Title: Trace formulas from Supersymmetric localizations

Speakers: Changha Choi

Series: Quantum Fields and Strings

Date: September 13, 2022 - 2:00 PM

URL: https://pirsa.org/22090076

Abstract: We present a new class of supersymmetric localization principles in the context of supersymmetric quantum mechanics. Contrary to the standard localization, this new principle deals with non-supersymmetric observables. We apply this to provide a path integral understanding of trace formulas in mathematics. Our examples will contain both integrable and chaotic quantum systems, namely trace formulas on compact Lie groups by Eskin, and generic locally symmetric space by Selberg.

Zoom Link: https://pitp.zoom.us/j/99332291865?pwd=c0kyUUlWN1AxeXpvNmRkNHRSRXNXdz09

Trace formulas from Supersymmetric localization

Changha Choi

In collaboration with Leon Takhtajan
arXiv:2112.07942, 22xx.xxxxx

PI, 09/13/2022

Table of Contents

Supersymmetry

Supersymmetry, which relates bosons and fermions, is one of the richest global symmetry in physics.

Regardless of its phenomenological role, supersymmetry stands as an essential concept in theoretical physics because of its own internal justifications. Most prominently,

- It deepens our understanding of strongly coupled quantum systems from the exact computations of a large class of non-trivial observables in SQFTs.
- It also has a fertile interaction with pure mathematics, which provides a unique and deep physical perspective even at the quantum mechanical level.

Supersymmetric localization in a nutshell

Supersymmetric localization is an outcome of looking SUSY path integral through the lens of (infinite-dimensional) equivariant cohomology.

Given a supersymmetry δ_Q which admits such interpretation, then the supersymmetric path integral defined by $\delta_Q S = \delta_Q[D\mu] = 0$ are invariant under the following deformation

$$\int [D\mu] e^{-S} = \int [D\mu] e^{-S - s\delta_Q V} \tag{1}$$

Whenever $\delta_Q^2 V = 0$ (and a globally defined δ_Q^2 and V).

Witten index

On the other hand, the most basic and important concept in such mathematical structure has been the Witten index [Witten 82]

$$I = \mathrm{Str}_{\mathcal{H}}[e^{-\beta \hat{H}}] = \mathrm{Tr}_{\mathcal{H}}[(-1)^F e^{-\beta \hat{H}}]$$

Which exhibits an invariance under β from its only sensitivity to ground states. Such deformation invariance relates supersymmetry to topology, namely from the relation between the (graded) cohomology of supercharge $\hat{Q}$ and the Hilbert space of ground states.

$$\mathrm{Ker}(\hat{Q})/\mathrm{Im}(\hat{Q})|_{\mathcal{H}} \simeq \mathcal{H}_{g.s.}$$

Q as a Dirac operator

From now on, we mainly consider minimal SQM with a single real supercharge ($N = 1/2$). Remarkably, cohomological property of $\hat{Q}$ together with the supersymmetry algebra ($\{(-1)^F, \hat{Q}\} = 0$, $\{\hat{Q}, \hat{Q}\} = 2H$) allow an interpretation of $\hat{Q}$ as a Dirac type operator [Witten 82], and hence Witten index becomes an analytic index of such operator.

SUSY and Atiyah-Singer index theorem (1)

One monumental outcome from these insights is the physical proof of the Atiyah-Singer index theorem [Álvarez-Gaumé 83], which considered following $N = 1/2$ sigma model on spin manifold M

$$\mathcal{L} = \frac{1}{2}\langle \dot{x}, \dot{x} \rangle + \frac{i}{2}\langle \psi, \nabla_{\dot{x}} \psi \rangle$$

Where classical supersymmetry is realized as

$$\delta_Q x^\mu = i\psi^\mu, \ \delta_Q \psi^\mu = -\dot{x}^\mu$$

And the supercharge operator $\hat{Q}$ acting on the Hilbert space is realized as a standard Dirac operator D associated with Levi-Civita connection acting on $\mathcal{S}$.

