Title: Classical Physics - Lecture 220916

Speakers:

Collection: Classical Physics (2022/2023)

Date: September 16, 2022 - 9:00 AM

URL: https://pirsa.org/22090050

YESTERDAY: MAXWELL'S EQS (1860)

$$\nabla \times B - \frac{\partial E}{\partial t} = 4\pi j \quad (1)$$
$$\nabla \times E + \frac{\partial B}{\partial t} = 0 \quad (2)$$
$$\nabla \cdot E = 4\pi \rho \quad (3)$$
$$\nabla \cdot B = 0 \quad (4)$$

- IF WE SET

$$\boxed{\begin{aligned} B &= \nabla \times A \\ E &= -\nabla\varphi - \frac{\partial A}{\partial t} \end{aligned}} \Rightarrow$$

(2) & (4) AUTOMATICALLY SATISFIED

- GAUGE FREEDOM

$$\boxed{\begin{aligned} \varphi &\to \varphi - \partial_t \Lambda \\ A &\to A + \nabla \Lambda \end{aligned}}$$

- ALLOWS TO IMPOSE LORENZ GAUGE

$$\boxed{\partial_t \varphi + \nabla \cdot A = 0}$$

RENZ

• (3) $\nabla\cdot E = \nabla\cdot\left(-\nabla\varphi - \frac{\partial A}{\partial t}\right) = 4\pi\rho$

$$-\underbrace{\nabla^2}_{\Delta}\varphi - \frac{\partial}{\partial t}\underbrace{\nabla\cdot A}_{-\partial_t\varphi} = 4\pi\rho$$

$$\boxed{\Box\varphi = -4\pi\rho\ , \quad \Box = -\frac{\partial^2}{\partial t^2} + \Delta}$$

• (1). $\nabla\times(\nabla\times A) - \frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{\partial A}{\partial t}\right) = 4\pi j$

$$\overbrace{\nabla\nabla\cdot A - \Delta A}^{\shortparallel} - \underbrace{\frac{\partial}{\partial t}\left(-\nabla\varphi\right.}_{+\nabla\frac{\partial\varphi}{\partial t}}\left. - \frac{\partial A}{\partial t}\right) = 4\pi j$$

RENZ

• (3) $\nabla\cdot E = \nabla\cdot\left(-\nabla\varphi - \frac{\partial A}{\partial t}\right) = 4\pi\rho$

$$-\underbrace{\nabla^2}_{\Delta}\varphi - \frac{\partial}{\partial t}\underbrace{\nabla\cdot A}_{-\partial_t\varphi} = 4\pi\rho$$

$$\boxed{\Box\varphi = -4\pi\rho\,, \quad \Box = -\frac{\partial^2}{\partial t^2} + \Delta}$$

• (1). $\nabla\times(\nabla\times A) - \frac{\partial}{\partial t}\left(-\nabla\varphi - \frac{\partial A}{\partial t}\right) = 4\pi j$

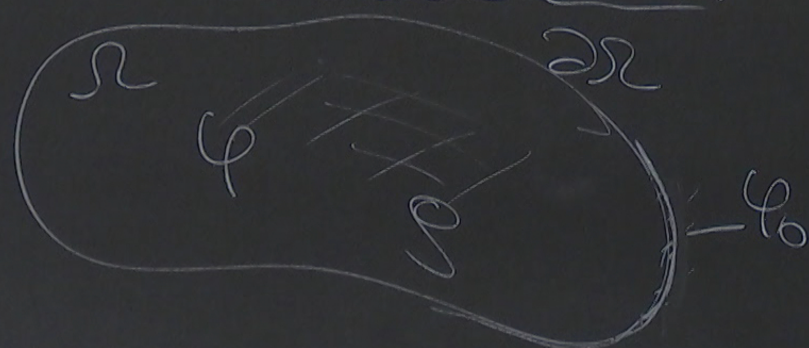
$$\nabla\nabla\cdot A - \Delta A - \underbrace{\frac{\partial}{\partial t}\left(-\nabla\varphi\right.}_{+\nabla\frac{\partial\varphi}{\partial t} = -\nabla\nabla\cdot A}\left. - \frac{\partial A}{\partial t}\right) = 4\pi j$$

