Title: Generalized Lense-Thirring spacetimes: higher curvature corrections and solutions with matter

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Series: Strong Gravity

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Abstract: The Lense-Thirring spacetime describes a 4-dimensional slowly rotating approximate solution of vacuum Einstein equations valid to a linear order in rotation parameter. It is fully characterized by a single metric function of the corresponding static (Schwarzschild) solution. We shall discuss a generalization of the Lense-Thirring spacetimes to the case that is not necessarily fully characterized by a single (static) metric function. This generalization lets us study slowly rotating spacetimes in various higher curvature gravities as well as in the presence of non-trivial matter such as non-linear electrodynamics. In particular, we construct slowly multiply-spinning solutions in Lovelock gravity and notably show that in four dimensions Einstein gravity is the only non-trivial theory amongst all up to quartic curvature gravities that
admits a Lense-Thirring solution characterized by a single metric function. We will also discuss a `magic square' version of our ansatz and show that it can be cast in the Painlevé-Gullstrand form (and thence is manifestly regular on the horizon) and admits a tower of exact rank-2 and higher rank Killing tensors that rapidly grows with the number of dimensions.

Zoom Link: TBD

Generalized Lense-Thirring spacetimes: higher curvature corrections and solutions with matter

David Kubizňák
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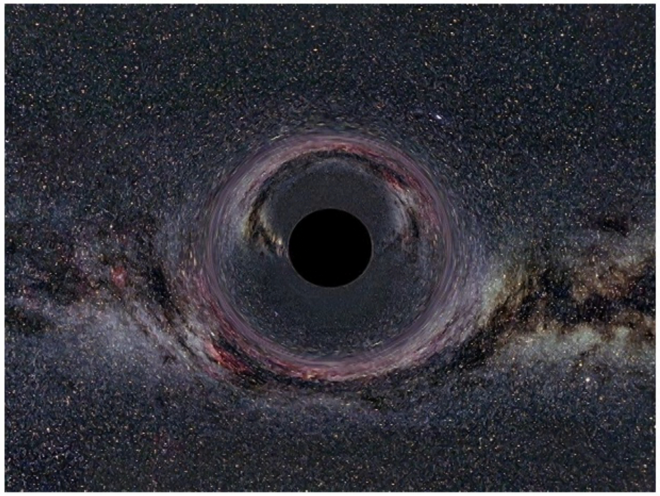


Strong Gravity Seminar
Perimeter Institute, Waterloo, Canada
Sep 8, 2022

Plan of the talk

I. **Lense-Thirring (LT) spacetime**

II. **Generalized LT metrics**

   a) Newmann-Janis algorithm & LT

   b) Higher curvature theories & Einstein uniqueness

   c) Matter & non-linear electrodynamics

   d) Higher dimensions & multiple rotations

III. "**Upgraded LT**"**:** PG form and hidden symmetries

IV. Summary

Based on:

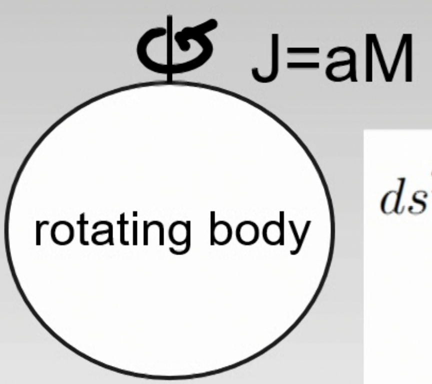
- Paper 1: F. Gray, DK, *Slowly rotating black holes with exact Killing tensor symmetries,* ArXiv:2110:14671.
- Paper 2: F. Gray, R.A. Hennigar, DK, R.B. Mann, M. Srivastava, *Generalized Lense-Thirring metrics: higehr-curvature corresctions and solutions with matter*; Arxiv:2112.07649.
- Paper 3: DK, T. Tahamtan, O. Svitek, *Slowly rotating black holes in non-linear electrodynamics*, ArXiv:2203:01919.

and

- J. Baines, T. Berry, A. Simpson, M. Visser, *Killing tensor and carter constant for Painleve-Gullstrand form of Lense-Thirring spacetime*, Arxiv:2110.01814.

I) Lense-Thirring spacetime

Lense-Thirring spacetime (1918)



$$ds^2 = -f\,dt^2 + \frac{dr^2}{f} + 2a(f-1)\sin^2\theta\, dt d\phi + r^2(\sin^2\theta\, d\phi^2 + d\theta^2)\,, \quad f = 1 - \frac{2M}{r}$$

- Spacetime outside a **slowly rotating body**
- **Approximate** (linear in a) vacuum solution of EE
- Linear in a approximation to **Kerr** (1963)
- Encodes **gravitomagnetic effects**

Frame dragging (gravitomagnetism)

= general-relativistic effect due to the **motion** (in particular rotation) **of matter** and gravitational waves, analogous in a way to electromagnetic induction.

