Title: Flat asymptotics, charges and dual charges -- what the Cotton can do

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Abstract: In this talk I will explore the role of the boundary Cotton tensor in the reconstruction of the solution space of four-dimensional asymptotically AdS and mostly asymptotically flat spacetimes. I will discuss charges from a purely boundary perspective, which emerge in sets of electric and magnetic towers, not necessarily conserved and possibly including subleading components.

Flat asymptotics, charges and dual charges

What the Cotton can do

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Highlights

1. Plan & motivations
2. Carrollian geometries & conservation properties
3. Asymptotics, reconstruction and AdS
4. Back to Ricci-flat spacetimes
5. Outlook

Questions & cues

Why asymptotic symmetries and charges? [Komar '59; ADM '60; BMS '62]

- Universal features of solutions to Einstein's equations
- Hints to holography and in particular flat holography

Irrespective of holographic correspondence...

...a solution is captured by a set fields defined on a conformal boundary and obeying conformal boundary dynamics [Penrose '63]

Can we compute charges from a boundary perspective?

Yes as a synthesis of bry. symmetry and bry. dynamics

Here Carrollian dynamics – Why?

Asymptotically flat spacetimes → null boundary → Carrollian geometry

What is the Cotton? [Émile Cotton, 1899]

- Covariant derivative of the Einstein tensor in Riemannian geometries – remarkable in 3 dimensions (no Weyl)
- Admits Carrollian relatives on Carrollian geometries

The main message for 4-dim Ricci-flat spactimes

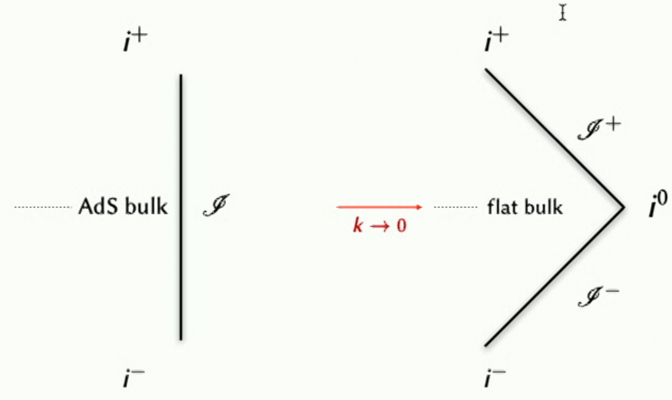
Boundary energy and momenta & Carrollian Cotton tensors

- generate infinite *dual towers of charges* determined from a boundary account (e.g. mass vs. nut)
- carry part of the *infinite Chthonian information* required for reconstructing the bulk

The Carrollian null boundary

From AdS$_n$ to flat$_n$ asymptotics

$\Lambda = -\frac{(n-1)(n-2)}{2} k^2 \to 0$



$k \equiv$ boundary velocity of light $\leftrightarrow$ $k \to 0$ Carrollian limit

The null bry. $\mathscr{I}^\pm$ is a Carrollian geometry in $n-1$ dimensions

Carrollian geometry [Lévy-Leblond '65; Sen Gupta '66; Duval et al. '14; Bekaert et al. '16; see also N. Obers' & G. Oling' talks]

Basic ingredients in $d+1$ dimensions (coordinates $t, \mathbf{x}$)

- degenerate metric: $\mathrm{d}s^2 = 0 \times \left(\Omega \mathrm{d}t - b_i \mathrm{d}x^i\right)^2 + a_{ij}\mathrm{d}x^i \mathrm{d}x^j$
- field of observers: $\frac{1}{\Omega}\partial_t$ (t should be spelled u)
- clock form: $\mu = \Omega \mathrm{d}t - b_i \mathrm{d}x^i$ (Ehresmann connection)

General covariance (in the present parameterization)

Carrollian diffeomorphisms: $t' = t'(t, \mathbf{x}) \quad \mathbf{x}' = \mathbf{x}'(\mathbf{x})$

Consequences of the bry. Carrollian structure

I – Ricci-flat spacetime reconstruction in n dimensions

should be *Weyl invariant and Carrollian covariant* wrt the $n-1$ -dim conformal bry. – gauges as Bondi, Newmann–Unti are not

II – Flat holography - *if it exists*

calls for a *Carrollian conformal field theory* on an $n-1$-dim bry.

