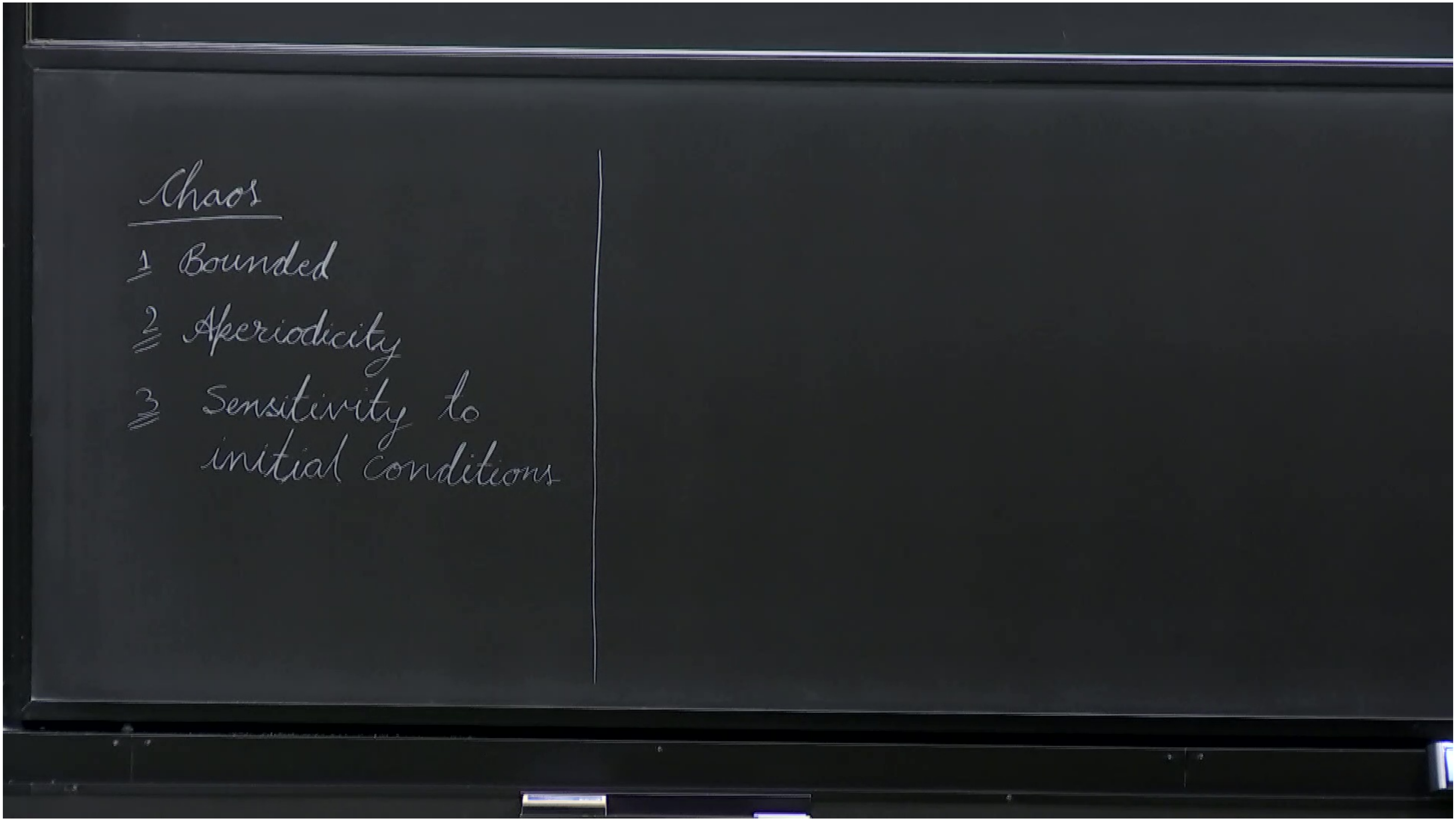
Title: Classical Physics - Lecture 220923

Speakers: Meenu Kumari

Collection: Classical Physics (2022/2023)

Date: September 23, 2022 - 9:00 AM

URL: https://pirsa.org/22090052


Chaos
1 Bounded
2 Aperiodicity
3 Sensitivity to
initial conditions

Chaos

1. Bounded
2. Aperiodicity
3. Sensitivity to initial conditions

1 + 2 − 3 — 2 uncoupled H.O with $\frac{\omega_1}{\omega_2}$ = irrational (quasi-periodic motion)

2 + 3 − 1 —

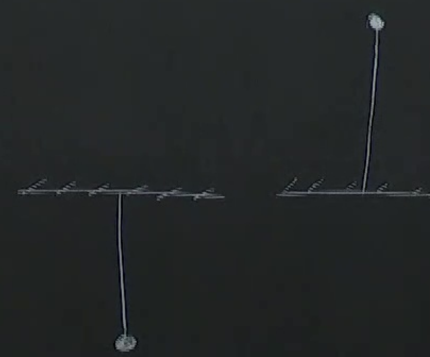
$$\dot{x} = 2x$$
$$\Rightarrow x_t = c\, e^{2t}$$
$$\dot{x} = 0$$
$$\Rightarrow x^* = 0$$