- Already in the **weak field approximation**

$$T_{\mu\nu} = \begin{pmatrix} \rho & \vec{j} \\ \vec{j} & 0 \end{pmatrix}\,, \quad j_\mu = T_{0\mu} = (\rho, \vec{j})$$

$$g_{00} = -1 - 2\phi\,, \quad g_{0i} = -4A_i\,, \quad g_{ij} = (1 - 2\phi)\delta_{ij}$$

- Einstein & geodesic equations then yield

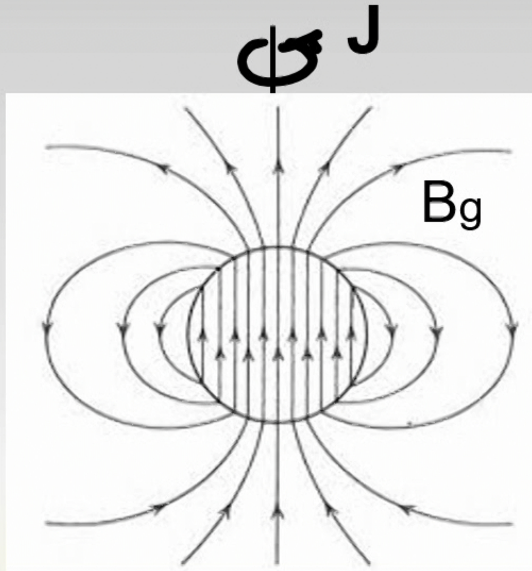
$$\Box A^\mu = -4\pi j^\mu\,, \quad \partial_\mu A^\mu = 0$$

$$\frac{d^2x^i}{dt^2} = E^i - 4F^i{}_j v^j$$

Moved from electrostatics to **electrodynamics**

Frame dragging (gravitomagnetism)

- **Lense-Thirring (1918)** (slowly rotating body)



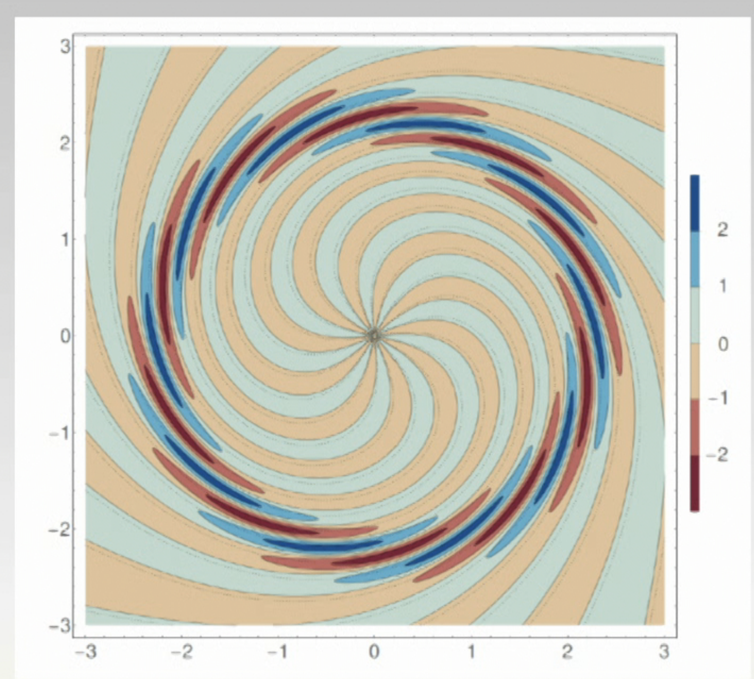
- "radially infalling geodesic" experiences "**Coriolis type force**"

$$r\frac{d^2\varphi}{dt^2} = -\underbrace{\frac{2J}{r^3}}_{2\omega(r)}\frac{dr}{dt}$$

(spacetime outside is dragged along the rotation of the body)

- **Frame dragging** inside the hollow shell

Rotating gravitational waves and shells

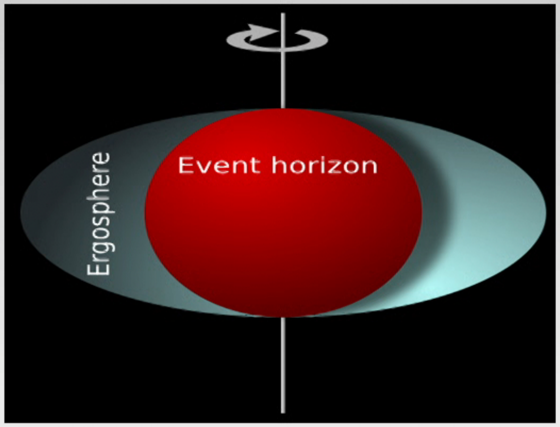


- W. Barker, T. Ledvinka, D. Lynden-Bell, J. Bičák, Rotation of inertial frames by angular momentum of matter and waves, CQG 34 (2017) 20, 205006; Arxiv:1710.10360.

- J. Bičák, J. Katz, T. Ledvinka, D. Lynden-Bell, Effects pf rotating gravitational waves, Phys. Rev D85 (2012), 124003; Arxiv:1207.2957.

Frame dragging (gravitomagnetism)

