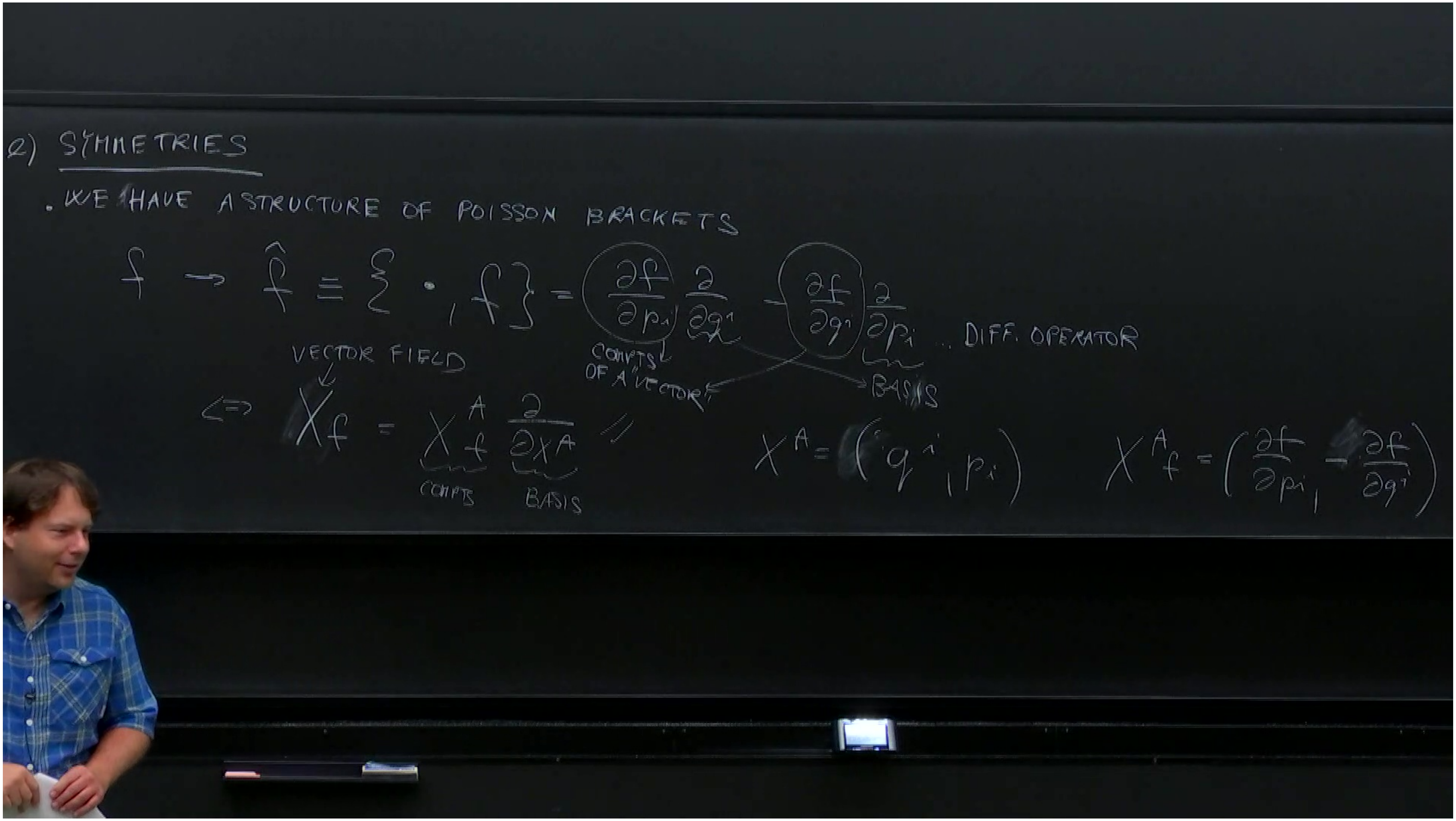
Title: Classical Physics - Lecture 220912