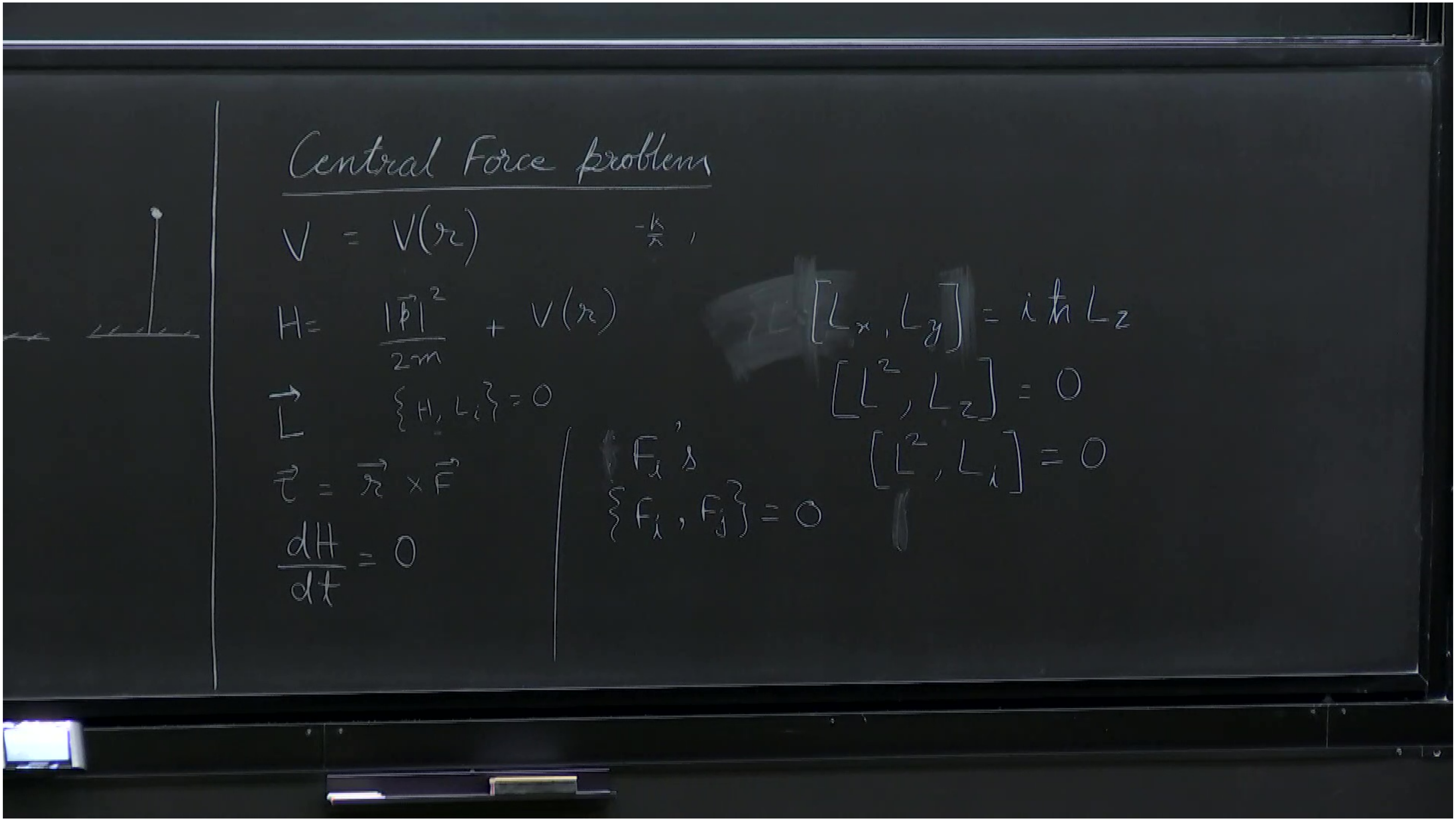
1 + 3 − 2

1 + 2 − 3 — 2 uncoupled H.O
with $\frac{\omega_1}{\omega_2}$: irrational
(quasi-periodic motion)

2 + 3 − 1 — $\dot{x} = 2x$
$\Rightarrow x_t = c\,e^{2t}$
$\dot{x} = 0$
$\Rightarrow x^* = 0$

1 + 3 − 2




Central Force problem
$V = V(r)$
$-\frac{k}{r}$,
$H = \frac{|\vec{p}|^2}{2m} + V(r)$
$\vec{L}$
$\{H, L_i\} = 0$
$\vec{\tau} = \vec{r} \times \vec{F}$
$\frac{dH}{dt} = 0$
F_i's
$\{F_i, F_j\} = 0$
$[L_x, L_y] = i\hbar L_z$
$[L^2, L_z] = 0$
$[L^2, L_i] = 0$

