Title: Geometric actions for BMS4

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Abstract: After reviewing the constrained Hamiltonian analysis of geometric actions, the construction is applied to the case of the BMS group in four dimensions, where it yields two plus one dimensional BMS4 invariant field theories. (Based on work done in collaboration with K. Nguyen and R. Ruzziconi)

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Geometric actions for gravity

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Collaboration with H. Gonzalez, P. Salgado, K. Nguyen, R. Ruzziconi

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- Dirac analysis
- Applications to gravity

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- Suppose G group of symmetries known
  (not necessarily fundamental theory)
- Construct model that can be quantized
  & has G as global symmetry group
  (Noether charges, current algebras)

$$S[g] = -\int d^4x \; \mathrm{Tr}\left[(\partial_\mu g\, g^{-1})(\partial^\mu g\, g^{-1})\right]$$

$$S[g] = +\int dt \; \mathrm{Tr}\left[(\dot{g}\, g^{-1})(\dot{g}\, g^{-1})\right]$$

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- global right invariance $\quad g \to g\, h_R$

$$\kappa = d(g\, h_R)\, h_R^{-1} g^{-1}$$ RI Maurer-Cartan form

- Exercise: express Lie group G & algebra $\mathfrak{g}$ theory in local coordinates

g^{μ} "Euler angles" arbitrary (abstract index notation)

generators of right/left translations = left/right vector fields

$$g \underbrace{\frac{d}{dt} h_R(t)}_{X} \;/\; \underbrace{\frac{d}{dt} h_L(t)}_{X} g \qquad \vec{L}_\mu = L^\nu{}_\mu(g) \frac{\partial}{\partial g^\nu} \;/\; \vec{R}_\mu = R^\nu{}_\mu \frac{\partial}{\partial g^\nu}$$

$$[\vec{L}_\mu, \vec{L}_\nu] = f_{\mu\nu}^{\ \ \rho} \vec{L}_\rho, \quad [\vec{R}_\mu, \vec{R}_\nu] = -f_{\mu\nu}^{\ \ \rho} \vec{R}_\rho, \quad [\vec{L}_\mu, \vec{R}_\nu] = 0$$

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left/right MC forms $\theta = g^{-1}dg \;/\; K = dg g^{-1}$

$$\theta = e_\mu (L^{-1})^\mu{}_\nu \, dg^\nu \;/\; K = e_\mu (R^{-1})^\mu{}_\nu \, dg^\nu$$

$$d\theta + \tfrac{1}{2}[\theta,\theta] = 0 \qquad dK - \tfrac{1}{2}[K,K] = 0$$

Adjoint representation $\mathrm{Ad}_g\, e_\mu = g e_\mu g^{-1}$

$$\mathrm{Ad}_g\, e_\mu = e_\nu (R^{-1\,\nu}{}_\rho L^\rho{}_\mu)$$

$$S[g^\mu] = \int dt \; \tfrac{1}{2} \underbrace{g_{\rho\nu} (R^{-1})^\rho{}_\mu (R^{-1})^\nu{}_\sigma}_{G_{\mu\nu}(g)} \dot g^\mu \dot g^\sigma$$

$g_{\mu\nu}$: Killing metric

Global sym & Noether charges

$$\delta_X g^\mu = L^\mu{}_\nu X^\nu$$

$$Q_X = \langle i_\gamma K, \mathrm{Ad}_g X \rangle$$

geodesic flow

$$\ddot g^\mu + \tfrac{1}{2} \Gamma_{(G)}{}^\mu{}_{\nu\rho}\, \dot g^\nu \dot g^\rho = 0$$

Theorem: (Arnold) $\Leftrightarrow$ $\boxed{\dot{\pi} = -\mathrm{ad}^*_{g^{-1}\dot{g}}\, \pi}$, $\pi \in \mathfrak{g}^*$

Proof: = Hamiltonian analysis $\{g^\mu, p_\nu\} = \delta^\mu{}_\nu$

$$p_\alpha = \frac{\partial L}{\partial \dot{g}^\alpha} = g_{\mu\nu} (R^{-1})^\mu{}_\rho (R^{-1})^\nu{}_\alpha \dot{g}^\rho \qquad \{g^\mu, g^\nu\} = 0 = \{p_\mu, p_\nu\}$$

$$\Rightarrow \underbrace{R^\alpha{}_\nu p_\alpha}_{\pi_\nu} = g_{\mu\nu} (R^{-1})^\mu{}_\rho \dot{g}^\rho \Longleftrightarrow \dot{g}^\rho = R^\rho{}_\mu g^{\mu\nu} \pi_\nu$$

Non-Darboux coordinates $\{\pi_\mu, \pi_\nu\} = f^\rho_{\mu\nu} \pi_\rho$ KKS bracket

$$\{g^\mu, g^\nu\} = 0 , \quad \{g^\mu, \pi_\nu\} = R^\mu{}_\nu$$

$$S_H = \int dt \; [\pi_\mu (R^{-1})^\mu{}_\nu \, \dot{g}^\nu - H] \qquad H = \frac{1}{2} \pi_\mu g^{\mu\nu} \pi_\nu$$

$$\boxed{\dot{\pi}_\mu = \{\pi_\mu, H\} = f^\rho{}_{\mu\nu} \, g^{\nu\sigma} \pi_\sigma}$$

$$\dot{g}^\mu = \{g^\mu, H\} = R^\mu{}_\nu \, g^{\nu\rho} \pi_\rho$$ $\Leftrightarrow$ def. of momentum

Noether charges $Q^\pi_X = \langle \pi, \mathrm{Ad}_g X \rangle$

Remarks: 1) More general Hamiltonians "inertia operators"

$A : \mathfrak{g} \to \mathfrak{g}^*$ invertible $\qquad H = \frac{1}{2} \pi_\mu A^{\mu\nu} \pi_\nu$

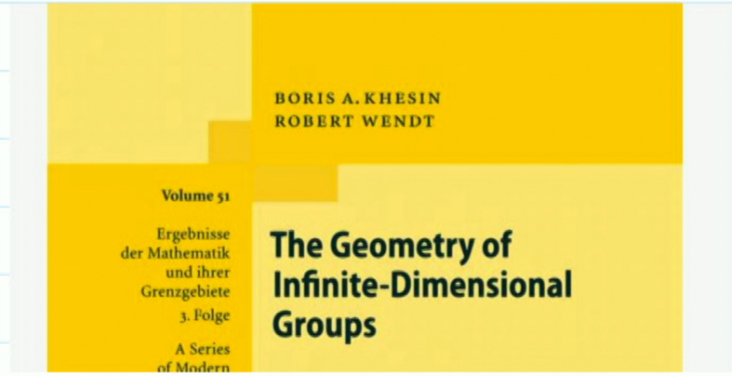
$X^\mu \mapsto A_{\mu\nu} X^\nu$

$SO(3)$: Euler top

34 I. Preliminaries

Group	*Metric*	*Equation*
SO(3)	$\langle \omega, A\omega \rangle$	Euler top
$\mathrm{SO}(3) \ltimes \mathbb{R}^3$	quadratic forms	Kirchhoff equation for a body in a fluid
SO(n)	Manakov's metrics	n-dimensional top
Diff(S^1)	L^2	Hopf (or, inviscid Burgers) equation
Virasoro	L^2	KdV equation
Virasoro	H^1	Camassa–Holm equation
Virasoro	$\dot{H}^1$	Hunter–Saxton (or Dym) equation
SDiff(M)	L^2	Euler ideal fluid
SDiff(M)	H^1	averaged Euler flow
$\mathrm{SDiff}(M) \ltimes \mathrm{SVect}(M)$	$L^2 + L^2$	Magnetohydrodynamics
Maps(S^1, SO(3))	H^{-1}	Heisenberg magnetic chain

Table 4.1: Euler equations related to various Lie groups.



