Title: Classical Physics - Lecture 220914

Speakers:

Collection: Classical Physics (2022/2023)

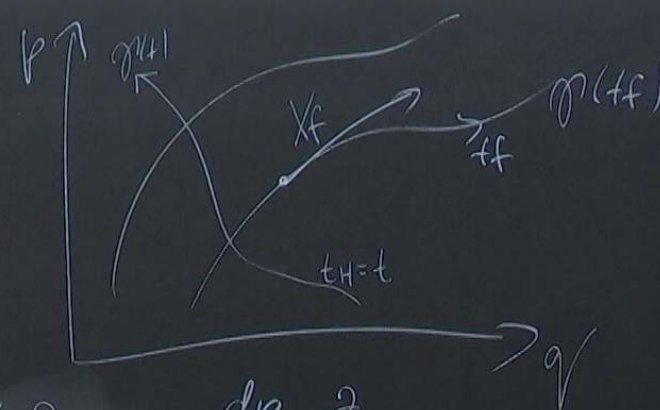
Date: September 14, 2022 - 9:00 AM

URL: https://pirsa.org/22090048

• HAVE f .. PHASE SPACE FUNCTION

$$X_f \underset{\uparrow \text{DEF}}{=} \{\cdot, f\} \underset{\uparrow}{=} \frac{d}{dt_f}$$

TANGENT VECTOR to $\gamma(t_f)$



$$= \frac{\partial f}{\partial p}\frac{\partial}{\partial q} - \frac{\partial f}{\partial q}\frac{\partial}{\partial p} = \frac{dq}{dt_f}\frac{\partial}{\partial q} + \frac{dp}{dt_f}\frac{\partial}{\partial p}$$

$$\boxed{\frac{\partial f}{\partial p} = \frac{dq}{dt_f}, \quad -\frac{\partial f}{\partial q} = \frac{dp}{dt_f}}$$

CANONICAL EQS

IF $f = H$ $(t_f = t)$

f) CONSTRAINTS

LET NOT ALL q's AND p's BE INDEPENDENT.

$$\boxed{\Phi(q^i, p_i) = 0}$$ CONSTRAINT.

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LET NOT ALL q's AND p's BE INDEPENDENT.

$$\boxed{\phi(q^i, p_i) = 0}$$ CONSTRAINT.

VALID AT ANY INSTANT OF TIME

"CONSERVED QUANTITY" $\simeq$ CORRESPONDS TO SOME KIND OF SYMMETRY.

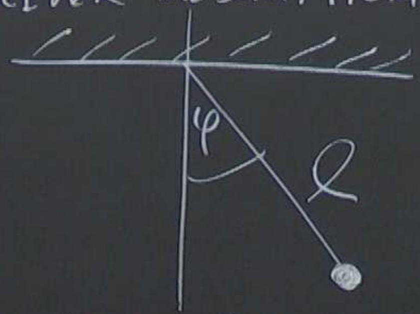
• IF $\Phi(q^i)=0$

ONLY q^i ...

HOLONOMIC CONSTRAINTS

EXAMPLE: PENDULUM

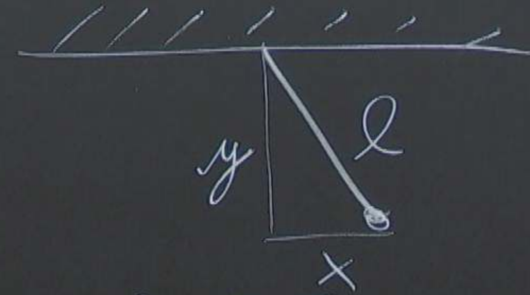
"CLEVER DESCRIPTION"



• $n=1$ ... DOF
• 2D PHASE SPACE (φ, p_φ)
• NO CONSTRAINTS

ETRY.

"CONSTRAINT SYSTEM DESCRIPTION"



• $n=2$ ... DOF
• PHASE SPACE (x, p_x, y, p_y)
• CONSTRAINT

$$\Phi(x,y) = x^2 + y^2 - \ell^2 = 0$$

(2) WHY TO CHOOSE CONSTRAINED DESCRIPTION ?