Dynamics

General-covariant action and energy–momentum tensor

Pseudo-Riemannian spacetimes in $d+1$ dim $T^{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g_{\mu\nu}}$

- Weyl invariance $\to T^{\mu}{}_{\mu} = 0$
- general covariance ($\xi = \xi^{\mu}(t, \mathbf{x})\partial_{\mu}$ diffeos) $\to \nabla_{\mu} T^{\mu\nu} = 0$
- ξ conformal Killing $\to I^{\mu} = \xi_{\nu} T^{\mu\nu}$ $\quad Q_{\xi} = \int_{\Sigma_d} *\mathrm{I}$ conserved

Carrollian-covariant action, energy and momentum

[Ciambelli, Marteau '18; comparison with G. Oling' talk]

$$\begin{cases} \Pi^{ij} = \frac{2}{\sqrt{a}\Omega}\frac{\delta S}{\delta a_{ij}} & \text{energy–stress tensor} \\ \Pi^{i} = \frac{1}{\sqrt{a}\Omega}\frac{\delta S}{\delta b_{i}} & \text{energy flux} \\ \Pi = -\frac{1}{\sqrt{a}}\left(\frac{\delta S}{\delta \Omega} + \frac{b_i}{\Omega}\frac{\delta S}{\delta b_i}\right) & \text{energy density} \end{cases}$$

In Carrollian spacetimes: conservation equations

- Weyl covariance $\rightarrow \Pi^{i}{}_{i} = \Pi$
- Carrollian covariance ($\xi = \xi^t(t, \mathbf{x})\partial_t + \xi^i(\mathbf{x})\partial_i$ diffeos)

$$\rightarrow \begin{cases} \frac{1}{\Omega}\hat{\mathscr{D}}_t\Pi + \hat{\mathscr{D}}_i\Pi^i + \Pi^{ij}\xi_{ij} = 0 & \text{time} \\ \left(\frac{1}{\Omega}\hat{\mathscr{D}}_t\delta^i_j + \xi^i{}_j\right) P_i + \hat{\mathscr{D}}_i\Pi^i{}_j + 2\Pi^i\varpi_{ij} = 0 & \text{space} \end{cases}$$

$\rightarrow$ momentum P_i

Conserved currents and charges

- Carrollian current: Carrollian scalar κ and vector K^i
- Carrollian divergence: $\mathcal{K} = \frac{1}{\Omega}\hat{\mathscr{D}}_t\kappa + \hat{\mathscr{D}}_j K^j$
- Charge: $Q_K = \int_{\Sigma_d} \mathrm{d}^d x \sqrt{a}\left(\kappa + b_i K^i\right)$ conserved if $\mathcal{K} = 0$

Carrollian conformal isometries $\xi = \xi^t \partial_t + \xi^i \partial_i$

Conformal Killings via $\mathscr{L}_\xi a_{ij}$ and $\mathscr{L}_\xi \frac{1}{\Omega}\partial_t$ *but not* $\mathscr{L}_\xi \mu$

- $\kappa = \xi^i P_i - \xi^{\hat{t}} \Pi$ and $K^i = \xi^j \Pi_j{}^i - \xi^{\hat{t}} \Pi^i$ $\quad \xi^{\hat{t}} = \xi^t - \xi^i \frac{b_i}{\Omega}$
- not conserved: $\mathcal{K} = -\Pi \cdot \mathscr{L}_\xi \mu$ [Petkou, Petropoulos, Rivera-Betancour, Siampos '22]

Remarkable property [Ciambelli, Leigh, Marteau, Petropoulos '19]