- **Existence of a BH ergosphere**

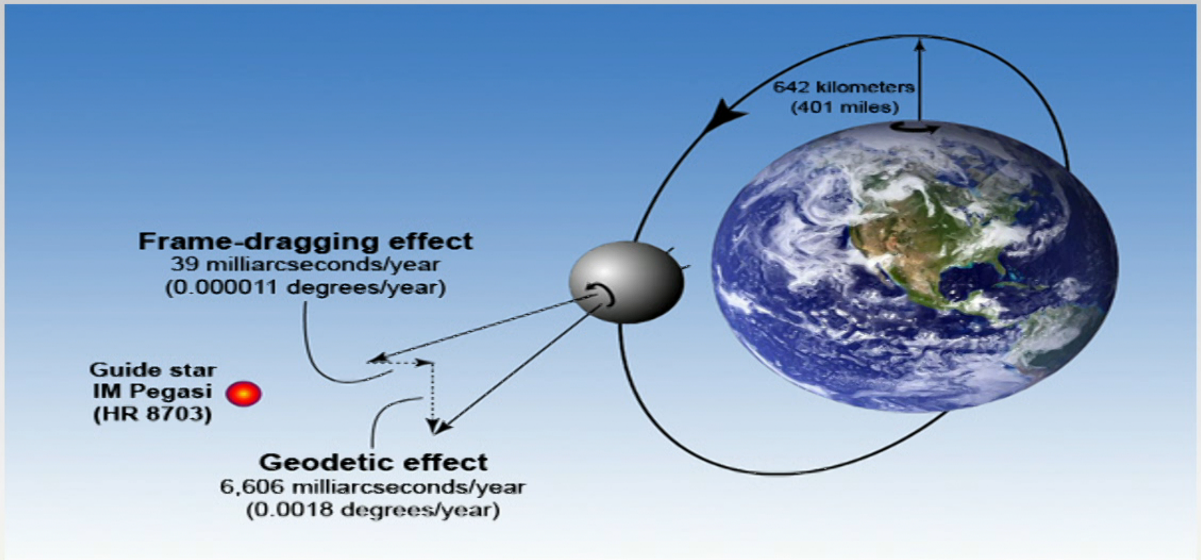


Example of extremal frame dragging: particle inside the ergosphere has to corotate with the black hole

- **Astrophysical applications**
  - **Bardeen-Petterson effect** – aligning of the accretion disc along the black hole spin axis

The Gravity Probe B Experiment

Everitt; et al. *"Gravity Probe B: Final Results of a Space Experiment to Test General Relativity".* Phys. Rev. Lett. **106** (22): 221101 (2011)



	Measured	Predicted
Geodetic precession (mas)	6602 ± 18	6606
Frame-dragging (mas)	37.2 ± 7.2	39.2

https://physics.aps.org/articles/v4/43

a) Newmann-Janis trick

= **algorithm** how to generate rotating solutions from the static ones by a "**complex coordinate transformation**".

1) **Start from** a spherical spacetime and write it ingoing Finkelstein coordinates:

$$ds^2 = -f dt^2 + \frac{dr}{f} + g d\Omega^2 \qquad du = dt - \frac{dr}{f}$$

or, alternatively in terms of the null frame:

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \bar{m}^\nu + m^\nu \bar{m}^\mu$$

$$l = \partial_r \,, \quad n = \partial_u - \frac{f}{2}\partial_r \,, \quad m = \frac{1}{\sqrt{2g}}\Big(\partial_\theta + \frac{i}{\sin\theta}\Big)\partial_\phi$$

Newmann-Janis trick

2) **Perform** the complex coordinate transformation:

$$u \to u - ia\cos\theta\,, \quad r \to r + ia\cos\theta$$

Together with

$$f \;\to F(r,a,\theta) \;\text{ and }\; g \;\to\; \Sigma(r,a,\theta).$$

$$l \to \partial_r\,, \quad n \to \partial_u - \frac{F}{2}\partial_r\,,$$
$$m \to \frac{1}{\sqrt{2\Sigma}}\left(\partial_\theta + ia\sin\theta(\partial_u - \partial_r) + \frac{i}{\sin\theta}\right)\partial_\phi$$

3) **Return back** to the Boyer-Lindquist coordinates, by

$$du = dt + \lambda(r)dr\,, \quad d\varphi = d\varphi + \chi(r)dr$$

Newmann-Janis trick

- Requiring that metric diagonal except $g_{t\varphi}$ results in

$$ds^2 = -\frac{\Delta}{\Sigma}\left(dt - a\sin^2\theta d\phi\right)^2 + \frac{\Sigma}{\Delta}dr^2 + \Sigma d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left[(r^2+a^2)d\phi - adt\right]^2$$

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad 2\rho = r(1-f)$$
$$\Delta = r^2 f + a^2 = r^2 - 2\rho r + a^2 \ .$$

Which yields the following **standard LT spacetime** in the linear O(a) expansion:

Standard LT spacetime

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + 2a(f-1)\sin^2\theta dt d\phi$$
$$+ r^2(\sin^2\theta d\phi^2 + d\theta^2)\,,$$

- Completely characterized by the static (spherical) **metric function f**
- In particular, for vacuum Einstein gravity: $f = 1 - \dfrac{2M}{r}$

- How generic is this **ansatz**? Can we apply it also for **modified gravity theories**, or for **non-vacuum** solutions?
- (Can the **Newmann-Janis trick** be used to generate the corresponding rotating solutions?)

b) Beyond Einstein gravity: higher-curvature corrections

$$\mathcal{L} = \frac{1}{16\pi}\Big(\frac{(d-1)(d-2)}{\ell^2} + R + \sum_i \alpha_i \mathcal{R}_i^{(2)} + \sum_i \beta_i \mathcal{R}_i^{(3)} + \sum_i \gamma_i \mathcal{R}_i^{(4)}\Big)$$

where

$$\mathcal{R}_1^{(2)} = R_{abcd}R^{abcd}\,, \quad \mathcal{R}_2^{(2)} = R_{ab}R^{ab}\,, \quad \mathcal{R}_3^{(2)} = R^2$$

$$\mathcal{R}_1^{(3)} = R_a{}^c{}_b{}^d R_c{}^e{}_d{}^f R_e{}^a{}_f{}^b\,, \quad \mathcal{R}_2^{(3)} = R_{ab}{}^{cd} R_{cd}{}^{ef} R_{ef}{}^{ab}\,, \quad \mathcal{R}_3^{(3)} = R_{abcd}R^{abc}{}_e R^{de}$$
$$\mathcal{R}_4^{(3)} = R_{abcd}R^{abcd}R\,, \quad \mathcal{R}_5^{(3)} = R_{abcd}R^{ac}R^{bd}\,, \quad \mathcal{R}_6^{(3)} = R_a{}^b R_b{}^c R_c{}^a\,,$$
$$\mathcal{R}_7^{(3)} = R_a{}^b R_b{}^a R\,, \quad \mathcal{R}_8^{(3)} = R^3\,,$$