- CAN ELIMINATE CONSTRAINT BY

$$x = l\cos\varphi, \quad y = l\sin\varphi$$

- EXTRA SYMMETRY (MANY TIMES VERY USEFUL)
- CAN CALCULATE "CONSTRAINT FORCES" (E.G. TENSION OF STRING)

• RECIPE HOW TO DEAL WITH HOLONOMIC CONSTRAINTS

$$\boxed{\underset{\substack{\uparrow \\ \text{TOTAL LAGRANGIAN}}}{L_T} = L - \underset{\substack{\uparrow \\ \text{ALL CONSTRAINTS}}}{\sum_{j=1}^{m}} \underset{\substack{\uparrow \\ \text{LAGRANGE MULTIPLIERS}}}{\lambda_j} \underset{\substack{\uparrow \\ \text{CONSTRAINTS}}}{\phi_j}}$$

$$\lambda_j = \lambda_j(t)$$

$$L_T = L_T(\underset{\substack{\uparrow \\ n}}{q^i}, \underset{\substack{\uparrow \\ m}}{\lambda_j})$$

$(n+m)$ DIMENSIONAL CONFIG. SPACE.

$$L_T = L - \sum_{j=1}^{m} \lambda_j \phi_j$$

TOTAL LAGRANGIAN

ALL CONSTRAINTS

LAGRANGE MULT

CONSTRAINTS

$$L_T = L_T(\underbrace{q^i}_{n}, \underbrace{\lambda_j}_{m}) = L_T(q^i, \dot{q}^i, \lambda_j)$$

$(n+m)$ DIMENSIONAL CONFIG. SPACE

$\lambda_j = \lambda_j($

$$\delta S_T = \int \delta L_T \, dt = \int \left(\delta L - \delta\lambda_j \phi_j - \lambda_j \delta\phi_j \right) dt$$

$$= \int \left(\underbrace{\left[\left(\frac{\partial L_T}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot{q}} \right) \right) \right]}_{0} \delta q \, dt - \underbrace{\phi_j}_{0} \delta \lambda_j \, dt \right)$$

STANDARD E-L BUT NOW FOR L_T INSTEAD OF $\underline{L}$.

$\phi_j = 0$

ORIGINAL CONSTRAINTS

$) dt$

$\phi_J \delta \lambda_J dt)$

$\emptyset$

$\phi_J = 0$

ORIGINAL CONSTRAINTS

m

ENDED UP WITH $(n+m)$ EOM

FOR UNKNOWN $\{q^i, \lambda_J\} \ldots \underline{n+m}$

$$\frac{\partial L_T}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot{q}}\right) = 0$$

$$\frac{\partial L}{\partial q^i} - \lambda_J \frac{\partial \phi_J}{\partial q^i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) + \emptyset$$

"FORCE"

