Title: BMS-like symmetries in cosmology

Speakers: Beatrice Bonga

Series: Quantum Gravity

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Abstract: There exists a solid framework to study gravitational waves in full, non-linear general relativity when the spacetime is asymptotically flat. An essential aspect of this framework is asymptotic symmetries. These symmetries are the BMS symmetries. The situation for cosmological spacetimes is different, however. Expanding spacetimes, whose expansion is decelerating such as matter- or radiation-dominated universes, share some similarities with the asymptotically flat case nonetheless. Not surprisingly, their asymptotic symmetries also share similarities with the BMS symmetries, but are not the same.

Zoom Link: https://pitp.zoom.us/j/98168915236?pwd=ejFtaVpVZXAxT2NIdDhtMUNSbmRvQT09

# BMS-like symmetries in cosmology

Béatrice Bonga – 15 September 2022

*Online QG Seminar @ Perimeter Institute*

[work in collaboration with Kartik Prahbu - PRD 102]

Radboud University

# Invaluable tool: perturbation theory

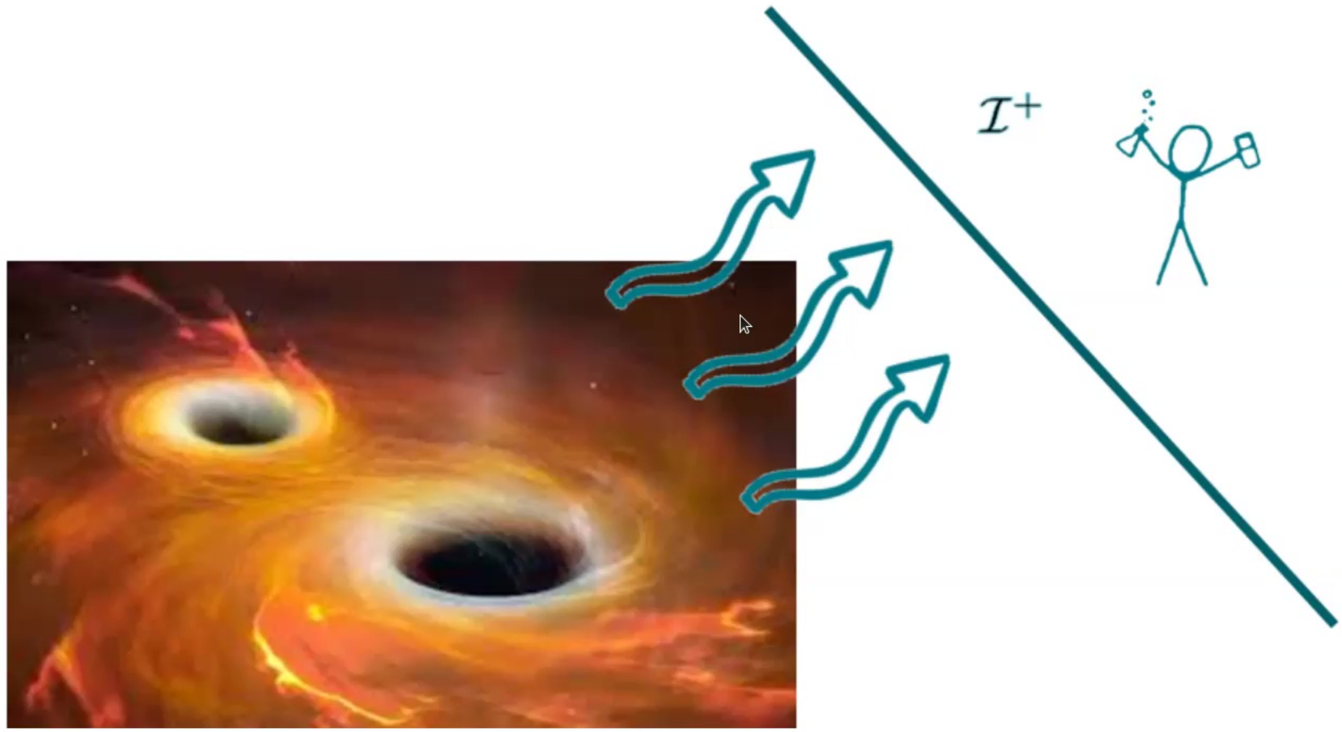
$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h_{\mu\nu}$$

Minkowski spacetime
Kerr spacetime
FLRW spacetime
⋮

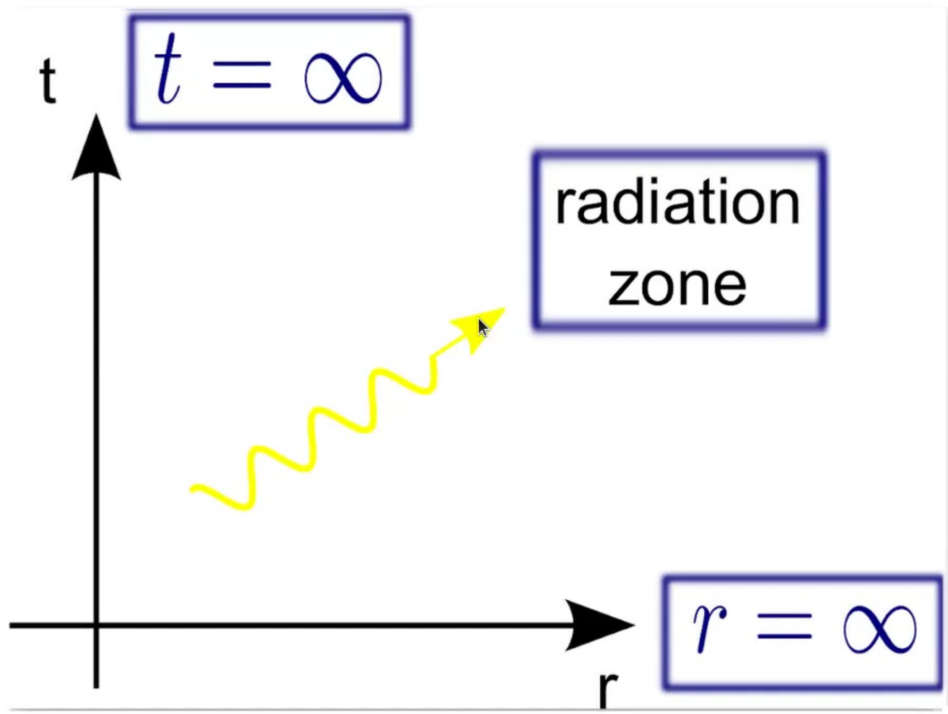
# But there is no canonical split!

