Title: Matrix quantization of gravitational edge modes

Speakers: Antony Speranza

Collection: Quantum Gravity Around the Corner

Date: October 05, 2022 - 11:00 AM

URL: https://pirsa.org/22100016

Abstract: The phase space of gravity restricted to a subregion bounded by a codimension-2 corner possesses an infinite-dimensional symmetry algebra consisting of diffeomorphisms of the 2-sphere and local SL(2,R) transformations of the normal planes. I will describe a deformation of a subalgebra preserving an area form on the sphere, and show that it leads to the finite dimensional algebra SU(N,N), reminiscent of older results concerning the fuzzy sphere, in which area-preserving diffeomorphisms are deformed to SU(N). This deformation is conjectured to be relevant to the quantization of the local gravitational phase space, and I will further demonstrate that the representation of SU(N,N) appearing in the quantization can be determined by matching the Casimir operators of the deformed algebra to classical phase space invariants. Based on 2012.10367 and upcoming work with W. Donnelly, L. Freidel, and S.F. Moosavian.

Matrix quantization of gravitational edge modes

Antony J. Speranza

UNIVERSITY OF ILLINOIS URBANA-CHAMPAIGN

Quantum Gravity Around the Corner
Perimeter Institute
October 5, 2022

2012.10367, 221x.xxxxx, with W. Donnelly, L. Freidel, F. Moosavian

Boundary charges in gravity

In gravity, diffeomorphisms are pure gauge, but in the presence of boundaries become physical symmetries with nontrivial charges.

Classical applications:

- Quasilocal energy for subregions
- Global charges (ADM mass, BMS charges) for asymptotic boundaries
- Brown-York stress tensor and applications to holography

Quantum gravity applications:

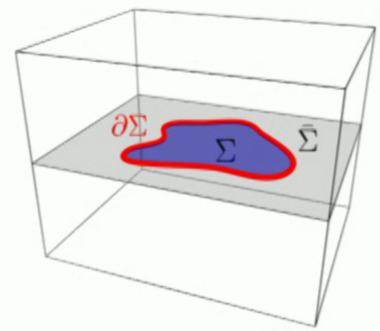
- Hilbert space factorization:
  $\mathcal{H}_{\mathsf{phys}} \neq \mathcal{H}_{\Sigma} \otimes \mathcal{H}_{\bar{\Sigma}}$ due to gauge constraints. Adding boundary charges restores factorization.
- Relevance to entanglement entropy in gravity:
  extended Hilbert space entropy formula [Donnelly 2012]

$$S_{\mathsf{ext}} = \langle S_{\mathsf{vN}} \rangle + S_{\mathsf{Shannon}} + \langle \log \dim R \rangle$$

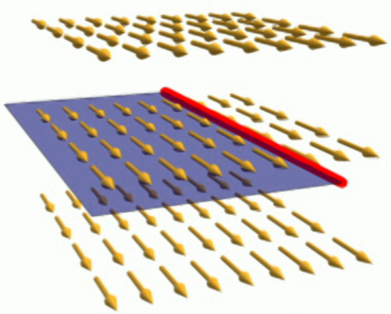
Surface-preserving charges

Charges for diffeomorphisms preserving codimension-2 surface $S = \partial\Sigma$

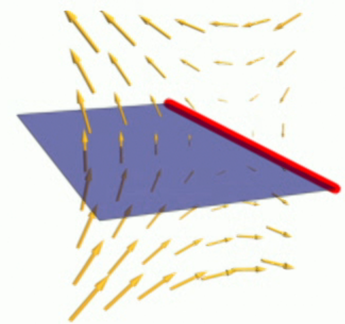
[Donnelly, Freidel 2016; AJS 2017]



Universal symmetry group: $\mathrm{Diff}(S) \ltimes SL(2,\mathbb{R})^S$



$\mathrm{Diff}(S)$
Charge: generalization of angular momentum



$SL(2,\mathbb{R})$
Charge: local area element

Classification of invariants

Characterize representations using coadjoint orbits
parallels classification of Poincaré reps [Donnelly, Freidel, Moosavian, AJS 2020]

Group	$SO(3,1) \ltimes \mathbb{R}^{3,1}$	$\mathrm{Diff}(S) \ltimes SL(2,\mathbb{R})^S$
Homogeneous Casimir	Mass $m^2 = -P_\mu P^\mu$	Area $A = \int_S \sqrt{q}$
Little group	$SO(3)$	$\mathrm{SDiff}(S)$
Little group generator	Pauli-Lubanski vector $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$	Outer curvature $W_{AB} = n^{ij}(R_{ijAB} + 2K_{iA}{}^{C} K_{jCB})$
Little group Casimirs	Spin $W_\mu W^\mu = -m^2 s(s+1)$	Generalized enstrophies $C_k = \int_S \sqrt{q}(\star W)^k, \quad k = 2, 3, \ldots$