EOM $$\mathcal{E}_{ab} = P_a{}^{cde} R_{bcde} - \frac{1}{2} g_{ab}\mathcal{L} - 2\nabla^c\nabla^d P_{acdb} = 0$$

$$P^{abcd} \equiv \frac{\partial \mathcal{L}}{\partial R_{abcd}}$$

Einstein-like Lense-Thirring spacetime

$$ds^2 = -N f dt^2 + \frac{dr^2}{f} + 2ah \sin^2\theta dt d\phi + r^2(\sin^2\theta d\phi^2 + d\theta^2)\,,$$

impose $N = 1$ $\quad h = f - 1$

- The first requirement has nothing to do with rotation – requires that static solutions are characterized by N=1. The corresponding theories are known as **generalized quasi-topological gravities**.

Quadratic order: Gauss-Bonnet

$$\alpha_1 = \alpha\,, \quad \alpha_2 = -4\alpha\,, \quad \alpha_3 = \alpha$$

Generalized quasi-topological gravities

Cubic order:

- **Quasi-topological gravity** (EOM on SSS are 2nd-order)

  R. C. Myers and B. Robinson, Black Holes in Quasi-topological Gravity, JHEP 1008 (2010) 067, [1003.5357].

- **Einsteinian Cubic Gravity** (active already in 4D)

  P. Bueno and P. A. Cano, Einsteinian cubic gravity, Phys. Rev. D94 (2016) 104005,

- **Single metric function** (coincides with Einsteinian in 4D)

  R. A. Hennigar, D. Kubiznak and R. B. Mann,Generalized quasitopological gravity, Phys.Rev. D95(2017) 104042, [1703.01631]

- **4-parametric** theories (of (3+8)-parametric theories)

Quartic order:

J. Ahmed, R.A. Hennigar, R.B. Mann, M.Mir, Quintessential Quasitopological Quartet, JHEP 05 (2017) Phys.Rev. D95(2017) 104042, [1703.11007]

- **19-parametric** theories (of (3+8+25)-parametric theories)

Einstein uniqueness

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + 2a(f-1)\sin^2\theta dt d\phi + r^2(\sin^2\theta d\phi^2 + d\theta^2)\,,$$

Theorem: In D=4, Einstein gravity is the only non-trivial gravitational theory up to powers quartic in curvature, whose vacuum Lense-Thirring solutions are of the above form (Paper 2)

- **Collorary:** Newman-Janis trick cannot be used to generate rotating solutions in other than Einstein gravity.

D. Hansen, N. Yunes, Applicability of the Newmann-Janis algorithm to black hole solutions of modified gravity theories, Phys. Rev. D88 (2013) 10, 104020 [Arxiv:1308.6631].

The proof:

- **Idea:** To eliminate all theories, it is enough to consider asymptotic solutions at large r:

$$f = 1 - \frac{2M}{r} + \sum_{i=2} \frac{a_i}{r^i}\,, \quad h = -\frac{2M}{r} + \sum_{i=2} \frac{b_i}{r^i}\,.$$

- Solving order by order constraints the couplings, to yield only trivial topological terms. $\blacksquare$

Interestingly: no longer true in 5d

Theorem: In D=5, there exists a 5-parametric generalized quasi-topological theory that admits the Einstein-like Lense-Thirring solutions. (Paper 2)

c) Lense-Thirring spacetimes with matter

- **Maxwell-AdS case**: the simple form **remains valid**

$$h = f - 1 \qquad f = 1 - \frac{2M}{r} + \frac{q^2}{r^2} + \frac{r^2}{\ell^2}$$

$$A = -\frac{q}{r}\left(dt - a\sin^2\theta d\phi\right)$$

(Kerr-Newmann solution can be generated by Newman-Janis algorithm)

- **Kerr-Sen case**: the simple form **is not sufficient**

$$\mathcal{L} = \frac{e^{\Phi}}{16\pi}\left(R + g^{ab}\partial_a\Phi\partial_b\Phi - F_{ab}F^{ab} - \frac{1}{12}H_{abc}H^{abc}\right)$$

$$f = 1 - \frac{2(M-b)}{r}, \quad N = \left(1 + \frac{2b}{r}\right)^{-2}, \quad h = N\left(f - 1 - \frac{2b}{r}\right)$$

Non-linear electrodynamics

$$\mathcal{L} = \mathcal{L}(\mathcal{S}, \mathcal{P})$$

$$\mathcal{S} = \frac{1}{2} F_{\mu\nu} F^{\mu\nu}\,, \quad \mathcal{P} = \frac{1}{2} F_{\mu\nu} (*F)^{\mu\nu}$$

- **NLE** (generalized Maxwell) **equations**

$$d * E = 0\,, \quad dF = 0$$

$$E_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial F^{\mu\nu}} = 2\Big(\mathcal{L}_{\mathcal{S}} F_{\mu\nu} + \mathcal{L}_{\mathcal{P}} * F_{\mu\nu}\Big)$$