$) dt$

$\phi_J \delta \lambda_J dt \Big)$

$\phi_J = 0$

RIGINAL CONSTRAINTS

m

ENDED UP WITH $(n+m)$ EOM

FOR UNKNOWN $\{q^i, \lambda_J\} \ldots \underline{n+m}$

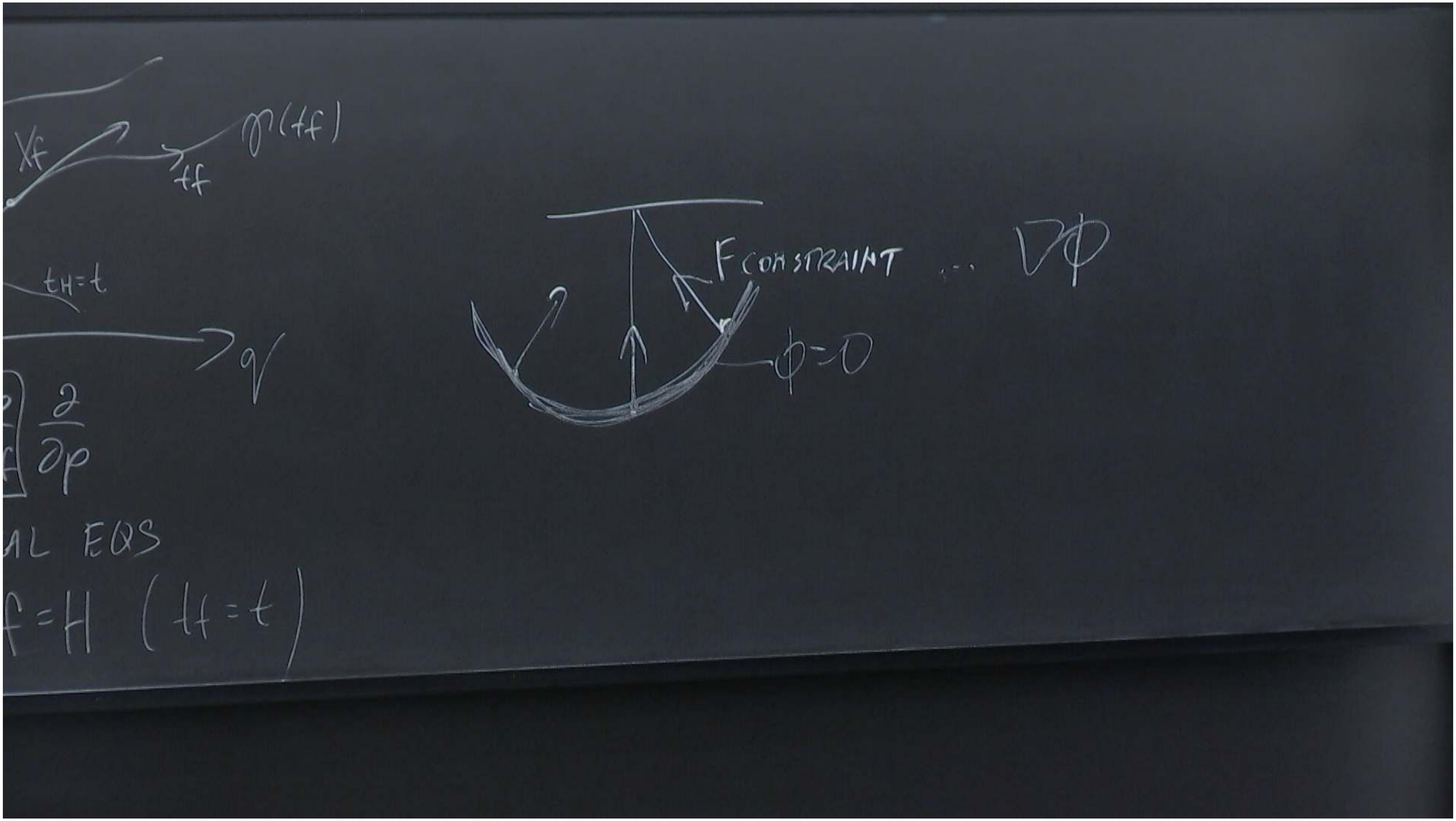
$$\frac{\partial L_T}{\partial q} - \frac{d}{dt}\left(\frac{\partial L_T}{\partial \dot q}\right) = 0$$

$$\frac{\partial L}{\partial q^i} - \underbrace{\lambda_J \frac{\partial \phi_J}{\partial q^i}} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q^i}\right) + \cancel{0}$$

"FORCE"

"CONSTRAINT FORCES"