Phase space : $\{x, p_x, y, p_y, z, p_z\}$

$\vec{L} = \vec{r} \times \vec{p}$

$L_x = y p_z - z p_y$

$L_y = z p_x - x p_z$

$L_z = x p_y - y p_x$

$L^2 = L_x^2 + L_y^2 + L_z^2$

$\{H, L^2, L_x, L_y, L_z\}$

Let $A_x := \{(x, p_x, y, p_y, z, p_z)$

$y, p_y, z, p_z\}$

$H, L^2, L_x, L_y, L_z\}$

Let $A_x := \{(x, p_x, y, p_y, z, p_z)$ such that $L_x = L_{x0}\}$

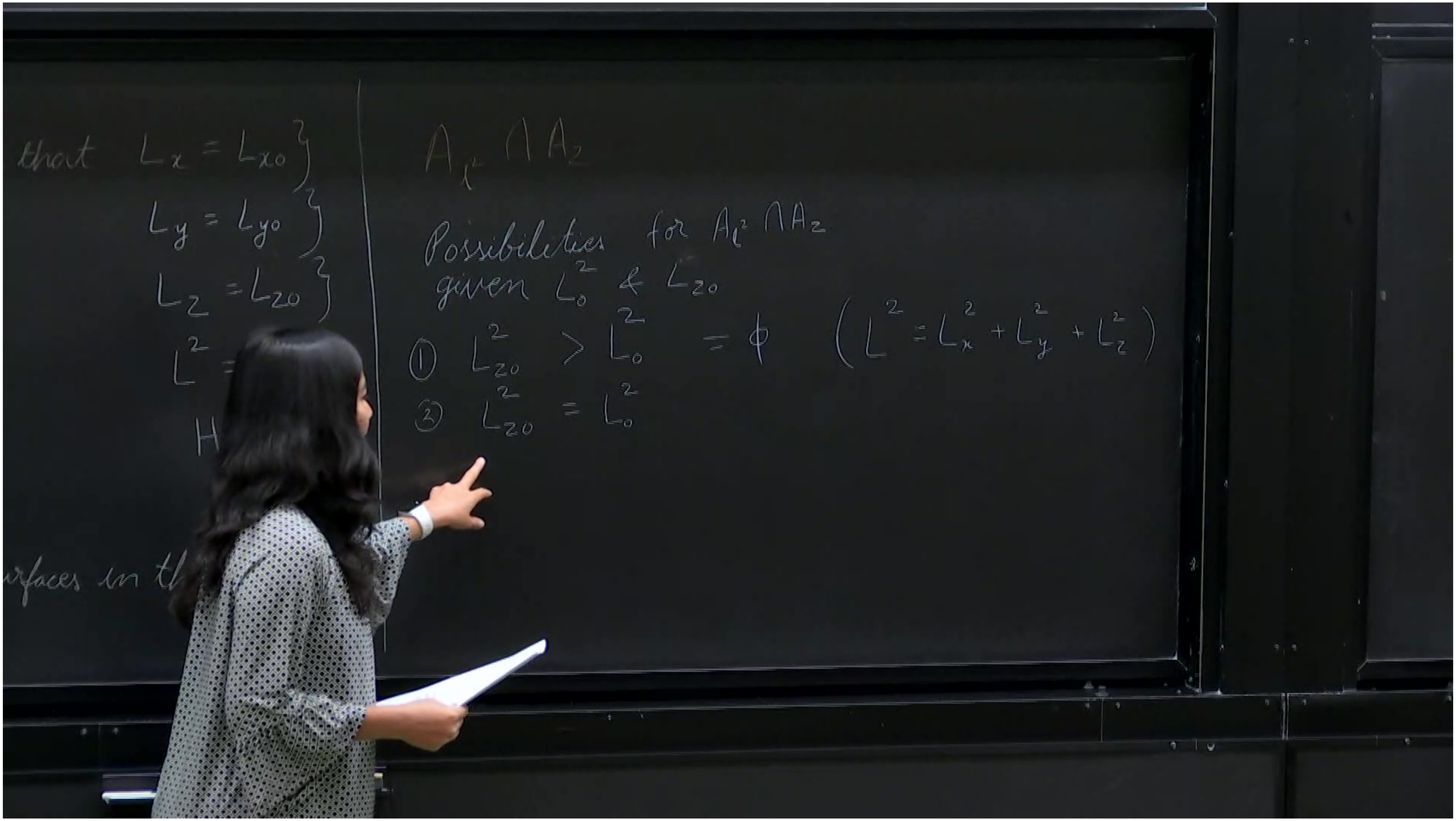
$A_y := \{ \quad L_y = L_{y0}\}$

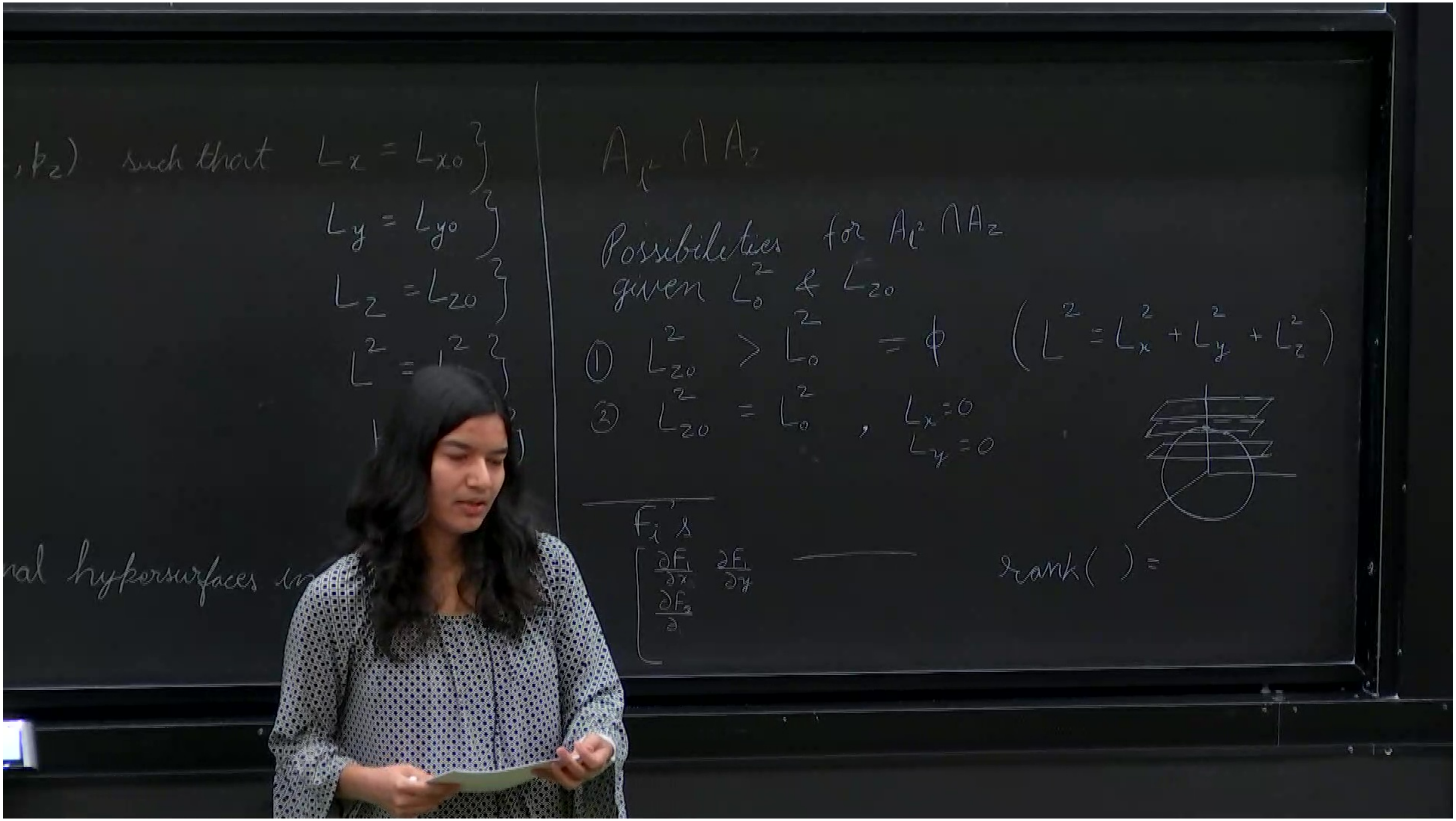
$A_z := \{ \quad L_z = L_{z0}\}$

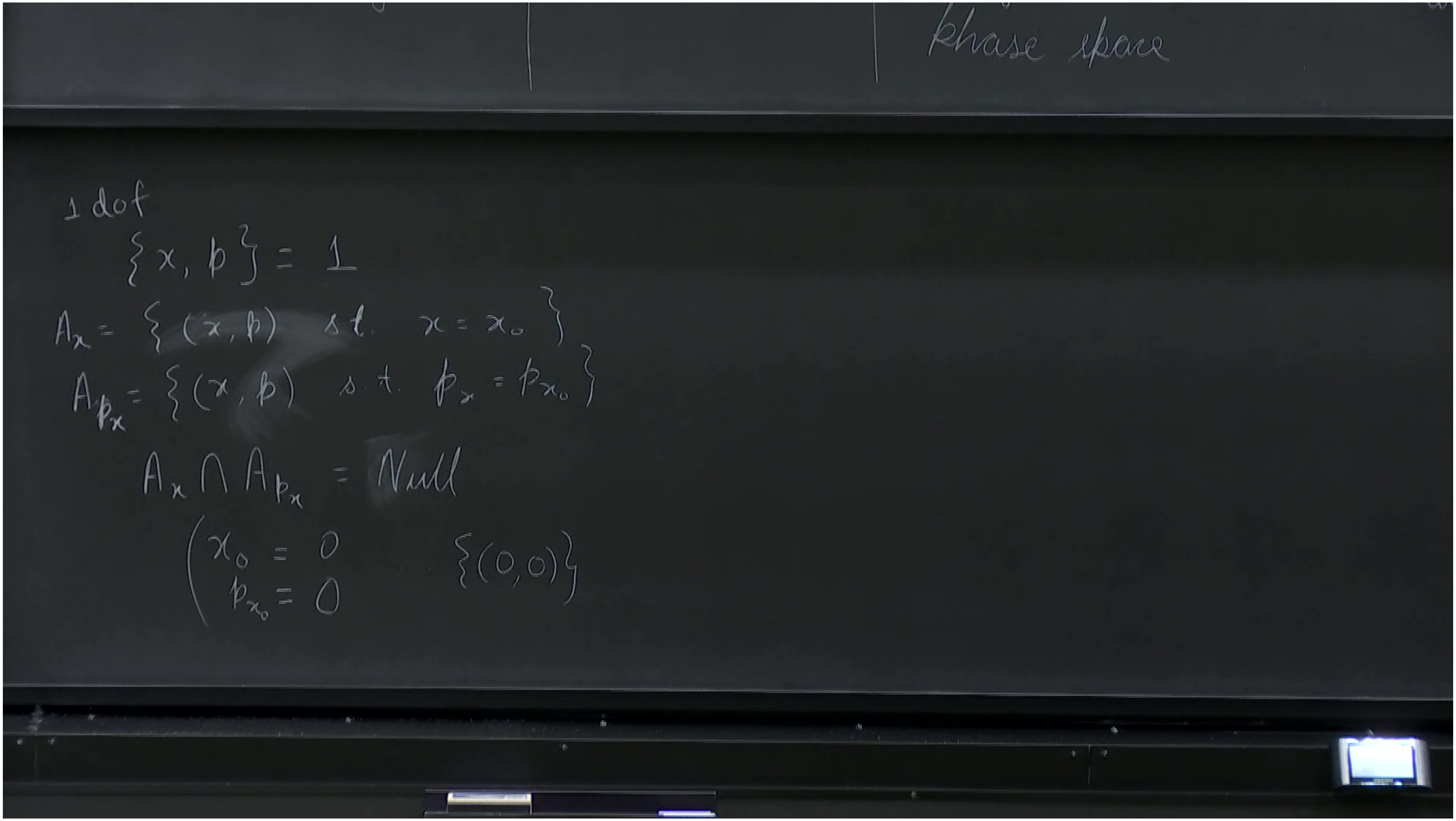
$A_{l^2} := \{ \quad L^2 = L_0^2\}$

$A_{Energy} := \{ \quad H = E_0\}$

$A_x, A_y \ldots\ldots$ are 5-dimensional hypersurfaces in the 6-dim phase space