SUSY and Atiyah-Singer index theorem (2)

The topological side of the Atiyah-Singer index theorem comes from the path integral side. Path integral representation of the supertrace requires Ramond spin structure on thermal circle,

$$I = \int [Dx][D\psi] e^{-\int_0^\beta d\tau \frac{1}{2}\langle \dot{x}, \dot{x}\rangle + \frac{1}{2}\langle \psi, \nabla_{\dot{x}}\psi\rangle}$$

Changing β is equivalent to the localization deformation and in the high-temperature limit, path integral localizes to constant modes and reduces to a finite dimensional integral which establishes

$$I = \mathrm{ind}(D) = \int_M \hat{A}(M)$$

Trace Formulas

It is an interesting historical fact that in mathematics, various non-trivial results dealing with full trace of elliptic kernels were established [Jacobi-Poisson 1828, Selberg 1956, Eskin 1964] independently of the advent of Dirac operator and index theorems [Atiyah, Singer 1963].

Surprisingly, while both are still active and important topics in modern mathematics, the clear physical understanding of the former has not been accessible.

The most basic type of trace formula is about the Laplace operator on Riemannian constant curvature space M

$$\mathrm{Tr}_{L^2(M)}[e^{-\beta\Delta}]$$

Trace formulas and QM

The existence of such identities are physically striking since these are exact statements about the canonical partition function of bosonic NLSM on M ($\mathcal{L} = \frac{1}{2}\langle \dot{x}, \dot{x} \rangle_g$)

$$Z_M(\beta) = \mathrm{Tr}_{L^2(M)}[e^{-\beta\Delta/2}]$$

Representative Trace Formulas

The more amusing thing is that it deals with both integrable and chaotic systems. Specifically, we consider

- $M = G$ [Jacobi-Poisson, Eskin]
- $M = \Gamma \backslash H^2$ [Selberg]

Trace Formula on $G = S^1$: Jacobi-Poisson

The simplest and famous example is the trace formula on S^1, which is nothing but the Jacobi-Poisson formula

$$\mathrm{Tr}_{L^2(S^1)}[e^{-\beta\Delta/2}] = \sum_{n\in\mathbb{Z}} e^{-2\pi^2\beta n^2} = \frac{1}{\sqrt{2\pi\beta}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{n^2}{2\beta}\right)$$

Trace formula on G: Eskin

$$\mathrm{Tr}_{L^2(G)} e^{-\frac{1}{2}\beta\Delta + i\langle h,\hat{r}\rangle} = \sum_{\lambda\in \mathrm{Irrep}\, G} d_\lambda\, \chi_\lambda(h) e^{-\frac{1}{2}\beta C_2(\lambda)}.$$

$$= \frac{V_G e^{\frac{1}{2}\beta\langle\rho,\rho\rangle}}{(2\pi\beta)^{n/2}} \sum_{\gamma\in\Gamma} \prod_{\alpha\in R_+} \frac{\frac{1}{2}\langle\alpha, h+\gamma\rangle}{\sin\frac{1}{2}\langle\alpha, h+\gamma\rangle} e^{-\frac{1}{2\beta}\langle h+\gamma, h+\gamma\rangle}$$

where $h \in \mathfrak{t}$ is a regular Cartan element and $\Gamma = \{\gamma \in \mathfrak{t} : e^\gamma = 1\}$ is the characteristic lattice.

Trace formula on $\Gamma\backslash H^2$: Selberg

$$
\begin{aligned}
&\mathrm{Tr}_{L^2(\Gamma\backslash H^2)}[e^{-\beta\Delta/2}] \\
=&\frac{\mathrm{Area}(\Gamma\backslash H^2)}{(2\pi\beta)^{3/2}}\int_0^\beta \frac{s}{\sinh\frac{s}{2}} e^{-\frac{s^2}{2\beta}-\frac{\beta}{8}} ds \\
&+\sum_{[\gamma]} \frac{l(\gamma_0)}{2\sinh\frac{l(\gamma)}{2}} \frac{e^{-\frac{l(\gamma)^2}{2\beta}-\frac{\beta}{8}}}{(2\pi\beta)^{1/2}} \\
&+\sum_{[\alpha]} \frac{1}{(2\pi\beta)^{1/2} m(\alpha)\sin\theta(\alpha)} \int_{-\infty}^{\infty} e^{-\beta u^2/2} \frac{\cosh(\pi u/2-\theta(\alpha)u)}{\cosh(\pi u/2)} du
\end{aligned}
$$