$\frac{1}{2\Omega} a^{ik} \left(\partial_t a_{kj} - a_{kj} \partial_t \ln \sqrt{a}\right) = \xi^i{}_j = 0 \Leftrightarrow a_{ij}(t, \mathbf{x}) = e^{\sigma(t,\mathbf{x})} \bar{a}_{ij}(\mathbf{x}) \Leftrightarrow$
conformal Carroll isometries $\equiv$ conf$[\bar{a}_{ij}(\mathbf{x})] \ltimes$ supertranslations

$$d + 1 = 3 \rightarrow \mathfrak{so}(3, 1) \ltimes \text{supertranslations} \equiv \text{BMS}_4$$

generally $\text{BMS}_{d+2} \equiv \mathfrak{ccarr}(d + 1)$

In summary

Main messages

- Null boundaries in asymptotically flat spacetimes are *Carrollian geometries* – zero speed of light
- Carrollian geometries with $\xi^{ij} = 0$ have an *infinite tower of conformal Killings*
- An infinite tower of conformal-Killing charges exist but *not all are always conserved*

Highlights

Pure gravity – asymptotically flat or AdS

Basic field in pure gravity: metric G_{AB} in $n = d + 2$ dim

$\{r, t, x^i\}$, $i = 1, \ldots, d$ plus *gauge fixing* ($n = d + 2$ conditions)
$\rightarrow$ find the solutions as $O(1/r^n)$ with coefficients $f(t, \mathbf{x})$

Gauge choice: twofold goal

- reach *manifest Weyl invariance & general covariance & possibly local Lorentz invariance* for the dynamics on the $n-1$-dim conformal boundary (admitting $k \rightarrow 0$ limit – excludes WFG [Ciambelli, Leigh '20])
- define charges from a *purely boundary perspective*

Einstein spacetimes covariantly reconstructed

Solution space with covariant incomplete modified Newman–Unti gauge i.e. $\beta = \beta_0(t, \mathbf{x})$ & $G_{ri} \neq 0$ [see M. Geiller' talk]

- $\frac{n(n-1)+2}{2}$ Einstein's equations $\rightarrow n^2 - 3$ functions of $(t, \mathbf{x})$ $\rightarrow$ boundary data $\mu, \nu, \ldots \in \{0, 1, \ldots, n-2 = d\}$
  - $g_{\mu\nu}$ symmetric $\leftarrow \frac{n(n-1)}{2}$
    boundary metric (no Dirichlet bc!)
  - $T_{\mu\nu}$ symmetric and traceless $\leftarrow \frac{n(n-1)}{2} - 1$
    conformal boundary energy–momentum tensor
  - $u^\mu \leftarrow n - 2$ (due to the gauge incompleteness $G_{ri} \neq 0$)
    boundary normalized vector field [see A. Delfante' talk]
- remaining $n - 1$ Einstein's equations $\boxed{\nabla_\mu T^{\mu\nu} = 0}$

On arbitrary (boundary) geometry $g_{\mu\nu}$ of dim $d+1$

$$T^{\mu\nu} = \varepsilon \frac{u^\mu u^\nu}{k^2} + \frac{\varepsilon}{d} h^{\mu\nu} + \tau^{\mu\nu} + \frac{u^\mu q^\mu}{k^2} + \frac{u^\nu q^\mu}{k^2}$$

- $\|u\|^2 = -k^2 \quad h^{\mu\nu} = g^{\mu\nu} + \frac{u^\mu u^\nu}{k^2}$
- q^μ, $\tau^{\mu\nu}$ transverse plus $\tau^\mu{}_\mu = 0$

$\rightarrow$ relativistic Weyl-covariant "fluid"

In 4-dim bulk (3-dim boundary – $d = 2$): the Cotton tensor

[de Haro '08; Mansi et al. '09; de Freitas et al. & Bakas et al. '14; Gath et al. '15; Ciambelli et al. '18]

$C_{\mu\nu} = \eta_\mu{}^{\rho\sigma} \nabla_\rho \left(R_{\nu\sigma} - \frac{R}{4} g_{\nu\sigma} \right)$ symmetric, traceless, $\nabla_\mu C^{\mu\nu} = 0$

decomposed along u^μ: c, c^μ, $c^{\mu\nu}$ transverse plus $c^\mu{}_\mu = 0$

$$C_{\mu\nu} = c \frac{u^\mu u^\nu}{k^2} + \frac{c}{2} h^{\mu\nu} + c^{\mu\nu} + \frac{u^\mu c^\mu}{k^2} + \frac{u^\nu c^\mu}{k^2}$$

