partial_-}R_i\frac{1}{\partial_-} \\ - e^\phi\gamma^{ij}\left(\partial_i\partial_j\phi + \frac{1}{2}\partial_i\phi\partial_j\phi - \partial_i\phi\partial_j\psi - \frac{1}{4}\partial_i\gamma^{kl}\partial_j\gamma_{kl} + \frac{1}{2}\partial_i\gamma^{kl}\partial_k\gamma_{jl}\right)$$

where

$$R_i \equiv e^\psi \left(\frac{1}{2}\partial_-\gamma^{jk}\partial_i\gamma_{jk} - \partial_-\partial_i\phi - \partial_-\partial_i\psi + \partial_i\phi\partial_-\psi \right) + \partial_k(e^\psi\gamma^{jk}\partial_-\gamma_{ij})$$



Light-cone Hamiltonian for gravity

- Conjugate momenta

$$\pi = \frac{\delta \mathcal{L}}{\delta(\partial_+ h)} = -\partial_- \bar{h}, \quad \bar{\pi} = \frac{\delta \mathcal{L}}{\delta(\partial_+ \bar{h})} = -\partial_- h$$

$(\pi, \bar{\pi})$ not independent variables \Rightarrow Half the d.o.f than in the ADM formalism

- Light-cone Hamiltonian for gravity

$$\mathcal{H} = \partial \bar{h} \bar{\partial} h + 2\kappa \partial_-^2 \bar{h} \left(h \frac{\bar{\partial}^2}{\partial_-^2} h - \frac{\bar{\partial}}{\partial_-} h \frac{\bar{\partial}}{\partial_-} h \right) + c.c. + \mathcal{O}(\kappa^2)$$

- Poisson brackets

$$[h(x), \bar{h}(y)] = \epsilon(x^- - y^-) \delta^{(2)}(x - y), \quad [h(x), h(y)] = [\bar{h}(x), \bar{h}(y)] = 0.$$

[Scherk-Schwarz '75, Bengtsson-Cederwall-Lindgren '83]

Symmetries in light-cone gravity

- Strict invariance of action: $\delta_X S[h, \bar{h}] = 0$
- Transformation = Poisson bracket with a generator $G_X[h, \bar{h}]$,

$$\delta_X h = [h, G_X]_{PB}$$

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BMS symmetry from residual gauge invariance

Is there any residual reparameterization freedom $x^\mu \rightarrow x^\mu + \xi^\mu$ left?

- First gauge condition holds

$$g_{--} = 0 \stackrel{\text{I}}{\Rightarrow} \partial_- \xi^+ = 0 \Rightarrow \xi^+ = f(x^+, x^j)$$

Second gauge condition $g_{-i} = 0$ gives

$$\partial_- \xi^j g_{ij} + \partial_i \xi^+ g_{+-} = 0$$

Fourth gauge condition fixes x^+ dependence of $f(x^+, x^j)$

- Residual reparameterizations

$$\xi^+ = f = \frac{1}{2} x^+ \partial_i Y^i + T(x^k)$$

$$\xi^i = -\partial_k f \frac{1}{\partial_-} (g_{-+} g^{ik}) + Y^i(x^k)$$

$$\xi^- = -\partial_i Y^i x^- + (\partial_+ \xi_i) x^i$$



BMS algebra in light-cone gravity

- BMS transformation law (on the initial surface $x^+ = 0$),

$$\begin{aligned} \delta_{\bar{Y}, \bar{Y}, T} h &= Y(x) \bar{\partial} h + \bar{Y}(\bar{x}) \partial h + (\partial \bar{Y} - \bar{\partial} Y) h + T \frac{\partial \bar{\partial}}{\partial_-} h \\ &\quad - 2\kappa T \partial_- \left(h \frac{\bar{\partial}^2}{\partial_-^2} h - \frac{\bar{\partial}}{\partial_-} h \frac{\bar{\partial}}{\partial_-} h \right) - 2\kappa T \frac{1}{\partial_-} \left(\frac{\partial^2}{\partial_-^2} \bar{h} \partial_-^2 h \right) \\ &\quad - 2\kappa T \frac{\partial^2}{\partial_-^3} (\bar{h} \partial_-^2 h) + 4\kappa T \frac{\partial}{\partial_-^2} \left(\frac{\partial}{\partial_-} \bar{h} \partial_-^2 h \right) + \mathcal{O}(\kappa^2) \end{aligned}$$