Physical interpretation of trace formulas

Physical interpretation of the right-hand side of trace formulas is even more interesting since they resembles the stationary phase approximation of path integral

$$Z_M(\beta) = \int_{x(0)=x(\beta)} [Dx] e^{-\int_0^\beta \frac{1}{2}\langle \dot{x}, \dot{x} \rangle} \simeq \sum_{\text{closed geodesics}} \frac{1}{\sqrt{\det \nabla_t^2}} e^{-S_{cl}}$$

This is obviously expected in S^1. While for G [Schulman 68, Dowker 70, Marinov , Terentyev 79] and $\Gamma \backslash H^2$ [Gutzwiller 80], it's remarkable that there are such similar suggestive interpretations.

Trace Formula on $G = S^1$: Jacobi-Poisson

The simplest and famous example is the trace formula on S^1, which is nothing but the Jacobi-Poisson formula

$$\mathrm{Tr}_{L^2(S^1)}[e^{-\beta\Delta/2}] = \sum_{n\in\mathbb{Z}} e^{-2\pi^2\beta n^2} = \frac{1}{\sqrt{2\pi\beta}} \sum_{n\in\mathbb{Z}} \exp\left(-\frac{n^2}{2\beta}\right)$$

Physical motivation for trace formulas

Can there be a unifying framework that explains trace formulas?

Intriguingly, the phenomena that path integrals effectively get contributions from significantly smaller subset of paths what usually happens in supersymmetric localization.

Can we really pursue this analogy further?

Trace Formulas from SUSY localization: Objections

There are tons of reasonable objections from the standard wisdom

- To our best knowledge, supersymmetric path integral has been only localized along the solutions of 1st order PDE.
- The trace formulas are purely bosonic but we need a supersymmetry (and hence fermionic a d.o.f.) to utilize a supersymmetric localization.
- Hilbert space in the SUSY system and original bosonic one are different and also the quantum Hamiltonian is changed.
- Supersymmetric localization can only handle supertrace but not the trace.
- Unlike the index theorems, trace formulas deal with full-spectrum.

Table of Contents

Supersymmetric observables

Typical observables in SQFT which admit SUSY localizations are 'supersymmetric' ones.

$$\langle \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots\rangle = \int [D\mu]\mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots e^{-S},$$

such that $\langle [Q, \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\ldots]\rangle = 0$

SUSY observables and standard localization

The proof is standard since

$$\partial_s \int [D\mu]\mathcal{O}(x_1,\ldots,x_n)e^{-S-s\delta_Q V},$$
$$= -\int [D\mu]V\delta_Q\mathcal{O}(x_1,\ldots x_n)e^{-S-s\delta V}$$

Now what if we consider non-supersymmetric $\mathcal{O}$?

SUSY observables and standard localization

The proof is standard since

$$\partial_s \int [D\mu]\mathcal{O}(x_1,\ldots,x_n)e^{-S-s\delta_Q V},$$
$$= -\int [D\mu]V\delta_Q\mathcal{O}(x_1,\ldots x_n)e^{-S-s\delta V}$$

Now what if we consider non-supersymmetric $\mathcal{O}$?

New supersymmetric localization principle (1)

Counter-intuitively, we claim that there is a new class of supersymmetric localization which applies to non-supersymmetric $\mathcal{O}$. We require that SQFT and observable to satisfy

(i) There are fermionic zero modes $\chi_1, \ldots, \chi_m$.

(iii) $\int \delta\mathcal{O} d\chi_1 \ldots d\chi_m = 0$

New supersymmetric localization principle (2)

Then, it can be shown that such Euclidean path integral admits a special type of invariant deformation of the form $S + s\delta V$ which satisfies following conditions

(A) V is invariant,

$$\delta^2 V = 0.$$

(B) $V, \delta V \perp \chi_\mu$, i.e.

$$\int V d\chi_\mu = \int \delta V d\chi_\mu = 0, \quad \mu = 1, \ldots, n.$$

Note that the condition (A) is standard, while condition (B) is a completely new requirement. It is rather constraining and naturally forces V to have a new structure.