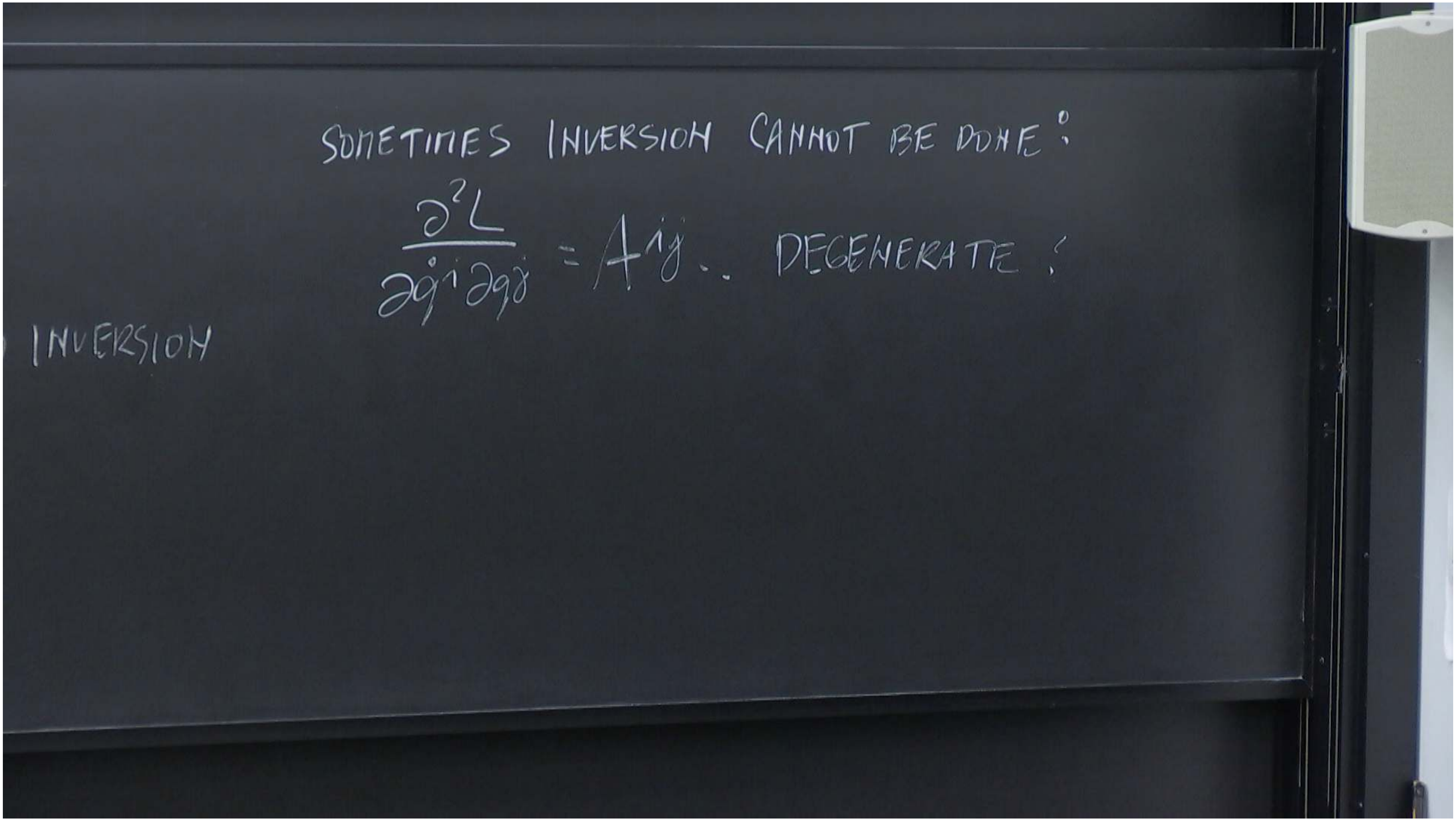
(PEPPENDICULAR TO CONSTRAINTS


Xf
tf
γ(tf)
tH=t
q
∂/∂p
AL EQS
f=H (tf=t)
F CONSTRAINT ... ∇φ
φ=0

- REMARK: "GAUGE CONSTRAINTS"
  - LET'S GO FROM LAGRANGIAN DESCR. → HAMILTONIAN.

CANONICAL MOMENTA $\boxed{p = \frac{\partial L}{\partial \dot{q}}}$ NEED INVERSION

$$H = p\dot{q} - L \Big|_{\dot{q} = \dot{q}(q,p)}$$


SOMETIMES INVERSION CANNOT BE DONE:
$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = A^{ij}$ .. DEGENERATE !
INVERSION

SOMETIMES INVERSION CANNOT BE DONE:

$$\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} = A^{ij} \ldots \text{ DEGENERATE!}$$

ON

... $\underline{m}$ ZERO EIGENVALUES

$$\dot{q}^i = \dot{q}^i(q^j, p_j, \lambda)$$

ARBITRARY FUNCTIONS

• DEFINE TOTAL HAMILTONIAN

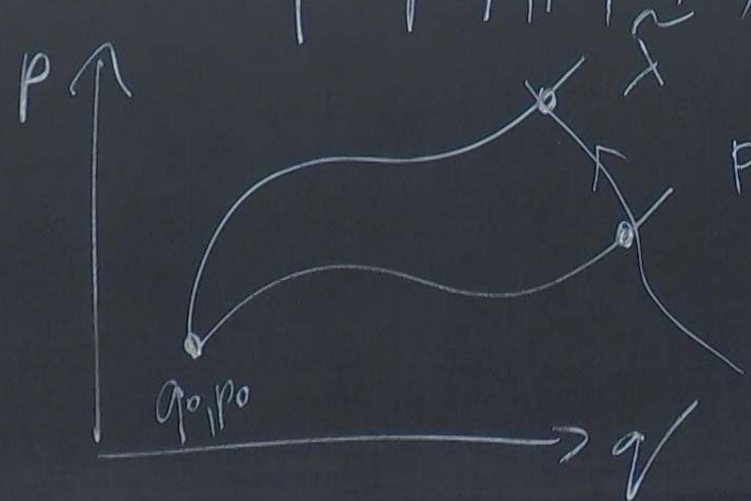
$$\boxed{H_T = H + \sum_{j=1}^{m} \lambda_j \phi_j(q^i, p_i)}$$

CONSTRAINTS
(CAN BE IDENTIFIED
↓
DIRAC'S PROCEDURE")