- **Einstein equations** $G_{\mu\nu} - 8\pi T_{\mu\nu} = 0$

$$T^{\mu\nu} = -\frac{1}{4\pi}\Big(2F^{\mu\sigma} F^{\nu}{}_{\sigma} \mathcal{L}_{\mathcal{S}} + \mathcal{P}\mathcal{L}_{\mathcal{P}} g^{\mu\nu} - \mathcal{L} g^{\mu\nu}\Big)$$

Non-linear electrodynamics

- **Static solutions** characterized by single metric function f

$$N = 1$$

T. Jacobson, *When is g_tt g_rr=-1?* CQG24 (2007) 5717 [0707.3222].

- **Rotating solutions** not known yet!

1) **Obviously wrong solutions** (satisfy only half of equations)

2) **Newmann-Janis** generated metrics

3) **Other solutions** that are not AF black holes (e.g. Taub-NUT)

Non-linear electrodynamics

- In paper 3, we have shown how to construct slowly rotating solutions in simplified NLE

$$\mathcal{L} = \mathcal{L}(\mathcal{S})$$

- For example, **magnetically charged** solutions:

$$ds^2 = -fdt^2 + \frac{dr^2}{f} + 2ah\sin^2\theta dt d\phi + r^2(\sin^2\theta d\phi^2 + d\theta^2)\,,$$

$$A = Q_m \cos\theta\Big(d\varphi - \frac{a\omega}{r^2}dt\Big)$$

Theorem: The above ansatz with h=(f-1) is inconsistent with any NLE but the **Maxwell theory**. In particular, all solutions generated by Newmann-Janis trick are incorrect at every order in a. (Paper 3)

d) Higher-dimensional Lense-Thirring spacetimes

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + \sum_{i=1}^{m} \mu_i^2 a_i h_i dt d\phi_i$$
$$+r^2(\sum_{i=1}^{m} d\mu_i^2 + \mu_i^2 d\phi_i^2) + r^2 \epsilon d\nu^2 \,,$$

- **Here** we have allowed for

$$m = \lfloor \frac{d-1}{2} \rfloor$$ rotation parameters

- **Polar coordinates** obey the following constraint:

$$\sum_{\mu=1}^{m} \mu_i^2 + \epsilon \nu^2 = 1$$

Einstein-like form

$$N = 1\,, \quad h_i = f - 1$$

- **Einstein-Maxwell-AdS** case:

$$\mathcal{L}_M = \frac{1}{16\pi}\Big(R - F_{ab}F^{ab} + \frac{(d-1)(d-2)}{\ell^2}\Big)$$

  - Solution given by

$$h_i = f - 1 = -\frac{m}{r^{d-3}} + \frac{q^2}{r^{2(d-3)}} + \frac{r^2}{\ell^2}$$

$$A = -\sqrt{\frac{d-2}{2(d-3)}}\frac{q}{r^{d-3}}\Big[dt - \sum_{i=1}^{m} a_i \mu_i^2 d\phi_i\Big]$$

  - Newmann-Janis works at O(a) order but not higher

Einstein-like form

$$N = 1\,, \quad h_i = f - 1$$

- **Multiply-spinning Lovelock solutions**:

$$\mathcal{L}_{\rm L} = \frac{1}{16\pi}\left(\frac{(d-1)(d-2)}{\ell^2} + R + \sum_{n=2}^{\lfloor (d-1)/2 \rfloor} \lambda_n \frac{(d+1-2n)!}{(d-1)!}(-1)^n \ell^{2n-2}\mathcal{X}_{2n}\right)$$

$$\mathcal{X}_{2n} = \frac{1}{2^n}\delta^{\mu_1 \dots \mu_{2n}}_{\nu_1 \dots \nu_{2n}} R^{\nu_1 \nu_2}_{\mu_1 \mu_2} \cdots R^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}}$$

- Solution determined by the Wheeler polynomial

$$\frac{16\pi M \ell^2}{(d-2)\Omega_{d-2} r^{d-1}} = 1 - \psi + \sum_{n=2}^{\lfloor (d-1)/2 \rfloor} \lambda_n \psi^n\,, \quad \psi \equiv \frac{\ell^2 (f-1)}{r^2}$$

Einstein-like form

$$N = 1\,, \quad h_i = f - 1$$

- **Does not work for D=5 minimal supergravity:**

$$\mathcal{L} = \frac{1}{16\pi}\Big(*(R - 2\Lambda) - \frac{1}{2}F \wedge *F + \frac{1}{3\sqrt{3}}F \wedge F \wedge A\Big)$$

- Chern-Simons charge **mixes** rotation parameters

$$f = 1 - \frac{2m}{r^2} + \frac{q^2}{r^4} + \frac{r^2}{\ell^2}$$

$$h_a = f - 1 + \frac{bq}{ar^2}\,, \quad h_b = f - 1 + \frac{aq}{br^2}$$

III) Upgraded LT: PG form and hidden symmetries

Let us complete the square

- instead

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + 2ap\sin^2\theta dt d\phi + r^2(\sin^2\theta d\phi^2 + d\theta^2)\,,$$

- **Consider** the following **"exact spacetime"**:

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2\sin^2\theta\left(d\phi + \frac{ap}{r^2}dt\right)^2 + r^2 d\theta^2$$