2) Bi-Hamiltonian (integrable) system

$$\{\Pi_\mu, \Pi_\nu\}_0 = f_{\mu\nu}^{\rho}\, b_{0\rho}$$ (2nd compatible PB)

Digression: Scalar sector of $\mathcal{N}=4, d=4$ SUGRA

$SL(2,\mathbb{R})/SO(2)$ symmetric space

$sl(2,\mathbb{R})$: $H=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, $E_+=\begin{pmatrix}0&1\\0&0\end{pmatrix}$, $E_-=\begin{pmatrix}0&0\\1&0\end{pmatrix}$

$$g = e^{\psi(E_+-E_-)}\underbrace{e^{\phi/2\, H}\, e^{\chi E_+}}_{g_B} \quad \text{Borel gauge} \qquad g^\mu = (\psi, \phi, \chi) \qquad g_B^\mu = (\phi, \chi)$$

$$P_\mu = \frac{1}{2}\left[(\partial_\mu g\, g^{-1}) + (\partial_\mu g g^{-1})^T\right], \qquad g \to k(x)\, g\, h$$

$SO(2) \circlearrowleft$ local $\qquad h \in SL(2,\mathbb{R})$ global

$$P_\mu \to k P_\mu k^{-1}$$

$$S_{SL(2,\mathbb{R})/SO(2)}[g_\Sigma^\mu] = -\int d^4x \, \mathrm{Tr}\, P_\mu P^\mu = -\frac{1}{2}\int d^4x \, [\partial_\mu\phi\,\partial^\mu\phi + e^{2\phi}\partial_\mu\chi\,\partial^\mu\chi]$$

axion-dilaton

$$\delta_{E_+} g_\Sigma = g_\Sigma E_+ \iff \delta_{E_+}\chi = 1, \quad \delta_{E_+}\phi = 0$$

$$\delta_H g_\Sigma = g_\Sigma H \iff \delta_H \phi = 2, \quad \delta_H \phi = -2\chi$$

$$\delta_{\delta h, E_-} g_\Sigma = \delta h \, g_\Sigma + g_\Sigma E_- \iff \delta_{\delta h, E_-}\phi = 2\chi, \quad \delta_{\delta h, E_-}\chi = e^{-2\phi} - \chi^2$$

compensating gauge transformation, "hidden" symmetries

non-linear realizations

Geometric actions

No Killing metric ?

- use fixed coadjoint vector $b_0 \in \mathfrak{g}^*$ to built first order action

$$S_G[g^\mu; b_0, X_0] = \int dt \left[\langle b_0, \frac{dg}{dt} g^{-1} \rangle - \langle b_0, \mathrm{Ad}_g X_0 \rangle \right]$$

$$= \int dt \left[b_{0\mu} (R^{-1})^\mu{}_\nu \dot{g}^\nu - b_{0\mu} (R^{-1}L)^\mu{}_\nu X_0^\nu \right]$$

$$\begin{cases} \delta_X g = g X \\ \delta_X g^\mu = L^\mu{}_\nu X^\nu \end{cases} \quad \Leftrightarrow \quad \frac{dX}{dt} = -[X_0, X]$$

- first order $\qquad a = \langle b_0, \kappa \rangle \quad , \quad \sigma = \delta a = \langle b_0, \frac{1}{2}[\kappa, \kappa] \rangle$

presymplectic kinetic term

- no path integral quantization because of gauge invariance

$\delta_{\epsilon(t)}\, g = \epsilon(t)\, g$ little algebra $\mathcal{H}_{b_0}$, $ad^*_{\epsilon(t)}\, b_0 = 0$

- are these all gauge transformations ?

- How many models $S[g^*; b_0, X_0]$ to study ?

$b_0' = Ad^*_{h^{-1}}\, b_0 \qquad S[g\,; b_0', X_0] = S[g'; b_0, X_0]$

$g' = hg$ field redefinition : equivalent

Virasoro :
Lazutkin & Pankratova
Kirillov, Witten,
Balog et al.

- only 1 representative needed per orbit of partition of $\mathfrak{g}^*$

Constrained Hamiltonian analysis

$\{g^\mu, p_\nu\} = \delta^\mu_\nu \qquad p_\mu = \frac{\partial L}{\partial \dot{g}^\mu} = b_{0\,\nu}(R^{-1})^\nu{}_\mu$ primary constraints

$p_\mu \leftrightarrow \pi_\mu = R^\nu{}_\mu\, p_\nu \qquad \phi^{b_0}_\mu = \pi_\mu - b_{0\mu} \approx 0$

$$S^H_G[g^\mu, \pi_\nu; b_0, X_0] = \int dt\, \left[\pi_\mu (R^{-1})^\mu{}_\nu \dot{g}^\nu - H_c - u^\mu \phi^{b_0}_\mu\right]$$

$$H_c = \langle b_0, \mathrm{Ad}_g X_0\rangle \approx \langle \pi, \mathrm{Ad}_g X_0\rangle$$

Dynamics: $\dot{g}^\mu = \{g^\mu, H_c + u^\nu \phi^{b_0}_\nu\} = L^\mu{}_\nu X_0^\nu + R^\mu{}_\nu u^\nu$

$\dot{\pi}_\mu = \{\pi_\mu, H_c + u^\nu \phi^{b_0}_\nu\} = \pi_\rho f^\rho{}_{\mu\nu} u^\nu$

Noether charges: $Q^\pi_X = \langle \pi, \mathrm{Ad}_g X\rangle, \quad \{Q^\pi_{X_1}, Q^\pi_{X_2}\} = Q^\pi_{[X_1, X_2]}$

Secondary constraints ?

$$\dot{\phi}^{b_0}_{\mu} \approx 0 \iff \underbrace{b_{0\rho} f^{\rho}{}_{\mu\nu}}_{C_{\mu\nu}} u^{\nu} = 0 \quad (*)$$ No, only restrictions on Lagrange multipliers

$\Updownarrow$

complete set of null vectors $e_a{}^{\mu}$, $\quad u^{\mu} = e_a{}^{\mu} v^a$ ← arbitrary

adapted basis $e_a{}^{\mu}, e_A{}^{\mu}, e^a{}_{\mu}, e^A{}_{\mu}$

$$e_a{}^{\mu} e^b{}_{\mu} = \delta_a^b, \quad e_A{}^{\mu} e^b{}_{\mu} = 0, \quad e_A{}^{\mu} e^B{}_{\mu} = \delta_A{}^B, \quad e_a{}^{\mu} e^a{}_{\nu} + e_A{}^{\mu} e^A{}_{\nu} = \delta^{\mu}_{\nu}$$

$$\Rightarrow f^C{}_{ab} = 0, \quad C_{ab} = 0 = C_{aB}, \quad C_{AB} = b_{0c} f^c{}_{AB} + b_{0C} f^C{}_{AB}$$ invertible

$$(C^{-1})^{AB} C_{BC} = \delta^A{}_C \qquad (*) \iff v^A = 0, \quad v^a$$ arbitrary

Dirac brackets:

$$\begin{array}{c|cc} & g^{\nu} & \pi_b \\ \hline g^{\mu} & C_{AB}(R^{-1})^{A}{}_{\mu}(R^{-1})^{B}{}_{\nu} & -(R^{-1})^{b}{}_{\mu} \\ \pi_a & +(R^{-1})^{a}{}_{\nu} & 0 \end{array} \quad \begin{array}{c|cc} & g^{\rho} & \pi_c \\ \hline g^{\nu} & R^{\nu}{}_{C}(C^{-1})^{CD}R^{\rho}{}_{D} & R^{\nu}{}_{c} \\ \pi_b & -R^{\rho}{}_{b} & 0 \end{array} = \begin{pmatrix} \delta^{\mu}{}_{\nu} & 0 \\ 0 & \delta^{a}{}_{b} \end{pmatrix}$$

π^R

$$\{g^{\nu}, g^{\rho}\}^{*} = R^{\nu}{}_{C}(C^{-1})^{CD}R^{\rho}{}_{D} \qquad \{g^{\nu}, \pi_c\}^{*} = R^{\nu}{}_{c}$$

$$\{\pi_b, \pi_c\}^{*} = f^{d}_{bc}\,\pi_d \approx 0$$

- solve first class constraints in the action $\longrightarrow$ $S_G[g^{\mu}; X_0, b_0]$ (degenerate)
- $\widehat{\mathrm{Diff}}(S^1)$ typical little groups $U(1)$ π_a
  $SL(2,\mathbb{R})$ at most 3d
  g^{μ} $\sim$ dim

Unconstrained model: drop all constraints $\phi^{(0)}_{\mu} = 0$

$$S^{U}_{G}[g^{\mu}, \pi_{\nu}; X_0] = \int dt \left[\pi_{\mu} (R^{-1})^{\mu}{}_{\nu} \dot{g}^{\nu} - \underbrace{\pi_{\mu} (R^{-1}L)^{\mu}{}_{\nu} X_0^{\nu}}_{H^{\pi}_{X_0}}\right]$$

$$\dot{g}^{\mu} = \{g^{\mu}, H^{\pi}_{X_0}\} = L^{\mu}{}_{\nu} X_0^{\nu} \iff \frac{dg}{dt} g^{-1} = \mathrm{Ad}_g X_0$$

$$\dot{\pi}_{\mu} = \{\pi_{\mu}, H^{\pi}_{X_0}\} = 0$$

conserved charges: Q^{π}_{X}, π_{μ}

level sets of $\pi_{\mu} = \ell_{0\mu}$

Hamiltonian reduction $\rightarrow$ do previous analysis in reverse

Gravity models: • AdS_3 gravity

general solution with Brown-Henneaux boundary conditions

$$ds^2 = \frac{\ell^2}{r^2} dr^2 - \left(r\, dx^+ - \frac{8\pi G \ell}{r} b^- dx^-\right)\left(r\, dx^- - \frac{8\pi G \ell}{r} b^+ dx^+\right)$$

$$x^\pm = \frac{t}{\ell} \pm \varphi \qquad b^\pm(x^\pm + 2\pi) = b^\pm(x^\pm)$$

conformal transformations $\quad x^\pm \to f^\pm(x^\pm), \quad f^\pm(x^\pm + 2\pi) = f^\pm(x^\pm) + 2\pi$

$$\tilde{b}^\pm = Ad^*_{f^{-1}} b^\pm = (\partial_\pm f^\pm)^2\, b^\pm \circ f^\pm - c^\pm S_{x^\pm}[f^\pm]$$

$$c^\pm = \frac{3\ell}{2G} \qquad S_x[f] = \frac{1}{24\pi}\left[\partial_x^2 (\ln \partial_x f) - \frac{1}{2}(\partial_x \ln \partial_x f)^2\right]$$