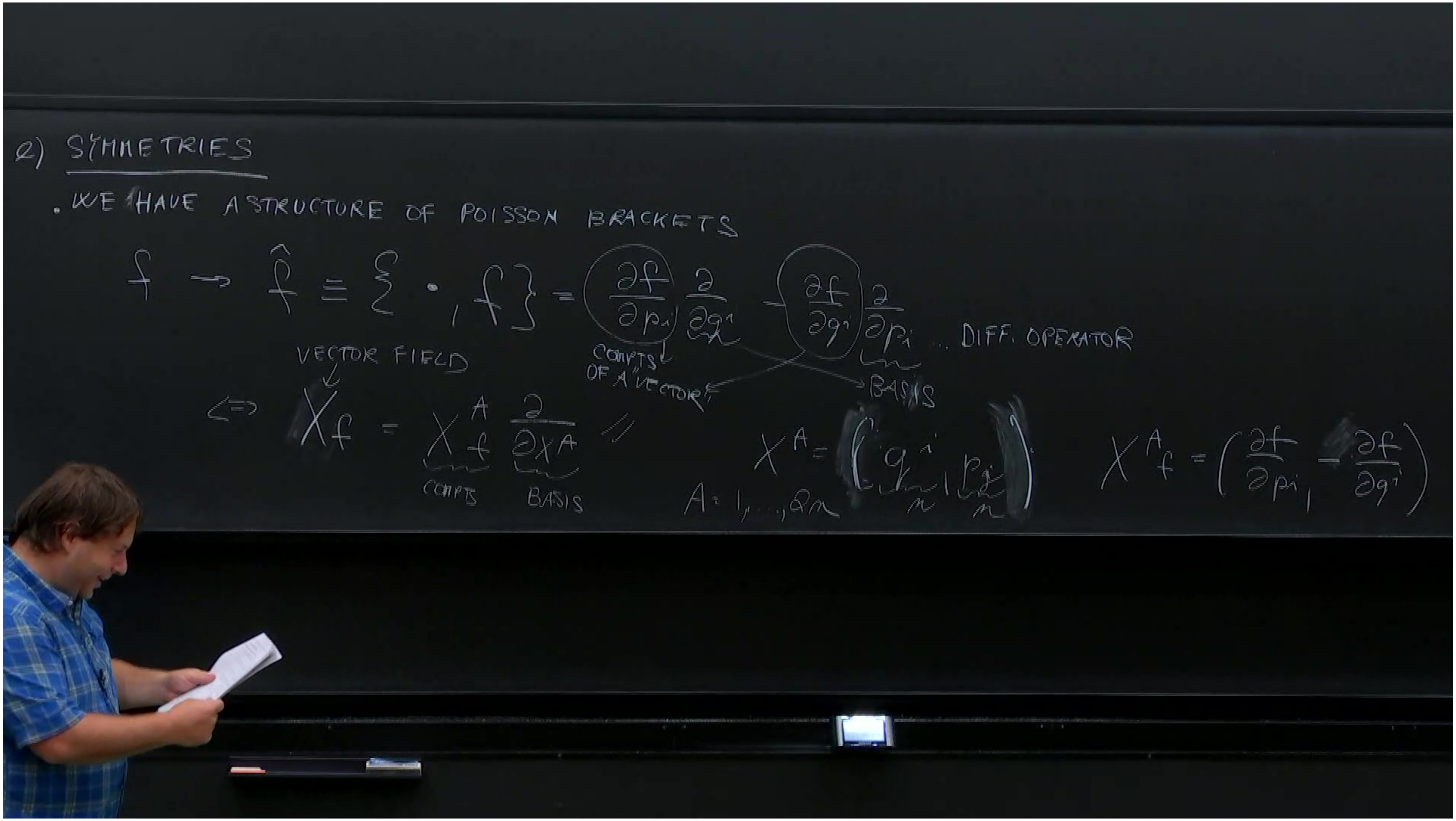
Speakers:

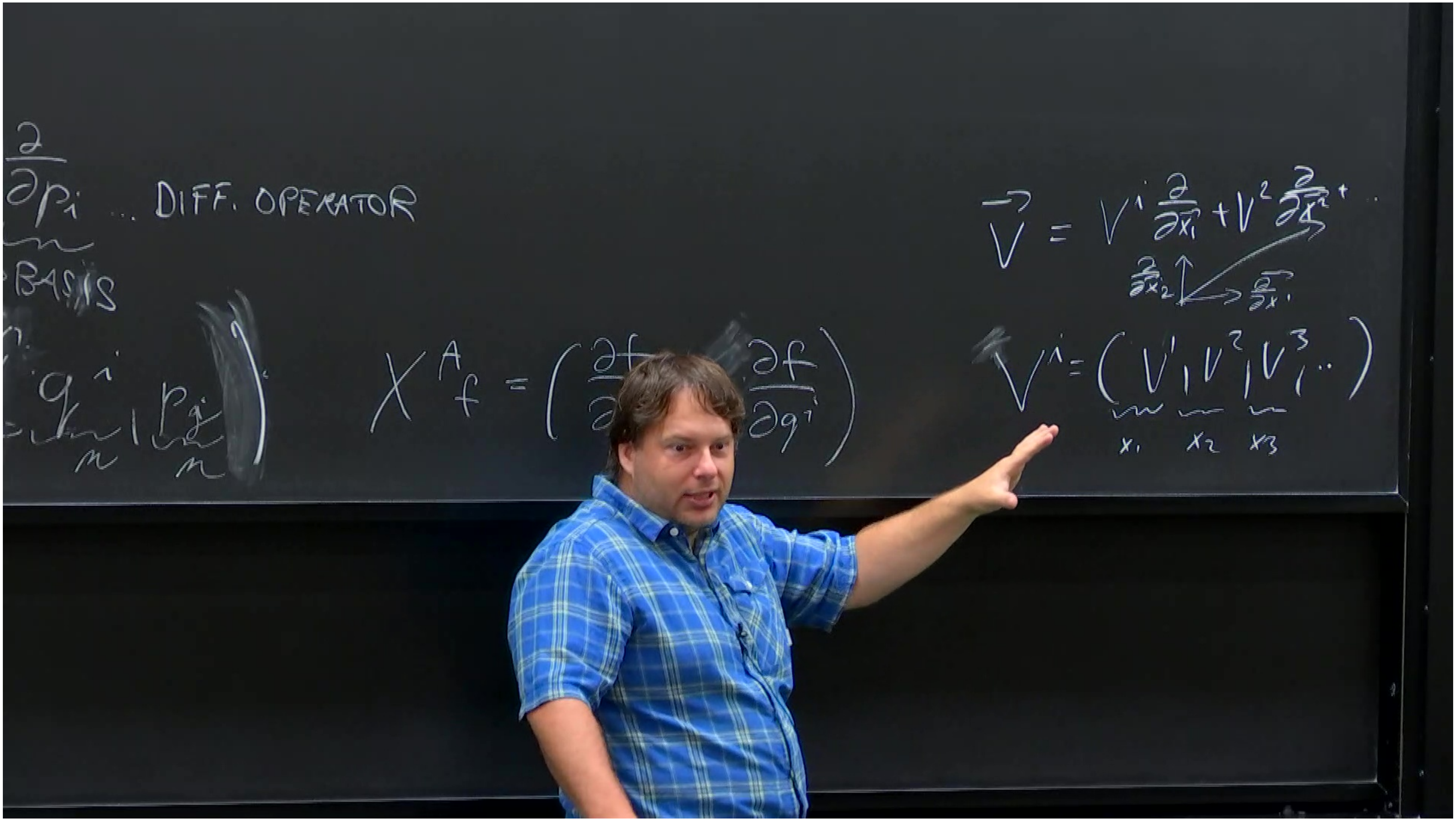
Collection: Classical Physics (2022/2023)