$$\boxed{\Box A = -4\pi j}$$

• FIRST CONSIDER ELECTROSTATICS

$$\boxed{\Delta\varphi = -4\pi\rho}$$

CONSIDER DIRECHLET PROBLEM

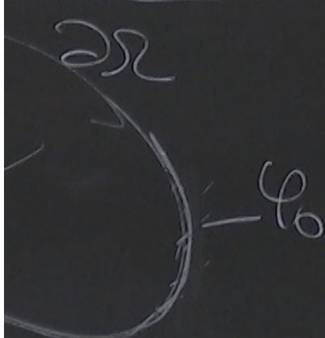


• WE HAVE $\varphi = \varphi_0$ ON $\partial\Omega$

STEP 1:

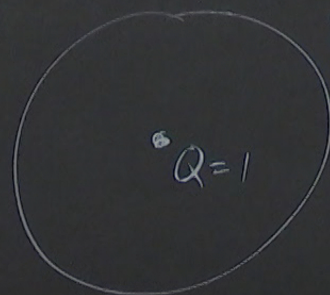
OSTATICS

PROBLEM



• WE HAVE $\varphi = \varphi_0$ ON $\partial\Omega$

STEP 1:



EMPTY SPACE

LET'S SEEK φ IN TERMS OF GREENS F.

$$\varphi(\vec{r}) = -4\pi \int G_3(\vec{r}, \vec{r}_0)\, \rho(\vec{r}_0)\, d^3 r_0$$

3D FREE SPACE GREEN F.

φ IN

EENS F.

$\vec{r}, \vec{r}_0)\, \rho(\vec{r}_0)\, d^3 r_0$

E SPACE GREEN F.

THIS WILL WORK PROVIDED

$$\Delta G_3 = \delta(\vec{r} - \vec{r}_0)$$

PROOF: $$\Delta \varphi = -4\pi \int \underbrace{(\Delta G_3)}_{\delta(r - r_0)} \rho(r_0)\, d^3 r_0$$
$$= -4\pi \rho$$

$\delta(\vec{r}-\vec{r}_0))$

$= -4\pi\rho$

$G_3(\vec{r},\vec{r}_0)\,\rho(\vec{r}_0)\,d^3 r_0$

↑

3D FREE SPACE GREEN F.

$$G_3(\vec{r},\vec{r}_0) = G_3(|\underbrace{\vec{r}-\vec{r}_0}_{\vec{R}}|)$$

• G_3 ... RELATED TO POTENTIAL OF POINT CHARGE

$$\int_{\Omega} \Delta G_3 \, dV = \int_{\Omega} \nabla \cdot \nabla G_3 \, dV = \int_{\partial\Omega} \nabla G_3 \cdot dS$$

$$= \int_{\partial\Omega} \frac{dG_3}{dR} R^2 \, d\cos\theta \, d\varphi = 4\pi R^2 \frac{dG_3}{dR}$$

TATICS

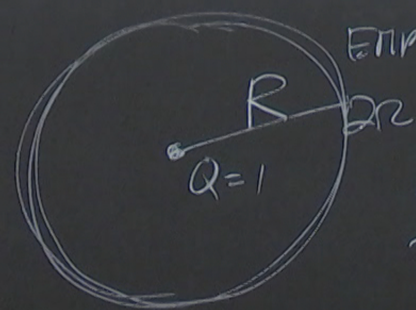
OBLEM

$\partial\Omega$

φ_0

• WE HAVE $\varphi = \varphi_0$ ON $\partial\Omega$

STEP 1:



EMPTY SPACE

LET'S SEEK φ IN TERMS OF GREENS F.

$$\varphi(\vec{r}) = -4\pi \int G_3(\vec{r}, \vec{r}_0)\, \rho(\vec{r}_0)\, d^3 r_0$$

3D FREE SPACE GREEN F.