IN GENERAL TREAT $H_T = H_T(q,p)$ ... λ's EXTERNAL FUNCTIONS

- ONLY n DOF ($2n$-DIM PHASE SPACE)
- AFTER SOLVING HAMILTON'S EQS FOR $\underline{q}$ & $\underline{p}$

$$q = q(q_0, p_0, \lambda, t)$$
$$p = p(q_0, p_0, \underset{\sim}{\lambda}, t)$$



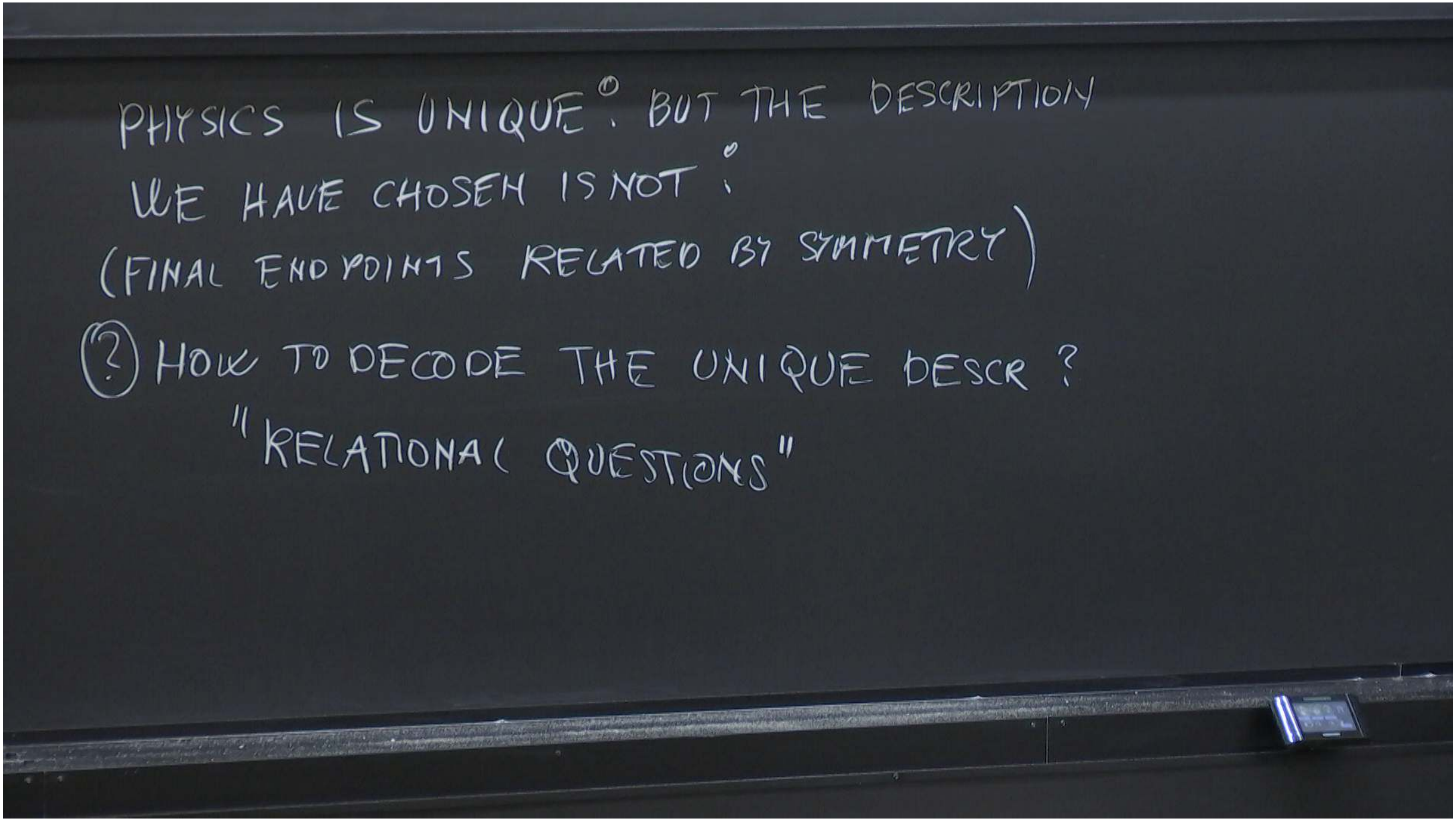
FIXED λ

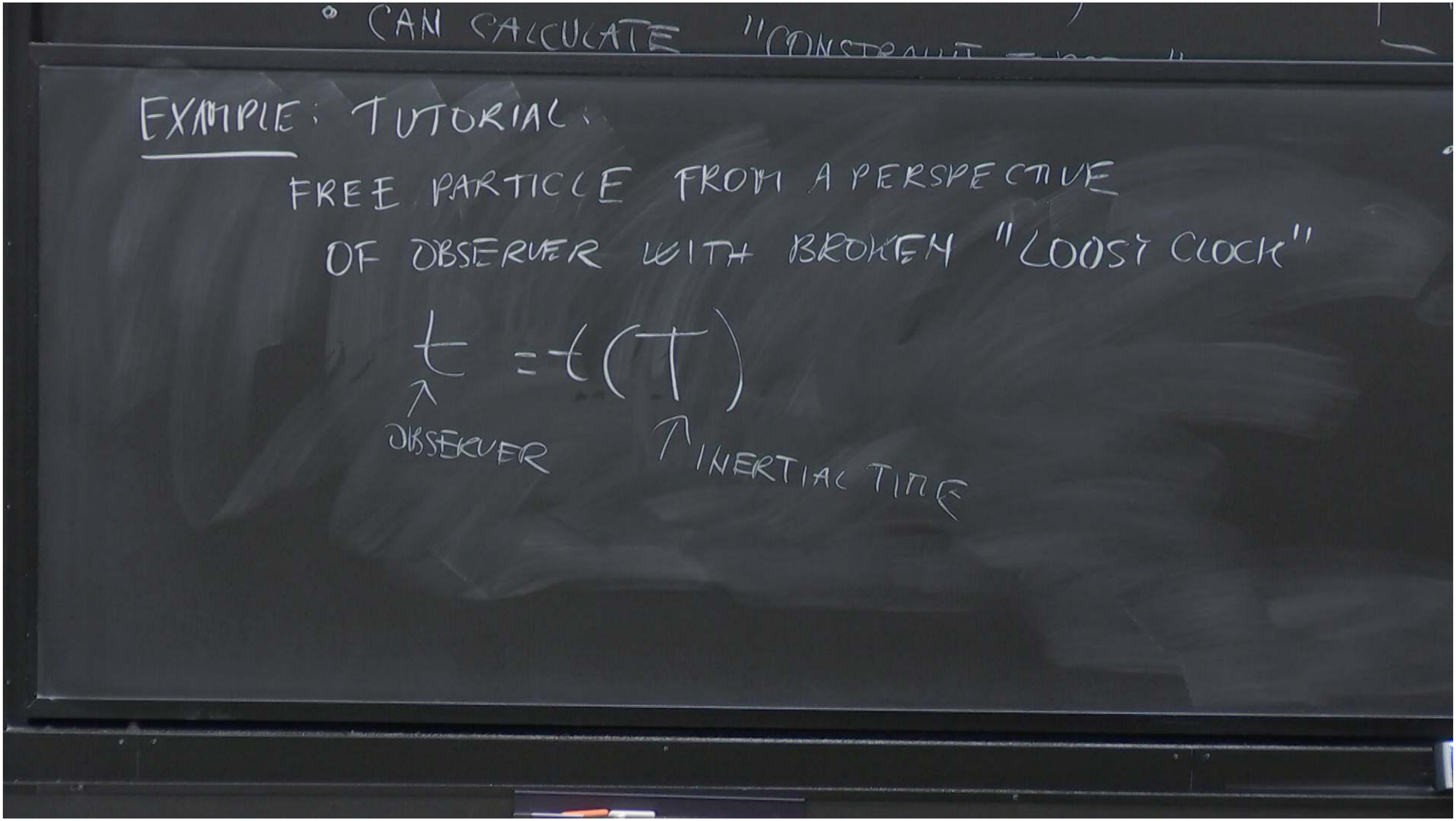
λ- DEPENDENT SOLUTION

RELATED BY THE SYMMETRY CORRESPONDING TO CONSTRAINT.

NAC

ONS


PHYSICS IS UNIQUE (1). BUT THE DESCRIPTION
WE HAVE CHOSEN IS NOT (1):
(FINAL ENDPOINTS RELATED BY SYMMETRY)
(2) HOW TO DECODE THE UNIQUE DESCR ?
"RELATIONAL QUESTIONS"


• CAN CALCULATE "CONSTRAINT
EXAMPLE: TUTORIAL:
FREE PARTICLE FROM A PERSPECTIVE
OF OBSERVER WITH BROKEN "LOOSE CLOCK"
t = t(T)
OBSERVER
INERTIAL TIME

• WE FIND THAT

$$\boxed{H = 0}$$ (BY NAIVE LEGENDRE TRANSF)