In $n = 4$ dimensions $\Lambda = -3k^2$

General solution: $6 + 5 + 2$ arbitrary boundary data

- $ds^2 = -k^2 \left(\Omega dt - b_i dx^i\right)^2 + a_{ij} dx^i dx^j \to \{c, c^{\mu}, c^{\mu\nu}\}$
- $T_{\mu\nu} \to \{\varepsilon = 2p, q^{\mu}, \tau^{\mu\nu}\}$
- $\mathbf{u} = u_{\mu} dx^{\mu} \to \left\{\sigma^{\mu\nu}, \omega^{\mu\nu}, \mathrm{A} = \frac{1}{k^2}\left(\mathbf{a} - \frac{\Theta}{2}\mathbf{u}\right), \mathscr{D}_{\mu}\right\}$

$$
\begin{aligned}
ds^2_{\text{Einstein}} = \; & 2\frac{\mathbf{u}}{k^2}(dr + r\mathrm{A}) + r^2 ds^2 - 2\frac{r}{k^2}\sigma_{\mu\nu} dx^{\mu} dx^{\nu} + \frac{\mathrm{S}}{k^4} \\
& + \frac{8\pi G}{k^4 r}\left[\varepsilon \mathbf{u}^2 + \frac{4\mathbf{u}}{3}\left(\mathbf{q} - \frac{1}{8\pi G} * \mathbf{c}\right)\right. \\
& \left. + \frac{2k^2}{3}\left(\tau + \frac{1}{8\pi G k^2} * \mathbf{c}\right)\right] + \frac{1}{r^2}\left(c\gamma \frac{\mathbf{u}^2}{k^4} + \cdots\right) \\
& + \mathrm{O}\left(1/r^3\right)
\end{aligned}
$$

$$S_{\mu\nu} = 2u_{(\mu}\mathscr{D}_{\lambda}\left(\sigma_{\nu)}{}^{\lambda} + \omega_{\nu)}{}^{\lambda}\right) - \frac{\mathscr{R}}{2} u_{\mu} u_{\nu} + 2\omega_{(\mu}{}^{\lambda}\sigma_{\nu)\lambda} + \left(\sigma^2 + k^4\gamma^2\right) h_{\mu\nu}$$

$$\gamma^2 = \frac{1}{2k^4}\omega_{\alpha\beta}\omega^{\alpha\beta}$$

What can the Cotton do in AdS_4 asymptotics?

$C_{\mu\nu} \neq 0 \Leftrightarrow$ non-conformally flat bry. $\leftrightarrow$ asymptotically *locally* AdS bulk (ex. Taub–NUT)

ξ bry. conformal Killing $\rightarrow I^{\mu} = \xi_{\nu} T^{\mu\nu}$ and $I^{\mu}_{\text{Cot}} = \xi_{\nu} C^{\mu\nu}$

$$Q_{\xi} = \int_{\Sigma_2} *\mathrm{I} \quad \text{and} \quad Q_{\text{Cot}\xi} = \int_{\Sigma_2} *\mathrm{I}_{\text{Cot}}$$

electric and *magnetic* dual conserved charges (bulk mass vs. nut)

- $Q_{\text{Cot}\xi} \sim$ magnetic Komar charges: off–shell conservation
- Limitation in AdS: at most 10 conformal Killings ($d + 1 = 3$)

Extendable in Ricci-flat spacetimes – more interesting

Highlights

Ricci-flat in covariant Newman–Unti gauge

Full solution space in $n = 4$ [Brussels & Paris groups]

$\mathrm{d}s^2_{\text{Ricci-flat}}$ described in terms of $2+1$ *Carrollian boundary data*

- Carrollian geometry (6) with zero geometrical shear ξ_{ij}
  - degenerate metric (3) $\mathrm{d}\ell^2 = a_{ij}\mathrm{d}x^i\mathrm{d}x^j$
  - Ehresmann connection (3) $\mu = \Omega \mathrm{d}t - b_i \mathrm{d}x^i$
- Carrollian conformal "fluid" (5) [see L. Freidel' talk]
  - energy (1) ε
  - momenta – heat current (2) and stress tensor (2) π_i & E_{ij}
- Carrollian-fluid "velocity" (2) hydro-frame freedom
- Carrollian dynamical shear (2) $\mathscr{C}_{ij}$
- *infinite* number of further Carrollian data obeying Carrollian dynamics – at every $\mathrm{O}\left(1/r^n\right)$: *Chthonian*