TH

PROOF:

• G_3 ... RELATED TO POTENTIAL OF POINT CHARGE

$$\int_\Omega \Delta G_3 \, dV = \int_\Omega \nabla \cdot \nabla G_3 \, dV = \int_{\partial\Omega} \nabla G_3 \cdot dS$$

$$= \int_{\partial\Omega} \frac{dG_3}{dR} R^2 \, d\cos\theta \, d\varphi = 4\pi R^2 \frac{dG_3}{dR}$$

$$= \int_\Omega \delta(r - r_0) \, dV_0 = 1$$

$$\boxed{G_3 = -\frac{1}{4\pi} \frac{1}{|\vec{R}|}}$$

TERMS OF GREENS F.

$$\varphi(\vec{r}) = -4\pi \int G_3(\vec{r},\vec{r}_0)\,\rho(\vec{r}_0)\,d^3r_0$$

3D FREE SPACE GREEN F.

φ_0

$$\boxed{\int_\Omega d\omega = \int_{\partial\Omega} \omega}$$

$\Delta\varphi = -4\pi \int (\underbrace{\Delta G_3}_{\delta(r-r_0)}$

$= -4\pi\rho$

$G_3(\vec{r},\vec{r}_0) = G_3(\underbrace{\vec{r}-\vec{r}_0}_{\vec{R}}$

F POINT CHARGE

$\nabla G_3\, dV = \int_{\partial\Omega} \nabla G_3 \cdot dS$

$d\cos\theta\, d\varphi = 4\pi R^2 \frac{dG_3}{dR}$

$= 1$

$$\boxed{G_3 = -\frac{1}{4\pi}\frac{1}{|\vec{R}|}}$$

STEP 2: GREEN'S IDENTITIES, CONSIDER ϕ, χ

$$\int_\Omega \nabla\cdot(\phi\nabla\chi)\,dV = \int_{\partial\Omega}(\phi\nabla\chi)\cdot dS$$

STEP 2: GREEN'S IDENTITIES, CONSIDER ϕ, χ (SOME FUNCTIONS)

$$\int_{\Omega} \nabla\cdot(\phi\nabla\chi)\,dV \overset{\text{GAUSS}}{=} \int_{\partial\Omega} (\phi\vec{\nabla}\chi)\cdot d\vec{S} \qquad \vec{dS} = \hat{n}\,dS$$

$$\overset{\text{LEIBNITZ}}{=} \int_{\Omega} (\nabla\phi\cdot\nabla\chi)\,dV + \int_{\Omega} \phi\,\nabla^2\chi\,dV$$

1ST GREEN'S ID.

NOTE $\nabla\chi\cdot dS = \nabla\chi\cdot\hat{n}\,dS$

$$= \underset{n\cdot\nabla\chi\,dS}{\nabla_n\chi\,dS} = \underline{\frac{\partial\chi}{\partial n}\,dS}$$

$$\boxed{\varphi(r) = -4\pi \int \rho\, G_3\, dV_0 + \int_{\partial\Omega} \left(\varphi \frac{\partial G_3}{\partial n} - G_3 \frac{\partial \varphi}{\partial n} \right) dS}$$

SEEMS LIKE VICTORY.

WE DON'T KNOW
(CANNOT KNOW.. ILL POSED)

STEP 3: $G_3 \to G$ ... DIRICHLET GREEN FUNCTION

$$\Delta G = \delta(r - r_0)$$

$$G\big|_{\partial\Omega} = \emptyset$$

$$G = G_3 + H \qquad \Delta H = 0$$

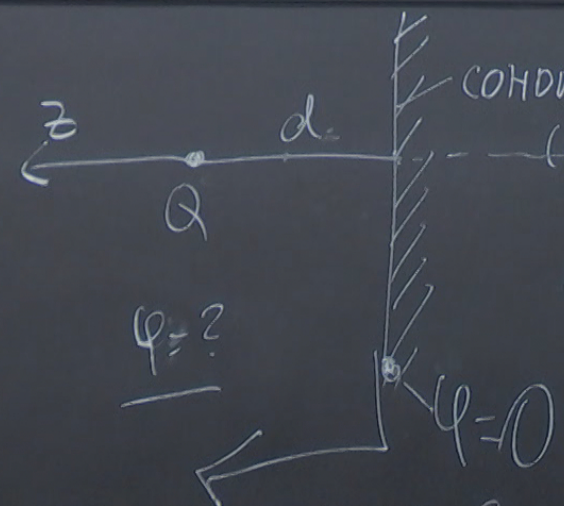
- IF WE CAN FIND THIS G

$$\boxed{\varphi(r) = -4\pi \int G \rho \, dV_0 + \int_{\partial\Omega} \varphi \frac{\partial G}{\partial n} \, dS}$$

UPGRADE OF PREVIOUS FORMULA.