- Cures the O(a^2) divergence of invariants on the "horizon"
- Admits the **Painleive-Gullstrand (PG) form**
- Admits the **exact irreducible Killing tensor**

$$K = \frac{1}{\sin^2\theta}(\partial_\phi)^2 + (\partial_\theta)^2 \quad \left(K = L_x^2 + L_y^2 + L_z^2 \right)$$

J. Baines, T. Berry, A.Simpson, M. Visser, Arxiv:2110.01814.

Painleve-Gullstrand (PG) form

$$ds^2 = - \quad f dt^2 + \frac{dr^2}{f} + r^2 \sin^2\theta \Big(d\phi + \frac{ap}{r^2} dt \Big)^2 + r^2 d\theta^2$$

- Radially infalling observer (from rest at "infinity")

$$u = dt + \frac{\sqrt{1-f}}{f} dr \quad \Rightarrow \quad dt = dT - \frac{\sqrt{1-f}}{f} dr$$

$$\begin{aligned} ds^2 = & -dT^2 + (dr + \sqrt{1-f} dT)^2 + r^2 d\theta^2 \\ & + r^2 \sin^2\theta \Big(d\phi + \frac{a(f-1)}{r^2} dT - \frac{a(f-1)\sqrt{1-f}}{r^2 f} dr \Big)^2 \end{aligned}$$

- Finally set $d\phi = d\Phi + \dfrac{a(f-1)\sqrt{1-f}}{r^2 f} dr$

$$\begin{aligned} ds^2 = & -dT^2 + (dr + \sqrt{1-f} dT)^2 + r^2 d\theta^2 \\ & + r^2 \sin^2\theta \Big(d\Phi + \frac{a(f-1)}{r^2} dT \Big)^2 , \end{aligned}$$

K. Martel, E. Poisson, gr-qc/0001069.

Killing tensors

=totally symmetric tensors obeying

$$\nabla^{(a_1} K^{a_2 a_3 \ldots a_{p+1})} = 0.$$

Generate ***constants of geodesic motion*** of degree p

M. Walker and R. Penrose, Comm. Math. Phys. 18 , 265 (1970).

$$\mathcal{K}_p = K^{a_1 \ldots a_p} \, p_{a_1} \cdots p_{a_p}$$

Poisson commute with the Hamiltonian generating geodesic flow

$$\mathcal{H} = \frac{1}{2} g^{ab} p_a p_b$$

Reducibility

$$K_{(1)}^{(a} K_{(2)}^{bc)}, \quad \text{or} \quad K_{(3)}^{(a} K_{(4)}^{b} K_{(5)}^{c)},$$

Algebra of Killing tensors

Killing tensors form an algebra with respect to (symmetric) ***Schouten-Nijenhuis brackets***:

$$[K_p, K_q]^{a_1 a_2 \ldots a_{p+q-1}}$$

$$\{\mathcal{K}_p, \mathcal{K}_q\} = \frac{\partial \mathcal{K}_p}{\partial q^i}\frac{\partial \mathcal{K}_q}{\partial p_i} - \frac{\partial \mathcal{K}_q}{\partial q^i}\frac{\partial \mathcal{K}_p}{\partial p_i}$$
$$\equiv [K_p, K_q]_{\mathrm{SN}}^{a_1 a_2 \ldots a_{p+q-1}} p_{a_1} p_{a_2} \cdots p_{a_{p+q-1}} .$$

For example:

$$\left[K_{(i)}, K_{(j)}\right]_{\mathrm{SN}}^{abc} \equiv K_{(i)}^{e(a} \nabla_e K_{(j)}^{bc)} - K_{(j)}^{e(a} \nabla_e K_{(i)}^{bc)}$$
$$\left[\partial_{\psi_j}, K_{(i)}\right]_{\mathrm{SN}}^{ab} \equiv \mathcal{L}_{\partial_{\psi_j}} K_{(i)}^{ab}$$

Examples of spacetimes with rank-2 KTs

1. Kerr geometry (in all dimensions)

   P. Krtouš, D. Kubizňák, D. N. Page, and V. P. Frolov, Killing-Yano Tensors, Rank-2 Killing Tensors, and Conserved Quantities in Higher Dimensions, JHEP 0702 (2007) 004

2. Taub-NUT space: generalization of Runge-Lenz vector

   G. W. Gibbons and P. J. Ruback, "The Hidden Symmetries Of Taub-NUT And Monopole Scattering," Phys. Lett. B 188 (1987) 226.

3. Various SUGRA black holes

   D.D. Chow, Symmetries of supergravity black holes, Class. Quant. Grav. 27, 205009 (2010) , arXiv:0811:1264.

Not known spacetimes with irreducible higher-rank Killing-Stackel tensors!

See, however: G. Gibbons, T. Houri, DK, C. Warnick, *Some spacetimes with higher-rank Killing tensors*, PLB700 (2011), 68.