Note: *last 2 items absent for algebraically special solutions*

2 + 1 dim Carrollian space → Carrollian Cotton tensors

expansion of c, c^{μ}, $c^{\mu\nu}$ in powers of k^2

- 4 sets of momenta Π_{Cot}, Π^{i}_{Cot}, P^{i}_{Cot}, Π^{ij}_{Cot}
- 4 sets of Carrollian conservation equations

only 2 sets if $\xi_{ij} = 0$ $(c, \psi_i, \chi_i, \Psi_{ij}, X_{ij})$

RICCI-FLAT SPACETIMES UP TO $O\left(1/r^3\right)$

$$
\begin{aligned}
\mathrm{d}s^2_{\text{Ricci-flat}} = {} & 2\mu \left[\mathrm{d}r - \frac{r\theta + \hat{\mathscr{K}}}{2}\mu + \left(r\varphi_i - *\hat{\mathscr{D}}_i * \varpi - \frac{1}{2}\hat{\mathscr{D}}_j \mathscr{C}^j{}_i\right)\mathrm{d}x^i\right] \\
& + \mathscr{C}_{ij}\left(r\mathrm{d}x^i\mathrm{d}x^j - *\varpi * \mathrm{d}x^i \mathrm{d}x^j\right) + \left(r^2 + *\varpi^2 + \frac{\mathscr{C}_{kl}\mathscr{C}^{kl}}{8}\right)\mathrm{d}\ell^2 \\
& + \frac{1}{r}\left[8\pi G\varepsilon\mu^2 - \frac{4}{3}\left(*\psi_i - 8\pi G\pi_i\right)\mathrm{d}x^i\mu\right] \\
& - \frac{16\pi G}{3r}E_{ij}\mathrm{d}x^i\mathrm{d}x^j + \frac{1}{r^2}\left(*\varpi c\mu^2 + \cdots\right) + O\left(1/r^3\right)
\end{aligned}
$$

- $4\pi G\varepsilon - \frac{1}{8}\mathscr{C}^{jk}\hat{\mathscr{N}}_{jk} = M$ Bondi mass aspect [see R. Ruzziconi' talk and refs.]
- $\hat{\mathscr{N}}^{ij} = \frac{1}{\Omega}\hat{\mathscr{D}}_t \mathscr{C}^{ij}$ covariant Bondi news
- $4\pi G\varepsilon + \frac{i}{2}c$ complex mass [see A.-M. Raclariu' talk $\mathcal{M} + i\tilde{\mathcal{M}}$]
- $*\psi_i - 8\pi G\pi_i = N_i$ angular momentum aspect
- possibly resummable (energy & momenta $\leftrightarrow$ Carrollian Cotton)

Salient features

- Weyl invariance & Carrollian covariance wrt boundary
- boundary Carrollian "fluid" with $\Pi, \Pi^i, P^i, \Pi^{ij}$ *under external force* – free if zero dynamical shear $\mathscr{C}_{ij}$
- always $\xi_{ij} = 0 \Leftrightarrow \infty$ conformal Carrollian group $\text{BMS}_4 \equiv$ asymptotic bulk symmetries
- boundary Carrollian Cotton in the form
  - $\Pi_{\text{Cot}}, \Pi^i_{\text{Cot}}, P^i_{\text{Cot}}, \Pi^{ij}_{\text{Cot}}\ (c, \chi_i, X_{ij})$
  - $\tilde{\Pi}_{\text{Cot}}, \tilde{\Pi}^i_{\text{Cot}}, \tilde{P}^i_{\text{Cot}}, \tilde{\Pi}^{ij}_{\text{Cot}}\ (\psi_i, \Psi_{ij})$

  obeying Carrollian dynamics
- part of the fluid data are determined by the Cotton data

Example of equation

Reminder Carrollian "fluid" energy equation

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\Pi + \hat{\mathscr{D}}_i\Pi^i + \Pi^{ij}\xi_{ij} = 0$$