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \epsilon\, h_{\mu\nu}$$

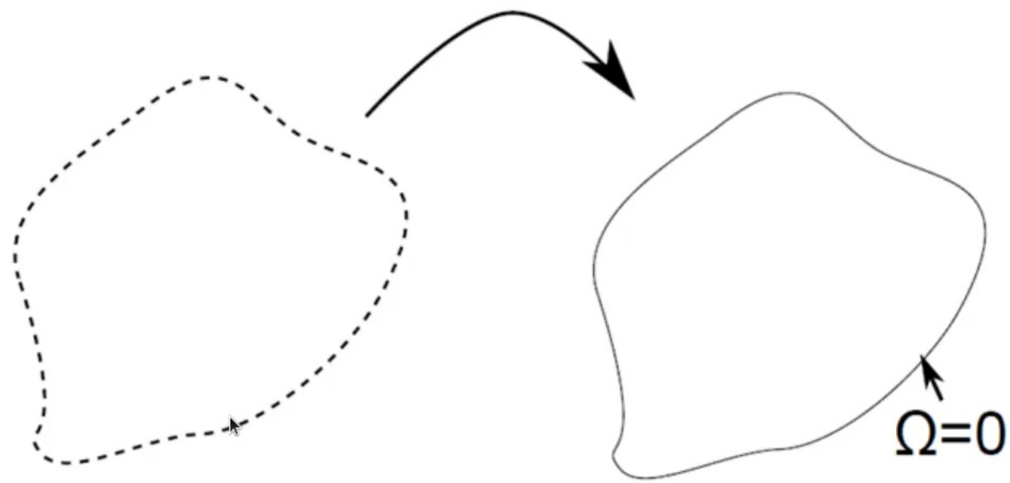
# From messy physics to peaceful realm



# Different infinities



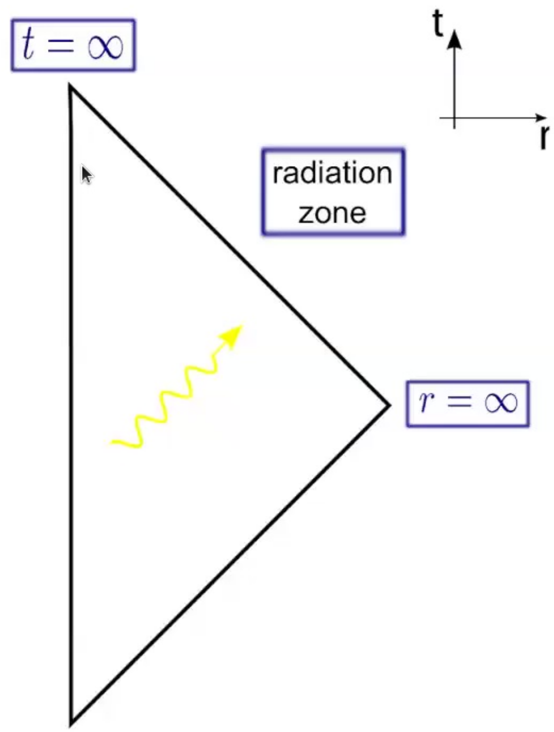
# Key idea: bring infinity to a finite distance



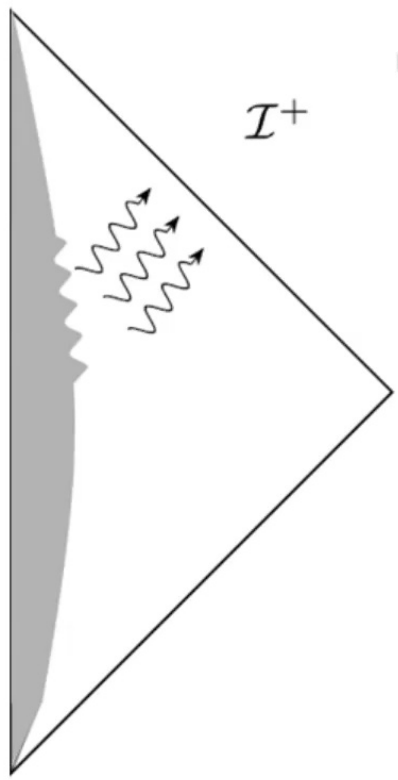

$(\hat{M}, \hat{g}_{ab})$ $(M, g_{ab} = \Omega^2 \hat{g}_{ab})$

Conformal completion

# Conformal diagram Minkowski



# Asymptotic flatness

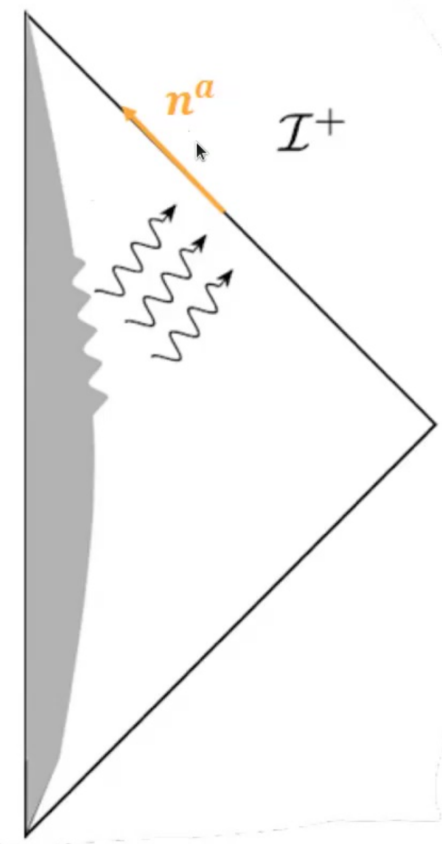



A physical spacetime $(\hat{M}, \hat{g}_{ab})$ is asymptotically flat if there exists a spacetime $(M, g_{ab})$ with boundary $\partial M \cong \mathcal{J} \cong \mathbb{R} \, x \, \mathbb{S}^2$ such that