Upgraded LT spacetimes

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2 \sum_{i=1}^{m} \mu_i^2 \Big(d\phi_i + \frac{a_i p_i}{r^2} dt\Big)^2 + r^2 (\sum_{i=1}^{m} d\mu_i^2) + \epsilon r^2 d\nu^2$$

- Horizon

$$f(r_+) = 0 \qquad \xi = \partial_t + \sum_{i=1}^{m} \Omega_i \partial_{\phi_i}\,, \quad \Omega_i = -\frac{a_i p_i}{r^2}\Big|_{r=r_+}$$

(ergosphere)

- PG form

$$ds^2 = -N dT^2 + \Big(dr + \sqrt{N(1-f)} dT\Big)^2 + r^2 \sum_{i=1}^{m} \mu_i^2 \Big(d\Phi_i + \frac{a_i p_i}{r^2} dT\Big)^2$$
$$+ r^2 (\sum_{i=1}^{m} d\mu_i^2) + \epsilon r^2 d\nu^2\,,$$

(horizon is manifestly regular)

- Explicit symmetries ∂_t ∂_{ϕ_i} (m+1)… can be enhanced

Towers of hidden symmetries

- Consider a set $S = \{1, .., m\}$

$I \in P(S)$ …. a member of the power set of S

- Define $2b^{(I)} \equiv r^2(dt + \sum_{i \in I} a_i \mu_i^2 d\phi_i)\,, \quad h^{(I)} \equiv db^{(I)}$

(relevant order expansion of PKY of Kerr-AdS)

- Construct

$$f^{(I)} \equiv \frac{1}{(|I|+1)!} * (\underbrace{h^{(I)} \wedge \cdots \wedge h^{(I)}}_{|I|+1 \text{ times}})$$

$$K^{(I)}_{\mu\nu} = (\prod_{i \in S} a_i)^{-2} (f^{(I)} \cdot f^{(I)})_{\mu\nu}$$

…rank-2 KTs

Towers of hidden symmetries

- Explicitly

$$K^{(I)} = \sum_{i \notin I}^{m-1+\epsilon} \left[\left(1 - \mu_i^2 - \sum_{j \in I} \mu_j^2\right)(\partial_{\mu_i})^2 - 2 \sum_{j \notin I \cup \{i\}} \mu_i \mu_j \, \partial_{\mu_i} \partial_{\mu_j} \right] + \sum_{i \notin I}^{m} \left[\frac{1 - \sum_{j \in I} \mu_j^2}{\mu_i^2} (\partial_{\phi_i})^2 \right]. \qquad (30)$$

- Of these

$$k = \sum_{i=0}^{m-2+\epsilon} \binom{m}{i} - \sum_{i=0}^{m-3} \binom{m}{i} = \frac{1}{2} m(m - 1 + 2\epsilon)$$

… are **irreducible**

- Note that this **tower grows quadratically** with number of dimensions (compared to linear for Kerr-AdS)
- For high enough d, provides an example of a spacetime with **more hidden** than explicit symmetries

Upgraded LT spacetimes

$$ds^2 = -Nf dt^2 + \frac{dr^2}{f} + r^2 \sum_{i=1}^{m} \mu_i^2 \Big(d\phi_i + \frac{a_i p_i}{r^2} dt\Big)^2 + r^2 (\sum_{i=1}^{m} d\mu_i^2) + \epsilon r^2 d\nu^2$$

- Horizon

$$f(r_+) = 0 \qquad \xi = \partial_t + \sum_{i=1}^{m} \Omega_i \partial_{\phi_i}\,, \quad \Omega_i = -\frac{a_i p_i}{r^2}\Big|_{r=r_+}$$

(ergosphere)

- PG form

$$ds^2 = -N dT^2 + \Big(dr + \sqrt{N(1-f)}dT\Big)^2 + r^2 \sum_{i=1}^{m} \mu_i^2 \Big(d\Phi_i + \frac{a_i p_i}{r^2} dT\Big)^2$$
$$+ r^2 (\sum_{i=1}^{m} d\mu_i^2) + \epsilon r^2 d\nu^2\,,$$

(horizon is manifestly regular)

Towers of hidden symmetries

- Explicitly

$$K^{(I)} = \sum_{i \notin I}^{m-1+\epsilon} \left[\left(1 - \mu_i^2 - \sum_{j \in I} \mu_j^2\right) (\partial_{\mu_i})^2 - 2 \sum_{j \notin I \cup \{i\}} \mu_i \mu_j \, \partial_{\mu_i} \partial_{\mu_j} \right] + \sum_{i \notin I}^{m} \left[\frac{1 - \sum_{j \in I} \mu_j^2}{\mu_i^2} (\partial_{\phi_i})^2 \right]. \quad (30)$$

- Of these

$$k = \sum_{i=0}^{m-2+\epsilon} \binom{m}{i} - \sum_{i=0}^{m-3} \binom{m}{i} = \frac{1}{2} m(m - 1 + 2\epsilon)$$

