Title: Review Talk: A primer on the covariant phase space formalism

Speakers: Adrien Fiorucci

Collection: Quantum Gravity Around the Corner

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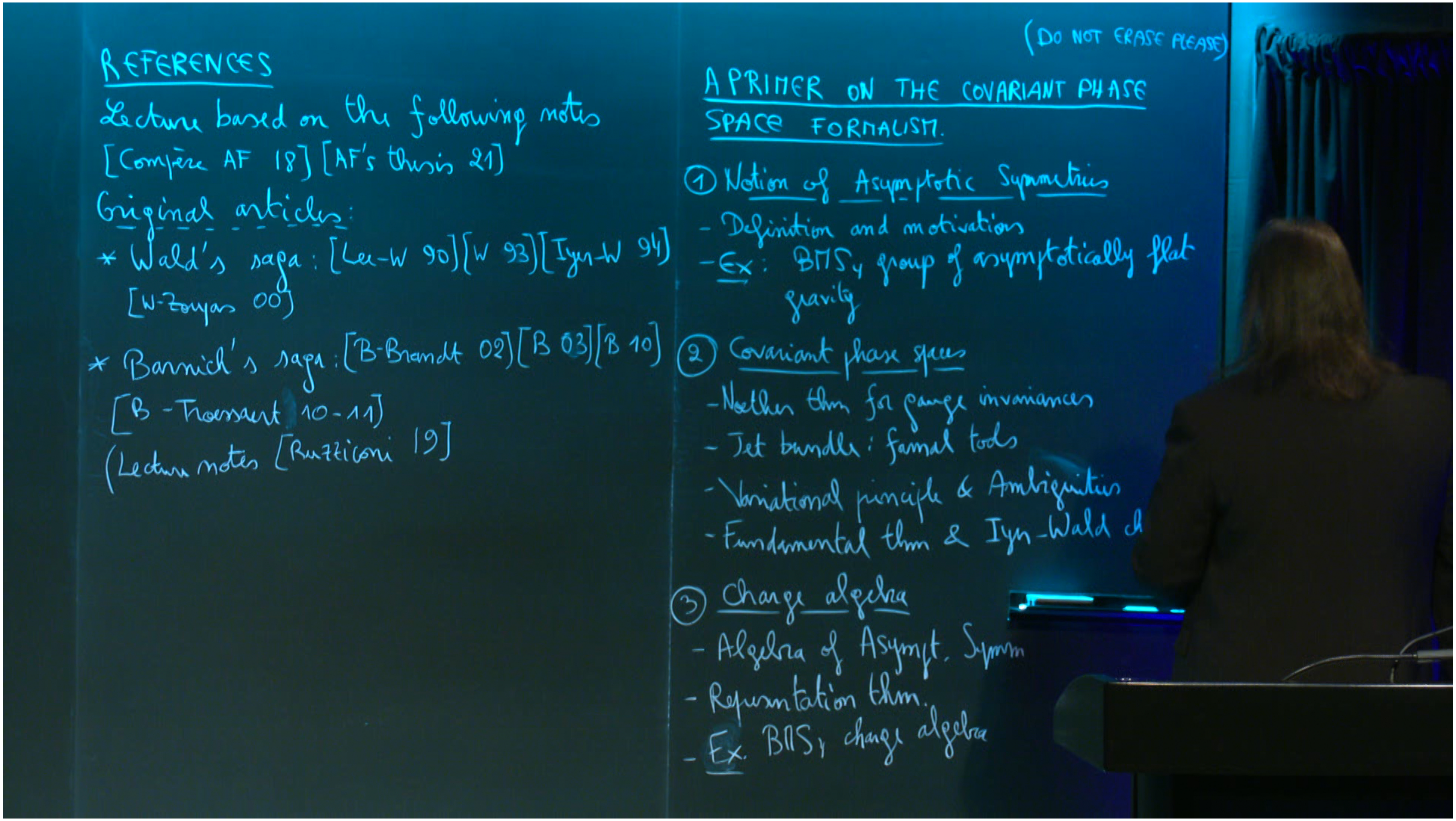
URL: https://pirsa.org/22100002

Abstract: This lecture aims at introducing the notion of asymptotic symmetries in gravity and the derivation of the related surface charges by means of covariant phase space techniques. First, after a short historical introduction, I will rigorously define what is meant by "asymptotic symmetry" within the so-called gauge-fixing approach. The problem of fixing consistent boundary conditions and the formulation of the variational principle will be briefly discussed. In the second part of the lecture, I will introduce the covariant phase space formalism, as conceived by Wald and coworkers thirty years ago, which adapts the Hamiltonian formulation of classical mechanics to Lagrangian covariant field theories. With the help of this fantastic tool, I will elaborate on the construction of canonical surface charges associated with asymptotic symmetries and address the crucial questions of their conservation and integrability on the phase space. In the third and last part, I will conclude with an analysis of the algebraic properties of the surface charges, describing in which sense they represent the asymptotic symmetry algebra in full generality, without assuming conservation or integrability. For pedagogical purposes, the theoretical concepts will be illustrated throughout in the crucial and well-known case of radiative asymptotically flat spacetimes in four dimensions, as described by Einstein's theory of General Relativity, and where many spectacular and unexpected features appear even in the simplest case of historical asymptotically Minkowskian boundary conditions. In particular, I will show that the surface charge algebra contains the physical information on the flux of energy and angular momentum at null infinity in the presence of gravitational radiation.


(DO NOT ERASE PLEASE)
REFERENCES
Lecture based on the following notes
[Compère AF 18] [AF's thesis 21]
Original articles:
* Wald's saga: [Lee-W 90] [W 93] [Iyer-W 94]
[W-Zoupas 00]
* Barnich's saga: [B-Brandt 02] [B 03] [B 10]
[B - Troessaert 10-11]
(Lecture notes [Ruzziconi 19]
A PRIMER ON THE COVARIANT PHASE SPACE FORMALISM.
1 Notion of Asymptotic Symmetries
- Definition and motivations
- Ex: BMS4 group of asymptotically flat gravity
2 Covariant phase space
- Noether thm for gauge invariances
- Jet bundle: formal tools
- Variational principle & Ambiguities
- Fundamental thm & Iyer-Wald
3 Charge algebra
- Algebra of Asympt. Symm
- Representation thm.
- Ex. BMS4 charge algebra

Fix the gauge (coordinates)
Boundary conditions (BC)
$\rightarrow$ A. S. = set of residual diffeos which preserve the B.C.