Schwarzian derivative

centrally extended group $\hat{G} = G \times \mathbb{R}$

$$(g_1, m_1) \cdot (g_2, m_2) = (g_1 \cdot g_2,\ m_1 + m_2 + \Xi(g_1, g_2))$$

group 2-cocycle

$$Ad_{(g,m)}(X, u) = (Ad_g X,\ u - \langle S(g), X\rangle)$$

Souriau cocycle $S(g): G \to \mathfrak{g}^*$

$$ad_{(X,u)}(Y, a) = (ad_X Y,\ \omega(X,Y))$$

Lie algebra cocycle

$$ad^*_{(X,u)}(b, c) = (ad^*_X b + c\, s(X),\ 0)$$

infinitesimal Souriau cocycle

MC: $(\theta, \theta_\pi), (K, K_\pi)$

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$$S_{\hat{G}}[(g,m);(b_0,c_0)] = I_G[g;b_0] + c_0\int\left(-\langle S(g),\theta\rangle + \theta_{\Xi}\right)$$

Virasoro $\widehat{\mathrm{Diff}}(S_1)$, $\Xi(f_1,f_2) = -\frac{1}{48\pi}\int_0^{2\pi} d\varphi\, \ln(\partial_\varphi f_1 \circ f_2)\, \partial_\varphi(\ln \partial_\varphi f_2)$

Bott-Thurston

$$S_{\widehat{\mathrm{Diff}}(S_1)}[f;b_0,c_0] = \int dt\, d\varphi\, \left[b_0(f) f' \dot{f} + \frac{c_0}{48\pi}\frac{\dot{f}''}{f'}\right]$$

Hamiltonian $X_0 = \frac{\partial}{\partial\varphi}$ in order to reproduce Gibbons-Hawking term

CS $\to$ cWZW $\to$ Liouville (chiral bosons)

$$S_{\widehat{\mathrm{Diff}}(S^1)}[f; b_0, c_0, X_0] = 2\int d\varphi\, dt \; b_0(f)\, f' \partial_- f + \frac{c_0}{48\pi}\left((\ln f')' (\partial_- \ln f')\right)$$

· relation to chiral boson actions, similar actions by Cotler-Jensen

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[7] A. Alekseev and S. L. Shatashvili. "From geometric quantization to conformal field theory". In: *Commun.Math.Phys.* 128 (1990), pp. 197–212. DOI: 10.1007/BF02097053.

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[9] G. W. Delius, P. van Nieuwenhuizen, and V. G. J. Rodgers. "The Method of Coadjoint Orbits: An Algorithm for the Construction of Invariant Actions". In: *Int. J. Mod. Phys. A* 5 (1990), pp. 3943–3984. DOI: 10.1142/S0217751X90001690.

Coadjoint representation & semi-direct product groups

adjoint $\mathfrak{g}$: $[e_a, e_b] = f^c_{ab}\, e_c \qquad (\mathrm{ad}\, e_a)^b{}_c = f^b_{ac} \iff \mathrm{ad}\, e_a(e_b) = f^c_{ab}\, e_c$

coadjoint $\mathfrak{g}^*$: $\langle e^b_*, e_a\rangle = \delta^b_a \qquad (\mathrm{ad}^* e_a) = -(\mathrm{ad}\, e_a)^T \iff \mathrm{ad}^* e_a(e^b_*) = -f^b_{ac}\, e^c_*$

group $\quad \mathrm{Ad}_g\, e_a = g\, e_a\, g^{-1}, \qquad \mathrm{Ad}^*_g = g\, e^b_*\, g^{-1}$

semi-direct product $\quad G \ltimes_\sigma A \; : \; (f,\alpha)\cdot(g,\beta) = (f\cdot g,\ \alpha + \sigma_f(\beta))$ $\qquad \sigma$: representation

ISO(3), ISO(3,1), $\qquad$ A : abelian ideal

BMS_3, BMS_4 ... $\qquad \mathfrak{g} \oplus_\Sigma A \; : \; [(X,\alpha),(Y,\beta)] = ([X,Y],\ \Sigma_X \beta - \Sigma_Y \alpha)$

$$\mathrm{Ad}_{(f,\alpha)}(X,\beta) = (\mathrm{Ad}_f X,\ \sigma_f\beta - \Sigma_{\mathrm{Ad}_f X}\,\alpha)$$

$$\mathrm{ad}_{(X,\alpha)}(Y,\beta) = ([X,Y],\ \Sigma_X \beta - \Sigma_Y \alpha)$$

dual space $\mathfrak{g}^* \oplus A^*$ $\quad \langle (j,p), (X,\alpha) \rangle = \langle j, X \rangle + \langle p, \alpha \rangle$

terminology $\quad j$: angular momentum $\quad p$: linear momentum $\quad$ BMS: add "super"

X : inf. rotation $\quad \alpha$: inf. translation

ingredients $\quad \times : A \oplus A^* \to \mathfrak{g}^* \quad : \quad \langle \alpha \times p, X \rangle = \langle p, \Sigma_X \alpha \rangle$

change in angular momentum due to a translation

$$\sigma^* : G \times A^* \to A^* \quad : \quad \langle \sigma_f^* p, \alpha \rangle = \langle p, \sigma_{f^{-1}} \alpha \rangle$$

coadjoint representation

$$\mathrm{Ad}^*_{(f,\alpha)} (j,p) = \left(\mathrm{Ad}^*_f j + \alpha \times \sigma^*_f p, \ \sigma^*_f p \right)$$

$$\mathrm{ad}^*_{(X,\alpha)} (j,p) = \left(\mathrm{ad}^*_X j + \alpha \times p, \ \Sigma^*_X p \right)$$

Geometric action for semi-direct product group

$$S = \int \left[\langle \pi, \kappa \rangle - du \langle \pi, \mathrm{Ad}_g Y_0 \rangle \right]$$

$$\langle j - \alpha \times p, \kappa_g \rangle + \langle p, d\alpha \rangle \qquad \langle j, \mathrm{Ad}_g X_0 \rangle + \langle p, \sigma_g \beta_0 - \Sigma_{\mathrm{Ad}_g X_0} \alpha \rangle$$

$$\widehat{BMS}_3 = \widehat{\mathrm{Diff}}(S_1) \ltimes \widehat{\mathrm{Vect}(S^1)}_{ab}$$

$$S_{\widehat{BMS}_3}[f, \alpha; p_0, j_0, c_1, c_2] = S_{\widehat{\mathrm{Diff}}(S_1)}[f; p_0, c_1]$$

$$+ \int d\varphi \int dt \; \dot{f} f' \left(p_0' \alpha + 2 p_0 \alpha' - \frac{c_2}{24\pi} \alpha''' \right) \circ f + \left(p_0 \frac{d\alpha}{dt} \right)$$

Hamiltonian $X_0 = (0, \partial_\varphi) \leftrightarrow \partial_u$

$$H = \int_0^{2\pi} d\varphi \left[f'^2 p_0(f) + \frac{c_2}{48\pi} \frac{f''^2}{f'} \right]$$

Asymptotically flat 3d metrics

$$ds^2 = 2\left[8\pi G\, p\, du - dr + 8\pi G\,(j + u p')\, d\varphi\right] du + r^2 d\varphi^2$$

$p = p(\varphi)$, $j = j(\varphi)$

finite BMS_3 transf.
$$\begin{cases} \tilde{p} = (f')^2\, p \circ f - c_2\, S_\varphi[f] \\ \tilde{j} = (f')^2\left[j + \alpha p' + 2\alpha' p - \frac{c_2}{24\pi}\alpha'''\right]\circ f - c_1\, S_\varphi[f] \end{cases}$$

EH : $c_1 = 0$, $c_2 = \frac{3}{G\pi}$

same actions as previously constructed from

$CS_{ISO(2,1)} \to cWZW \to BMS_3$ inv chiral boson action

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Asymptotically flat 3d metrics

$$ds^2 = 2\left[8\pi G\, p\, du - dr + 8\pi G\,(j + u p')\, d\varphi\right] du + r^2\, d\varphi^2$$

$$p = p(\varphi)\,, \quad j = j(\varphi) \qquad \begin{cases} \tilde{p} = (f')^2\, p \circ f - c_2\, S_\varphi[f] \\ \tilde{j} = (f')^2\left[j + \alpha p' + 2\alpha' p - \frac{c_2}{24\pi}\alpha'''\right]\circ f - c_1\, S_\varphi[f] \end{cases}$$

finite BMS_3 transf.