… are **irreducible**

Towers of hidden symmetries

- Even more remarkably, while

$$[K^{(I_1)}, K^{(I_2)}]_{\mathrm{SN}} = 0 \quad \text{if } I_1 \cap I_2 = I_1$$

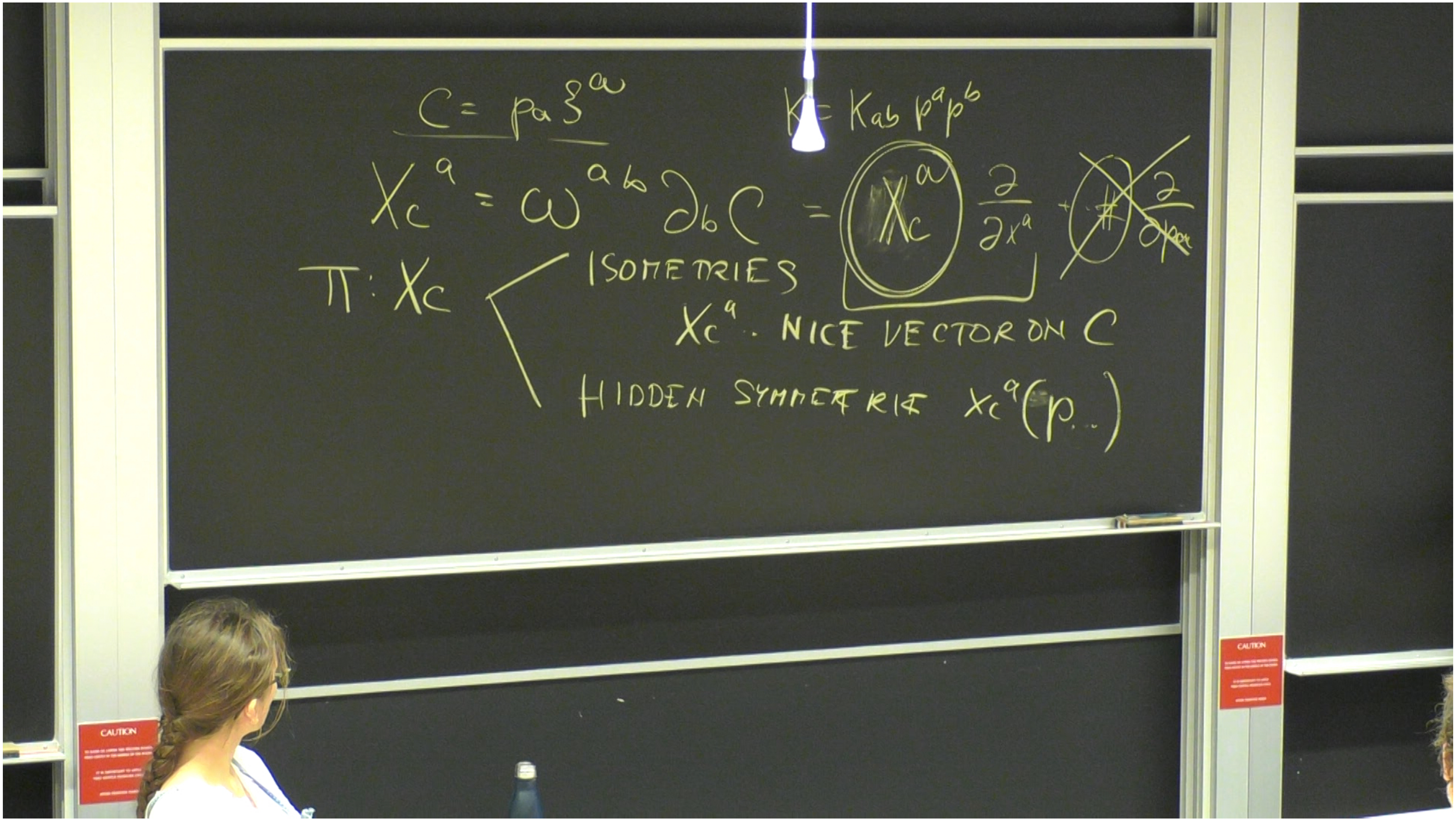
Otherwise a new (potentially irreducible) **higher-rank Killing tensor** is generated

- For example, in 6d: $[K^{(\emptyset)}, K^{(1)}]_{\mathrm{SN}} = 0 = [K^{(\emptyset)}, K^{(2)}]_{\mathrm{SN}}$

$M = [K^{(1)}, K^{(2)}]_{\mathrm{SN}}$ … irreducible rank-3 KT

- Further brackets with K1 and K2 generate rank 4 KT's, and so on …. does the tower **terminate** at some point?

- Example of a **physically motivated** spacetime with **higher rank Killing tensors** (is energy-momentum unphysical?)


C = pa ξ^a
K = Kab p^a p^b
Xc^a = ω^ab ∂b C = Xc^a ∂/∂x^a
π : Xc
ISOMETRIES
Xc^a .. NICE VECTOR ON C
HIDDEN SYMMETRIES Xc^a(p...)
CAUTION
CAUTION

Summary

1) **Lense-Thirring metric** describes an approximate **slowly rotating spacetime** outside a rotating body/BH.

$$ds^2 = -f dt^2 + \frac{dr^2}{f} + 2a(f-1)\sin^2\theta dt d\phi + r^2(\sin^2\theta d\phi^2 + d\theta^2)\,, \quad f = 1 - \frac{2M}{r}$$

It captures interesting effects such as **frame dragging**!

This has been measured by **gravity probe B.**

2) The above **standard (Einstein-like) form** is intrinsically linked to **Newmann-Janis algorithm** (truncated at linear order in a)

Summary

3) We have considered generalizations to **higher-curvature gravities**, spacetimes with **matter** (in particular NLE), and **higher dimensions**.

Theorem: In D=4, Einstein gravity is the only non-trivial gravitational theory up to powers quartic in curvature, whose vacuum Lense-Thirring solutions are of the above form (characterized by static f).

⇨ Newmann-Janis cannot be used beyond Einstein gravity.

Theorem: The above ansatz is inconsistent with any NLE but the **Maxwell theory**. (Excludes “all known” rotating solutions in these theories.)

Summary

4) **Interesting** (new) **examples** include:
   - Charged-AdS slowly rotating black holes
   - Multiply-spinning Lovelock black holes
   - Slowly rotating NLE solutions

5) **Magic square** LT version**:**
   - Admits PG form (manifestly regular on horizon)
   - Admits remarkable tower of rank-2 and higher-rank Killing tensors – deserves further study
   - Price to pay: The corresponding energy momentum may violate energy conditions.

6) **Full solutions?** In less than 50 years?