- TO CONSTRUCT G .. POWERFUL METHOD.. METHOD OF IMAGES

EX:



"IMAGE CHARGE"

$\varphi = ?$

NEED G

WE HAVE DIRICHLET PROBLEM

$$\underbrace{\varphi\big|_{z=0}}_{\text{PLATE}} = 0 = \varphi\big|_{\infty}$$

$$\Delta \varphi = -4\pi Q\, \delta(\vec{r} - \vec{r}_0), \quad \vec{r}_0 = (0, 0, d)$$

$$G = \underset{\substack{\uparrow \\ Q}}{G_3} + H = -\frac{1}{4\pi}\frac{1}{|r-d|} + \frac{1}{4\pi}\frac{1}{|r+d|}$$

$$\Rightarrow \underline{\varphi(r) = -4\pi Q\, G(d, r)}$$

$$\int \Delta G_3 \, dV = \int \nabla \cdot \nabla G_3 \, dV = \oint \nabla G_3 \cdot dS$$

- RETURN DYNAMICS BACK:

$$\boxed{\Box \varphi = -4\pi \rho}$$

LET'S ONLY DO EMPTY SPACE

$$\varphi = \varphi_S + \varphi_H$$

SPECIAL

$$\Box \varphi_H = 0$$

$\int \nabla^2 G_3 \, dV = \int \nabla G_3 \cdot dS$ $\qquad$ $\int \nabla \cdot (\phi \nabla \psi)\, dV \overset{\text{GAUSS}}{=} \int (\phi \nabla$

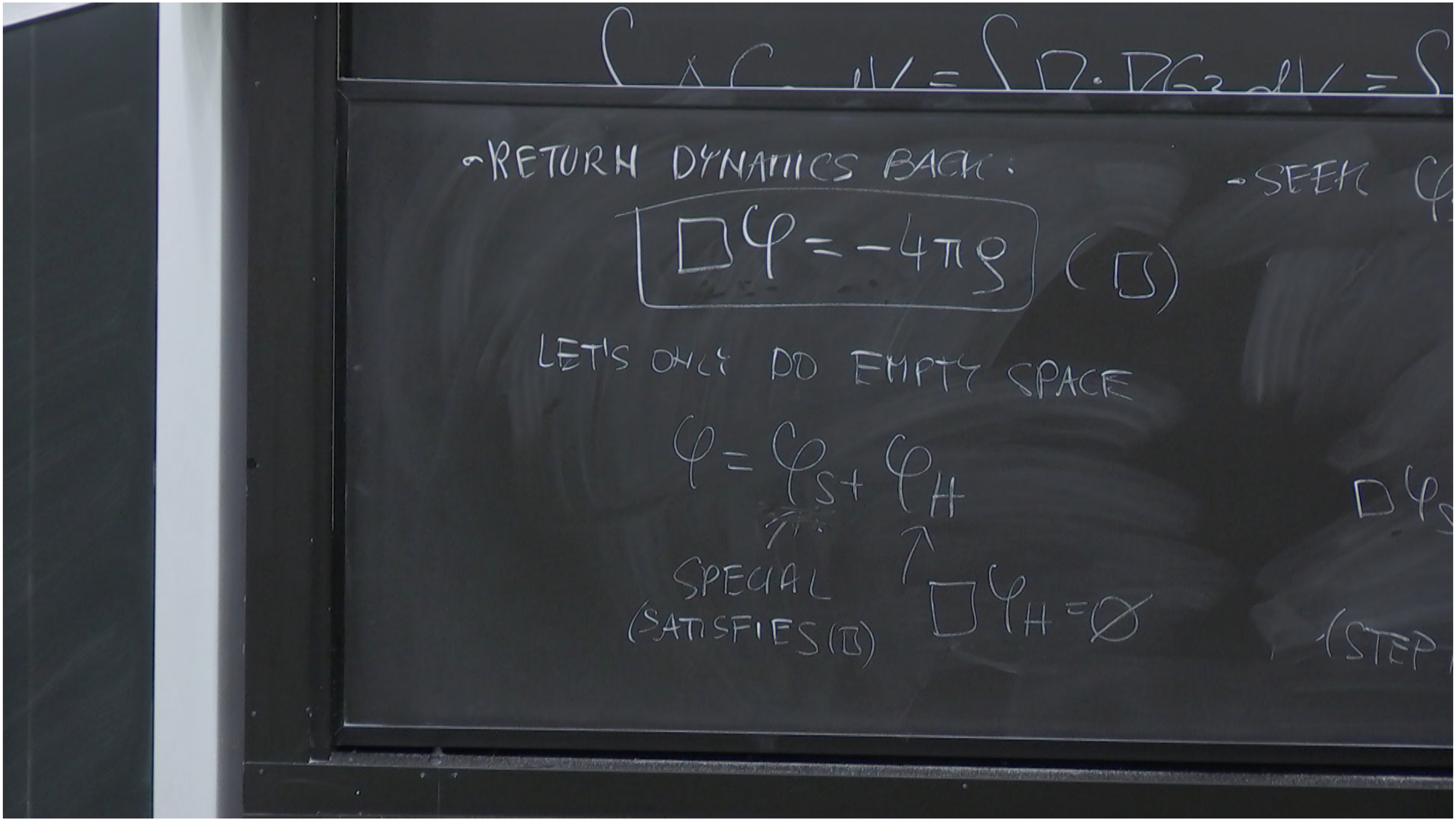
- SEEK $\varphi_S(\vec{r},t) = \int \underbrace{G(\vec{r},\vec{r}',t,t')} 4\pi \rho(\vec{r}',t')\, d\vec{r}'\, dt'$