EH : $\quad c_1 = 0\,, \quad c_2 = \frac{3}{G}$

same actions as previously constructed from

$$CS_{ISO(2,1)} \to cWZW \to BMS_3 \text{ inv chiral boson action}$$

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Coadjoint representation of BMS_4: general structure

2d conformally flat S aim: unified description for sphere & punctured plane

$$ds^2 = -2(P\bar{P})^{-1} d\xi\, d\bar{\xi} \quad \begin{cases} \xi' = \xi'(\xi) \quad \bar{\xi}' = \bar{\xi}'(\bar{\xi}) & \text{conformal coordinate trsf.} \\ P'(x) = P(x) e^{-E(x)}, \quad \bar{P}'(x) = \bar{P}(x) e^{-\bar{E}(x)} & \text{complex Weyl rescaling} \begin{cases} E_R & \text{standard Weyl} \\ iE_I & \text{local rotation} \end{cases} \end{cases}$$

$x = (\xi, \bar{\xi})$

zweibeins $\quad ds^2 = e^a_\mu dx^\mu \eta_{ab} e^b_\nu dx^\nu \quad \eta_{ab} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad e_1^\mu \frac{\partial}{\partial x^\mu} = P \frac{\partial}{\partial \xi} \quad e_2^\mu \frac{\partial}{\partial x^\mu} = \bar{P} \frac{\partial}{\partial \bar{\xi}}$

conformal fields $\quad \phi'_{h,\bar{h}}(x') = \left(\frac{\partial \xi}{\partial \xi'}\right)^h \left(\frac{\partial \bar{\xi}}{\partial \bar{\xi}'}\right)^{\bar{h}} \phi_{h,\bar{h}}(x) \qquad P'(x') = P(x) e^{-E(x')} \frac{\partial \xi'}{\partial \xi}$ combined trsf.

weighted scalars $\quad \eta'^{s,w}(x') = e^{w E_R(x')} e^{-is E_I(x')} \eta^{s,w}(x)$

Held, Posadas, Newman JMP 1970

Lorentz group & sphere

D'Hoker, Phong Rev. Mod. Phys. 1988

interpolation map $\quad \eta^{s,w} = P^h \bar{P}^{\bar{h}} \phi_{h,\bar{h}}$

$s = h - \bar{h}, \quad w = -(h + \bar{h}) \qquad h = \frac{s-w}{2}, \quad \bar{h} = -\frac{s+w}{2}$

covariant derivative $\quad \nabla : \quad \Gamma^{\xi}_{\xi\xi} = -\partial \ln(P\bar{P}) \qquad \Gamma^{\bar{\xi}}_{\bar{\xi}\bar{\xi}} = -\bar{\partial} \ln(P\bar{P})$

$$\Gamma'^{\xi'}_{\xi'\xi'}(x') = \Gamma^{\xi}_{\xi\xi}(x)\frac{\partial\xi}{\partial\xi'} + \frac{\partial\xi'}{\partial\xi}\frac{\partial^2\xi}{\partial\xi'\partial\xi'} + 2\partial' E_R(x')$$

introduce Weyl connection $\quad D: \quad W'(x') = \frac{\partial\xi}{\partial\xi'}W + 2\partial' E_R(x'), \quad \bar{W}'(x') = \frac{\partial\bar{\xi}}{\partial\bar{\xi}'}\bar{W}(x) + 2\bar{\partial}' E_R(x')$

$$\underbrace{D\phi_{h,\bar{h}}}_{(h+1,\bar{h})} = [\nabla + hW]\phi_{h,\bar{h}}, \quad \underbrace{\bar{D}\phi_{h,\bar{h}}}_{(h,\bar{h}+1)} = [\bar{\nabla} + \bar{h}\bar{W}]\phi_{h,\bar{h}}$$

$\partial, \nabla, D = \partial_\xi, \nabla_\xi, D_\xi$

$\bar{\partial}, \bar{\nabla}, \bar{D} = \partial_{\bar{\xi}}, \nabla_{\bar{\xi}}, D_{\bar{\xi}}$

weighted scalars

$$\eth\, \eta^{s,w} = P^{h+1}\bar{P}^{\bar{h}}\nabla\phi_{h,\bar{h}} = P\bar{P}^{-s}\partial(\bar{P}^{s}\eta^{s,w}), \qquad \bar{\eth}\,\eta^{s,w} = P^{h}\bar{P}^{\bar{h}+1}\bar{\nabla}\phi_{h,\bar{h}} = \bar{P}P^{s}\bar{\partial}(P^{-s}\eta^{s,w})$$

Weyl covariant

$$\underbrace{\mathscr{D}\eta^{s,w}}_{[s+1,w-1]} = P^{h+1}\bar{P}^{\bar{h}} D\phi_{h,\bar{h}} = \left(\eth + \left(\frac{s-w}{2}\right)PW\right)D\phi_{h,\bar{h}}, \qquad \underbrace{\bar{\mathscr{D}}\eta^{s,w}}_{[s-1,w-1]} = P^{h}\bar{P}^{\bar{h}+1}\bar{D}\phi_{h,\bar{h}} = \left[\bar{\eth} - \left(\frac{w+s}{2}\right)\bar{P}\bar{W}\right]\phi_{h,\bar{h}}$$

$$[\mathscr{D}, \bar{\mathscr{D}}]\eta^{s,w} = -\frac{s}{2}R_s\,\eta^{s,w} - P\bar{P}\left(\frac{s-w}{2}\bar{\partial}W + \frac{s+w}{2}\partial\bar{W}\right)\eta^{s,w}$$

$\mathscr{R}_s$: scalar curvature

bms$_4$ algebra $\quad [(\mathcal{Y}_1,\bar{\mathcal{Y}}_1,\mathcal{T}_1),(\mathcal{Y}_2,\bar{\mathcal{Y}}_2,\mathcal{T}_2)] = (\hat{\mathcal{Y}},\hat{\bar{\mathcal{Y}}},\hat{\mathcal{T}})$

$$\hat{\mathcal{Y}} = \mathcal{Y}_1\partial\mathcal{Y}_2 - \mathcal{Y}_2\partial\mathcal{Y}_1 \qquad \hat{\mathcal{T}} = \mathcal{Y}_1\partial\mathcal{T}_2 - \frac{1}{2}\partial\mathcal{Y}_1\mathcal{T}_2 - (1\leftrightarrow 2) + c.c.$$

subalgebra g $\quad (\mathcal{Y},\bar{\mathcal{Y}},0) \qquad (\tilde{\mathcal{Y}},\tilde{\bar{\mathcal{Y}}},0)$

(Lorentz, Witt⊕Witt)

representation of g on $\eta^{s,w}$ $\qquad$ on $\phi_{h,\bar{h}}$

$$\mathcal{Y}\cdot\eta^{s,w} = \mathcal{Y}\partial\eta^{s,w} + \frac{s-w}{2}\partial\mathcal{Y}\,\eta^{s,w} \qquad \tilde{\mathcal{Y}}\cdot\phi_{h,\bar{h}} = \tilde{\mathcal{Y}}\partial\phi_{h,\bar{h}} + h\,\partial\tilde{\mathcal{Y}}\,\phi_{h,\bar{h}}$$

$$\bar{\mathcal{Y}}\cdot\eta^{s,w} = \bar{\mathcal{Y}}\bar{\partial}\eta^{s,w} - \frac{s+w}{2}\bar{\partial}\bar{\mathcal{Y}}\,\eta^{s,w} \qquad \tilde{\bar{\mathcal{Y}}}\cdot\phi_{h,\bar{h}} = \tilde{\bar{\mathcal{Y}}}\bar{\partial}\phi_{h,\bar{h}} + \bar{h}\,\bar{\partial}\tilde{\bar{\mathcal{Y}}}\,\phi_{h,\bar{h}}$$

$$\mathcal{E}_{x\,\alpha} = (\mathcal{Y},\bar{\mathcal{Y}})\cdot\mathcal{T}^{[0,1]} \qquad \mathcal{E}_{x\,\alpha} = [\tilde{\mathcal{Y}},\tilde{\bar{\mathcal{Y}}})\cdot\tilde{\mathcal{T}}^{(-1/2,-1/2)}$$

action of inf rotation on translations

$\mathfrak{bms}_4^*$ dual space $([J],[\bar J], P)$ $\quad ([\tilde J],[\hat{\bar J}],\tilde P)$