Classification of invariants

Characterize representations using coadjoint orbits parallels classification of Poincaré reps [Donnelly, Freidel, Moosavian, AJS 2020]

Group	$SO(3,1) \ltimes \mathbb{R}^{3,1}$	$\text{Diff}(S) \ltimes SL(2,\mathbb{R})^S$
Homogeneous Casimir	Mass $m^2 = -P_\mu P^\mu$	Area $A = \int_S \sqrt{q}$
Little group	$SO(3)$	$\text{SDiff}(S)$
Little group generator	Pauli-Lubanski vector $W^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} J_{\nu\rho} P_\sigma$	Outer curvature $W_{AB} = n^{ij}(R_{ijAB} + 2K_{iA}{}^{C} K_{jCB})$
Little group Casimirs	Spin $W_\mu W^\mu = -m^2 s(s+1)$	Generalized enstrophies $C_k = \int_S \sqrt{q}(\star W)^k, \quad k = 2, 3, \dots$

Matrix Regularization

Regularized algebra:

- Expect quantum effects to deform the algebra if a deformation exists
- Regulated symmetry algebra needed in entanglement entropy computation (e.g. lattice regulator for Maxwell theory)
- Planckian fuzziness expected from quantum gravity effects
- Practical motivation: representation theory for continuum algebra is not well-developed

Matrix Regularization

Regularized algebra:

- Expect quantum effects to deform the algebra if a deformation exists
- Regulated symmetry algebra needed in entanglement entropy computation (e.g. lattice regulator for Maxwell theory)
- Planckian fuzziness expected from quantum gravity effects
- Practical motivation: representation theory for continuum algebra is not well-developed

Will identify a deformation of the area-preserving subaglebra SDiff$(S) \ltimes SL(2,\mathbb{R})^S$, and sketch how to utilize the representation theory of the deformed algebra in the quantization of the classical phase space.

Fuzzy Sphere

Deformation motivated by the fuzzy sphere

[Hoppe 1989; Pope, Stelle 1989; Madore 1992]

S^2 is a phase space with symmetry algebra $\mathfrak{sdiff}(S^2)$. In the spherical harmonic basis Y_α, $\alpha = (A, a)$, structure constants are

$$Y_\alpha Y_\beta = E_{\alpha\beta}{}^{\gamma} Y_\gamma, \qquad \{Y_\alpha, Y_\beta\} = C_{\alpha\beta}{}^{\gamma} Y_\gamma$$

Quantization sends each harmonic to an $N \times N$ matrix $[\hat{Y}_\alpha]_i{}^j$

Commutators of $\hat{Y}_\alpha$ generate $\mathfrak{su}(N)$ Lie algebra

Structure constants [Freidel, Krasnov 2002; Alekseev, Recknagel, Schomerus 1999]

$$\hat{Y}_\alpha \cdot \hat{Y}_\beta = \hat{M}_{\alpha\beta}{}^{\gamma} \hat{Y}_\gamma$$

$$\hat{M}_{\alpha\beta\gamma} = \frac{\sqrt{N}}{(-1)^{2J}} \begin{pmatrix} A & B & C \\ a & b & c \end{pmatrix} \begin{Bmatrix} A & B & C \\ J & J & J \end{Bmatrix}, \qquad N = 2J + 1$$

Fuzzy Sphere

Using an asymptotic expansion of $6j$-symbol [Nomura 1989], can show

$$\hat{M}_{\alpha\beta\gamma} = E_{\alpha\beta\gamma} + \frac{i}{N} C_{\alpha\beta\gamma} + \mathcal{O}(N^{-2})$$

Hence, matrix product gives the expected form of a Moyal product with $\hbar = \frac{A_S}{2\pi N}$.

Subleading terms in the $6j$-symbol expansion give the expected corrections in the Moyal product.

Fuzzy Sphere

Using an asymptotic expansion of $6j$-symbol [Nomura 1989], can show

$$\hat{M}_{\alpha\beta\gamma} = E_{\alpha\beta\gamma} + \frac{i}{N} C_{\alpha\beta\gamma} + \mathcal{O}(N^{-2})$$

Hence, matrix product gives the expected form of a Moyal product with $\hbar = \frac{A_S}{2\pi N}$.