Table of Contents

Trace formula of S^1 from SQM

We first show that the trace formula on S^1 (Jacobi-Poisson formula) is indeed a consequence of the new type of localization principle. We consider $N = 1/2$ quantum mechanical sigma model on S^1 on Euclidean time

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi}, \ x \simeq x + 1$$

Where SUSY is given by

$$\delta_Q x = \psi, \ \delta_Q \psi = -\dot{x}$$

Non-SUSY observable for S^1 trace formula

We claim that the following non-supersymmetric observable admits a new class of supersymmetric localizations.

$$I(\beta) = \mathrm{Tr}[(-1)^F \hat{\psi} e^{-\beta \hat{H}}] = \int_{\Lambda L S^1} [Dx][D\psi]\psi(0)e^{-S}$$

This is sensitive to the full spectrum and actually equal to the $Z_{S^1}(\beta) = \mathrm{Tr}_{\mathcal{H}_B}(e^{-\beta \Delta_{S^1}})$ since

$$I(\beta) = \mathrm{Tr}_{\mathcal{H}_B}(e^{-\beta \Delta_{S^1}})\mathrm{Str}_{\mathcal{H}_F}(\psi) = \mathrm{Tr}_{\mathcal{H}_B}(e^{-\beta \Delta_{S^1}})$$

Non-SUSY observable for S^1 trace formula

We claim that the following non-supersymmetric observable admits a new class of supersymmetric localizations.

$$I(\beta) = \mathrm{Tr}[(-1)^F \hat{\psi} e^{-\beta \hat{H}}] = \int_{\Lambda L S^1} [Dx][D\psi]\psi(0)e^{-S}$$

This is sensitive to the full spectrum and actually equal to the $Z_{S^1}(\beta) = \mathrm{Tr}_{\mathcal{H}_B}(e^{-\beta \Delta_{S^1}})$ since

$$I(\beta) = \mathrm{Tr}_{\mathcal{H}_B}(e^{-\beta \Delta_{S^1}})\mathrm{Str}_{\mathcal{H}_F}(\psi) = \mathrm{Tr}_{\mathcal{H}_B}(e^{-\beta \Delta_{S^1}})$$

Trace formula of S^1 from SQM

We first show that the trace formula on S^1 (Jacobi-Poisson formula) is indeed a consequence of the new type of localization principle. We consider $N = 1/2$ quantum mechanical sigma model on S^1 on Euclidean time

$$\mathcal{L} = \frac{1}{2}\dot{x}^2 + \frac{1}{2}\psi\dot{\psi}, \ x \simeq x + 1$$

Where SUSY is given by

$$\delta_Q x = \psi, \ \delta_Q \psi = -\dot{x}$$

New class of invariant deformations for trace formula (1)

Remarkably, we claim that above observable has an invariant 'higher-derivative' deformation. Namely, if we consider localizing one-form $V = -\ddot{x}\dot{\psi}$, than we found a following invariant deformation

$$S_E(\lambda) = S_E + \lambda\delta_Q V, \quad \delta_Q V = (\ddot{x})^2 + \dot{\psi}\ddot{\psi}$$

And hence the localization reduces the path integral onto closed geodesics $\ddot{x} = 0$, $x \in LS^1$!

Eskin's trace formula on G

As a more non-trivial application of the new localization principle, we now consider a trace formula when the manifold is a compact semisimple Lie group G [Eskin 64].

$$\mathrm{Tr}_{L^2(G)} e^{-\frac{1}{2}\beta\Delta + i\langle h, \hat{r}\rangle} = \sum_{\lambda \in \mathrm{Irrep}\, G} d_\lambda\, \chi_\lambda(h) e^{-\frac{1}{2}\beta C_2(\lambda)}.$$

$$= \frac{V_G e^{\frac{1}{2}\beta\langle\rho,\rho\rangle}}{(2\pi\beta)^{n/2}} \sum_{\gamma\in\Gamma} \prod_{\alpha\in R_+} \frac{\frac{1}{2}\langle\alpha, h+\gamma\rangle}{\sin\frac{1}{2}\langle\alpha, h+\gamma\rangle} e^{-\frac{1}{2\beta}\langle h+\gamma, h+\gamma\rangle}$$