Date: September 12, 2022 - 9:00 AM

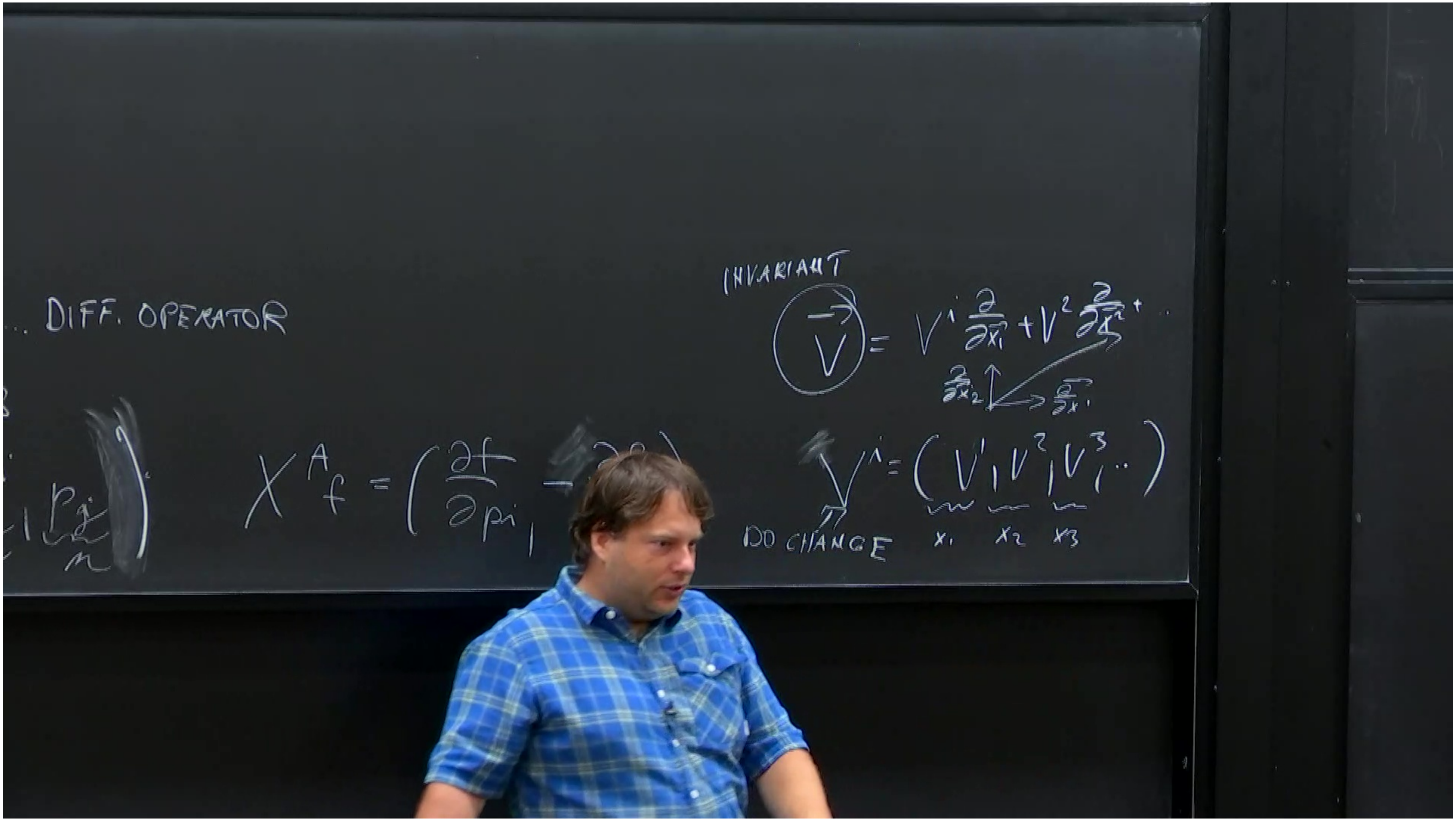
URL: https://pirsa.org/22090047


2) SYMMETRIES
. WE HAVE A STRUCTURE OF POISSON BRACKETS
$f \to \hat{f} \equiv \{\cdot, f\} = \underbrace{\frac{\partial f}{\partial p_i}}_{\text{COMPTS OF A "VECTOR"}} \underbrace{\frac{\partial}{\partial q^i}}_{\text{BASIS}} - \frac{\partial f}{\partial q^i} \frac{\partial}{\partial p_i}$ ... DIFF. OPERATOR
VECTOR FIELD
$\Leftrightarrow X_f = \underbrace{X^A_f}_{\text{COMPTS}} \underbrace{\frac{\partial}{\partial X^A}}_{\text{BASIS}}$ //
$X^A = (q^i, p_i)$
$X^A_f = \left(\frac{\partial f}{\partial p_i}, -\frac{\partial f}{\partial q^i}\right)$