that $L_x = L_{x0}$
$L_y = L_{y0}$
$L_z = L_{z0}$
$L^2 =$
$A_{l^2} \cap A_z$
Possibilities for $A_{l^2} \cap A_z$
given L_0^2 & L_{z0}
① $L_{z0}^2 > L_0^2 \quad = \phi \quad (L^2 = L_x^2 + L_y^2 + L_z^2)$
② $L_{z0}^2 = L_0^2$


such that
$L_x = L_{x0}$
$L_y = L_{y0}$
$L_z = L_{z0}$
Possibilities for $A_{l^2} \cap A_z$
given L_0^2 & L_{z0}
① $L_{z0}^2 > L_0^2 = \phi$
$(L^2 = L_x^2 + L_y^2 + L_z^2)$
② $L_{z0}^2 = L_0^2$, $L_x = 0$, $L_y = 0$
F_i's
rank( ) =
al hypersurfaces in


phase space
1 dof
$\{x, p\} = 1$
$A_x = \{(x, p) \text{ s.t. } x = x_0\}$
$A_{p_x} = \{(x, p) \text{ s.t. } p_x = p_{x_0}\}$
$A_x \cap A_{p_x} = \text{Null}$
$x_0 = 0$
$p_{x_0} = 0$
$\{(0,0)\}$