1. $\Omega$ and $g_{ab} = \Omega^2 \; \hat{g}_{ab}$ are smooth on $M$, $\Omega \hat{=} 0$ and $n_a = \nabla_a \Omega$ is nowhere vanishing on $\mathcal{J}$
2. Einstein's equations are satisfied with $\hat{T}_{ab}$ such that $\Omega^{-2} \hat{T}_{ab}$ has a smooth limit to $\mathcal{J}$

# Consequences

- Einstein's equation ⟹ $n^a$ is null on $\mathcal{J}$
- $q_{ab}$ = induced metric on $\mathcal{J}$ is degenerate: 0 + +

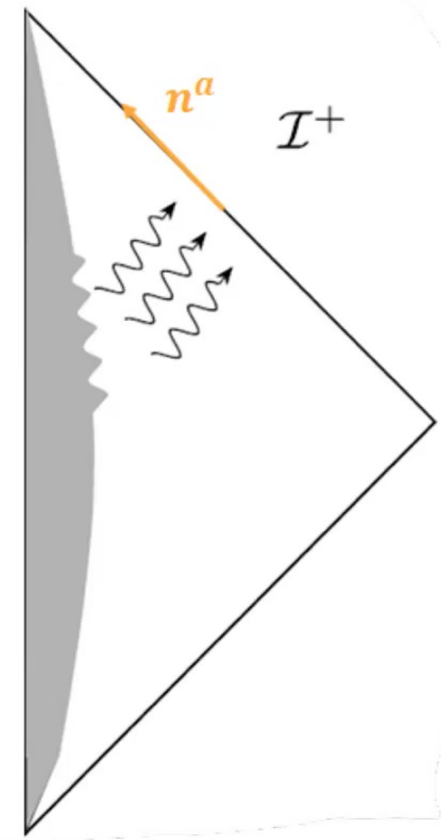


# Consequences

- Einstein's equation ⟹ $n^a$ is null on $\mathcal{I}$
- $q_{ab}$ = induced metric on $\mathcal{I}$ is degenerate: 0 + +

**Conformal freedom**
$$\Omega \to \Omega' = \omega\,\Omega$$
$$g_{ab} \to g'_{ab} = \omega^2 g_{ab}$$
$$n^a \to n^{a\prime} = \omega^{-1} n^a$$

$$n^a = g^{ab}\nabla_b \Omega$$
$$n'^a = g'^{ab}\nabla_b\Omega' = \omega^{-2} g^{ab}\nabla_b(\omega\Omega)$$
$$= \omega^{-1} g^{ab}\nabla_b\Omega + \mathcal{O}(\Omega)$$
$$\hat{=}\ \omega^{-1} n^a$$



# Universal structure

This is common to ***all*** asymptotically flat spacetimes

$$\{ q_{ab}, n^a \} = \{ \omega^2 q_{ab}, \omega^{-1} n^a \}$$

*Gravitational radiation is encoded in the next-order structure and differs from spacetime to spacetime*

# Key points of asymptotics

Nowhere in this construction did we introduce a split of the background and "gravitational waves".

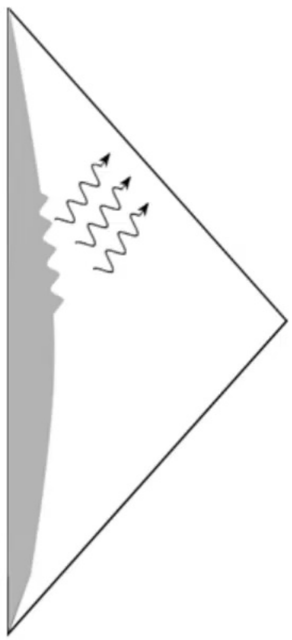
The split occurs naturally at null infinity:

- universal structure is like a background,
- first order structure contains gravitational radiation,

and it is **fully non-linear**!

# Generic asymptotically flat spacetime

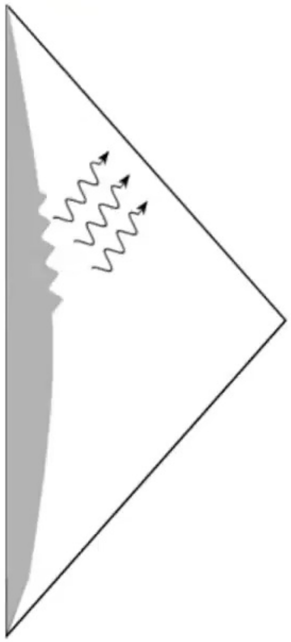
$$d\hat{s}^2 = -UV\,du^2 - 2U\,dudr + \gamma_{AB}(r\,d\theta^A + W^A\,du)(r\,d\theta^B + W^B\,du)$$



$$\begin{aligned} U &= 1 + B/r^2 + O(r^{-3}), \\ V &= 1 - 2M/r + N/r^2 + O(r^{-3}), \\ W^A &= A^A/r + B^A/r^2 + O(r^{-3}), \\ \gamma_{AB} &= \Omega_{AB} + f_{AB}/r + \tfrac{1}{4}f^2\Omega_{AB}/r^2 + O(r^{-3}) \end{aligned}$$

# Generic asymptotically flat spacetime

$$d\hat{s}^2 = -UV\,du^2 - 2U\,du\,dr + \gamma_{AB}(r\,d\theta^A + W^A\,du)(r\,d\theta^B + W^B\,du)$$



$$U = 1 + B/r^2 + O(r^{-3}),$$
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$$W^A = A^A/r + B^A/r^2 + O(r^{-3}),$$
$$\gamma_{AB} = \Omega_{AB} + f_{AB}/r + \tfrac{1}{4}f^2\Omega_{AB}/r^2 + O(r^{-3})$$