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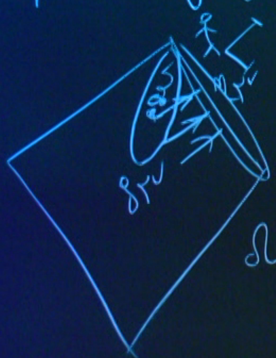
Asymptotically flat S.T.



Time: u

$\Omega(u)$: energy

$$\Omega(u_f)-\Omega(u_i)=\int_{u_i}^{u_f} du \oint_{S^2} -\frac{1}{32\pi G}\, d^2\Omega\, N_{AB}N^{AB}$$

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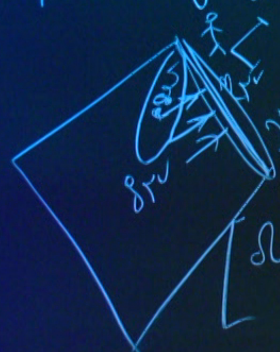

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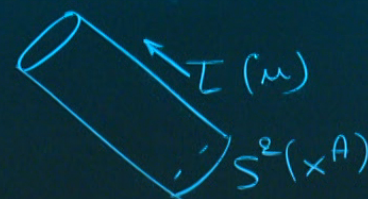
$u = t - r$

$$ds^2 = -du^2 - 2du\,dr + r^2 \mathring{q}_{AB} dx^A dx^B$$

$x^A = (\theta, \varphi)$

$\mathcal{J}^+ = \{r \to \infty\}$ future null infinity



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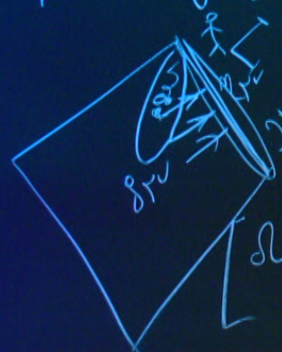

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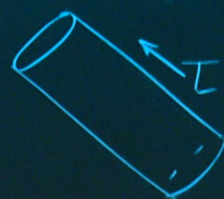

$$\xi^u = T(x^A) + \frac{u}{2}\mathring{D}_A \mathring{R}^A$$

$$\xi^r = -\frac{r}{2}\mathring{D}_A \mathring{R}^A + \mathcal{O}(r^0)$$

$$\xi^A = R^A + \mathcal{O}\left(\frac{1}{r}\right)$$

$R^A \in \mathrm{Diff}(S^2)$ obeying

$$\mathring{D}_A R_B + \mathring{D}_B R_A = \mathring{D}_C R^C \mathring{q}_{AB}$$

$$T \in C^\infty(S^2)$$

$$\mathring{D}_A \mathring{D}_B T - \frac{1}{2}\mathring{q}_{AB}\mathring{D}_C \mathring{D}^C T = 0$$

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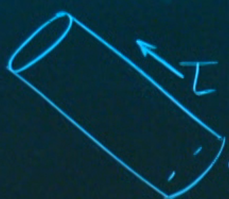

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$T \in C^\infty(S^2)$

$\mathring{D}_A \mathring{D}_B T - \frac{1}{2} q_{AB} \mathring{D}_C \mathring{D}^C T = 0 \leftarrow \nabla_{(A}\xi_{B)} = 0 \quad \mathcal{O}(r)$

$\left[-\frac{1}{32\pi G} \int d^2\Omega\, N_{AB} N^{AB} \right]$

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$$-\frac{1}{32\pi G} d^2\Omega\, N_{AB} N^{AB} \Big]$$

$$ISO(3,1) = SO(3,1) \ltimes T$$

$$\downarrow$$

$$\boxed{BMS_4} = SO(3,1) \ltimes S$$

$$T_{[1,2]} = R_1^A \partial_A T_2 + \frac{1}{2} T_1 \mathring{D}_A R^A - (1 \leftrightarrow 2)$$