Physical interpretation of Eskin's formula

Quantum mechanically, such formula describes a propagator of free bosonic particle on G, who's (naive!) path integral representation is given by

$$k_\beta(e^h) = \int_{g(0)=1, g(\beta)=e^h} [Dg] e^{-\int_0^\beta d\tau \frac{1}{2}\langle g^{-1}\dot{g}, g^{-1}\dot{g}\rangle}$$

And the Eskin formula exhibits a remarkable (almost) exactness of the semiclassical approximation [Schulman 68, Dowker 70, Marinov, Terentyev 79, Picken 89, Camporesi 90], schematically

$$k_\beta(e^h) = e^{-\frac{\beta R}{12}} \sum_{geodesics} \frac{1}{\sqrt{\det D^2 S / Dg^2}} e^{-S_{cl}} \tag{2}$$

Remarks on DeWitt term

However the system is not entirely semi-classical since the overall non-trivial β dependent factor $e^{-\frac{\beta R}{12}}$ is not captured by the semiclassical approximations. Mathematically, such term is related to short time asymptotics of heat kernel which is proportional to scalar curvature.

Physically, this term is known as a DeWitt term [DeWitt 57], which is a notorious quantum ambiguity ($O(\hbar^2)$) in the formulation of a generic quantum mechanical path integral on curved space.

Construction of SUSY model on G (1)

To facilitate the localization, we need to choose a supersymmetric system closely related to the bosonic trace on G.

One natural candidate would be the standard $N = 1/2$ SUSY sigma model on G

$$S = \int_0^\beta d\tau \frac{1}{2}\langle g^{-1}\dot{g}, g^{-1}\dot{g}\rangle + \frac{i}{2}\langle \psi, \nabla_{\dot{x}}\psi\rangle$$

But in this case, we are obviously changing the spectral side of the trace since the Hamiltonian becomes Dirac-Laplacian.

A natural question is therefore if there is some SUSY system such that which contains a full information about the purely bosonic case.

Construction of SUSY model on G (2)

It is striking that on G, one can realize such a goal with a minimal modification: namely, we add a G-invariant torsion on the Lagrangian by deforming the affine connection.

$$\nabla_X^{(\alpha)} Y = \nabla_X Y + \alpha[X, Y], \quad X, Y \in \mathfrak{g}$$

It is known that these connections are G invariant for any $\alpha \in \mathbb{R}$ [Cartan, Schouten 26] and we are going to choose $\alpha = -1/2$ which is called (-) parallelizing connection.

Remarkably, SUSY Lagrangian becomes effectively decoupled if we trivialize the fermions along the left-G invariant section of the frame bundle

$$S = \int d\tau \frac{1}{2}\langle g^{-1}\dot{g}, g^{-1}\dot{g}\rangle + \frac{i}{2}\langle \psi, \dot{\psi}\rangle$$

Construction of SUSY model on G (1)

To facilitate the localization, we need to choose a supersymmetric system closely related to the bosonic trace on G.

One natural candidate would be the standard $N = 1/2$ SUSY sigma model on G

$$S = \int_0^\beta d\tau \frac{1}{2}\langle g^{-1}\dot{g}, g^{-1}\dot{g}\rangle + \frac{i}{2}\langle \psi, \nabla_{\dot{x}}\psi\rangle$$

But in this case, we are obviously changing the spectral side of the trace since the Hamiltonian becomes Dirac-Laplacian.

A natural question is therefore if there is some SUSY system such that which contains a full information about the purely bosonic case.

Construction of SUSY model on G (3)

It is straightforward to canonically quantize such action. First, supercharge is unambiguously given by ($J = g^{-1}\dot{g}$)

$$\hat{Q} = \langle \hat{J}, \hat{\psi} \rangle + \frac{i}{3} \langle \hat{\psi}\hat{\psi}, \hat{\psi} \rangle$$

Therefore natural SUSY Hamiltonian uniquely governed by the SUSY algebra is [Braden 86]

$$\hat{H} = \frac{1}{2}\{\hat{Q}, \hat{Q}\} = \frac{1}{2}\Delta + \frac{R}{12}\hat{I}$$

Which remarkably captures the precise DeWitt term!