Flat Space | Mass aspect | Radiative modes | Angular momentum aspect

# Asymptotic symmetry algebra

Spacetime diffeomorphism that leave the universal structure at null infinity invariant

**Universal structure**

$$\mathcal{L}_\xi q_{ab} \hat{=} 2\,\alpha\, q_{ab} \text{ with } \mathcal{L}_n \alpha \hat{=} 0$$

$$\mathcal{L}_\xi n^a \hat{=} -\,\alpha\, n^a$$

**Coordinates**

$$\Omega^2 \mathcal{L}_\xi \hat{g}_{ab} \hat{=}\ 0$$

# Bondi-Metzner-Sachs algebra (BMS)

- Asymptotic symmetry algebra is ***bigger*** than Poincaré
- BMS = supertranslations & rotations

$$\xi^a \partial_a = \left( f(\theta, \varphi) + \frac{1}{2} u D_A Y^A \right) \partial_u + Y^A \partial_A$$

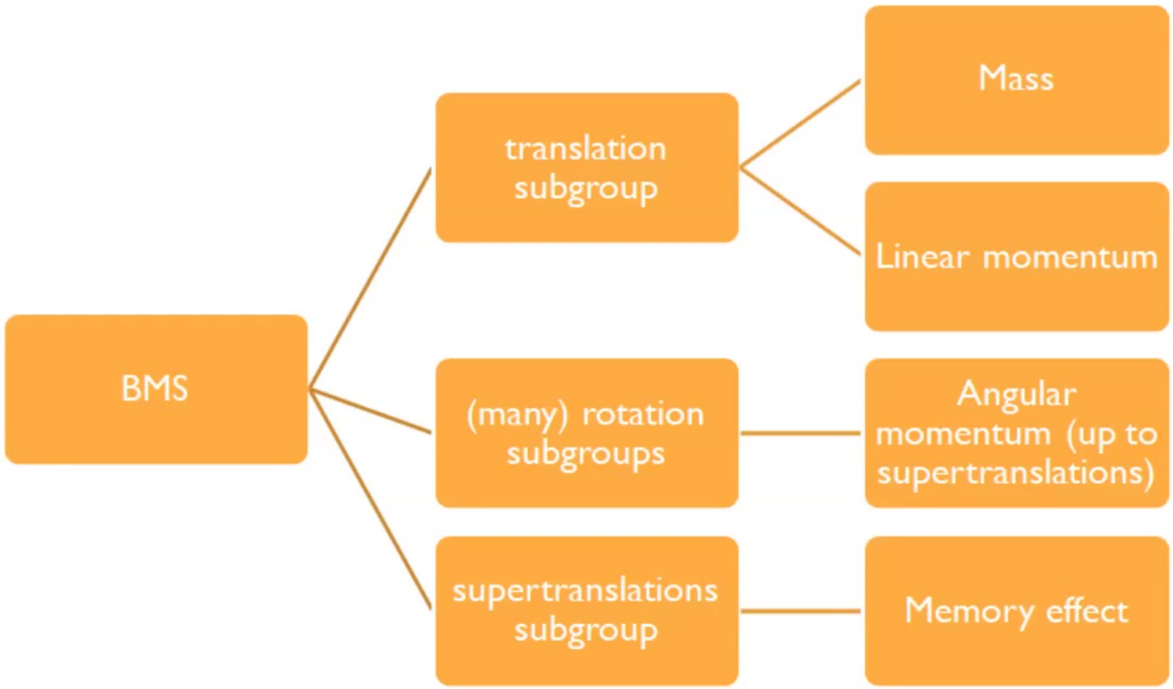
supertranslations

rotations

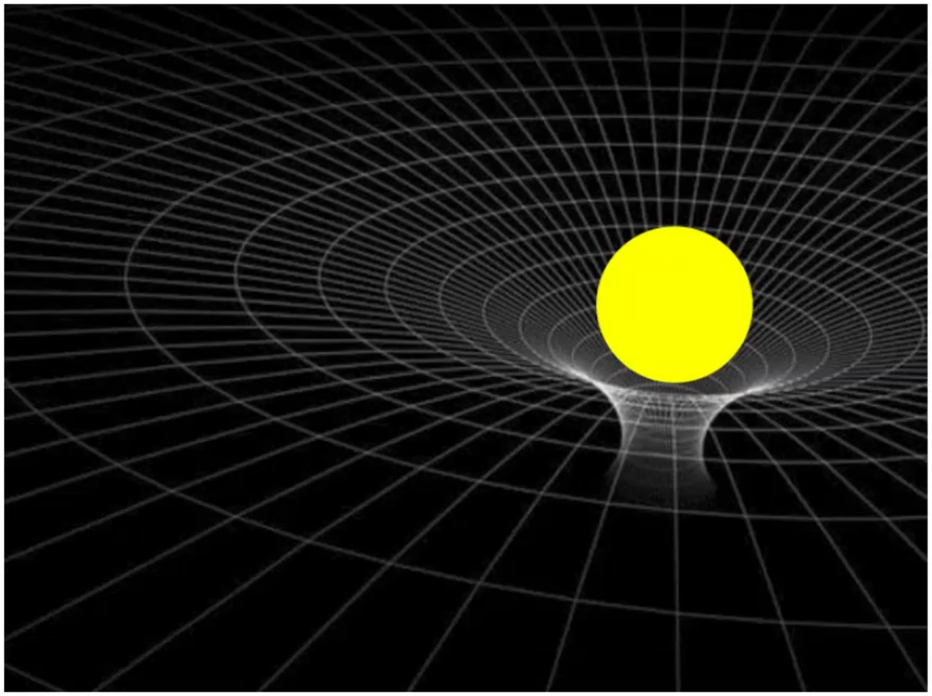
$$2D_{(A}Y_{B)} + q_{AB} D_C Y^C = 0$$

# What is BMS good for?

It provides quantities with a physical interpretation!