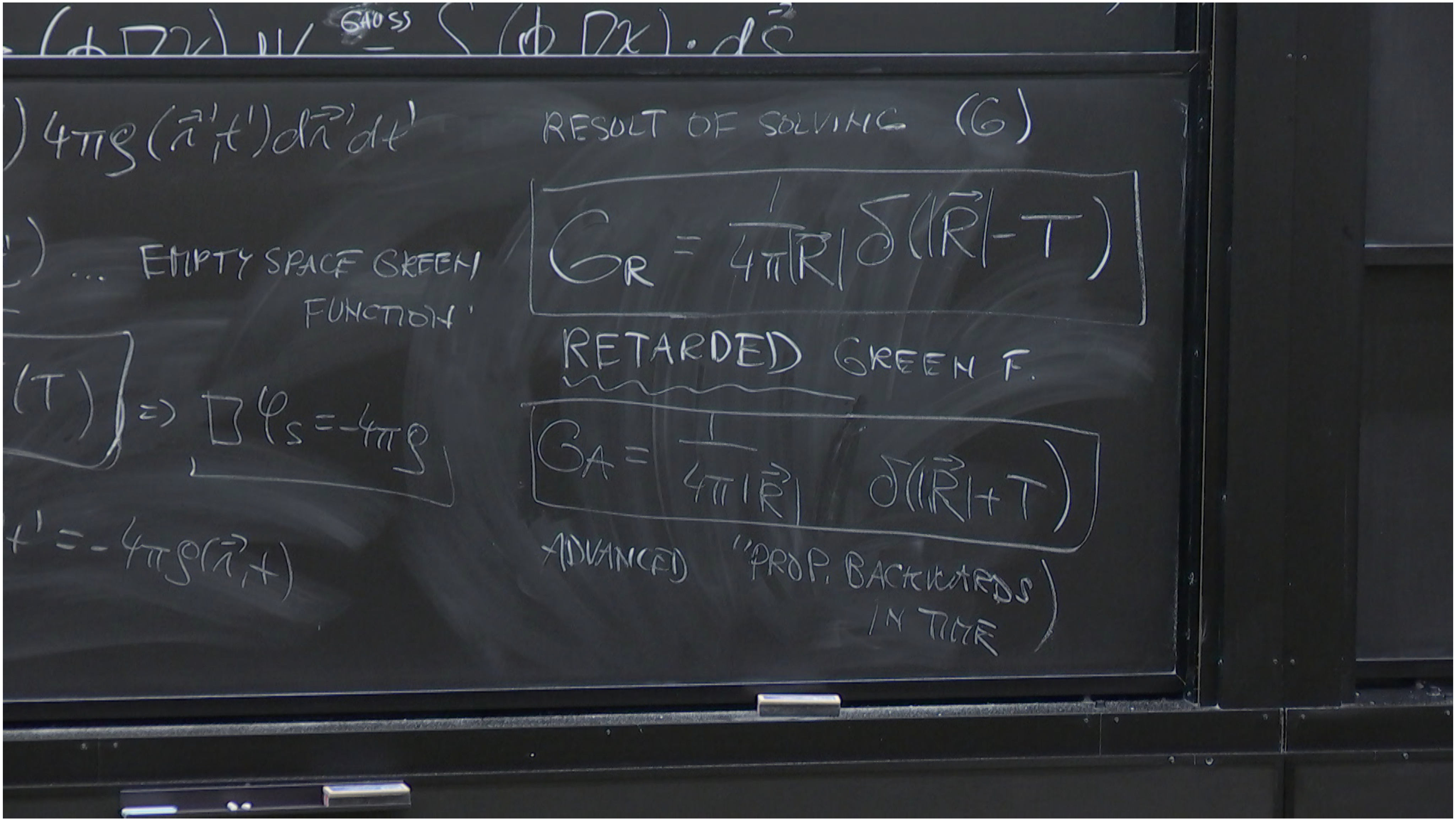
$G(\underbrace{\vec{r}-\vec{r}'}_{\vec{R}}, \underbrace{t-t'}_{T})$ ... EMPTY SPACE GREEN FUNCTION'