2) SYMMETRIES
. WE HAVE A STRUCTURE OF POISSON BRACKETS
$f \rightsquigarrow \hat{f} \equiv \{\cdot, f\} = \frac{\partial f}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial f}{\partial q^i}\frac{\partial}{\partial p_i}$ ... DIFF. OPERATOR
VECTOR FIELD
COMPTS OF A "VECTOR"
BASIS
$\Longleftrightarrow X_f = \underbrace{X^A_f}_{\text{COMPTS}} \underbrace{\frac{\partial}{\partial x^A}}_{\text{BASIS}}$
$X^A = (\underbrace{q^i}_{n}, \underbrace{p_i}_{n})$
$A = 1, \dots, 2n$
$X^A_f = \left(\frac{\partial f}{\partial p_i}, -\frac{\partial f}{\partial q^i}\right)$



DIFF. OPERATOR
BASIS
$\vec{V} = V^1 \frac{\partial}{\partial x_1} + V^2 \frac{\partial}{\partial x_2} + \dots$
$V^i = (V^1, V^2, V^3, \dots)$
x_1 x_2 x_3


DIFF. OPERATOR
INVARIANT
DO CHANGE

VECTOR FIELD

COMPTS OF A "VECTOR"

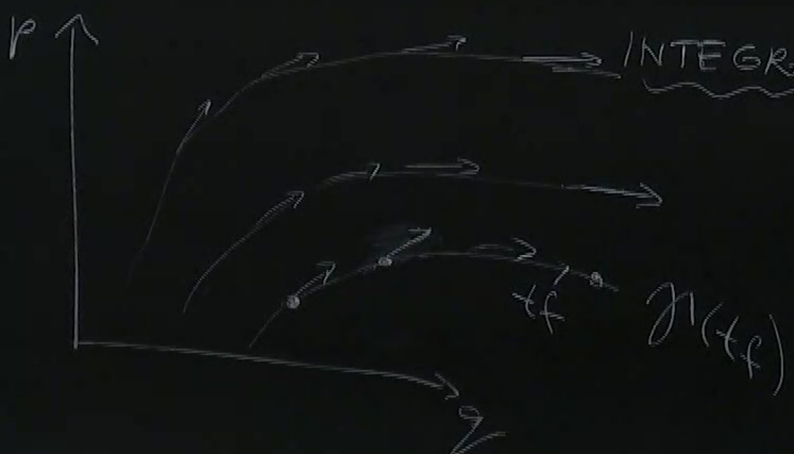
BASIS

$$\Leftrightarrow \quad X_f = \underbrace{X^A_f}_{\text{COMPTS}} \; \underbrace{\frac{\partial}{\partial X^A}}_{\text{BASIS}}$$