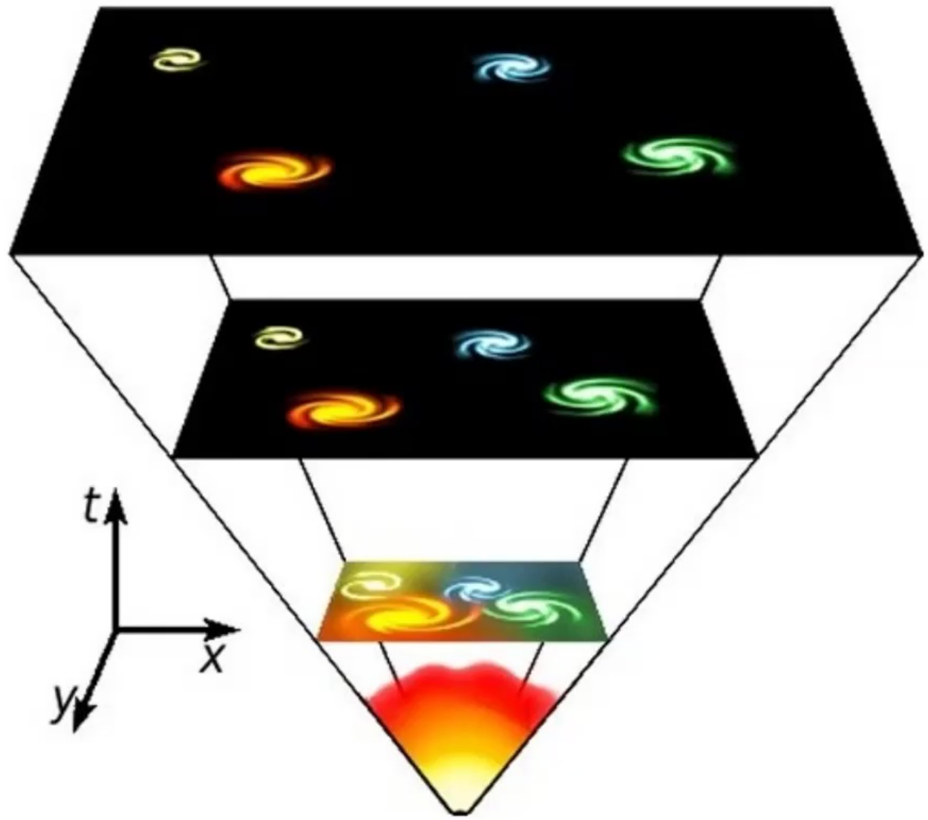


# Critical assumption



Move far away from sources:
'spacetime becomes flat'

# Expanding spacetimes are not asymptotically flat!



# Why assume asymptotic flatness?

P. G. BERGMANN:

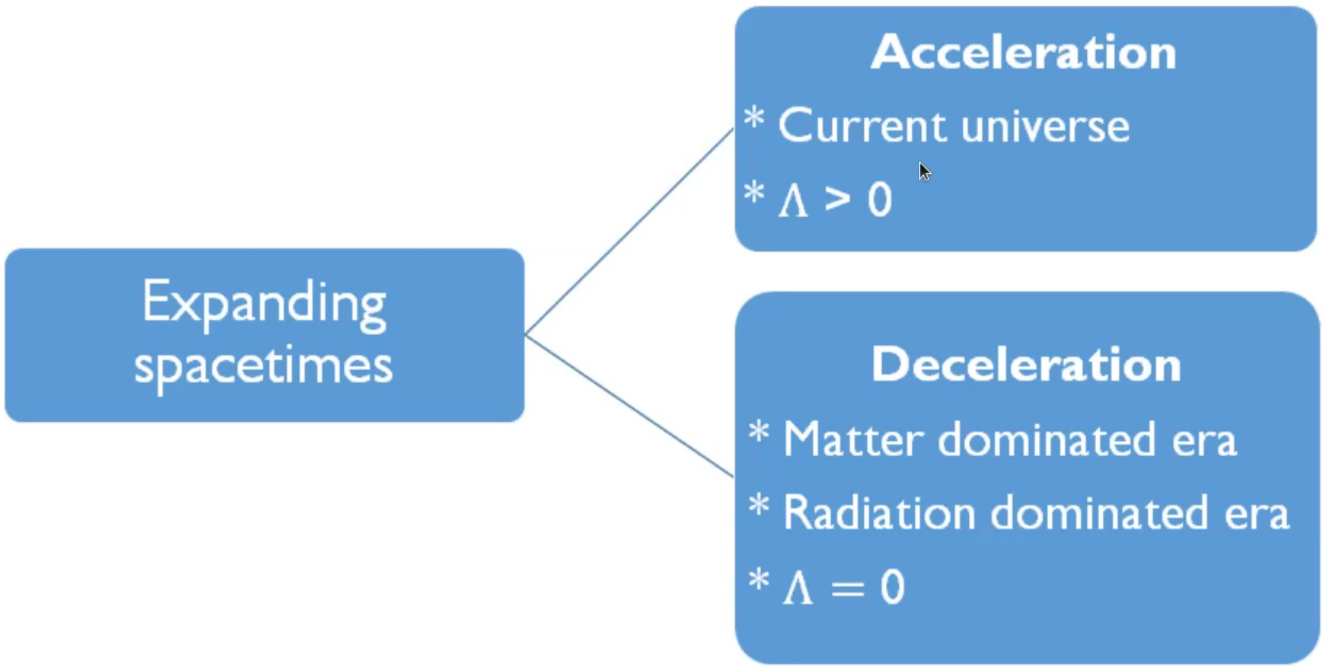
The only answer I can give is that the investigations date back less than two years, I believe, and that people have simply started with the mathematically simplest situation, or what they hoped was the simplest situation.