$$R^A_{[1,2]} = R_1^B \partial_B R_2^A - (1 \leftrightarrow 2)$$

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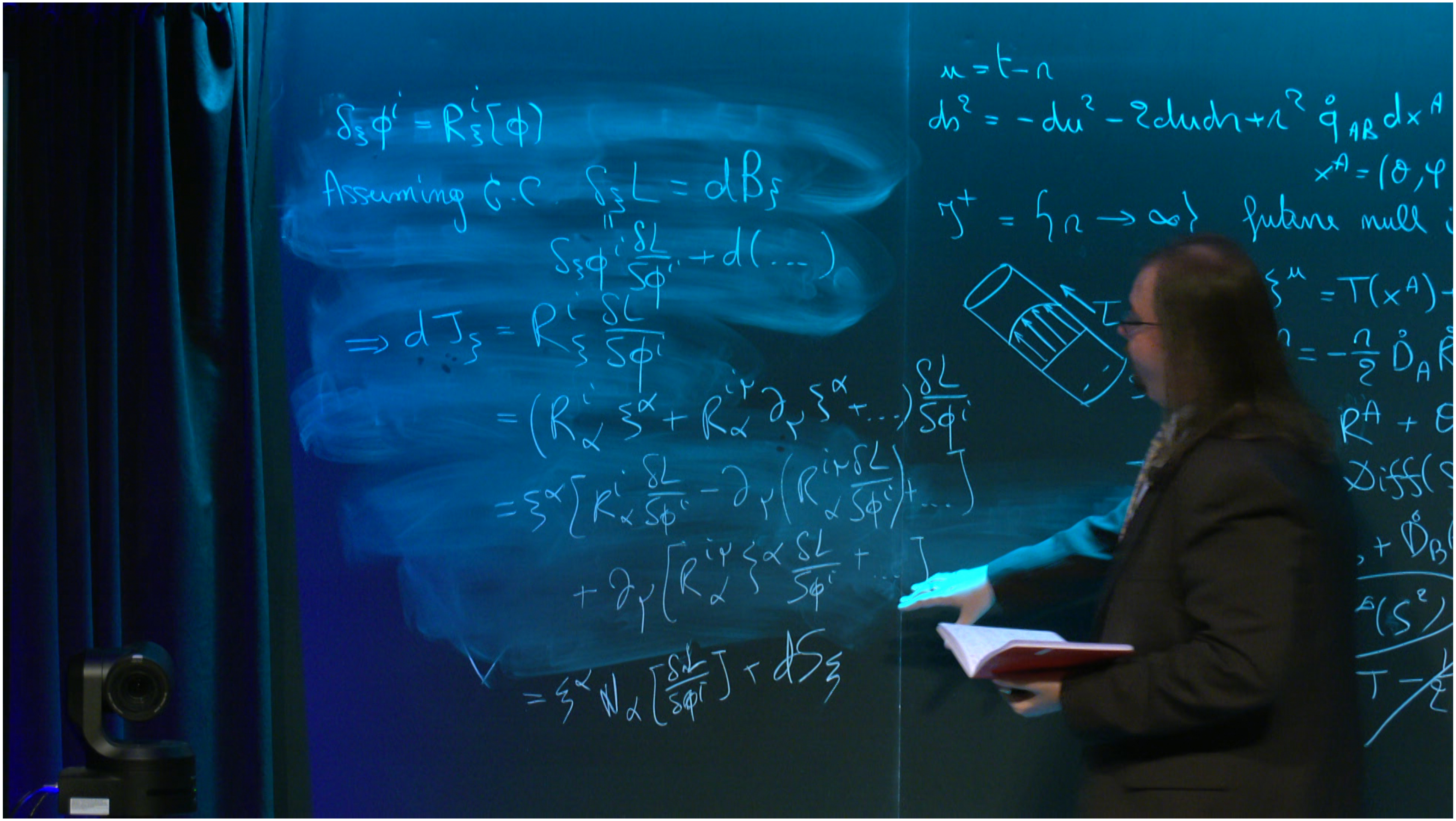
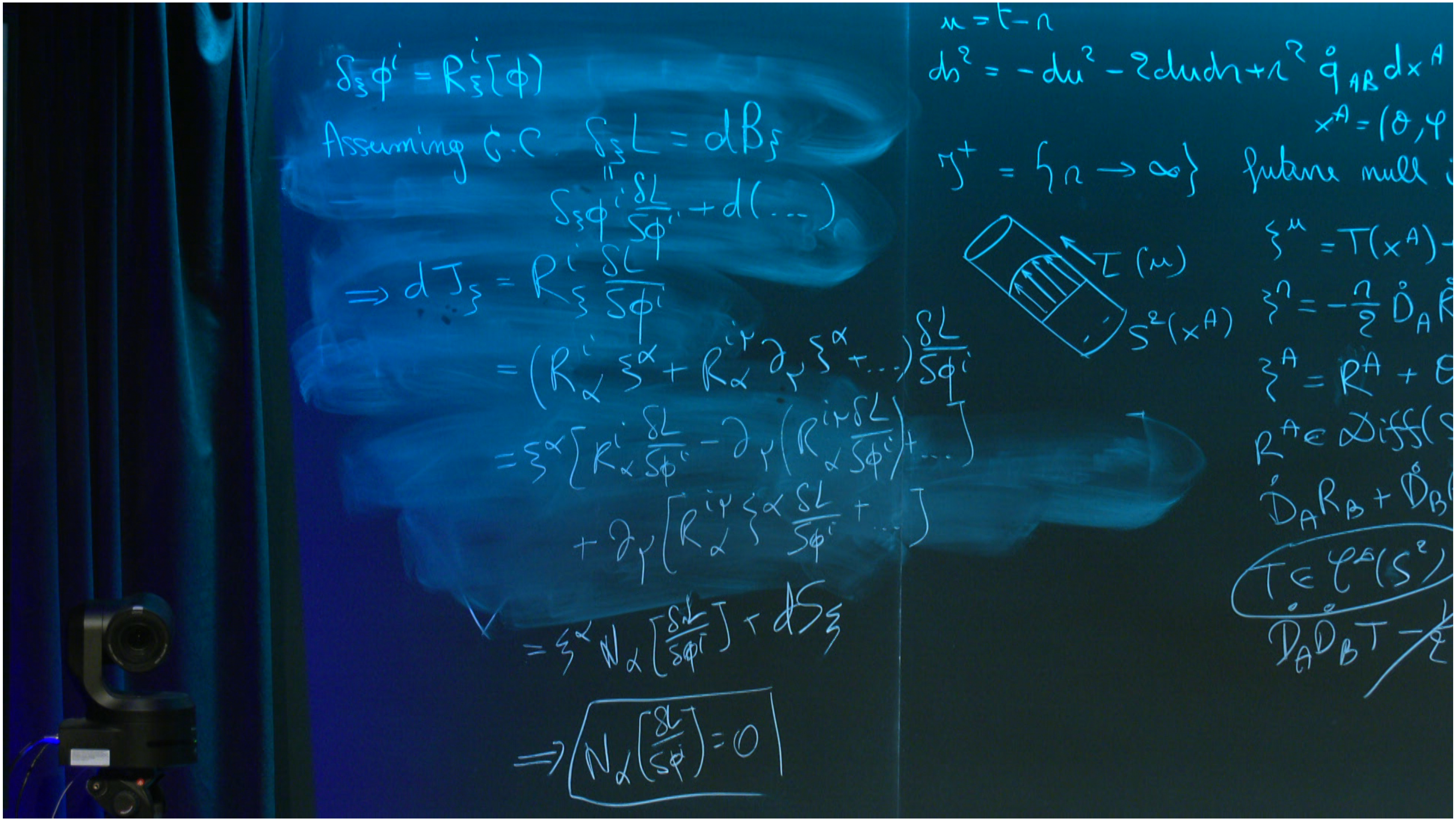