Mathematically, the supercharge $\hat{Q}$ is realized as the Kostant's Dirac operator on acting on $\mathbf{S}_G$ and the above SUSY algebra can be thought as a generalized version of the Lichnerowicz formula [Bismut 89, Kostant 99].

SUSY localization proof of Trace formula on G (1)

Another nice feature of the above action is that the existence of d_G number of fermionic zero modes $\chi^a = \frac{1}{\beta}\int_0^\beta \psi^a d\tau$.
We consider

$$\mathrm{Str}[\hat{\chi}^1 \dots \hat{\chi}^n e^{-\beta\hat{H}+i\langle h,\hat{r}\rangle}] = e^{-\frac{1}{12}\beta R}\mathrm{Tr}_{L^2(G)} e^{-\frac{1}{2}\beta\Delta+i\langle h,\hat{r}\rangle}.$$

On the other hand, the supertrace is given by the following path integral

$$\int_{\Pi TLG} [Dg][D\psi]\chi^1 \dots \chi^n e^{-S^h}$$

with

$$S^h = \frac{1}{2}\int_0^\beta (\langle J, J\rangle + \langle \psi, \dot{\psi}\rangle) d\tau + \frac{1}{\beta}\int_0^\beta \langle \mathrm{Ad}_{g^{-1}} h, J\rangle d\tau$$

SUSY localization proof of Trace formula on G (2)

Such action has a following supersymmetry which satisfies the condition (iii) ($J^h = J + \frac{1}{\beta}\mathrm{Ad}_{g^{-1}}h$)

$$\delta_Q g = g\psi, \quad \delta_Q \psi = -J^h - \psi\psi$$

And the following invariant deformation

$$V = -\frac{1}{2}\int_0^\beta \langle j^h, \dot{\psi}\rangle \, d\tau,$$

$$\delta_Q V = \frac{1}{2}\int_0^\beta (\langle j^h, j^h\rangle + \langle \dot{\psi}, (\partial_\tau + \mathrm{ad}_{J^h})\dot{\psi}\rangle) \, d\tau.$$

Which localizes the path integral onto the closed geodesics on G satisfying $j^h = 0$ (which are characterized by Γ) and produces the trace formula on G! $\square$

Table of Contents

Geometry of locally symmetric space

$X = \Gamma \backslash H^N$ is a simplest and most famous example of locally symmetric space whose spectrum is highly chaotic. Generically, it can be engineered as

$$X = \Gamma \backslash G / K \tag{3}$$

For example, $N = 2$ we have $G = SL(2, \mathbb{R})$ and $K = SO(2, \mathbb{R})$.

Trace formula on non-compact G

Our starting point is the analysis of [Krausz, Marinov 97] which showed that real time propgator on non-compact G admits a semi-classical interpretation similar to the compact case. This can be understood as a trace formula from the similar SUSY model on G with a starting point

$$\begin{aligned} & e^{-\frac{i\langle\rho,\rho\rangle T}{2}}\langle e^h|e^{-\frac{i\Delta T}{2}}|1_G\rangle \\ &= \mathrm{Tr}_{H_F}\left((-1)^F\hat{\chi}_1\ldots\hat{\chi}_{d_G}\langle 1_G|e^{-i\hat{H}t+i\langle h,\hat{r}\rangle}|1_G\rangle\right) \\ &= \int_{\Pi T\Omega G}\mathscr{D}g\mathscr{D}\psi\, e^{iS^h} \end{aligned} \tag{4}$$

SQM on $\Gamma\backslash G/K$

These consideration naturally provides a SQM on $\Gamma\backslash G/K$ as a SGLSM on $\Gamma\backslash G$.

$$\mathcal{L} = \frac{1}{2}\langle J_A, J_A\rangle + \frac{i}{2}\langle \psi, D_A\psi\rangle \tag{6}$$

Where A is a connection of principal K bundle.