H. BONDI:

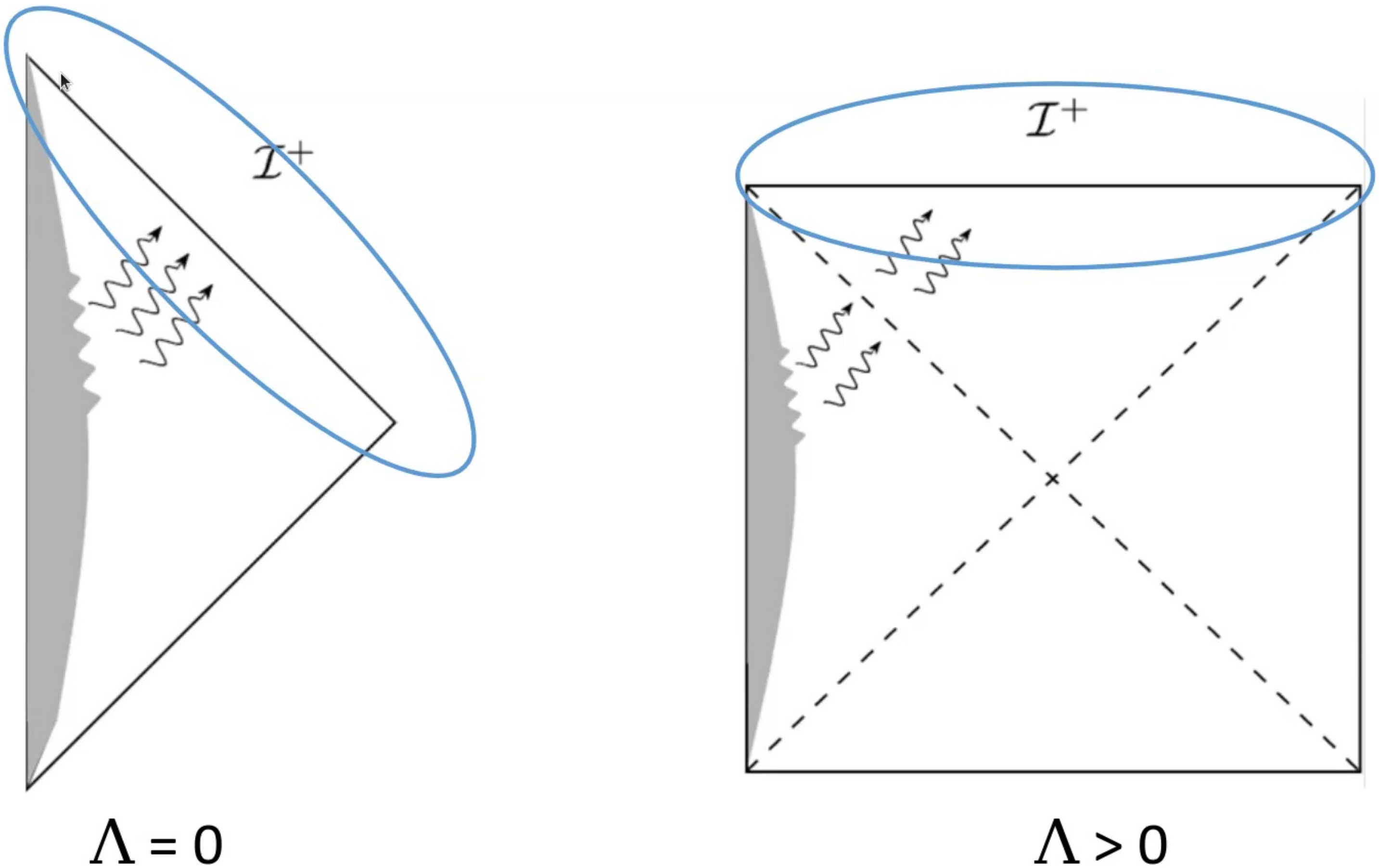
I regret it as much as you do, that we haven't yet got to the point of doing the Friedmann universe.

***Conference Warsaw 1962***

# Expansion rates



# Radiation zones



# Decelerating FLRW spacetimes

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 \delta_{AB} dx^A dx^B\right]$$

physical metric

$$a(\eta) = \left(\frac{\eta}{\eta_0}\right)^{\frac{s}{1-s}}$$

$$s = \frac{2}{3(1+w)}$$

$$0 \le s < 1$$

$$-1/3 < w < \infty$$

$$P = w\rho$$

| | |
|---|---|
| $w = 1$ | stiff fluid |
| $w = 1/3$ | radiation |
| $w = 0$ | dust |
| $w = -1$ | cosmological constant |

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 \delta_{AB}\, dx^A dx^B\right]$$

$$\eta = \frac{\sin T}{\cos R + \cos T} = \frac{\sin\left(\frac{V+U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

$$r = \frac{\sin R}{\cos R + \cos T} = \frac{\sin\left(\frac{V-U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

$$\left\{\begin{array}{l} U = T - R \\ V = T + R \end{array}\right\} \Longleftrightarrow \begin{array}{l} -\pi < U < \pi \\ |U| < V < \pi \end{array}$$

$$d\hat{s}^2 = a^2(\eta)\left[-d\eta^2 + dr^2 + r^2 \delta_{AB} dx^A dx^B\right]$$

$$\eta = \frac{\sin T}{\cos R + \cos T} = \frac{\sin\left(\frac{V+U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

$$r = \frac{\sin R}{\cos R + \cos T} = \frac{\sin\left(\frac{V-U}{2}\right)}{2\cos\frac{U}{2}\cos\frac{V}{2}}$$

$$\begin{cases} U = T - R \\ V = T + R \end{cases} \iff \begin{matrix} -\pi < U < \pi \\ |U| < V < \pi \end{matrix}$$

Choose $\Omega = 2\left(\cos\frac{U}{2}\cos\frac{V}{2}\right)^{\frac{1}{1-s}}\left(\sin\frac{U+V}{2}\right)^{-\frac{s}{1-s}}$

$$ds^2 = \Omega^2 d\hat{s}^2 = -dU\,dV + \sin\left(\frac{V-U}{2}\right)^2 \delta_{AB} dx^A dx^B$$