$\delta_\xi \phi^i = R^i_\xi[\phi]$
Assuming G.C. $\delta_\xi L = dB_\xi$
$\delta_\xi \phi^i \frac{\delta L}{\delta \phi^i} + d(\ldots)$
$\Rightarrow dJ_\xi = R^i_\xi \frac{\delta L}{\delta \phi^i}$
$= (R^i_\alpha \xi^\alpha + R^{i\mu}_\alpha \partial_\mu \xi^\alpha + \ldots)\frac{\delta L}{\delta \phi^i}$
$= \xi^\alpha \left[R^i_\alpha \frac{\delta L}{\delta \phi^i} - \partial_\mu\left(R^{i\mu}_\alpha \frac{\delta L}{\delta \phi^i}\right) + \ldots\right]$
$+ \partial_\mu\left[R^{i\mu}_\alpha \xi^\alpha \frac{\delta L}{\delta \phi^i} + \ldots\right]$
$= \xi^\alpha N_\alpha\left[\frac{\delta L}{\delta \phi^i}\right] + dS_\xi$
$u = t - r$
$ds^2 = -du^2 - 2du\,dr + r^2 \mathring{q}_{AB} dx^A$
$x^A = (\theta, \varphi$
$\mathscr{I}^+ = \{r \to \infty\}$ future null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$= \xi^\alpha [R^i_\alpha \frac{\delta L}{\delta \phi^i} - \partial_\mu (R^{i\mu}_\alpha \frac{\delta L}{\delta \phi^i}) + \dots]$
$+ \partial_\mu [R^{i\mu}_\alpha \xi^\alpha \frac{\delta L}{\delta \phi^i} + \dots]$
$= \xi^\alpha N_\alpha [\frac{\delta L}{\delta \phi^i}] + dS_\xi$
$\Rightarrow \boxed{N_\alpha (\frac{\delta L}{\delta \phi^i}) = 0}$
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$I(u)$
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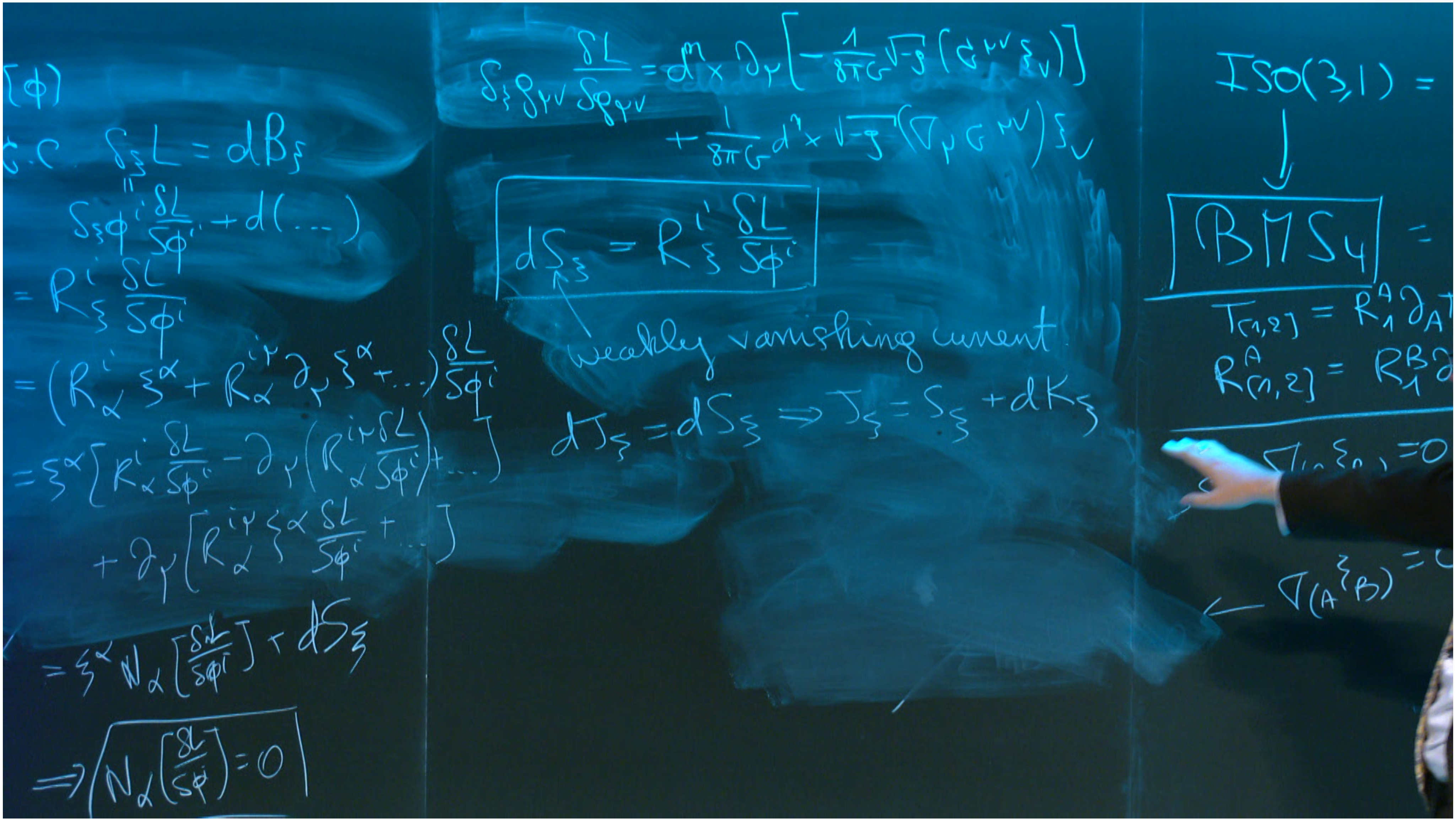