$(0,0)\ ;\ [0,-2]$

pairing $\langle ([J],[\bar J],P);(Y,\bar Y,T)\rangle = \int_S d\mu\, [\bar J Y + J\bar Y + PT]$, $\quad d\mu(\xi,\bar\xi) = \frac{iC}{P\bar P}\, d\xi\wedge d\bar\xi$

$$\langle([\tilde J],[\tilde{\bar J}],\tilde P),(\tilde Y,\tilde{\bar Y},\tilde T)\rangle = \int_S d\tilde\mu\, [\tilde{\bar J}\tilde Y + \tilde J \tilde{\bar Y} + \tilde P\tilde T] \qquad d\tilde\mu = iC\, d\xi\wedge d\bar\xi$$

assumption: pairing annihilates total $\eth, \bar\eth$ $(\partial,\bar\partial)$ derivatives
non-degenerate $\quad\rightarrow$ integrations by parts

$$ad^*_{(Y,\bar Y,T)} J = \bar Y \bar\eth J + 2\bar\eth\bar Y J + \underbrace{\eth(YJ)}_{= ad^*_{\bar Y} J \sim 0} + \underbrace{\tfrac12 T\bar\eth P + \tfrac32 \bar\eth T P}_{\alpha\times P},$$

$$ad^*_{(Y,\bar Y,T)} P = \underbrace{Y\eth P + \tfrac32 \eth Y P + c.c.}_{\mathcal{E}^*_X P}$$

work out formulas for the group ✓

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Ingredients

(super-)translation	$\mathcal{T} : [0,1]$	$\tilde{\mathcal{T}} : (-\frac{1}{2}, -\frac{1}{2})$	real
(super-) rotation	$\mathcal{Y} : [-1,1]$	$\tilde{\mathcal{Y}} : (-1,0)$	$\bar{\mathcal{D}} \mathcal{Y} = 0 \Leftrightarrow \bar{D} \tilde{\mathcal{Y}} = 0$
	$\bar{\mathcal{Y}} : [1,1]$	$\bar{\tilde{\mathcal{Y}}} : (0,-1)$	$\mathcal{D} \bar{\mathcal{Y}} = 0 \Leftrightarrow D \bar{\tilde{\mathcal{Y}}} = 0$
(super-) momentum	$\mathcal{P} : [0,-3]$	$\tilde{\mathcal{P}} : (\frac{3}{2}, \frac{3}{2})$	real
(super-) angular momentum	$\mathcal{J} : [-1,-3]$	$\tilde{\mathcal{J}} : (1,2)$	$\mathcal{J} \sim \mathcal{J} + \mathcal{D} \underset{[-2,-2]}{\mathcal{L}}$, $\tilde{\mathcal{J}} \sim \tilde{\mathcal{J}} + D \underset{(0,2)}{\tilde{\mathcal{L}}}$
	$\bar{\mathcal{J}} : [1,-3]$	$\bar{\tilde{\mathcal{J}}} : (2,1)$	$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar{\mathcal{D}} \underset{[2,-2]}{\bar{\mathcal{L}}}$, $\bar{\tilde{\mathcal{J}}} \sim \bar{\tilde{\mathcal{J}}} + \bar{D} \underset{(2,0)}{\bar{\tilde{\mathcal{L}}}}$

In all relations, weights/dimensions are such that Weyl connection drops out !

$\mathcal{D} \to \eth$ $\quad D \to \partial$ simplest description in terms of conformal fields

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bms_4^* dual space $\quad ([\mathcal{J}], [\bar{\mathcal{J}}], \mathcal{P}) \quad ([\tilde{\mathcal{J}}], [\hat{\bar{\mathcal{J}}}], \tilde{\mathcal{P}})$

$(0,0) \; ; \; [0,-2]$

pairing $\quad \langle([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P}); (\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})\rangle = \int_S d\mu \, [\bar{\mathcal{J}}\mathcal{Y} + \mathcal{J}\bar{\mathcal{Y}} + \mathcal{P}\mathcal{T}] \, , \; d\mu(\xi,\bar{\xi}) = \frac{iC}{P\bar{P}} d\xi \wedge d\bar{\xi}$

$\langle([\tilde{\mathcal{J}}],[\tilde{\bar{\mathcal{J}}}],\tilde{\mathcal{P}}), (\tilde{\mathcal{Y}},\tilde{\bar{\mathcal{Y}}},\tilde{\mathcal{T}})\rangle = \int_S d\tilde{\mu} \, [\tilde{\bar{\mathcal{J}}}\tilde{\mathcal{Y}} + \tilde{\mathcal{J}}\tilde{\bar{\mathcal{Y}}} + \tilde{\mathcal{P}}\tilde{\mathcal{T}}] \qquad d\tilde{\mu} = iC \, d\xi \wedge d\bar{\xi}$

assumption : pairing annihilates total $\eth, \bar{\eth}$ $(\mathcal{J}, \bar{\mathcal{J}})$ derivatives
non-degenerate $\longrightarrow$ integrations by parts

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{J} = \bar{\mathcal{Y}}\bar{\eth}\mathcal{J} + 2\bar{\eth}\bar{\mathcal{Y}}\mathcal{J} + \underbrace{\eth(\mathcal{Y}\mathcal{J})}_{=\mathrm{ad}^*_{\mathcal{Y}}\mathcal{J} \sim 0} + \underbrace{\tfrac{1}{2}\mathcal{T}\bar{\eth}\mathcal{P} + \tfrac{3}{2}\bar{\eth}\mathcal{T}\mathcal{P}}_{\alpha \times \mathcal{P}} \, ,$$

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{P} = \underbrace{\mathcal{Y}\eth\mathcal{P} + \tfrac{3}{2}\eth\mathcal{Y}\mathcal{P} + \text{c.c.}}_{\mathcal{E}^*_{\mathcal{X}}\mathcal{P}}$$

work out formulas for the group ✓

$\mathfrak{bms}_4^*$ dual space $([\mathcal{J}], [\bar{\mathcal{J}}], \mathcal{P})$ $([\tilde{\mathcal{J}}], [\hat{\bar{\mathcal{J}}}], \tilde{\mathcal{P}})$

$(0,0)$; $[0,-2]$

pairing $\langle([\mathcal{J}],[\bar{\mathcal{J}}],\mathcal{P});(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})\rangle = \int_S d\mu\,[\bar{\mathcal{J}}\mathcal{Y} + \mathcal{J}\bar{\mathcal{Y}} + \mathcal{P}\mathcal{T}]$, $d\mu(\zeta,\bar{\zeta}) = \frac{iC}{P\bar{P}}\, d\zeta\wedge d\bar{\zeta}$

$\langle([\tilde{\mathcal{J}}],[\tilde{\bar{\mathcal{J}}}],\tilde{\mathcal{P}}),(\tilde{\mathcal{Y}},\tilde{\bar{\mathcal{Y}}},\tilde{\mathcal{T}})\rangle = \int_S d\tilde{\mu}\,[\tilde{\bar{\mathcal{J}}}\tilde{\mathcal{Y}} + \tilde{\mathcal{J}}\tilde{\bar{\mathcal{Y}}} + \tilde{\mathcal{P}}\tilde{\mathcal{T}}]$ $d\tilde{\mu} = iC\, d\zeta\wedge d\bar{\zeta}$

assumption : pairing annihilates total $\eth, \bar{\eth}$ $(\partial, \bar{\partial})$ derivatives

non-degenerate $\rightarrow$ integrations by parts

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{J} = \bar{\mathcal{Y}}\bar{\eth}\mathcal{J} + 2\bar{\eth}\bar{\mathcal{Y}}\mathcal{J} + \underbrace{\eth(\mathcal{Y}\mathcal{J})}_{=\mathrm{ad}^*_{\bar{\mathcal{Y}}}\mathcal{J}\,\sim 0} + \underbrace{\tfrac{1}{2}\mathcal{T}\bar{\eth}\mathcal{P} + \tfrac{3}{2}\bar{\eth}\mathcal{T}\mathcal{P}}_{\alpha\times P},$$

$$\mathrm{ad}^*_{(\mathcal{Y},\bar{\mathcal{Y}},\mathcal{T})}\mathcal{P} = \underbrace{\mathcal{Y}\eth\mathcal{P} + \tfrac{3}{2}\eth\mathcal{Y}\,\mathcal{P} + \text{c.c.}}_{\mathcal{E}^*_X P}$$

work out formulas for the group ✓

adjoint repres. group $\quad \mathcal{Y}'(x') = e^{E_R(x')}\, e^{i E_I(x')}\, \mathcal{Y}(x)$

$$\beta'(x') = e^{E_R(x')}\left(\beta - (\mathcal{Y}\partial\alpha - \tfrac{1}{2}\partial\alpha\,\partial\mathcal{Y} + c.c.)\right)(x)$$

coadjoint repres. group $\quad \mathcal{J}'(x') = e^{-3E_R(x')}\, e^{i E_I(x')}\left(\mathcal{J} + \tfrac{1}{2}\partial\bar\partial\mathcal{P} + \tfrac{3}{2}\bar\partial\partial\mathcal{P}\right)(x)$

$$\mathcal{P}'(x') = e^{-3E_R(x')}\, \mathcal{P}(x)$$

in terms of conf. fields

$$\tilde{\mathcal{Y}}'(\xi') = (c\xi+d)^{-2}\,\tilde{\mathcal{Y}}(\xi)$$

$$\tilde\beta'(x') = (c\xi+d)^{-1}(\bar c\bar\xi+\bar d)^{-1}\left(\tilde\beta - \tilde{\mathcal{Y}}\partial\tilde\alpha - \tfrac{1}{2}\tilde\alpha\,\partial\tilde{\mathcal{Y}} + c.c.\right](x)$$

$$\tilde{\mathcal{J}}'(x') = (c\xi+d)^{2}(\bar c\bar\xi+\bar d)^{4}\left(\tilde{\mathcal{J}}(x) + \left(\tfrac{1}{2}\tilde{\bar\partial}\,\partial\,\tilde{\mathcal{P}} + \tfrac{3}{2}\partial\tilde{\bar\partial}\,\tilde{\mathcal{P}}\right)\right)(x)$$

$$\tilde{\mathcal{P}}'(x') = (c\xi+d)^{3}(\bar c\bar\xi+\bar d)^{3}\,\tilde{\mathcal{P}}(x)$$