$$X^A = (\underbrace{q^i}_{n}, \underbrace{p_i}_{n}) \qquad A = 1, \dots, 2n$$

$$X^A_f = \left(\frac{\partial f}{\partial p_i}, \right.$$

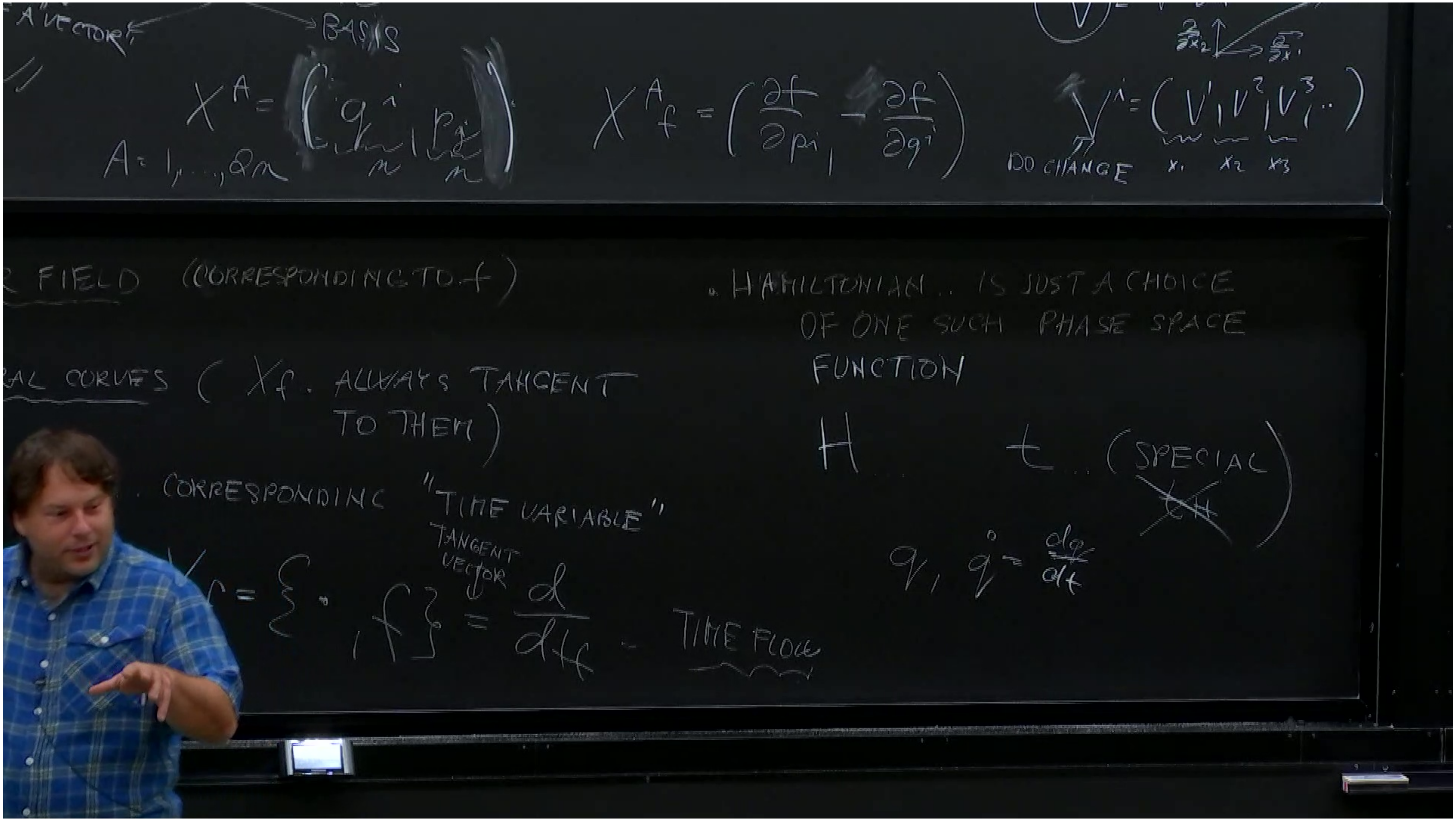
X_f ... HAMILTONIAN VECTOR FIELD (CORRESPONDING TO f)



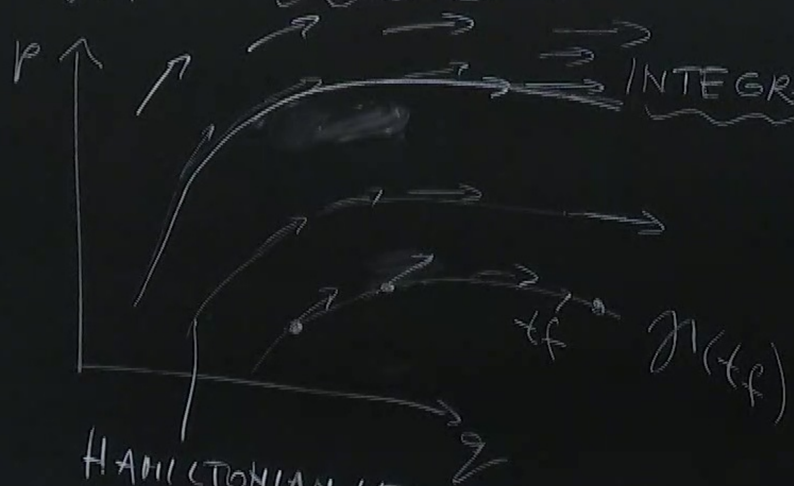
INTEGRAL CURVES (X_f ALWAYS TANGENT TO THEM)

t_f ... CORRESPONDING "TIME VARIABLE"

$$X_f = \{ \cdot , f \} = \frac{d}{dt_f} \quad \text{TIME FLOW}$$


A VECTOR
BASIS
$X^A = (q^i, p_i)$
$A = 1, \dots, 2n$
$X^A_f = \left(\frac{\partial f}{\partial p_i}, -\frac{\partial f}{\partial q^i}\right)$
$V^i = (V^1, V^2, V^3, \dots)$
DO CHANGE
x_1 x_2 x_3
FIELD (CORRESPONDING TO f)
AL CURVES (Xf . ALWAYS TANGENT TO THEM)
CORRESPONDING "TIME VARIABLE"
TANGENT VECTOR
$X_f = \{\cdot, f\} = \frac{d}{dt_f}$ - TIME FLOW
HAMILTONIAN .. IS JUST A CHOICE OF ONE SUCH PHASE SPACE FUNCTION
H
t .. (SPECIAL)
$q, \dot{q} = \frac{dq}{dt}$

X_f ... HAMILTONIAN VECTOR FIELD (CORRESPONDING TO f)