$[\phi]$
c.c. $\delta_\xi L = dB_\xi$
$\delta_\xi \phi^i \frac{\delta L}{\delta \phi^i} + d(\ldots)$
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$\Rightarrow \boxed{N_\alpha \left(\frac{\delta L}{\delta \phi^i} \right) = 0}$
$\delta_\xi g_{\mu\nu} \frac{\delta L}{\delta g_{\mu\nu}} = d^n x \, \partial_\mu \left[-\frac{1}{8\pi G} \sqrt{-g} \left(G^{\mu\nu} \xi_\nu \right) \right]$
$+ \frac{1}{8\pi G} d^n x \sqrt{-g} \left(\nabla_\mu G^{\mu\nu} \right) \xi_\nu$
$\boxed{dS_\xi = R^i_\xi \frac{\delta L}{\delta \phi^i}}$
weakly vanishing current
$dJ_\xi = dS_\xi \Rightarrow J_\xi = S_\xi + dK_\xi$
$ISO(3,1) =$
$\boxed{BMS_4} =$
$T_{[1,2]} = R^A_1 \partial_A$
$R^A_{[1,2]} = R^B_1 \partial$
$= 0$
$\nabla_{(A} \xi_{B)} =$

$\ldots x\, \partial_\mu \left[-\frac{1}{8\pi G}\sqrt{-g}\,(G^{\mu\nu}\xi_\nu)\right]$

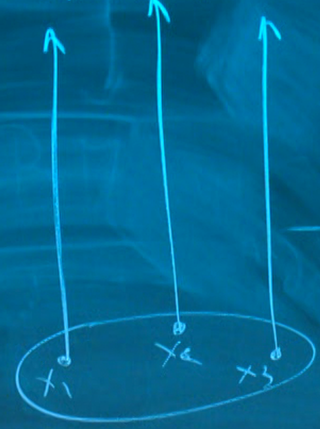
$-\frac{1}{8\pi G} d^n x \sqrt{-g}\,(\nabla_\mu G^{\mu\nu})\,\xi_\nu$

$\boxed{\xi^i \frac{\delta L}{\delta \phi^i}}$

... vanishing current

$\delta S_\xi \Rightarrow J_\xi = S_\xi + dK_\xi$

$\phi^i, \phi^i_\mu, \phi^i_{\mu\nu}, \ldots = \phi^i_{(\mu)}$



Ω

$\mathcal{J}$ = field space

$T^*\mathcal{J} = \mathrm{Span}\{\delta\phi^i_{(\mu)}\}$

Vertical derivative

$\delta = \sum_{(\mu)} \delta\phi^i_{(\mu)} \frac{\partial}{\partial\phi^i_{(\mu)}}$

$T^*\Omega = \mathrm{Span}\{dx^\mu\}$

Horizontal derivative

$d = dx^\mu \partial_\mu$

$\partial_\mu = \frac{\partial}{\partial x^\mu} + \sum_{(\nu)} \phi^i_{\mu(\nu)} \frac{\partial}{\partial\phi^i_{(\nu)}}$

REFERENCES

Lecture based on

[Compère AF 18] ...

Original article...

* Wald's saga: [W-Zoupas 00]

* Barnich's saga [B-Troessaert 10 ...

(Lecture notes [Ru...

$\int d^4x\, \partial_\mu \left[-\frac{1}{8\pi G}\sqrt{-g}\,(G^{\mu\nu}\xi_\nu)\right]$

$-\frac{1}{8\pi G} \int d^4x \sqrt{-g}\,(\nabla_\mu G^{\mu\nu})\,\xi_\nu$

$\boxed{\xi^i \frac{\delta L}{\delta \phi^i}}$

vanishing current

$\delta S_\xi \Rightarrow J_\xi = S_\xi + d\underset{\equiv}{K_\xi}$

$\phi^i, \phi^i_\mu, \phi^i_{\mu\nu}, \ldots = \phi^i_{(\mu)}$ $\mathcal{J}$ = field space

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δ_ξ : "Vertical Lie"