Here

$$\frac{1}{\Omega}\hat{\mathscr{D}}_t\varepsilon + \frac{1}{8\pi G}\hat{\mathscr{D}}_i * \chi^i =$$
$$\frac{1}{16\pi G}\left(\hat{\mathscr{D}}_i\hat{\mathscr{D}}_j\hat{\mathscr{N}}^{ij} + \mathscr{C}^{ij}\hat{\mathscr{D}}_i\hat{\mathscr{R}}_j + \frac{1}{2}\mathscr{C}_{ij}\frac{1}{\Omega}\hat{\mathscr{D}}_t\hat{\mathscr{N}}^{ij}\right)$$

Quotable

The structure of the source is mirroring the Carrollian Cotton

Spin off: towers of Carrollian charges [see A. Seraj' talk]

From the boundary Carrollian "fluid:" electric tower

Q^{e}_{ξ} conserved on-shell for zero shear and for $\mathscr{L}_{\xi}\mu = 0$

From the boundary Carrollian Cotton: magnetic tower

Q^{m}_{ξ} conserved for $\mathscr{L}_{\xi}\mu = 0$

Self-dual tower

Q^{sd}_{ξ} conserved $\forall\xi$

Other towers – electric & magnetic

From Chthonian Carrollian data associated with the *subleading* $\mathrm{O}\left({}^{1}/_{r^n}\right)$ terms in the bulk action – under construction

The leading towers

Reminder: $Q_K = \int_{\Sigma_2} \mathrm{d}^2x \sqrt{a} \left(\kappa + b_i K^i\right)$ $\qquad \xi \in \mathrm{BMS}_4$

ELECTRIC $\kappa = \xi^i \pi_i - \xi^{\hat{t}} \varepsilon \quad K^i = \frac{\varepsilon}{2}\xi^i - \frac{1}{8\pi G}\left(\xi^j * \Psi^i{}_j + \xi^{\hat{t}} * \chi^i\right)$

MAGNETIC $\kappa = \xi^i \psi_i - \xi^{\hat{t}} c \quad K^i = \frac{c}{2}\xi^i - \xi^j \Psi^i{}_j - \xi^{\hat{t}} \chi^i$

SELF-DUAL $\kappa = \xi^i \chi_i \quad K^i = -\xi^j X^i{}_j$

c, ψ_i, χ_i, Ψ_{ij}, X_{ij} – Carrollian Cotton tensors

ε – Bondi mass aspect

π_i – angular momentum aspect

Comparison: BMS standard approach [see e.g. L. Donnay' talk]

$Q_\xi \sim \int_{S^2} \mathrm{d}z \wedge \mathrm{d}\bar{z} \left(2\mathcal{M}\mathcal{T} + \mathcal{N}\bar{\mathcal{Y}} + \bar{\mathcal{N}}\mathcal{Y}\right) \to$ electric tower

In summary for $n = 4$

Main messages

- Bulk Ricci-flatness $\leftrightarrow$ *boundary covariant Carrollian dynamics* with *infinite* number of fields
- Conformal Carrollian isometry: BMS_4 – *infinite* and matches the asymptotic bulk symmetries
- *Multiple* infinite towers of *not-always-conserved* charges – *electric* vs. *magnetic*

Quotable facts

- The boundary Cotton appears everywhere – AdS, flat, leading, subleading …
  - in the bulk metric expansion
  - in the evolution equations (flux balance)
- Origin: expansion of the bulk Weyl
- Application: infinite towers of magnetic charges

Further investigation

- Further comparison with bulk approaches for towers of charges [Godazgar, Godazgar, Pope '18–21; see also A. Seraj' talk]
- Transverse (cross-tower) symmetry properties [Freidel et al.]
- Analysis of electric–magnetic (self-)duality properties – tuning the bulk Weyl, Ehlers' $SL(2,\mathbb{R})$ [Mittal, Petropoulos, Rivera, Vilatte]
- Search for possible role in $\text{flat}_4/\text{CCFT}_3$ holography – or in its $\text{flat}_4/\text{CFT}_2$ celestial emanation