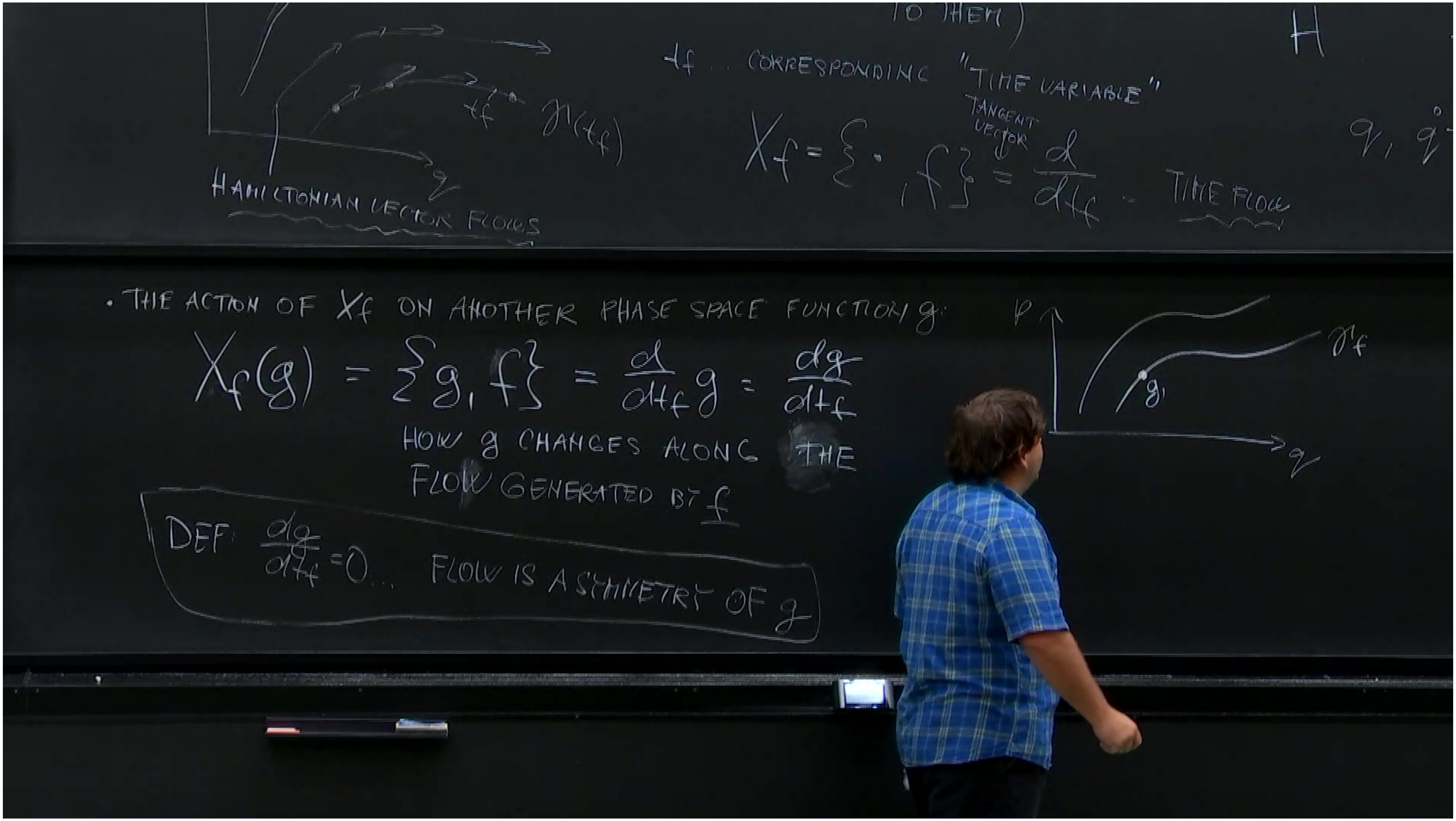
INTEGRAL CURVES (X_f . ALWAYS TANGENT TO THEM)

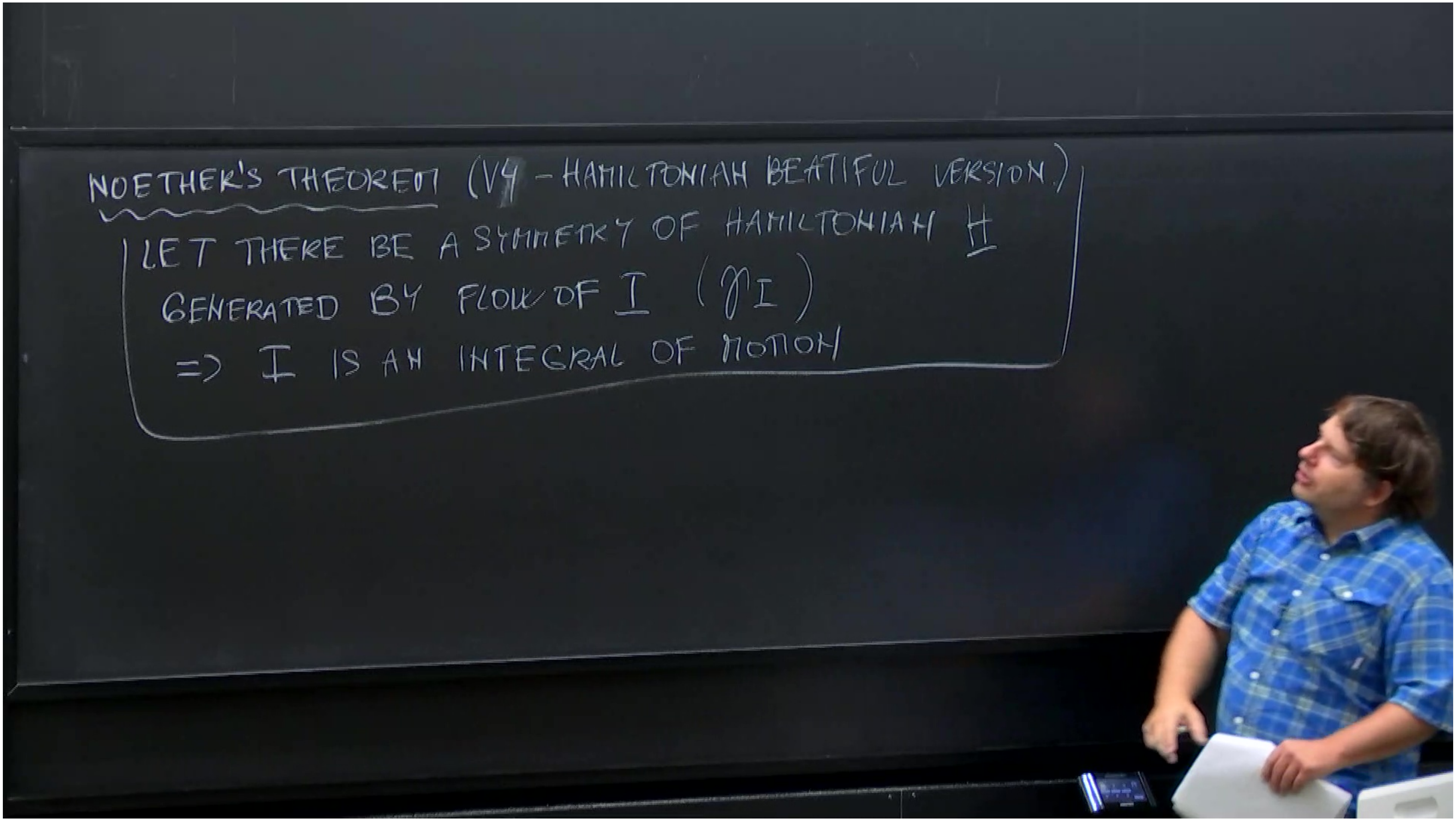
t_f ... CORRESPONDING "TIME VARIABLE"