Variation: $\delta_\xi \phi^i = R^i_\xi$

REFERENCES

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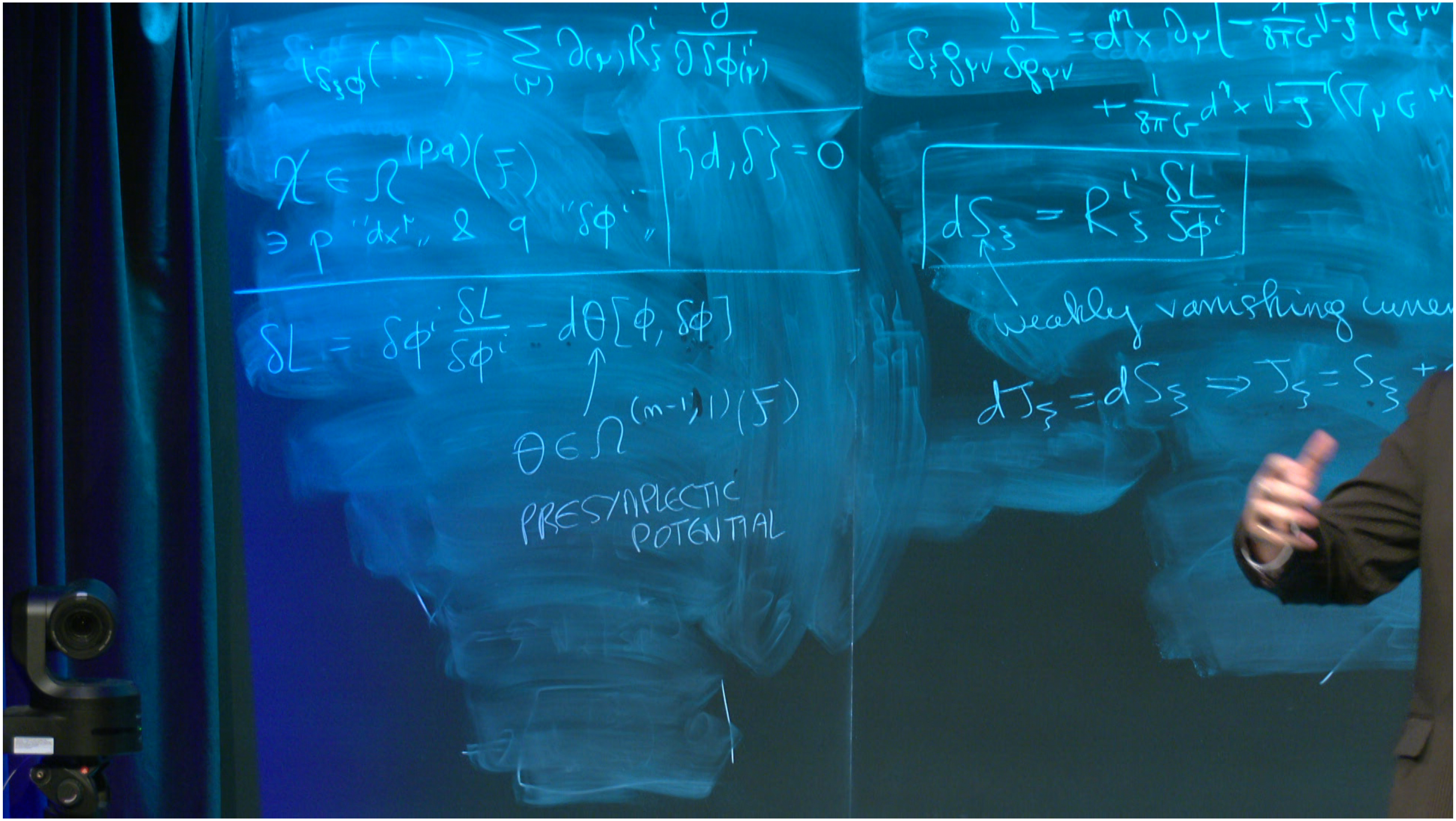
$i_{\delta_\xi \phi}(\ldots) = \sum_{(\mu)} \partial_{(\mu)} R^i_\xi \frac{\partial}{\partial \delta\phi^i_{(\mu)}}$
$\chi \in \Omega^{(p,q)}(F)$
$\ni p$ "dx^μ" & q "$\delta\phi^i$"
$\{d, \delta\} = 0$
$\delta L = \delta\phi^i \frac{\delta L}{\delta\phi^i} - d\Theta[\phi, \delta\phi]$
$\Theta \in \Omega^{(n-1,1)}(F)$
PRESYMPLECTIC POTENTIAL
$\delta_\xi g_{\mu\nu} \frac{\delta L}{\delta g_{\mu\nu}} = d^n x \, \partial_\mu \left[-\frac{1}{8\pi G}\sqrt{-g}\,(G^{\mu\nu}\right.$
$+ \frac{1}{8\pi G} d^n x \sqrt{-g}\,(\nabla_\mu G^{\mu\nu}$
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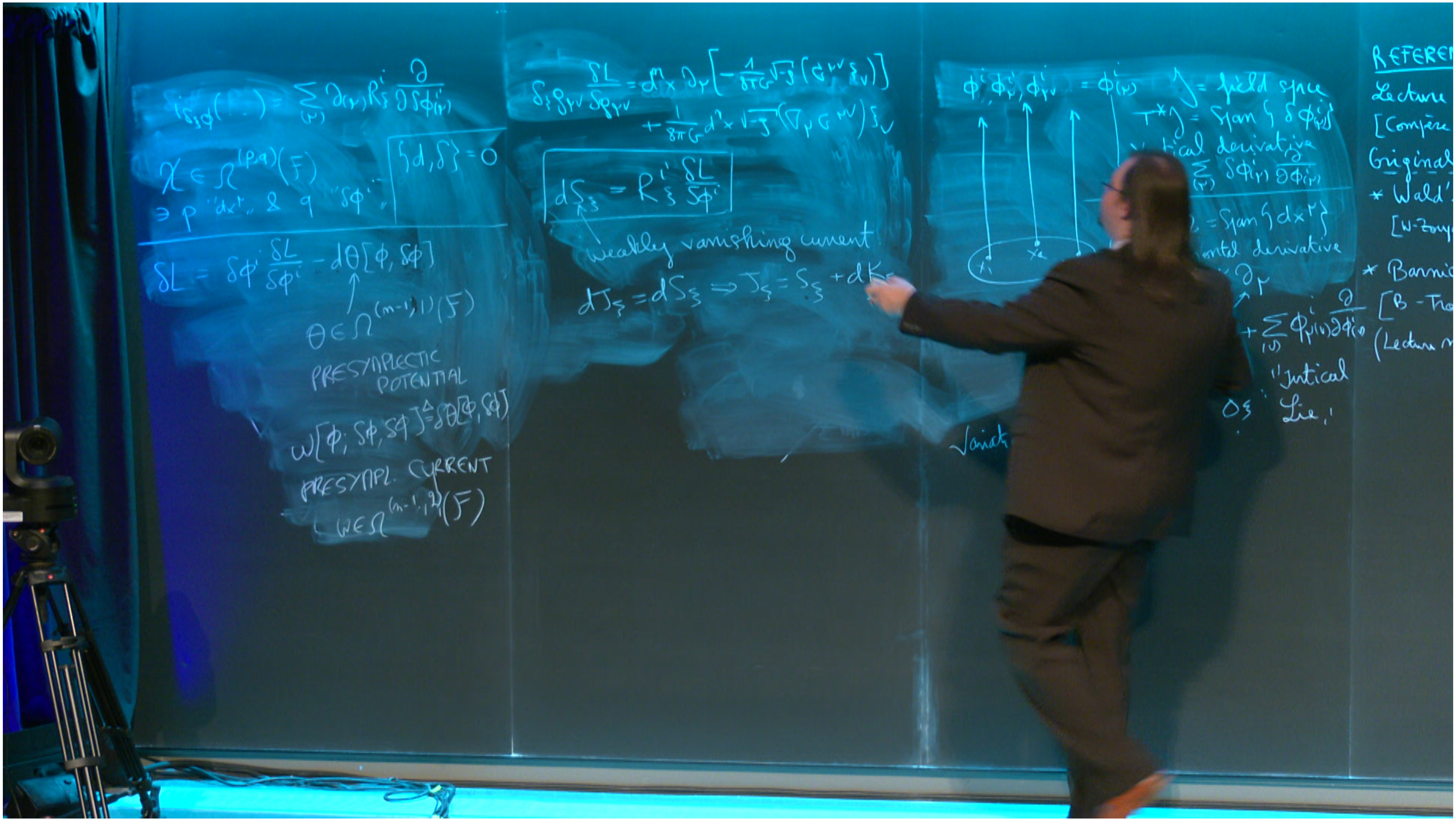
weakly vanishing
$dJ_\xi = dS_\xi \Rightarrow J_\xi = S_\xi +$



weakly vanishing current
PRESYMPLECTIC POTENTIAL
PRESYMPL. CURRENT
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PRESYMPLECTIC POTENTIAL

$$\omega[\phi; \delta\phi, \delta\phi] \triangleq \delta\Theta[\phi, \delta\phi]$$