... hypersurfaces in the 6-dim

phase space

$\frac{\partial F}{\partial x}$

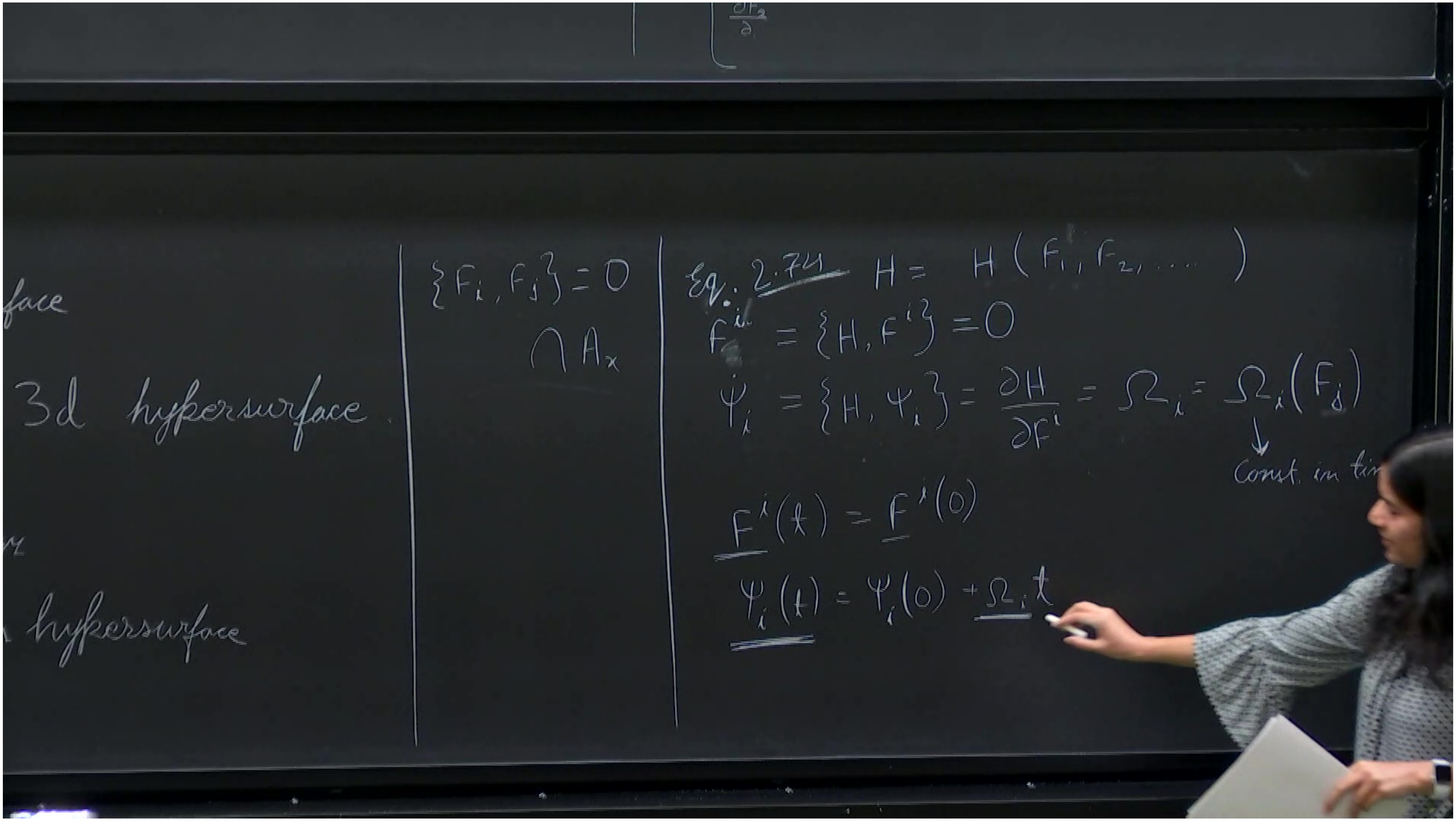
$\frac{\partial F_2}{\partial}$

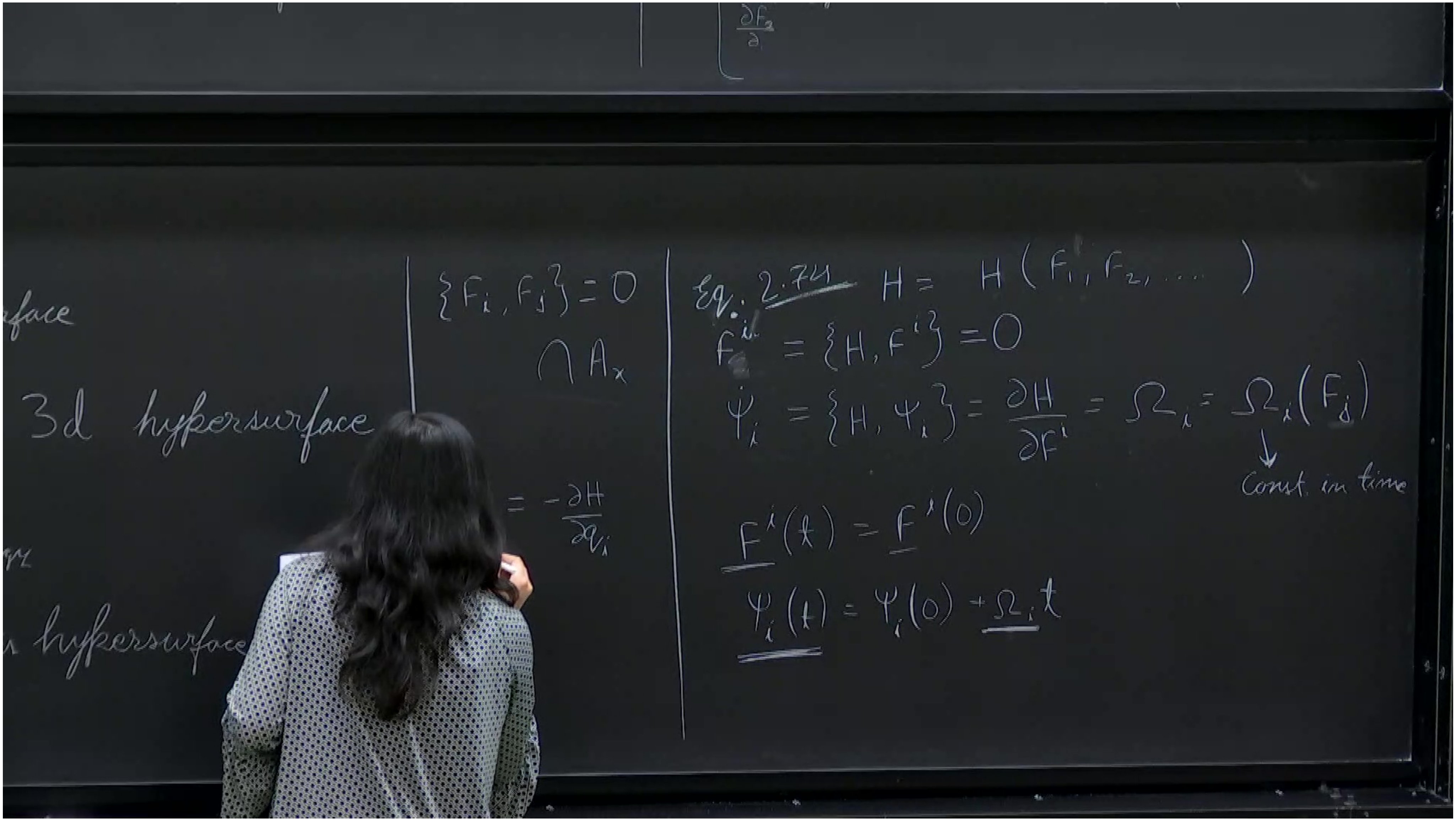
③ $L_{20}^2 < L_0^2$: 4-d hypersurface

$A_{l^2} \cap A_z \cap A_{Energy}$: $\leq 3d$ hypersurface.