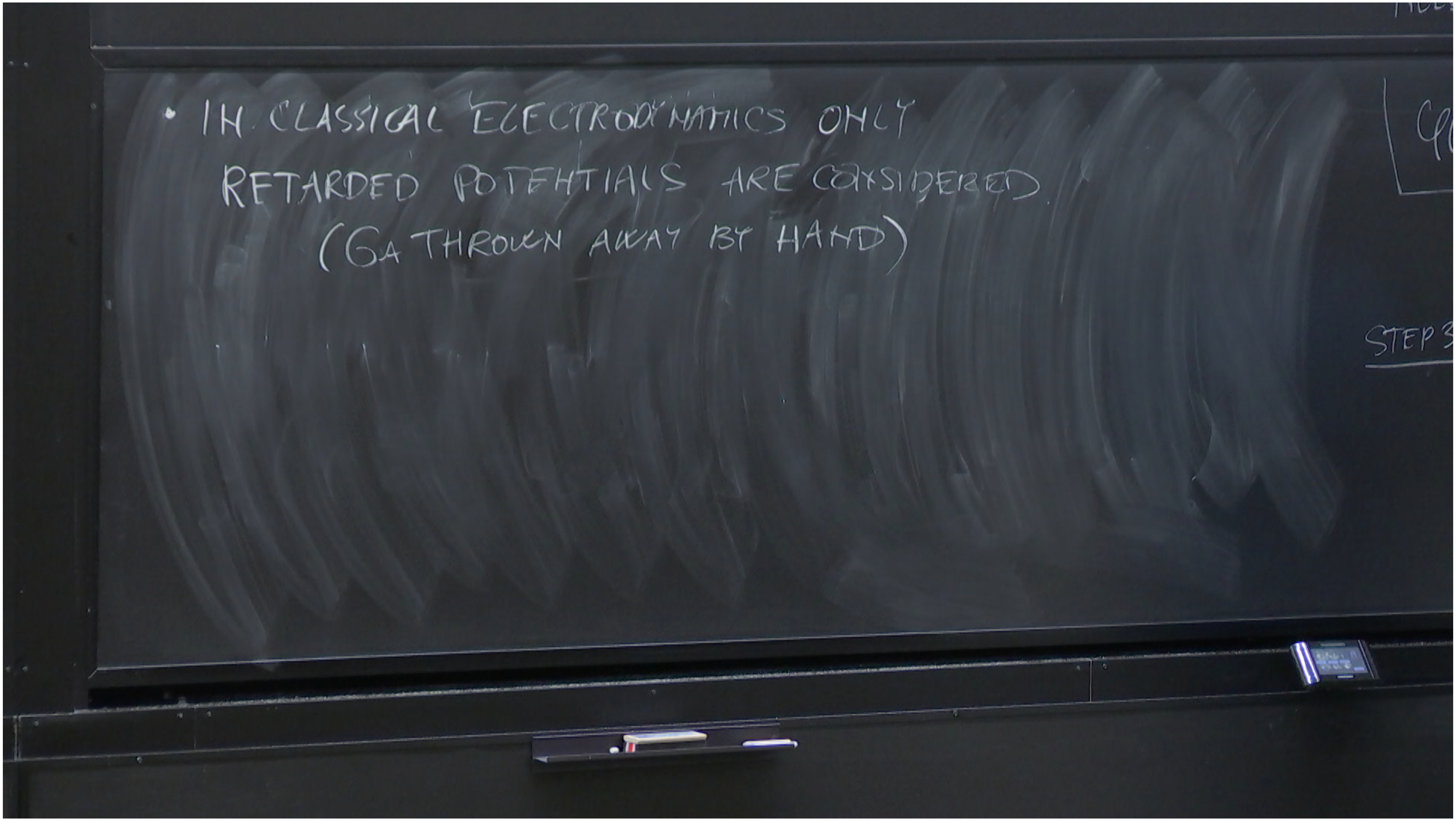
$\underbrace{\Box G(\vec{R},T) = -\delta(\vec{R})\,\delta(T)}$ $\Rightarrow$ $\boxed{\Box \varphi_S = -4\pi\rho}$