THERE IS A CONSTRAINT

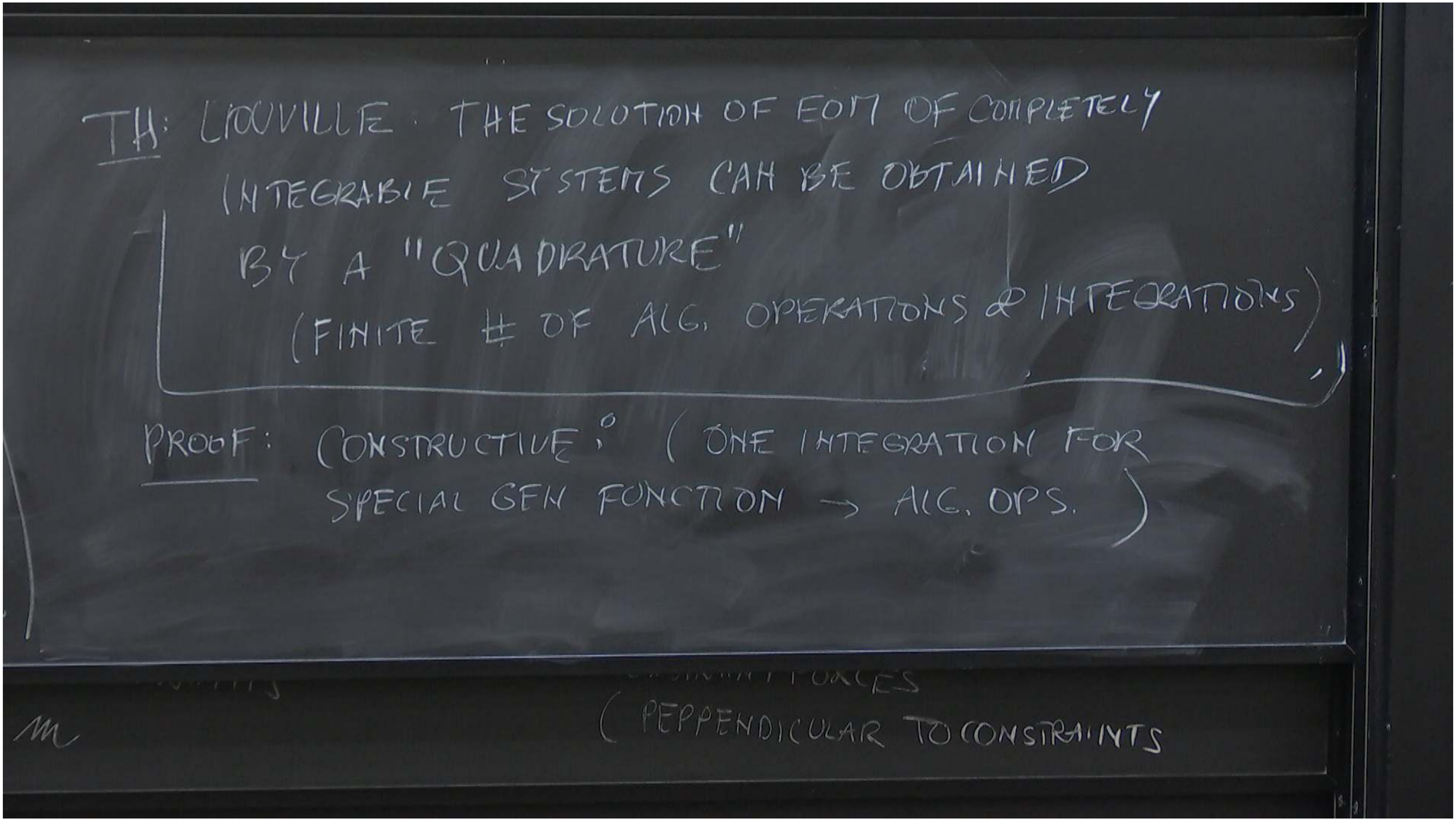
$$\phi = P_T + \frac{p_x^2}{2m} = 0$$

H_T .. NON-TRIVIAL ...... "TIME EVOLUTION"

g) INTEGRABLE SYSTEMS

DEF: SYSTEM WITH n DOF IS COMPLETELY INTEGRABLE IF IT ADMITS n INDEPENDENT INTEGRALS OF MOTION I_i ($\{H, I_i\} = 0$) THAT ARE IN INVOLUTION

$$\{I_i, I_j\} = 0 \quad \forall\, i, j = 1 \ldots n$$


TH: LIOUVILLE: THE SOLUTION OF EOM OF COMPLETELY
INTEGRABLE SYSTEMS CAN BE OBTAINED
BY A "QUADRATURE"
(FINITE # OF ALG. OPERATIONS & INTEGRATIONS)
PROOF: CONSTRUCTIVE: (ONE INTEGRATION FOR
SPECIAL GEN FUNCTION → ALG. OPS.)
FORCES
(PEPPENDICULAR TO CONSTRAINTS