$(x, p_x \ldots\ldots) \in A_{l^2} \cap A_z \cap A_{Energy}$

H


3d hypersurface
hypersurface
$\{F_i, F_j\} = 0$
$\cap A_x$
Eq. 2.74 $H = H(F_1, F_2, \ldots)$
$\dot{F}^i = \{H, F^i\} = 0$
$\dot{\psi}_i = \{H, \psi_i\} = \frac{\partial H}{\partial F^i} = \Omega_i = \Omega_i(F_j)$
Const. in
$F^i(t) = F^i(0)$
$\psi_i(t) = \psi_i(0) + \Omega_i t$



3d hypersurface
hypersurface
$\{F_i, F_j\} = 0$
$\cap A_x$
$= -\frac{\partial H}{\partial q_i}$
Eq.: 2.74 $H = H(F_1, F_2, \dots)$
$\dot{F}^i = \{H, F^i\} = 0$
$\dot{\psi}_i = \{H, \psi_i\} = \frac{\partial H}{\partial F^i} = \Omega_i = \Omega_i(F_j)$
Const. in time
$F^i(t) = F^i(0)$
$\psi_i(t) = \psi_i(0) + \Omega_i t$

face

3d hypersurface

rge

n hypersurface

$\{F_i, F_j\} = 0$

$\cap A_x$

$\dot{p}_i = -\frac{\partial H}{\partial q_i}$

$\dot{q}_i = \frac{\partial H}{\partial p_i}$

Eq.: 2.74 $H = H(F_1, F_2, \ldots)$

$\dot{F}^i = \{H, F^i\} = 0$

$\dot{\Psi}_i = \{H, \Psi_i\} = \frac{\partial H}{\partial F^i} = \Omega_i = \Omega_i(F_j)$

Const. in time

$\underline{F^i}(t) = \underline{F}^i(0)$

$\underline{\Psi_i(t)} = \Psi_i(0) + \underline{\Omega_i}\, t$

Resonant tori

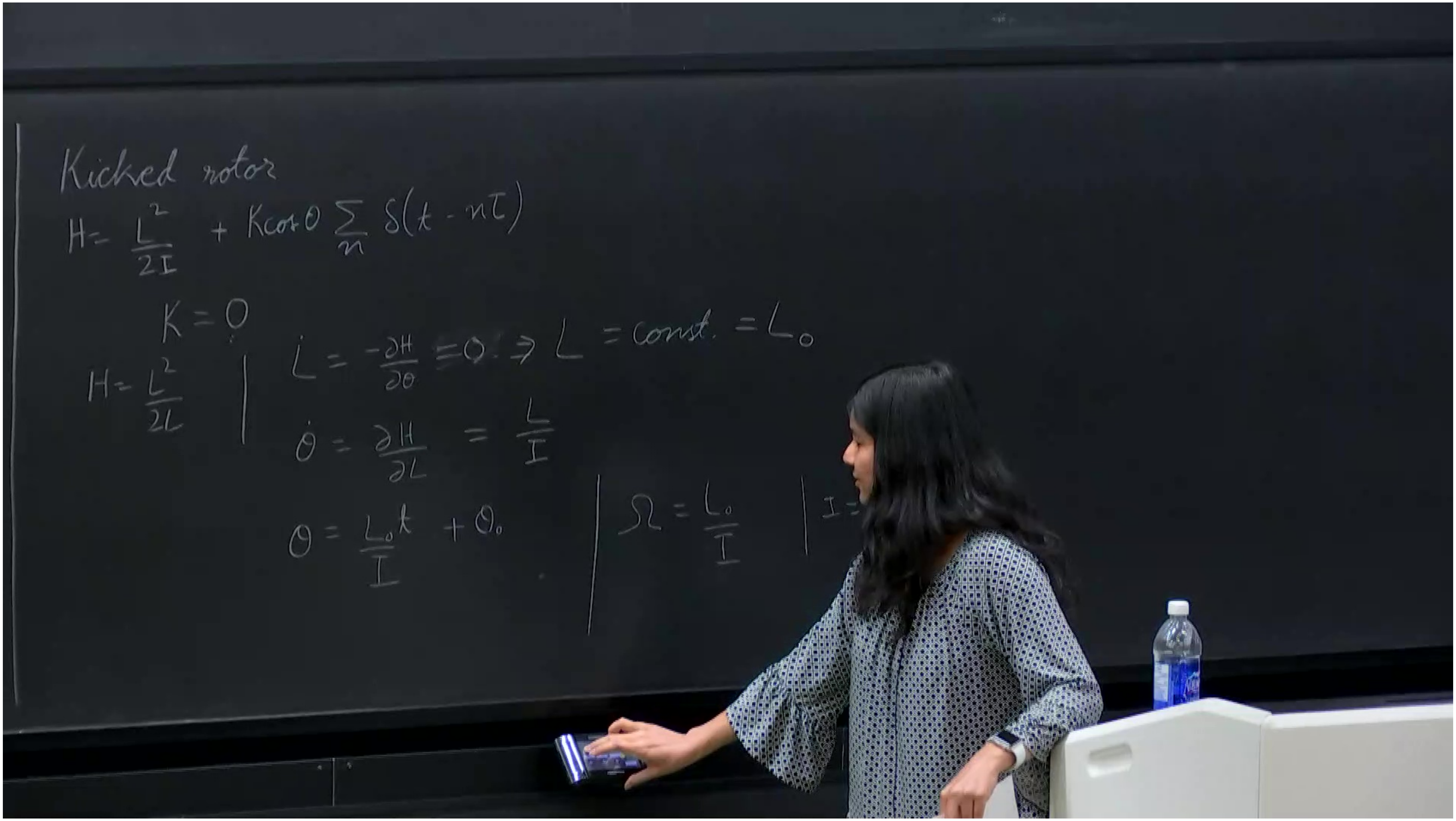
Given actions $J_1, J_2, \dots J_n$, the torus is called resonant if