Subleading terms in the $6j$-symbol expansion give the expected corrections in the Moyal product.

Conclusion: Structure constants for $\mathfrak{su}(N)$ in spherical harmonic basis approach those of $\mathfrak{sdiff}(S)$ as $N \to \infty$;
$\leftrightarrow$ $\mathfrak{sdiff}(S)$ admits a deformation to $\mathfrak{su}(N)$.

SDiff$(S) \ltimes SL(2,\mathbb{R})^S$ and $SU(N,N)$

Gravitational algebra $\mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ contains additional generators $Y_{a\alpha} = \tau_a \otimes Y_\alpha$, $\rightarrow \mathfrak{sl}(2,\mathbb{R})$-valued functions.

Lie bracket computed pointwise:

$$[Y_{a\alpha}, Y_{b\beta}] = E_{\alpha\beta}{}^{\gamma} \varepsilon_{ab}{}^{c} Y_{c\gamma}$$

Ansatz: Obtain deformed algebra by promoting Y_α to a fuzzy spherical harmonic: $\hat{Y}_{a\alpha} = \tau_a \otimes \hat{Y}_\alpha$, $2N \times 2N$ matrix.

Matrix commutators found to be

$$[\hat{Y}_{a\alpha}, \hat{Y}_{b\beta}] = i\varepsilon_{ab}{}^{c} \hat{E}_{\alpha\beta}{}^{\gamma} \hat{Y}_{\gamma c} - \frac{i}{2N} \eta_{ab} \hat{C}_{\alpha\beta}{}^{\gamma} \mathbb{1}_2 \otimes \hat{Y}_\gamma$$

which approach the classical structure constants above.

$\mathrm{SDiff}(S) \ltimes SL(2,\mathbb{R})^S$ and $SU(N,N)$

Gravitational algebra $\mathfrak{sdiff}(S) \oplus_{\mathcal{L}} \mathfrak{sl}(2,\mathbb{R})^S$ contains additional generators $Y_{a\alpha} = \tau_a \otimes Y_\alpha$, $\to$ $\mathfrak{sl}(2,\mathbb{R})$-valued functions.

Lie bracket computed pointwise:

$$[Y_{a\alpha}, Y_{b\beta}] = E_{\alpha\beta}{}^{\gamma}\varepsilon_{ab}{}^{c}Y_{c\gamma}$$

Ansatz: Obtain deformed algebra by promoting Y_α to a fuzzy spherical harmonic: $\hat{Y}_{a\alpha} = \tau_a \otimes \hat{Y}_\alpha$, $2N \times 2N$ matrix.

Matrix commutators found to be

$$[\hat{Y}_{a\alpha}, \hat{Y}_{b\beta}] = i\varepsilon_{ab}{}^{c}\hat{E}_{\alpha\beta}{}^{\gamma}\hat{Y}_{\gamma c} - \frac{i}{2N}\eta_{ab}\hat{C}_{\alpha\beta}{}^{\gamma}\mathbb{1}_2 \otimes \hat{Y}_\gamma$$

which approach the classical structure constants above.

Additional generators $\hat{Y}_{\bullet\alpha} \equiv \mathbb{1}_2 \otimes \hat{Y}_\alpha$ generate an $\mathfrak{su}(N)$ algebra, which was shown to approach $\mathfrak{sdiff}(S)$.

Quantization and Casimir matching

Hamiltonians $(J_\alpha, N_{a\alpha})$ generate the classical symmetry algebra on phase space via Poisson brackets.

Quantization replaces these with operators $(\hat{J}_\alpha, \hat{N}_{a\alpha})$ on a Hilbert space which furnish a unitary representation of the deformed symmetry algebra.

Commutators depend only on the algebra via the relation

$$[\cdot,\cdot] \to i\hbar\{\cdot,\cdot\} + \mathcal{O}(\hbar^2)$$

Full operator product depends on the choice of representation. Constrained by matching to the classical abelian product:

$$\hat{J}_\alpha \hat{J}_\beta \to \widehat{J_\alpha J_\beta} + \mathcal{O}(\hbar)$$

Simplest in practice to impose this relation on Casimir operators to determine the representation and deformation parameter N.