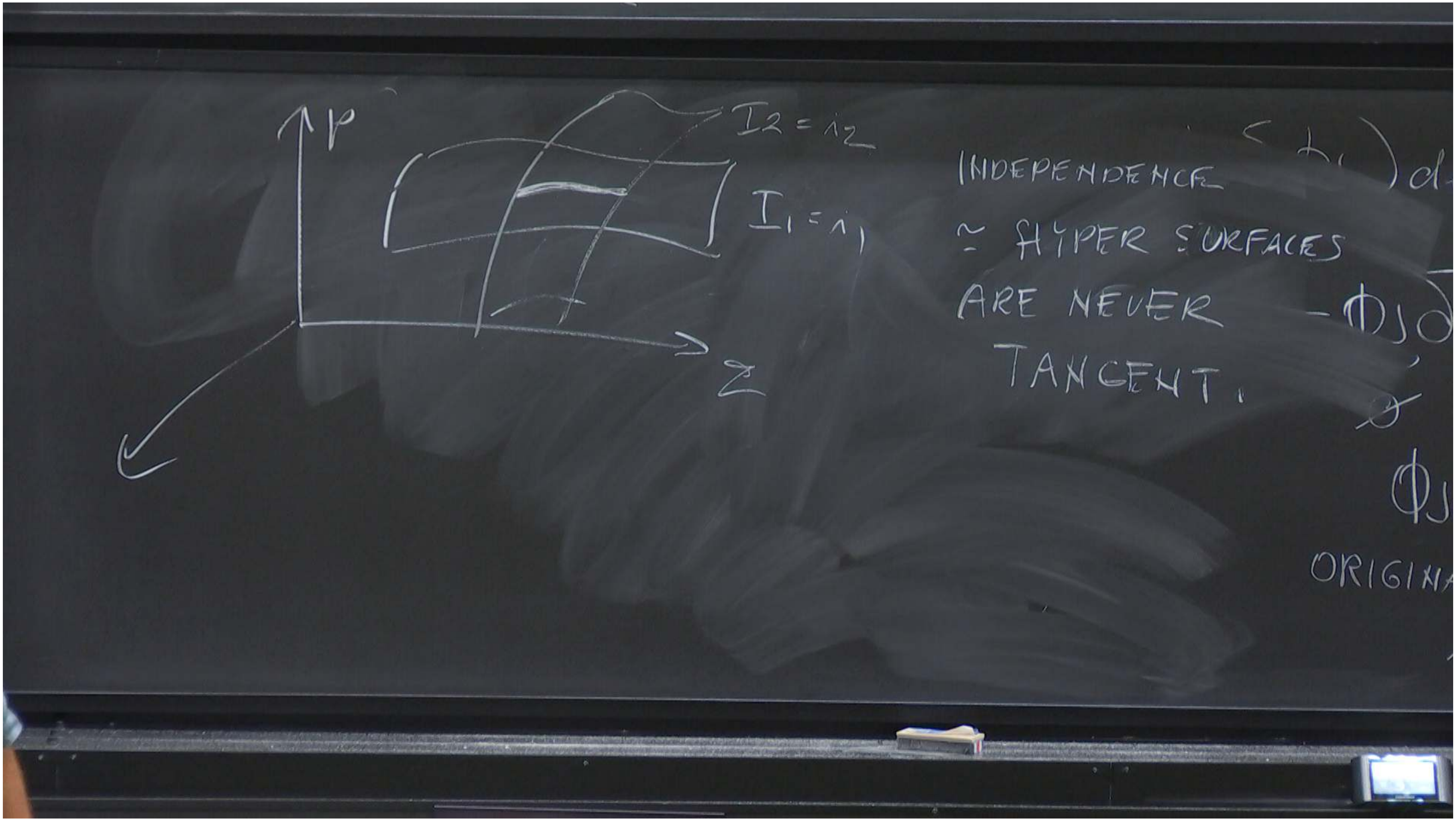
TH: LIOUVILLE: THE SOLUTION OF EOM OF COMPLETELY INTEGRABLE SYSTEMS CAN BE OBTAINED BY A "QUADRATURE" (FINITE # OF ALG. OPERATIONS & INTEGRATIONS).

PROOF: CONSTRUCTIVE: (ONE INTEGRATION FOR SPECIAL GEN FUNCTION → ALG. OPS.)

- $I_1 = \text{CONST} = I_1(q,p)$ $\qquad \boxed{\phi = I_1 - \lambda_1 = \emptyset}$

ENDED UP WITH $(n+m)$ EOM

FOR UNKNOWN $\{ \ldots \}$?


p
z
I2 = i2
I1 = i1
INDEPENDENCE
≃ HYPER SURFACES
ARE NEVER
TANGENT.