$\longrightarrow$ can add $V = -U$ & $V = \pi$, because this metric is smooth everywhere including at the boundaries

# The conformal factor

But near $\mathscr{I}$...

$$\Omega \sim \cos\frac{U}{2}\,(\pi - V)^{\frac{1}{1-s}} \longrightarrow \text{NOT smooth!}$$

$$\nabla_a \Omega \sim \cos\frac{U}{2}\,(\pi - V)^{\frac{s}{1-s}}\,\nabla_a V \longrightarrow \hat{=}\, 0 \text{ unless } s=0$$

Bad choice for $\Omega$?

$$\Omega' = \omega\,\Omega$$

with $\omega \sim (\pi - V)^{-\frac{s}{1-s}}$ $\Rightarrow$

- * $\Omega$ is smooth @ $\mathscr{I}$ ☑
- * $\nabla_a \Omega \not\hat{=}\, 0$ ☑

# The conformal factor

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Bad choice for $\Omega$?

What to do?

$$\Omega' = \omega\,\Omega$$

with $\omega \sim (\pi - V)^{-\frac{s}{1-s}}$ $\Rightarrow$

- * $\Omega$ is smooth @ $\mathscr{I}$ ☑
- * $\nabla_a \Omega \,\hat{\neq}\, 0$ ☑

but then

$$g'_{ab} = \Omega'^2\, \hat{g}_{ab} \sim (\pi - V)^{-\frac{2s}{1-s}}\, g_{ab}$$

THIS DIVERGES @ $\mathscr{I}$!

# Simple resolution

$\Omega^{1-s}$ is smooth @ $\mathscr{I}$

* $\Omega^{1-s} \hat{=} 0$
* $\nabla_a \Omega^{1-s} \not\hat{=} 0$

Define the normal to $\mathscr{I}$ using $\Omega^{1-s}$

$$\Rightarrow n_a := \frac{1}{1-s} \nabla_a \Omega^{1-s}$$

$$= \Omega^{-s} \nabla_a \Omega$$

$$\hat{=} -\frac{2^{-s}}{1-s} \left(\cos \frac{U}{2}\right)^{1-s} \nabla_a V$$

# Presence of matter

For asymptotically flat spacetimes, $\Omega^{-2}\hat{T}_{ab}$ should have a limit to $\mathcal{I}$ but FLRW spacetimes are homogeneous, so there is matter *everywhere*!

$$\lim_{\to \mathcal{I}} 8\pi G \, g^{ab}\hat{T}_{ab} = \frac{6s(1-s)}{(1-s)^2}\left(\sec\frac{U}{2}\right)^2 \longrightarrow \text{NON-VANISHING}$$

$$8\pi G \, \hat{T}_{ab} = \underbrace{2s\,\Omega^{2(s-1)}\, n_a n_b}_{\text{universal}} + 2s\,\Omega^{s-1}\, \tau_{(a} n_{b)} + \text{finite}$$

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$$8\pi G\, \hat{T}_{ab} = \underbrace{2s\,\Omega^{2(s-1)}\, n_a n_b}_{\text{universal}} + 2s\,\Omega^{s-1}\, \tau_{(a} n_{b)} + \text{finite}$$

depends on choice $\Omega$

$\tau_a \hat{=} \tan\frac{U}{2}(\nabla_a U + \nabla_a V)$

# Spacetimes with a cosmological null asymptote

A physical spacetime $\left(\hat{M}, \hat{g}_{ab}\right)$ admits a cosmological null asymptote if there exists a spacetime $(M, g_{ab})$ with boundary $\partial M \cong \mathcal{I} \cong \mathbb{R}\, x\, \mathbb{S}^2$ such that

(1) $\Omega \ \hat{=}\ 0\,, \Omega^{1-s}$ and $g_{ab} = \Omega^2\ \hat{g}_{ab}$ is smooth on $M$,

$n_a = \Omega^{-s}\nabla_a\Omega$ is nowhere vanishing on $\mathcal{I}$ (for $0 \leq s < 1$)

(2) Einstein's equations are satisfied with $\hat{T}_{ab}$ such that

$\lim_{\to \mathscr{I}} g^{ab}\hat{T}_{ab}$ exists

$$\lim_{\to \mathscr{I}} \Omega^{1-s}\left[8\pi\hat{T}_{ab} - 2s\Omega^{2(s-1)}n_a n_b\right] \hat{=}\ 2s\tau_{(a}n_{b)}$$

# Asymptotic symmetry algebra

All smooth vector fields that map

$$\{q_{ab}, n^a\} \longrightarrow \{q'_{ab} = \omega^2 q_{ab}, n'^a = \omega^{-1-s} n^a\}$$

$$\Longrightarrow \mathfrak{b}_s \cong \mathfrak{so}(1,3) \ltimes \mathfrak{S}_s$$

# In terms of coordinates on $\mathcal{I}$

$$\xi^a \partial_a = \left( f(\theta, \varphi) + \frac{1+s}{2} \, u \, D_A Y^A \right) \partial_u + Y^A \partial_A$$