Casimir matching for $SU(N)$

Classical generators

$$J_\alpha = \frac{A}{16\pi G}\left(\frac{Aw_0}{4\pi}\right)\int_S \nu_0\, Y_\alpha\left(\frac{-W}{w_0}\right)$$

Classical Casimirs (enstrophies)

$$C_k = \left(\frac{A}{16\pi G}\right)^k\left(\frac{Aw_0}{4\pi}\right)^k\int_S \nu_0\left(\frac{-W}{w_0}\right)^k$$

Deformed Casimirs

$$\hat{C}_k = \left(\frac{\hbar N}{2}\right)^k N^{k-1} c_k(R)$$

where $c_k(R)$ is the standard expression for $\mathfrak{su}(N)$ Casimirs in the representation R.

Casimir matching for $SU(N)$

Classical generators

$$J_\alpha = \frac{A}{16\pi G}\left(\frac{Aw_0}{4\pi}\right)\int_S \nu_0\, Y_\alpha\left(\frac{-W}{w_0}\right)$$

Classical Casimirs (enstrophies)

$$C_k = \left(\frac{A}{16\pi G}\right)^k \left(\frac{Aw_0}{4\pi}\right)^k \int_S \nu_0 \left(\frac{-W}{w_0}\right)^k$$

Deformed Casimirs

$$\hat{C}_k = \left(\frac{\hbar N}{2}\right)^k N^{k-1} c_k(R)$$

where $c_k(R)$ is the standard expression for $\mathfrak{su}(N)$ Casimirs in the representation R.

Large N scaling $c_k(R) \sim N^{k-1} n$, $n =$ #Young diagram boxes,

$$N \sim \left(\frac{A}{8\pi G\hbar}\right)^{\frac{1}{3}}, \qquad n \sim N^2.$$

Casimir matching for $SU(N,N)$

Classical charges

$$J_\alpha = \int_S \nu_0 \, Y_\alpha J, \qquad N_{a\alpha} = \int_S \nu_0 Y_\alpha N_a$$

Classical Casimir invariants

$$C_{2k} = \int_S \nu_0 (N_a N^a)^k$$
$$C_{2k+1} = \int_S \nu_0 \, (N^2)^k \left((2k+1)J + k \frac{\varepsilon_{abc}}{N^2} N^a \{N^b, N^c\} \right)$$

Casimir matching for $SU(N,N)$

Classical charges

$$J_\alpha = \int_S \nu_0 Y_\alpha J, \qquad N_{a\alpha} = \int_S \nu_0 Y_\alpha N_a$$

Classical Casimir invariants

$$C_{2k} = \int_S \nu_0 (N_a N^a)^k$$

$$C_{2k+1} = \int_S \nu_0 \, (N^2)^k \left((2k+1)J + k \frac{\varepsilon_{abc}}{N^2} N^a \{N^b, N^c\} \right)$$

Match to corresponding $\mathfrak{su}(N,N)$ Casimirs

$$\hat{C}_{2k} = \hbar^{2k} N^{2k-1} \tilde{c}_{2k}(R), \qquad \hat{C}_{2k+1} = \frac{\hbar^{2k+1} N^{2k}}{2} \tilde{c}_{2k+1}(R)$$

More detailed matching involves study of unireps of $\mathfrak{su}(N,N)$

Conclusion and future work

Summary:

Exhibited deformations symmetry groups appearing in gravitational phase space

- SDiff(S) $\longleftrightarrow$ $SU(N)$
- SDiff(S) $\ltimes SL(2,\mathbb{R})^S$ $\longleftrightarrow$ $SU(N,N)$
- SDiff(S) $\ltimes \mathbb{R}^S$ $\longleftrightarrow$ $SL(N,\mathbb{C}) \times \mathbb{R}$

Conclusion and future work

Summary:

Exhibited deformations symmetry groups appearing in gravitational phase space

- SDiff(S) $\longleftrightarrow$ $SU(N)$
- SDiff(S) $\ltimes SL(2,\mathbb{R})^S$ $\longleftrightarrow$ $SU(N,N)$
- SDiff(S) $\ltimes \mathbb{R}^S$ $\longleftrightarrow$ $SL(N,\mathbb{C}) \times \mathbb{R}$

Future Work:

- More detailed Casimir matching for $SU(N)$, relate Young diagram shape to outer curvature function W
- Explore effects of different surface topologies (e.g. torus)
- Examine representations of $SU(N,N)$ for detailed Casimir matching to SDiff(S) $\ltimes SL(2,\mathbb{R})^S$ invariants
- Explore deformations of full algebra Diff(S) $\ltimes SL(2,\mathbb{R})^S$, extended algebras Diff(S) $\ltimes (SL(2,\mathbb{R}) \ltimes \mathbb{R}^2)^S$
- Ultimately understand entropy in gravity from study of deformed symmetry representations