PRESYMPL. CURRENT

$$\omega \in \Omega^{(n-1,2)}(F)$$

$\iota_{\delta_\xi \phi}(\ldots) = \sum_{(\nu)} \partial_{(\nu)} R^i_\xi \frac{\partial}{\partial \delta\phi^i_{(\nu)}}$

$\chi \in \Omega^{(p,q)}(\mathcal{F})$

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PRESYMPLECTIC POTENTIAL

$\omega[\phi; \delta\phi, \delta\phi] \triangleq \delta\Theta[\phi, \delta\phi]$

PRESYMPL. CURRENT

$\omega \in \Omega^{(n-1,2)}(\mathcal{F})$

$W[\phi, \delta\phi, \delta\phi] = \int_\Sigma \omega[\phi, \delta\phi, \delta\phi]$

(PRESYMPL. FORM)

Ambiguities:

* $L \to L + dA$

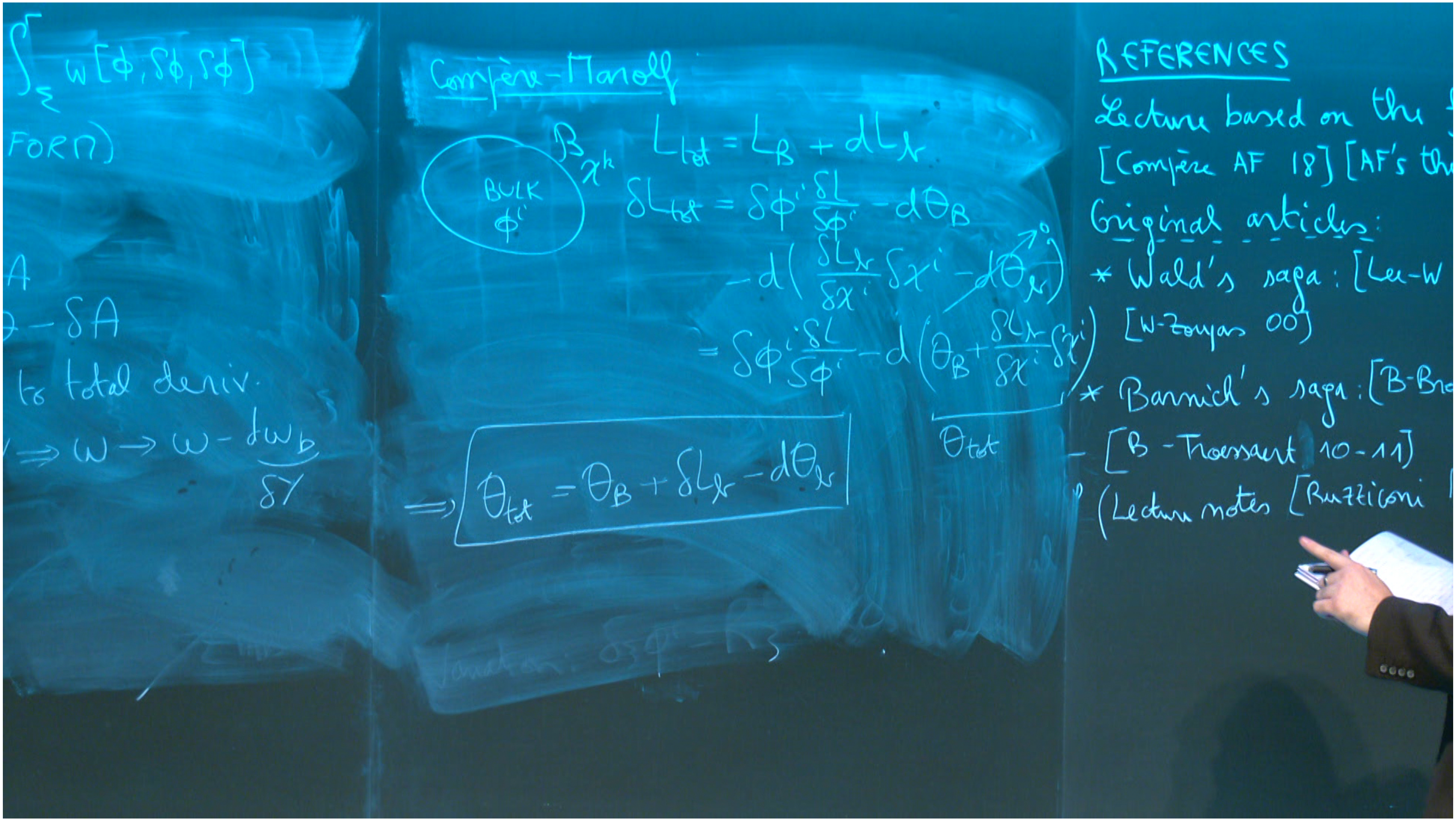
$\Rightarrow \Theta \to \Theta - \delta A$

* Θ defined up to total deriv.