Expansions : spin weighted spherical harmonics: ${}_sZ_{jm}$ unnormalized ${}_sY_{jm}$ normalized

Gelfand, Minlos, Shapiro (1958) ; Wu & Yang, Nucl. Phys. B (1976)
Newman, Penrose, JMP (1966) ; Thorne, Rev. Mod. Phys (1980)

conformal Killing eq. on S^2 $\quad \bar{\eth} \mathcal{Y}^{[-1,1]} = 0 = \eth \bar{\mathcal{Y}}^{[1,-1]}$

$$\mathcal{Y}_m = -R\sqrt{2}\ {}_{-1}Z_{1,m} \qquad m=-1,0,1 \qquad \mathcal{Y} = \sum_{m=-1}^{1} y_m \mathcal{Y}_m$$

$$\mathcal{T}_{jm} = {}_0Z_{j,m} \qquad \mathcal{T} = \sum_{j,|m|\leq j} t_{j,m}\mathcal{T}_{j,m}, \qquad \bar{t}_{jm} = (-1)^m t_{j,-m}$$

dual basis $\quad \mathcal{Y}^m_* = \dfrac{-6}{R\sqrt{2}(1+m)!(1-m)!}\ {}_{-1}Z_{1,m} \qquad \mathcal{J} = \sum_{m=-1}^{1} j_m \mathcal{Y}^m_*$

$$\mathcal{T}^{j,m}_* = \frac{(2j+1)!(2j)!}{j!j!(j+m)!(j-m)!}\ {}_0Z_{j,m} \qquad \mathcal{P} = \sum_{j,|m|\leq j} p_{j,m}\mathcal{T}^{j,m}_*, \quad \bar{p}_{j,m} = (-1)^m p_{j,-m}$$

NB: conformal fields : $\tilde{\mathcal{Y}}_m = \mathcal{Y}_m P_S = \zeta^{1-m} \Rightarrow [\tilde{\mathcal{Y}}_m, \tilde{\mathcal{Y}}_n] = (m-n)\tilde{\mathcal{Y}}_{m+n}$

→ all other structure constants can be worked out explicitly (ugly)

Remark (i) Penrose & Rindler Vol I, section 4.15

$w \geq |s|$ $\qquad \eth^{w+s+1}\, \eta^{s,w}$ $\qquad \bar{\eth}^{w+s+1}\, \eta^{s,w}$

$\qquad [w+1, s-1] \qquad [-w-1, -s-1]$

definite boost weight

In the study of spin-weighted spherical harmonics it is useful to contemplate the following array:

j \ s	$-\frac{5}{2}$	-2	$-\frac{3}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$
0						1					
$\frac{1}{2}$					2		2				
1	$\overset{\eth'}{\leftarrow}$			3		3		3		$\overset{\eth}{\rightarrow}$	
$\frac{3}{2}$			4		4		4		4		
2		5		5		5		5		5	
$\frac{5}{2}$	6		6		6		6		6		6
$\vdots$	$\cdot^{\cdot^{\cdot}}$										$\ddots$

$s = \ldots\ -\frac{5}{2}\ -2\ -\frac{3}{2}\ -1\ -\frac{1}{2}\ 0\ \frac{1}{2}\ 1\ \frac{3}{2}\ 2\ \frac{5}{2}\ \ldots$ (4.15.60)

The numbers in this triangular array (which extends indefinitely downwards) represent the complex *dimensions* of the various spaces of spin-weighted spherical harmonics, as discussed in (4.15.43) *et seq.* Each of these spaces is characterized by its values of s and j, as shown. The dimension *zero* is assigned wherever a blank space appears in the array. The operator $\eth$ carries us a step of one s-unit to the right and $\eth'$ one s-unit to the left. (From our earlier discussion, the j-value is not affected by $\eth$ or $\eth'$.) Whenever such a step carries us off the array, the result of the operator $\eth$ or $\eth'$ is zero. Note that the dimension remains constant whenever it does not drop to, or increase from, zero.

$\bar{\eth}\mathcal{Y} = 0 \iff \eth^3 \mathcal{Y} = 0$ $\qquad$ weight

$\eth\bar{\mathcal{Y}} = 0 \iff \bar{\eth}^3 \bar{\mathcal{Y}} = 0$ $\qquad$ same solutions

dual situation $\quad w \leq -|s| - 2$

$\eth^{s-w-1}\, \kappa^{w+1, s-1} \qquad \bar{\eth}^{-s-w-1}\, \kappa^{-w-1, -s-1}$

$[s, w] \qquad [s, w]$ $\qquad$ definite boost

$[2, -2] \qquad [-2, 0]$ $\qquad$ weight

$\mathcal{J} \sim \mathcal{J} + \bar{\eth}\,\bar{\mathcal{I}} \iff \tilde{\mathcal{J}} \sim \tilde{\mathcal{J}} + \eth^3 \mathcal{M}$

same equivalence classes

$j \leq w$: finite dim. repres of Lorentz, "heads"

$j > w$: ∞ dim , "tails"

Remark (ii) reduction to Poincaré

$\eth^2 \mathcal{J} = 0 = \bar{\eth}^2 \mathcal{J}$ $\qquad \mathcal{P} \sim \mathcal{P} + \eth^2 \mathcal{N} + \bar{\eth}^2 \bar{\mathcal{N}}$

$\qquad [-2, 1] \qquad [2, -1]$

Realization on punctured plane

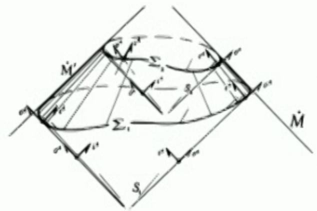
- Weyl trsf $e^{-\Phi(\xi,\bar\xi)} = \frac{\sqrt{2}}{1+\xi\bar\xi}$ $\quad \xi = R^{-1} z \quad ds^2 = -2\, dz\, d\bar z$

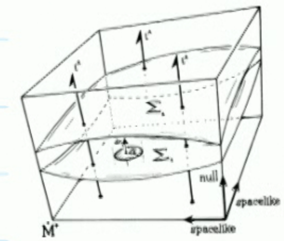
- 2-punctures: remove points at origin & infinity $\mathbb{C}_0$

- on the level of the algebra, look at the algebra of all infinitesimal local conformal trsf.

Not the Lie algebra of globally well-defined trsf.

For asymptotically flat spaces, $\dot M$ is in fact a null hypersurface [7]. The structure of $\dot M$ is essentially the same as for Minkowski space (Figure 4). We shall omit the three points I^-, I^0, I^+ here. Then $\dot M$ consists of two portions, each of which is topologically a "cylinder" $S^2 \times E^1$. We are concerned, here, only with the future portion $\dot M^+$, and by judicious choice of conformal factor Ω, we can ensure that the geometry of $\dot M^+$ is as simple as possible. In fact, by taking one generator of $\dot M^+$ "back to infinity" we can open out the cylinder into a space with Euclidean three-space topology. Furthermore, it turns out that we can also make this three-space metrically *flat* (Figure 6). This will simplify matters considerably.





Penrose 1967 AMS

$P=1 \Rightarrow$ weighted scalars = conformal fields

$$e^{\Phi(x')} = \frac{\partial z'}{\partial z} \qquad e^{\Phi_R(x')} = \left(\frac{\partial z'}{\partial z}\frac{\partial \bar z'}{\partial \bar z}\right)^{1/2}, \quad e^{i\Phi_I(x')} = \left(\frac{\partial z'/\partial z}{\partial \bar z'/\partial \bar z}\right)^{1/2}$$

gravity : sphere $\rightarrow$ plane

CFT : plane $\rightarrow$ sphere

Coulomb gas ?