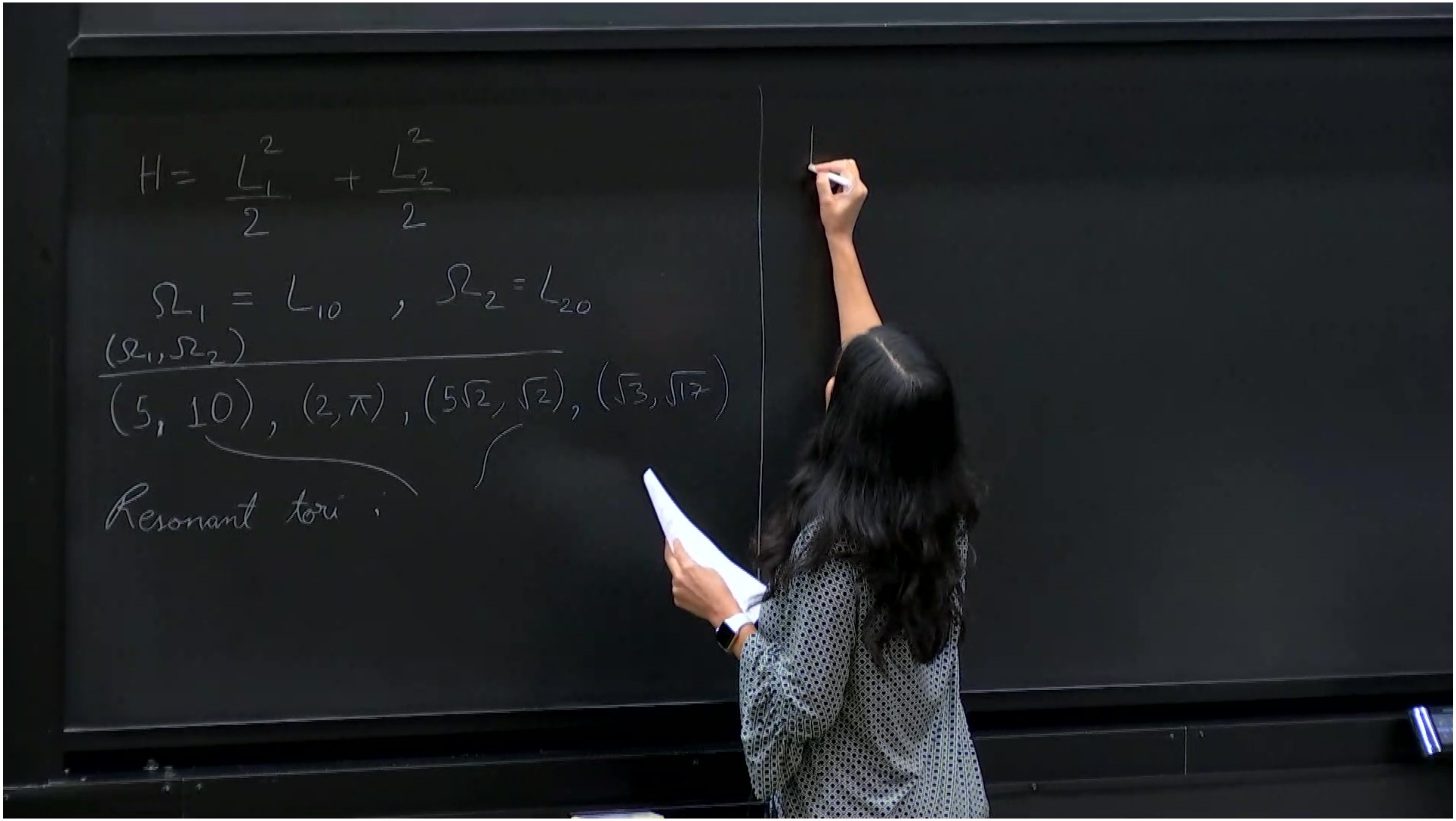
$$\sum_{i=1}^{n} k_i \Omega_i (J_1, J_2, \dots J_n) = 0$$