$\Box \varphi_S = \int \underbrace{(\Box G)}_{-\delta(r-r')\delta(t-t')} 4\pi \rho(\vec{r}',t')\, dr'\, dt' = -4\pi \rho(\vec{r},t)$

(STEP 1)


RETURN DYNAMICS BACK:
□φ = −4πρ (B)
LET'S ONLY DO EMPTY SPACE
φ = φS + φH
SPECIAL
(SATISFIES (B))
□φH = 0
SEEK
(STEP


RESULT OF SOLVING (G)
$G_R = \frac{1}{4\pi|\vec{R}|}\delta(|\vec{R}|-T)$
RETARDED GREEN F.
$G_A = \frac{1}{4\pi|\vec{R}|}\delta(|\vec{R}|+T)$
ADVANCED "PROP. BACKWARDS IN TIME"
EMPTY SPACE GREEN FUNCTION'
$\Rightarrow \Box \varphi_s = -4\pi\rho$


• IN CLASSICAL ELECTRODYNAMICS ONLY
RETARDED POTENTIALS ARE CONSIDERED
(GA THROWN AWAY BY HAND)
STEP

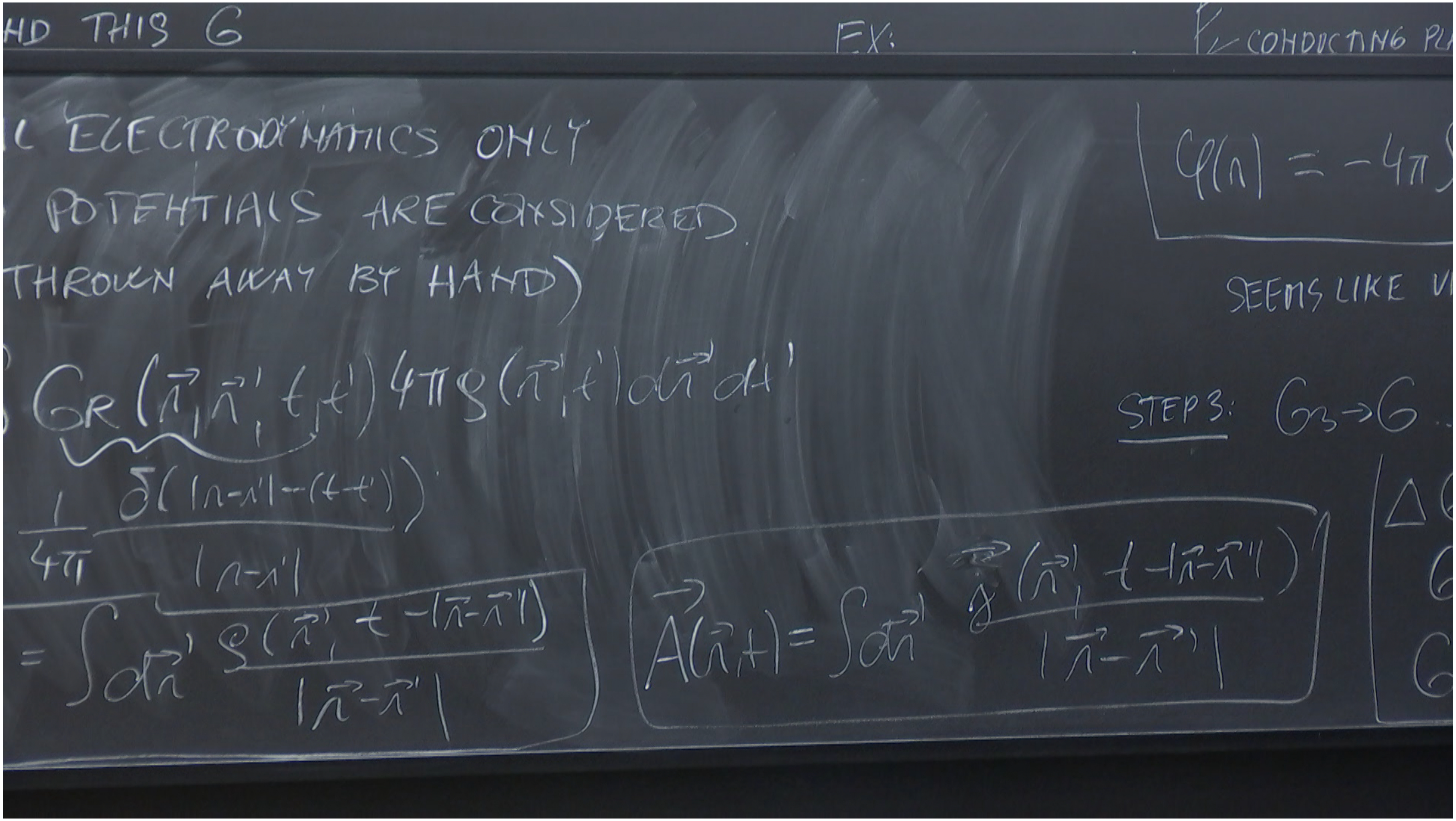
• IF WE CAN FIND THIS G

EX:

• IN CLASSICAL ELECTRODYNAMICS ONLY RETARDED POTENTIALS ARE CONSIDERED (G_A THROWN AWAY BY HAND)