Trace formula on $\Gamma\backslash G$

Remarkable thing of our SUSY model on G is that, it is invariant under the projection $\pi_\Gamma : G \to \Gamma\backslash G$. This means, we can effectively formulate real time trace formula on $L^2(\Gamma\backslash G)$ from SUSY as well.

$$\begin{aligned} & e^{-\frac{i\langle\rho,\rho\rangle T}{2}} \mathrm{Tr}_{L^2(\Gamma\backslash G)} \left(e^{-\frac{i\Delta T}{2}} \right) \\ & = \mathrm{Str}_H \left(\hat{\chi}_1 \cdots \hat{\chi}_{d_G} e^{-\frac{i\Delta T}{2}} \right) \\ & = \int_{\Pi TL(\Gamma\backslash G)} \mathscr{D}g \mathscr{D}\psi \, \chi_1 \cdots \chi_{d_G} e^{iS} \end{aligned} \tag{5}$$

Canonical quantization of $\Gamma\backslash G/K$ SQM

The model has r_K number of fermionic zero modes. Let's consider $X = \Gamma\backslash H^2$, then the Hilbert space from the canonical quantization is given by

$$\mathcal{H}_0 = L^2_{-1/2}(X) \oplus L^2_{1/2}(X) \oplus \mathcal{H}_{F,\mathfrak{k}} \tag{7}$$

So the model indeed describe a spinning particle on X.

Twisted SQM on $\Gamma\backslash G/K$

To have a Selberg trace formula, we need to find a way to extract only $L_0^2(X)$.
One natural idea is to insert a temporal Wilson line $\mathcal{W}_k = e^{k\int A}$ and quantize. Then, Hilbert space is twisted by

$$\mathcal{H}_k = L^2_{(k-1)/2}(X) \oplus L^2_{(k+1)/2}(X) \oplus \mathcal{H}_{F,\mathfrak{k}} \tag{8}$$

Which still, wouldn't give what we desire.

Towards the Selberg trace formula

Surprisingly, we claim that following non-supersymmetric observable made out of singular composite of Wilson lines gives the desired Lorentzian trace!

$$\mathrm{Tr}_{L_0^2(X)}(e^{iT\Delta/2}) = \langle \frac{1}{\mathcal{W}_1 - \mathcal{W}_{-1}} \psi_{\mathfrak{k}} \rangle \tag{9}$$

Moreover, this non-SUSY obervable admits invariant deformation, which eventually localizes to the (Lorentzian) Selberg trace formula!

Trace formula on $\Gamma\backslash H^2$: Selberg

$$
\begin{aligned}
&\mathrm{Tr}_{L^2(\Gamma\backslash H^2)}[e^{-\beta\Delta/2}] \\
=&\frac{\mathrm{Area}(\Gamma\backslash H^2)}{(2\pi\beta)^{3/2}}\int_0^\beta \frac{s}{\sinh\frac{s}{2}} e^{-\frac{s^2}{2\beta}-\frac{\beta}{8}} ds \\
&+\sum_{[\gamma]} \frac{l(\gamma_0)}{2\sinh\frac{l(\gamma)}{2}} \frac{e^{-\frac{l(\gamma)^2}{2\beta}-\frac{\beta}{8}}}{(2\pi\beta)^{1/2}} \\
&+\sum_{[\alpha]} \frac{1}{(2\pi\beta)^{1/2} m(\alpha)\sin\theta(\alpha)} \int_{-\infty}^{\infty} e^{-\beta u^2/2} \frac{\cosh(\pi u/2-\theta(\alpha)u)}{\cosh(\pi u/2)} du
\end{aligned}
$$

Towards trace formulas on general $\Gamma\backslash G/K$

In fact, the construction on $\Gamma\backslash H^2$ can be generalized to the arbitrary pair of (Γ, G, K). Note that explicit trace formula has only been established for rank-one case. Can physics give an access to higher-ranks?

Future directions

There are many interesting and challenging directions

- How far this higher-derivative localization would yield new insight in the path integrals? (Note that such deformation is quite universal.)
- Trace formulas for non-compact X.
- Trace formulas in QFT with $d \geq 2$.
- Can this shed a new light on the chaotic properties of quantum system?

Thanks for your attention!!