supertranslations with conformal weight $1 + s$

rotations

$$2D_{(A}Y_{B)} + q_{AB} D_C Y^C = 0$$

# The critical *s*-dependence

A supertranslation can again be written as

$$\xi^a \partial_a = f(\theta, \varphi)\, \partial_u$$

has conformal weight $1+s$

For fixed $\xi^a = f n^a \longmapsto \xi^a = f' n'^a$

$$= f' \omega^{-1-s} n^a$$

# The critical *s*-dependence

A supertranslation can again be written as

$$\xi^a \partial_a = f(\theta,\varphi)\,\partial_u$$

has conformal weight $1+s$

For fixed $\xi^a = f n^a \longmapsto \xi^a = f' n'^a$

$$= f' \omega^{-1-s} n^a$$

$$\Longrightarrow \boxed{f' = \omega^{1+s} f}$$

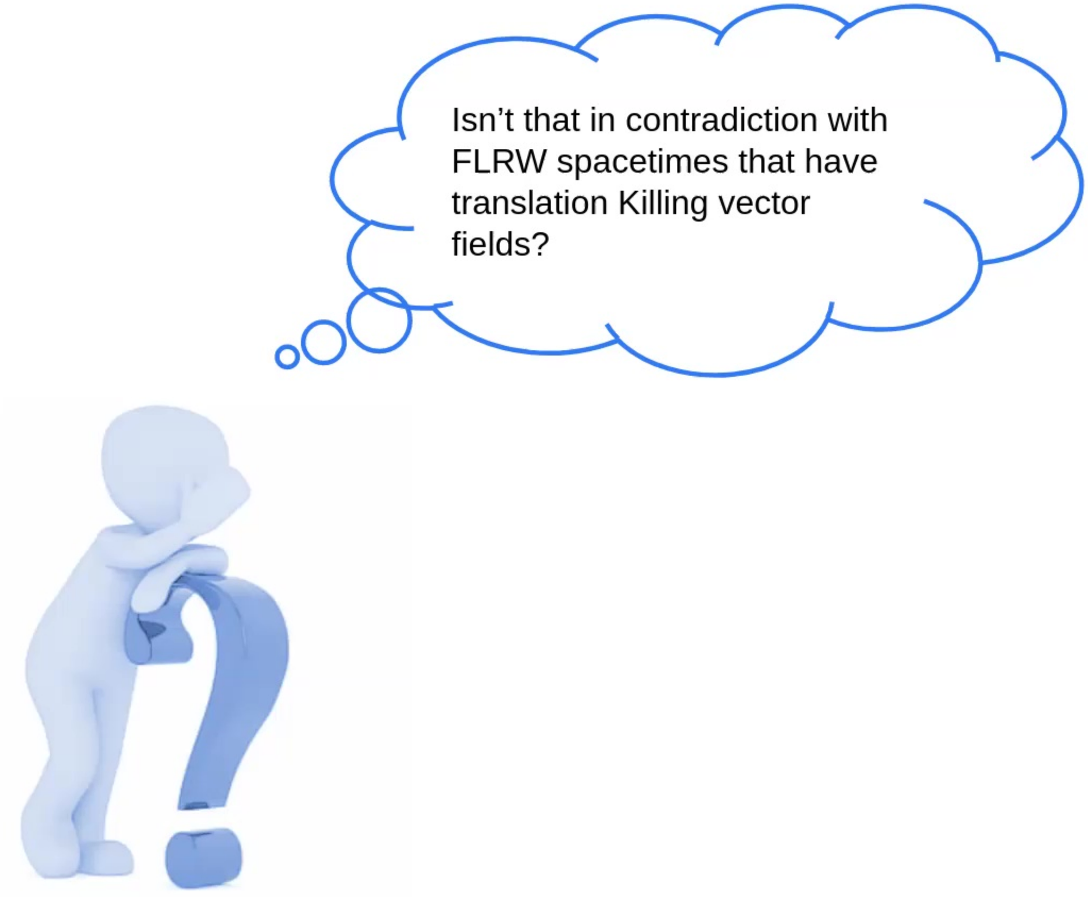
# Consequences s-dependence

- No preferred translation subalgebra

  notion of mass and linear momentum?

- Conformal Carroll algebra with N=2/(1+s)

# How about FLRW?



# How about FLRW?

Isn’t that in contradiction with FLRW spacetimes that have translation Killing vector fields?

No! This is similar to absence of a rotation subgroup for asymptotically flat spacetimes.

# No peeling: challenge for radiation

$$\Psi_4 := -C_{abcd}\overline{m}^a n^b \overline{m}^c n^d \mathrel{\widehat{=}} 0\,, \quad \tilde{\Omega}^{-1}\Psi_4 \mathrel{\widehat{=}} \left(\tfrac{1}{2}\partial_u^2 C_{AB} + s\partial_u \eth_A \tau_B + \tfrac{1}{2}s^2\tau_A\partial_u\tau_B\right)\overline{m}^A\overline{m}^B$$

$$\Psi_3 := -C_{abcd}l^a n^b \overline{m}^c n^d \mathrel{\widehat{=}} -\tfrac{s}{4}\partial_u\tau_A\overline{m}^A$$

$$\Psi_2 := -C_{abcd}l^a m^b \overline{m}^c n^d \mathrel{\widehat{=}} -\tfrac{1}{6}\left[W^{(2)} - 1 - s\left(\partial_u\tau + \tfrac{1}{2}\eth_A\tau^A + s\tau_A\tau^A + \tfrac{3}{2}i\epsilon^{AB}\eth_A\tau_B\right)\right]$$

# Any other examples?

$$A > 0\,, \quad \mathcal{L}_{\mathring{n}} A|_{\mathcal{I}} = 0$$

$$\mathring{g}_{ab}$$

$$\hat{g}_{ab} = \mathring{\Omega}^{-\frac{2s}{1+s}} A^{\frac{2s}{1+s}} \mathring{g}_{ab}$$

$$\hat{T}_{ab} = \mathring{T}_{ab} + f(n_a, \Omega^{1-s}, A)$$

$$g_{ab} = \mathring{\Omega}^2 \mathring{g}_{ab}$$

Class of spacetimes at least as big as asymptotically flat spacetimes!

# Exciting, but…

Linearization stability still open question!

# Conclusion

- Gravitational radiation can be studied using asymptotics in the full non-linear theory

- Asymptotic symmetry algebra provides charges and fluxes with a physical interpretation

  - Asymptotic flat spacetimes: BMS
  - Asymptotic cosmological null asymptotes: BMS-***like***, there is no preferred translation subalgebra!

\+ *ongoing work with Berend Schneider & Sk Jahanur Hoque*

# Future applications

- ❖ Next order structure
  - ➢ Study rigorously gravitational radiation produced by compact sources in cosmological spacetimes
  - ➢ Study the gravitational memory effect
  - ➢ Charges and fluxes
- ❖ Link with timelike future infinity
- ❖ … your favorite topic!

*Thank you for listening!*