- Symmetry algebra

$$\left[\delta(Y_1, \bar{Y}_1, T_1), \delta(Y_2, \bar{Y}_2, T_2) \right] h = \delta(Y_{12}, \bar{Y}_{12}, T_{12}) h,$$

with parameters

$$\begin{aligned} Y_{12} &\equiv Y_2 \bar{\partial} Y_1 - Y_1 \bar{\partial} Y_2 \\ \bar{Y}_{12} &\equiv \bar{Y}_2 \partial \bar{Y}_1 - \bar{Y}_1 \partial \bar{Y}_2 \\ T_{12} &\equiv [Y_2 \bar{\partial} T_1 + \bar{Y}_2 \partial T_1 + \frac{1}{2} T_2 (\bar{\partial} Y_1 + \partial \bar{Y}_1)] - (1 \leftrightarrow 2). \end{aligned}$$

→ BMS algebra from residual gauge invariance without reintroducing the zero modes



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(3+1): Asymptotic symmetries at spatial infinity

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- Boundary value problem on a Cauchy hypersurface
- **Spin 1:** Must include a surface dof $\bar{\Psi}$ to obtain full U(1) gauge symmetries
Setting $\bar{\Psi}$ to zero amounts to improper gauge fixing
- **Spin 2:** Supertranslations obtained without any extra surface degrees of freedom

[Henneaux-Troessaert '18]

(2+2): Residual gauge symmetries in light-cone formulation

- Symmetry \equiv invariance of light-cone action
- Characteristic initial value problem on a null hypersurface
- **Spin 1:** Must include the zero mode α to obtain all residual gauge symmetries
Setting α to zero amounts to residual gauge fixing
- **Spin 2:** Supertranslations obtained without reintroducing the zero modes



Some concluding remarks...

Connections with amplitudes

- Action in terms of helicity states - closer to on-shell physics
- Various applications- MHV Lagrangians , KLT relations , Double copy methods

[Gorsky-Rosly, Ananth-Theisen, Ananth-Kovacs-Parikh, ...]

Self-dual, Anti self-dual and all that

- Closely related to Chalmers-Seigel action, double copy construction for SD sectors

[Campiglia-Nagy '21]

- Double copy for BMS symmetries [work in progress]
- Newmann-Penrose formalism [work in progress], Weyl double copy, ...



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Formal (2+2) Hamiltonian analysis

- First-order formalism with physical dof
- Role of gauge constraints, zero modes, etc. [work in progress]
- Dictionary between residual gauge symmetries in (2+2) with asymptotic symmetries

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“Canonical formalism on a null surface: The scalar and the electromagnetic fields

[Nagarajan-Goldber

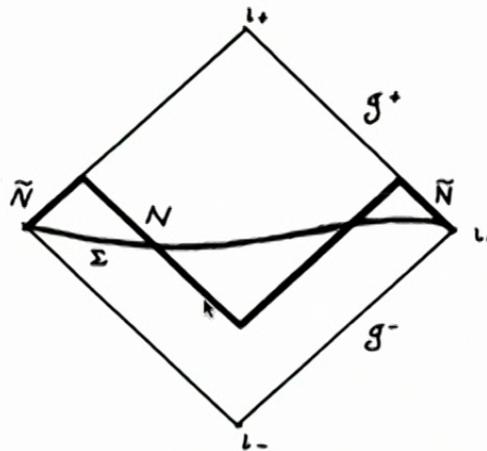


FIG. 1. The Cauchy surface Σ is replaced by the union of an outgoing null cone N and the section of \mathcal{J}^+ extending back to Σ at spacelike infinity. $M = N \cup \tilde{N}$.

Thank you for your attention!