Expansions

$$\phi_{h,\bar h}(z,\bar z) = \sum_{k,l} a_{k,l} \; {}_{h,\bar h}\tilde{z}_{k,l}\,, \qquad {}_{h,\bar h}\tilde{z}_{k,l} = z^{-h-k}\,\bar z^{-\bar h-l}$$

$h,\bar h \in \mathbb{N} \Rightarrow k,l \in \mathbb{Z}$

$h,\bar h \in \frac{\mathbb{N}}{2} \Rightarrow k,l \in \frac{1}{2}+\mathbb{Z}$

(NS)

Pairing $\langle \psi_{-\bar h+1,-h+1}, \phi_{h,\bar h} \tilde{\rangle} = \mathrm{Res}_z\, \mathrm{Res}_{\bar z} \left[\overline{\psi_{-\bar h+1,-h+1}}\; \phi_{h,\bar h} \right]$

assumptions ✓

$\mathrm{Res}_z(\partial\phi) = 0 = \mathrm{Res}_{\bar z}(\bar\partial\phi)$

adjoint repr. group $\tilde{y}'(z') = \left(\frac{\partial z}{\partial z'}\right)^{-1} \tilde{y}(z)$

$$\tilde{\beta}'(x') = \left(\frac{\partial z}{\partial z'}\right)^{-1/2}\left(\frac{\partial \bar z}{\partial \bar z'}\right)^{-1/2} \left(\tilde\beta - \left(\tilde{y}\,\partial\tilde\alpha - \tfrac{1}{2}\tilde\alpha\,\partial\tilde{y} + c.c.\right)\right)(x)$$

coadjoint repr group $J'(x') = \left(\frac{\partial z}{\partial z'}\right)^{1}\left(\frac{\partial \bar z}{\partial \bar z'}\right)^{2}\left(J + \frac{1}{2}\tilde{\mathfrak{J}}\,\bar\partial\,\tilde{\mathfrak{P}} + \frac{3}{2}\,\bar\partial\tilde{\mathfrak{J}}\,\tilde{\mathfrak{P}}\right)(x)$

$$\tilde{\mathfrak{P}}'(x) = \left(\frac{\partial z}{\partial z'}\right)^{3/2}\left(\frac{\partial \bar z}{\partial \bar z'}\right)^{3/2} \tilde{\mathfrak{P}}(x)$$

to be used for conformal mapping.

Realization on punctured plane

- Weyl trsf $e^{-\Phi(\xi,\bar\xi)} = \frac{\sqrt{2}}{1+\xi\bar\xi}$ $\quad \xi = R^{-1} z \quad ds^2 = -2\, dz\, d\bar z$

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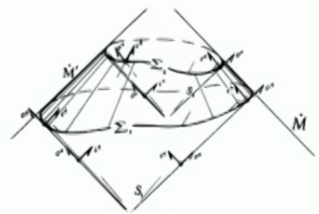

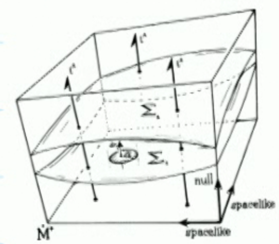

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gravity : sphere $\to$ plane
CFT : plane $\to$ sphere
Coulomb gas ?

Transformation of (relevant) free data at $\mathcal{J}^+$ $\qquad s=(\mathcal{Y},\bar{\mathcal{Y}},T)\,,\quad f=T+\frac{1}{2}u(\eth\mathcal{Y}+\bar{\eth}\bar{\mathcal{Y}})$

$$\delta_s\sigma^0=[f\partial_u+\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+\tfrac{3}{2}\eth\mathcal{Y}-\tfrac{1}{2}\bar{\eth}\bar{\mathcal{Y}}]\sigma^0-\eth^2 f$$

$$\delta_s\dot{\sigma}^0=[f\partial_u+\mathcal{Y}\eth+\bar{\mathcal{Y}}\bar{\eth}+2\eth\mathcal{Y}\qquad]\dot{\sigma}^0-\tfrac{1}{2}\eth^2(\underline{\underline{\eth\mathcal{Y}+\bar{\eth}\bar{\mathcal{Y}}}})$$

EM tensor
Schwarzian derivative

$$\delta_s\psi_2^0=[\ \text{"}\quad\text{"}\quad\text{"}\ +\tfrac{3}{2}\eth\mathcal{Y}+\tfrac{3}{2}\bar{\eth}\bar{\mathcal{Y}}]\psi_2^0+2\eth f\,\psi_3^0$$

$$\delta_s\psi_1^0=[\ \text{"}\quad\text{"}\quad\text{"}\ +2\eth\mathcal{Y}+\bar{\eth}\bar{\mathcal{Y}}]\psi_1^0+3\eth f\,\psi_2^0$$

broken current algebra

$$J_s=\frac{i}{R^2}\Big[(P_s\bar{P}_s)^{-1}J^u_s\,d\xi\wedge d\bar{\xi}+P_s^{-1}J^{\bar{\xi}}_s\,du\wedge d\xi-\bar{P}_s J^{\xi}_s\,du\wedge d\bar{\xi}\Big]$$

$$\delta_{s_1}J_{s_2}+\boxed{\Theta_{s_2}(\delta_{s_1}X)}\approx-J_{[s_1,s_2]}+d\,L_{s_1,s_2}$$

non-conservation

$$d\,J_s+\Theta_s(\delta_{(0,0,1)}X)\approx 0$$

$s_1:(\mathcal{Y},\bar{\mathcal{Y}},T)=(0,0,1)$ $\qquad \Theta_s(\delta X)\sim\dot{\sigma}^0,\dot{\bar{\sigma}}^0$ vanishes in the absence of news

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time components

$$J^u_s = -\frac{1}{8\pi G}\Big\{ \Big[\overbrace{\Psi^0_2 + \bar\Psi^0_2}^{\text{BH}} + \overbrace{\sigma^0\dot{\bar\sigma}^0 + \bar\sigma^0\dot\sigma^0}^{\text{gravitons}}\Big] f + \Big[\Psi^0_1 + \sigma^0 \eth\bar\sigma^0 + \tfrac{1}{2}\eth(\sigma^0\bar\sigma^0)\Big] \mathcal{Y} + \Big[\bar\Psi^0_1 + \bar\sigma^0\bar\eth\sigma^0 + \tfrac{1}{2}\bar\eth(\sigma^0\bar\sigma^0)\Big]\bar{\mathcal{Y}}\Big\}$$

$$\Theta^u_s(\delta\chi) = \frac{1}{8\pi G}\left[\dot{\bar\sigma}^0\delta\sigma^0 + \dot\sigma^0\delta\bar\sigma^0\right] f$$

charges $\quad Q_s = \int_{S^2,\, u=cte} \frac{i}{R^2}\frac{d\zeta d\bar\zeta}{P_s\bar P_s} J^u_s \qquad \Theta_s[\delta\chi] = \int_{S^2,\, u=cte} \frac{i}{R^2}\frac{d\zeta d\bar\zeta}{P_s\bar P_s}\Theta^u_s[\delta\chi]$

algebra $\quad \delta_{s_1} Q_{s_2} + \Theta_{s_2}[\delta_{s_1}\chi] = -Q_{[s_1,s_2]}$

(non-)conservation of BMS_4 charges

G.B. & C.Troessaert JHEP (2011)

JHEP (2013)

$$\frac{d}{du} Q_s = \underbrace{-\int_{S^2,\, u=cte}\frac{i}{R^2\, 8\pi G}\frac{d\zeta d\bar\zeta}{P_s\bar P_s}\left[\dot{\bar\sigma}^0\delta_s\sigma^0 + \dot\sigma^0\delta_s\bar\sigma^0\right]}_{\text{fluxes} \quad \text{generalizes mass loss}}$$

$\boxed{\text{non-radiative spacetimes (no news)}}$ $\sigma^0 = \sigma^0(\xi, \bar\xi, \cancel{u})$ $\left(\Rightarrow \dot\sigma^0 = 0 = \psi_3^0 = \psi_4^0 , \quad \mathcal{O}_s[\delta\chi] = 0 \right)$

compare "abstract" construction of $\mathfrak{bms}_4^*$

identification at $u=0$ $\quad P = -\frac{1}{2G}(\psi_2^0 + \bar\psi_2^0)$ $\quad \bar J = -\frac{1}{2G}\Big(\underbrace{\psi_1^0 + \sigma^0 \eth \bar\sigma^0 + \frac{1}{2}\eth(\sigma^0\bar\sigma^0)}_{\psi^0_{1\bar J}}\Big)$

super-momentum = Bondi mass aspect

~~super~~- angular momentum = Bondi angular momentum aspect

(pre)-momentum map: $\mathfrak{F}$. algebra of non-radiative free data

$\mathfrak{bms}_4$ representation δ_s , $[\delta_{s_1}, \delta_{s_2}] = \delta_{[s_1,s_2]}$

$$\mu : \mathfrak{F} \to \mathfrak{bms}_4^*$$

$$\mu\left(-\frac{1}{2G}(\psi_2^0 + \bar\psi_2^0)\right) = P , \quad \mu\left(-\frac{1}{2G}\psi^0_{1\bar J}\right) = [\bar J] , \quad \mu \circ \delta_s = ad_s^* \circ \mu$$