$$\varphi(\vec{r},t) = \int \underbrace{G_R(\vec{r},\vec{r}',t,t')}_{\frac{1}{4\pi}\frac{\delta(|r-r'|-(t-t'))}{|r-r'|}} 4\pi\rho(\vec{r}',t')\,d\vec{r}'\,dt'$$

$$\boxed{\varphi(\vec{r},t) = \int d\vec{r}'\,\frac{\rho(\vec{r}',t-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}}$$


HD THIS G
EX:
CONDUCTING PL
L ELECTRODYNAMICS ONLY
POTENTIALS ARE CONSIDERED
THROWN AWAY BY HAND)
$G_R(\vec{r}, \vec{r}', t, t') 4\pi \rho(\vec{r}', t') d\vec{r}' dt'$
$\frac{1}{4\pi} \frac{\delta(|r-r'|-(t-t'))}{|r-r'|}$
$= \int d\vec{r}' \frac{\rho(\vec{r}', t-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}$
$\vec{A}(\vec{r},t) = \int d\vec{r}' \frac{\vec{j}(\vec{r}', t-|\vec{r}-\vec{r}'|)}{|\vec{r}-\vec{r}'|}$
$\varphi(r) = -4\pi$
SEEMS LIKE
STEP 3: $G_3 \to G$