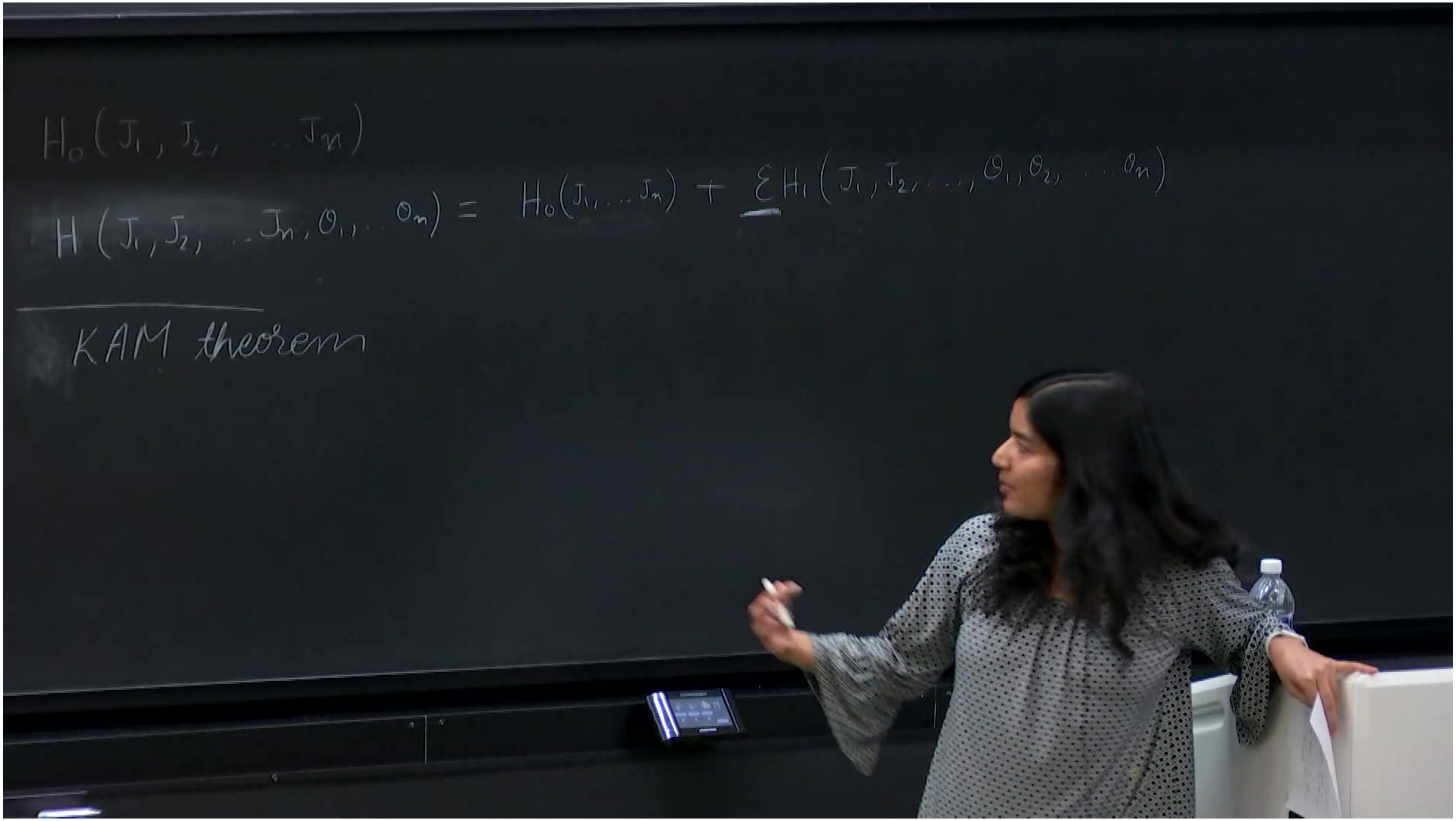
$k_i \in$ non-zero integers

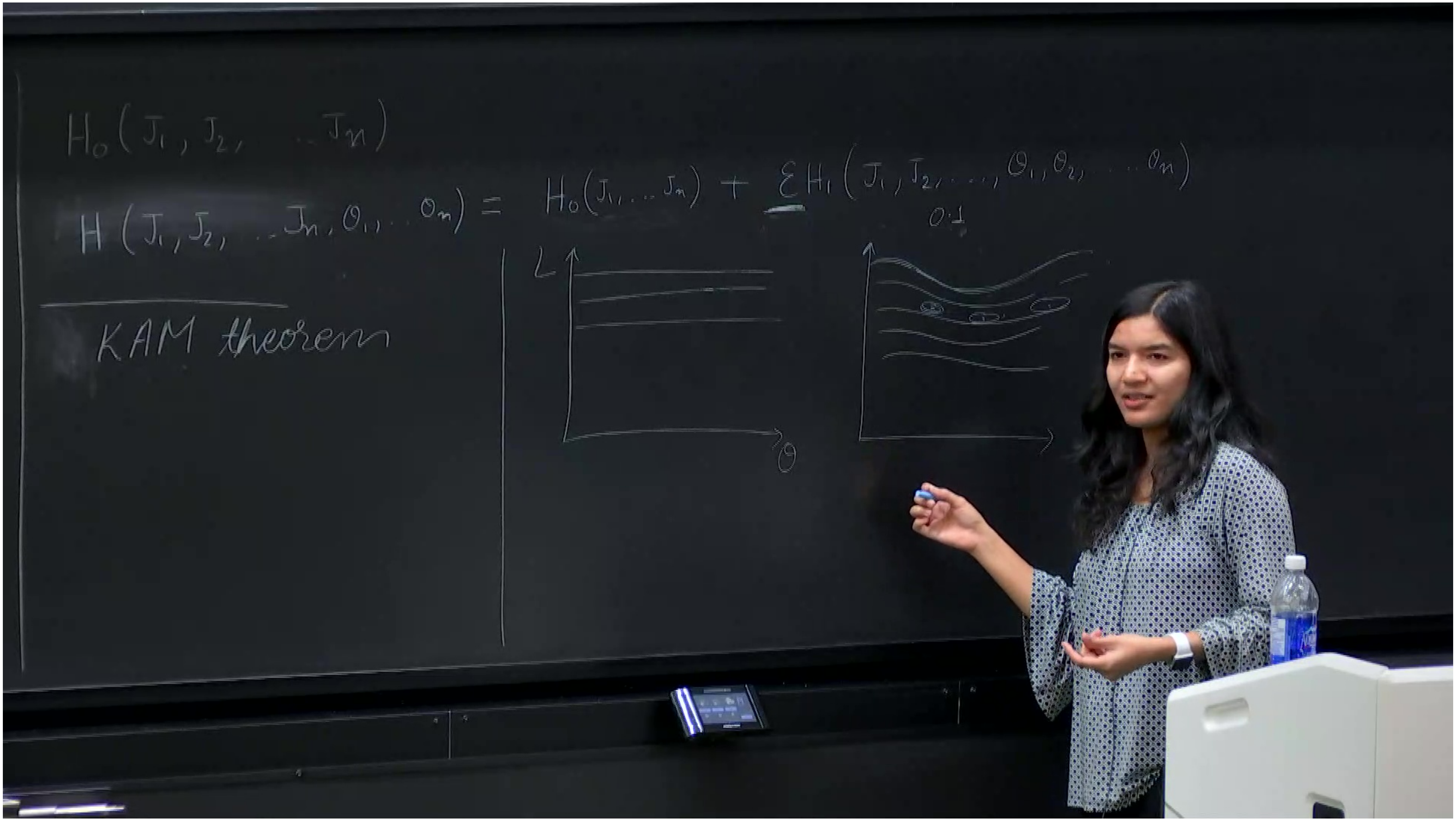
Ω_1, Ω_2

$$\frac{\Omega_1}{\Omega_2} = \text{Rational} = \frac{k_1}{k_2} \Rightarrow \Omega_1 k_2 - k_1 \Omega_2 = 0$$


Kicked rotor
$H = \frac{L^2}{2I} + K\cos\theta \sum_n \delta(t - n\tau)$
$K = 0$
$H = \frac{L^2}{2I}$
$\dot{L} = -\frac{\partial H}{\partial \theta} = 0 \Rightarrow L = \text{const.} = L_0$
$\dot{\theta} = \frac{\partial H}{\partial L} = \frac{L}{I}$
$\theta = \frac{L_0 t}{I} + \theta_0$
$\Omega = \frac{L_0}{I}$


$H = \frac{L_1^2}{2} + \frac{L_2^2}{2}$
$\Omega_1 = L_{10}$, $\Omega_2 = L_{20}$
(Ω_1, Ω_2)
$(5, 10)$, $(2, \pi)$, $(5\sqrt{2}, \sqrt{2})$, $(\sqrt{3}, \sqrt{17})$
Resonant tori :


$H_0(J_1, J_2, \dots J_n)$
$H(J_1, J_2, \dots J_n, \theta_1, \dots \theta_n) = H_0(J_1, \dots J_n) + \varepsilon H_1(J_1, J_2, \dots, \theta_1, \theta_2, \dots \theta_n)$
KAM theorem


$H_0(J_1, J_2, \ldots J_n)$
$H(J_1, J_2, \ldots J_n, \theta_1, \ldots \theta_n) = H_0(J_1, \ldots J_n) + \varepsilon H_1(J_1, J_2, \ldots, \theta_1, \theta_2, \ldots \theta_n)$
0.1
KAM theorem
L
θ