$\Theta \to \Theta + dY \Rightarrow \omega \to \omega - d\underbrace{\omega_b}_{\delta Y}$

ϕ^i

x_i


∫_Σ ω[φ, δφ, δφ]
FORM)
A
− δA
to total deriv.
⇒ ω → ω − dω_b
δY
Compère-Marolf
B_{χ^k}
BULK
φ^i
L_tot = L_B + dL_χ
δL_tot = δφ^i δL/δφ^i − dΘ_B
− d(δL_χ/δχ^i δχ^i − dΘ_χ)
= δφ^i δL/δφ^i − d(Θ_B + δL_χ/δχ^i δχ^i)
Θ_tot
⇒ Θ_tot = Θ_B + δL_χ − dΘ_χ
REFERENCES
Lecture based on the
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* Barnich's saga: [B-Bra
[B-Troessaert 10-11]
(Lecture notes [Ruzziconi

$$L = L_B + dL_\chi$$

$$= \delta\phi^i \frac{\delta L}{\delta\phi^i} - d\Theta_B$$

$$- d\left(\frac{\delta L_\chi}{\delta\chi^i}\delta\chi^i - \cancel{d\Theta_\chi}^{\,0}\right)$$

$$= \delta\phi^i \frac{\delta L}{\delta\phi^i} - d\left(\underbrace{\Theta_B + \frac{\delta L_\chi}{\delta\chi^i}\delta\chi^i}_{\Theta_{tot}}\right)$$

$$\boxed{L_\chi - d\Theta_\chi}$$

① ω is closed on-shell

$$d\omega = d\delta\Theta = -\delta d\Theta$$

$$= \delta\left(\cancel{\delta L}^{\,0} + \frac{\delta L}{\delta\phi^i}\delta\phi^i\right)$$

$$= \delta\left(\frac{\delta L}{\delta\phi^i}\right)\wedge\delta\phi^i = 0 \text{ on-shell}$$

$$\int_{\Sigma_2}\omega = \int_{\Sigma_1}\omega + \int_B \omega$$

$$\omega|_B = 0$$

② Conservative B.C

A PRIMER ON THE COVARI

SPACE FORMALISM.

① Notion of Asymptotic Sy

- Definition and motivations
- Ex: BMS_4 group of asym gravity

② Covariant phase spaces

- Noether thm for gauge invar
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③ Charge algebra

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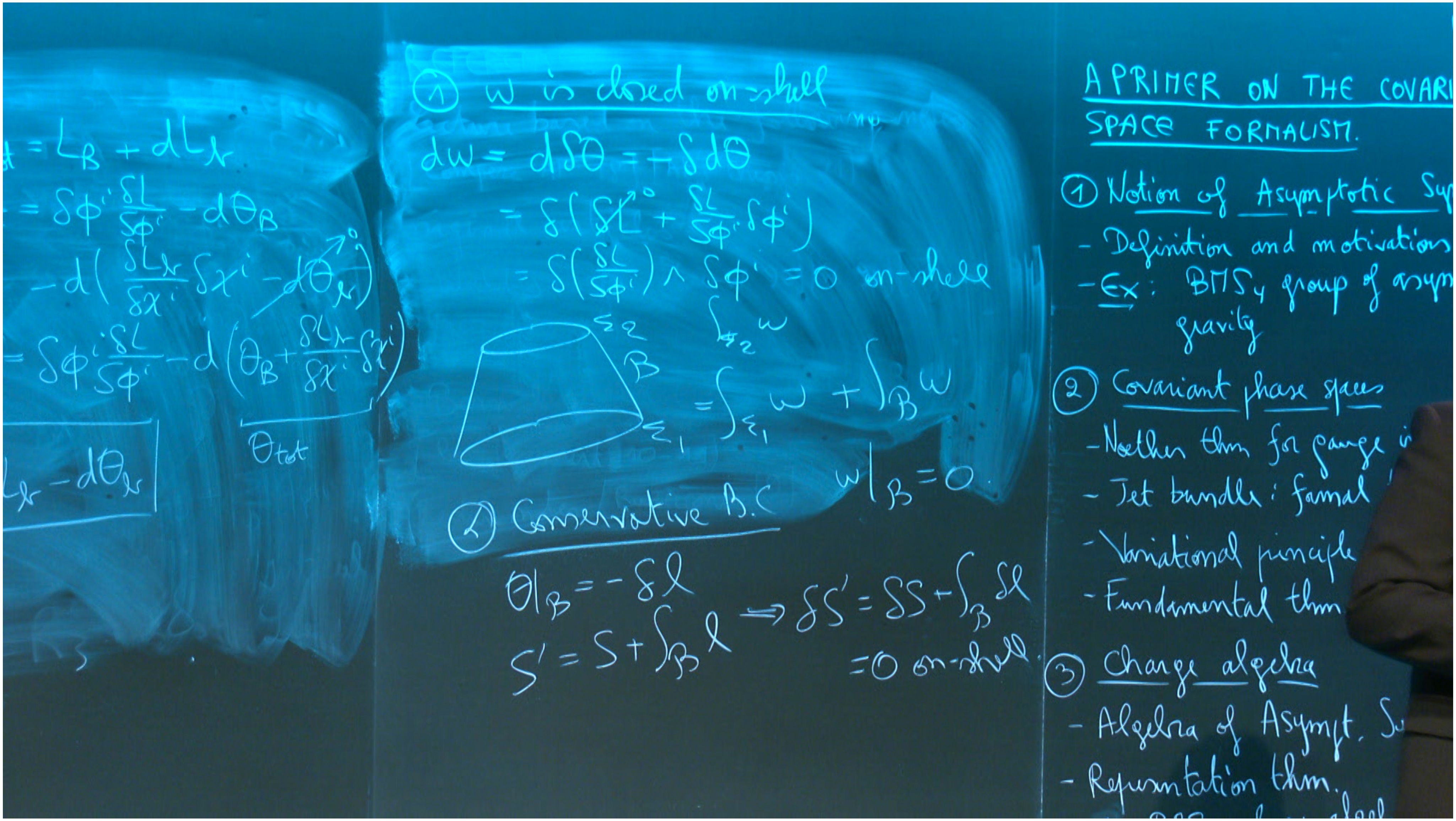
- Representation thm.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3 Charge algebra
- Algebra of Asympt. S
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1 ω is closed on-shell
dω = dδΘ = −δdΘ
= δ(δL + δL/δφ^i δφ^i)
= δ(δL/δφ^i) ∧ δφ^i = 0 on-shell
Σ2
B
Σ1
ω|B = 0
2 Conservative B.C
Θ|B = −δℓ
S′ = S + ∫B ℓ ⇒ δS′ = δS + ∫B δℓ
= 0 on-shell
= L_B + dL_χ
= δφ^i δL/δφ^i − dΘ_B
− d(δL_χ/δχ^i δχ^i − dΘ_χ)
= δφ^i δL/δφ^i − d(Θ_B + δL_χ/δχ^i δχ^i)
Θ_tot
L_χ − dΘ_χ