$\boxed{\text{non-radiative spacetimes (no news)}}$ $\sigma^0 = \sigma^0(\xi, \bar{\xi}, \cancel{u})$ $(\Rightarrow \dot{\sigma}^0 = 0 = \psi_3^0 = \psi_4^0, \quad \mathcal{O}_s[\delta \chi] = 0)$

compare "abstract" construction of $\mathfrak{bms}_4^*$

identification at $u=0$ $\quad \mathcal{P} = -\frac{1}{2G}(\psi_2^0 + \bar{\psi}_2^0) \quad \bar{\mathcal{J}} = -\frac{1}{2G}\big(\underbrace{\psi_1^0 + \sigma^0 \eth \bar{\sigma}^0 + \frac{1}{2}\eth(\sigma^0\bar{\sigma}^0)}_{\psi^0_{1\bar{\mathcal{J}}}}\big)$

super-momentum
= Bondi mass aspect

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(pre)-momentum map: $\mathfrak{F}$. algebra of non-radiative free data
$\mathfrak{bms}_4$ representation δ_s, $[\delta_{s_1}, \delta_{s_2}] = \delta_{[s_1, s_2]}$

$$\mu : \mathfrak{F} \longrightarrow \mathfrak{bms}_4^*$$

$$\mu\left(-\frac{1}{2G}(\psi_2^0 + \bar{\psi}_2^0)\right) = \mathcal{P}, \quad \mu\left(-\frac{1}{2G}\psi^0_{1\bar{\mathcal{J}}}\right) = [\bar{\mathcal{J}}], \quad \mu \circ \delta_s = \mathrm{ad}_s^* \circ \mu$$

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transformation laws at $u=0$ $\qquad \delta_s(\psi_2^0+\bar\psi_2^0) = (\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + \frac{3}{2}\eth\mathcal{Y} + \frac{3}{2}\bar\eth\bar{\mathcal{Y}})(\psi_2^0+\bar\psi_2^0)$ ✓

$$\delta_s\psi_1^0 = [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + 2\eth\mathcal{Y} + \bar\eth\bar{\mathcal{Y}}]\psi_1^0 + \frac{1}{2}\mathcal{T}\eth(\psi_2^0+\bar\psi_2^0+\bar\eth^2\cancel{\sigma}^0-\eth^2\cancel{\bar\sigma}^0)$$
$$+\frac{3}{2}\eth\mathcal{T}(\psi_2^0+\bar\psi_2^0+\bar\eth^2\cancel{\sigma}^0-\eth^2\cancel{\bar\sigma}^0)$$

$$\delta_s\psi_{1\mathcal{T}}^0 = [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth + 2\eth\mathcal{Y} + \bar\eth\bar{\mathcal{Y}}]\psi_{1\mathcal{T}}^0 + \frac{1}{2}\mathcal{T}\eth(\psi_2^0+\bar\psi_2^0) + \frac{3}{2}\eth\mathcal{T}(\psi_2^0+\bar\psi_2^0)$$

$$+\frac{1}{2}\bar\eth\left(\mathcal{T}\bar\eth\eth\sigma^0 - \bar\eth\mathcal{T}\eth\sigma^0 + 3\eth\mathcal{T}\bar\eth\sigma^0 - 3\bar\eth\eth\mathcal{T}\sigma^0 - \frac{3}{2}\mathcal{T}\sigma^0\right)\boxed{-\frac{1}{2}\eth^3(\mathcal{T}\bar\sigma^0)}$$

$\boxed{\text{trivial !}}$

Remark: electric case $\bar\eth^2\sigma_e^0 = \eth^2\bar\sigma_e^0 \iff \sigma_e^0 = \eth^2\chi_e$

$$\delta_s\chi_e = [\mathcal{Y}\eth + \bar{\mathcal{Y}}\bar\eth - \frac{1}{2}\eth\mathcal{Y} - \frac{1}{2}\bar\eth\bar{\mathcal{Y}}] - \mathcal{T}$$

Newman & Penrose JMP 1966

Strominger et al. 2013-

Compère et al. 2016

simplified pre-momentum map $\mu' : \mathcal{F}_e \longrightarrow \mathfrak{bms}_4^*$ (not physically relevant !)

$$\mu'[-\tfrac{1}{2G}(\psi_2^0+\bar\psi_2^0)] = \mathcal{T}, \quad \mu'[-\tfrac{1}{2G}\psi_1^0] = [\mathcal{Y}], \quad \mu'\circ\delta_s = \mathrm{ad}_s^*\circ\mu'$$

Geometric action for BMS_4^E (in progress)

$$\int du\,\langle\pi,\kappa\rangle_{\mathrm{BMS}_4^E} = \int du \int d\mu(x)\Big[\big(\bar{J} - (\frac{1}{2}T\partial P + \frac{3}{2}\partial T P)\big)\frac{df}{du}\circ f^{-1} + \text{c.c.} + P\frac{dT}{du}\Big],\ ($$

$$H_{(0,0,1)} = \int d\mu P([f'\bar{f}']^{\frac{1}{2}}\circ(f^{-1},\bar{f}^{-1})). \qquad (4.40)$$

In this case, the associated equations of motion are

$$\frac{df}{du}\circ f^{-1} = 0 = \frac{d\bar{f}}{du}\circ\bar{f}^{-1}, \quad \frac{dT}{du} = [f'\bar{f}']^{\frac{1}{2}}\circ(f^{-1},\bar{f}^{-1}), \qquad (4.41)$$

which means that $f, \bar{f}$ are u-independent, while $T = T^O(z,\bar{z}) + u[f'\bar{f}']^{\frac{1}{2}}\circ(f^{-1},\bar{f}^{-1})$, together with

$$\frac{dP}{du} = 0, \quad \frac{d\bar{J}}{du} = \bar{\partial}(\cdot), \quad \frac{dJ}{du} = \partial(\cdot). \qquad (4.42)$$

flux-balance equations

The Poisson brackets $\{\pi_\mu,\pi_\nu\} = f^\rho_{\mu\nu}\pi_\rho$ read explicitly

KKS Poisson bracket

$$\{\bar{J}(z,\bar{z}),P(w,\bar{w})\} = [\frac{3}{2}P(z,\bar{z})\partial_z\delta(z,w) + \frac{1}{2}\partial_z P(z,\bar{z})\delta(z,w)]\delta(\bar{z},\bar{w}),$$
$$\{J(z,\bar{z}),P(w,\bar{w})\} = \delta(z,w)[\frac{3}{2}P(z,\bar{z})\partial_{\bar{z}}\delta(\bar{z},\bar{w}) + \frac{1}{2}\partial_{\bar{z}}P(z,\bar{z})\delta(\bar{z},\bar{w})],$$
$$\{\bar{J}(z,\bar{z}),\bar{J}(w,\bar{w})\} = [2\bar{J}(z,\bar{z})\partial_z\delta(z,w) + \partial_z\bar{J}(z,\bar{z})\delta(z,w)]\delta(\bar{z},\bar{w}),$$
$$\{J(z,\bar{z}),J(w,\bar{w})\} = \delta(z,w)[2J(z,\bar{z})\partial_{\bar{z}}\delta(\bar{z},\bar{w}) + \partial_{\bar{z}}J(z,\bar{z})\delta(\bar{z},\bar{w})],$$
$$\{\bar{J}(z,\bar{z}),J(w,\bar{w})\} = J(z,\bar{z})\partial_z\delta(z,w)\delta(\bar{z},\bar{w}) = \bar{J}(w,\bar{w})\delta(z,w)\partial_{\bar{z}}\delta(\bar{z},\bar{w}),$$
$$\{P(z,\bar{z}),P(w,\bar{w})\} = 0.$$

"Euler angles" for SO(3,1)

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} e^{-E/2} & -\bar{A}e^{E/2} \\ -Be^{-E/2} & (1+\bar{A}B)e^{E/2} \end{pmatrix} \qquad (5.1)$$

with $E, A, B \in \mathbb{C}$

Relations:

$$\partial_z\delta(z,w) = -\partial_w\delta(z,w), \quad F(w)\partial_z\delta(z,w) = F(z)\partial_z\delta(z,w) + \partial_z F(z)\delta(z,w),$$

relevant for $BMS_4^{glob.}$

Perspectives

1) classify coadjoint orbits of BMS_4

2) coadjoint repres. of generalized BMS_4 Campiglia & Laddha Phys. Rev. 2014

$Diff(S^2) \ltimes C^\infty(S^2)$ on S^2 drop $\eth \bar{\mathcal{Y}} = 0 = \bar{\eth} \mathcal{Y}$ $\eth^3 \mathcal{Y} = 0 = \bar{\eth}^3 \bar{\mathcal{Y}}$

and also equivalence relations

$\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \bar{\eth} \bar{\mathcal{Y}}$, $\bar{\mathcal{J}} \sim \bar{\mathcal{J}} + \eth^3 M$

simply expand everything in spin-weighted spherical harmonics

3) apply to corner symmetry groups