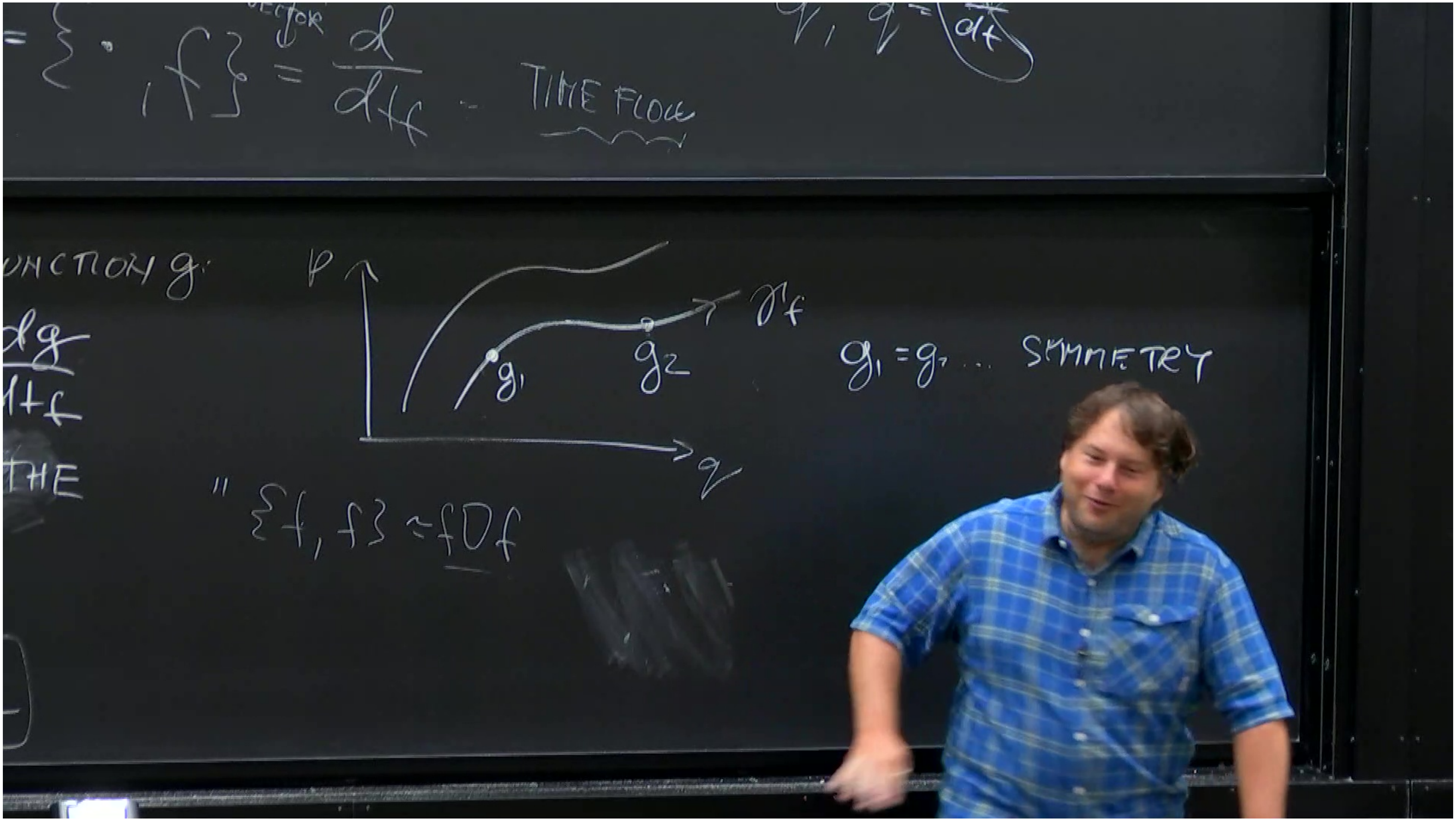
TANGENT VECTOR

$$X_f = \{\cdot\,, f\} = \frac{d}{dt_f} \; - \; \text{TIME}$$

HAMILTONIAN VECTOR FLOWS


HAMILTONIAN VECTOR FLOWS
t_f ... CORRESPONDING "TIME VARIABLE"
TANGENT VECTOR
$X_f = \{\cdot, f\} = \frac{d}{dt_f}$ ... TIME FLOW
H
• THE ACTION OF X_f ON ANOTHER PHASE SPACE FUNCTION g:
$X_f(g) = \{g, f\} = \frac{d}{dt_f} g = \frac{dg}{dt_f}$
HOW g CHANGES ALONG THE FLOW GENERATED BY f
DEF: $\frac{dg}{dt_f} = 0$ ... FLOW IS A SYMMETRY OF g


NOETHER'S THEOREM (V4 - HAMILTONIAN BEATIFUL VERSION)
LET THERE BE A SYMMETRY OF HAMILTONIAN H
GENERATED BY FLOW OF I (γ_I)
=> I IS AN INTEGRAL OF MOTION

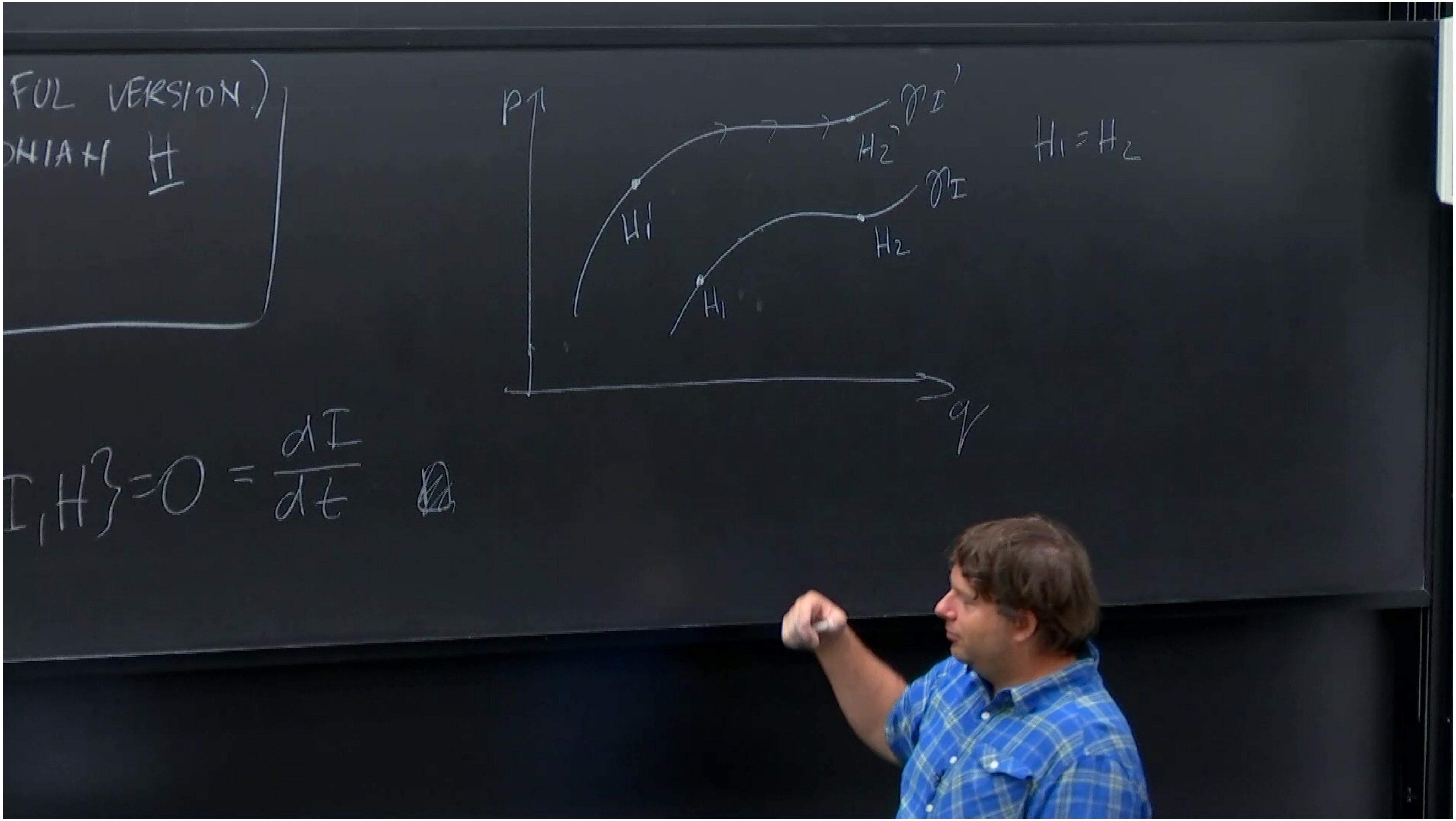

TIME FLOW
UNCTION g:
THE
SYMMETRY

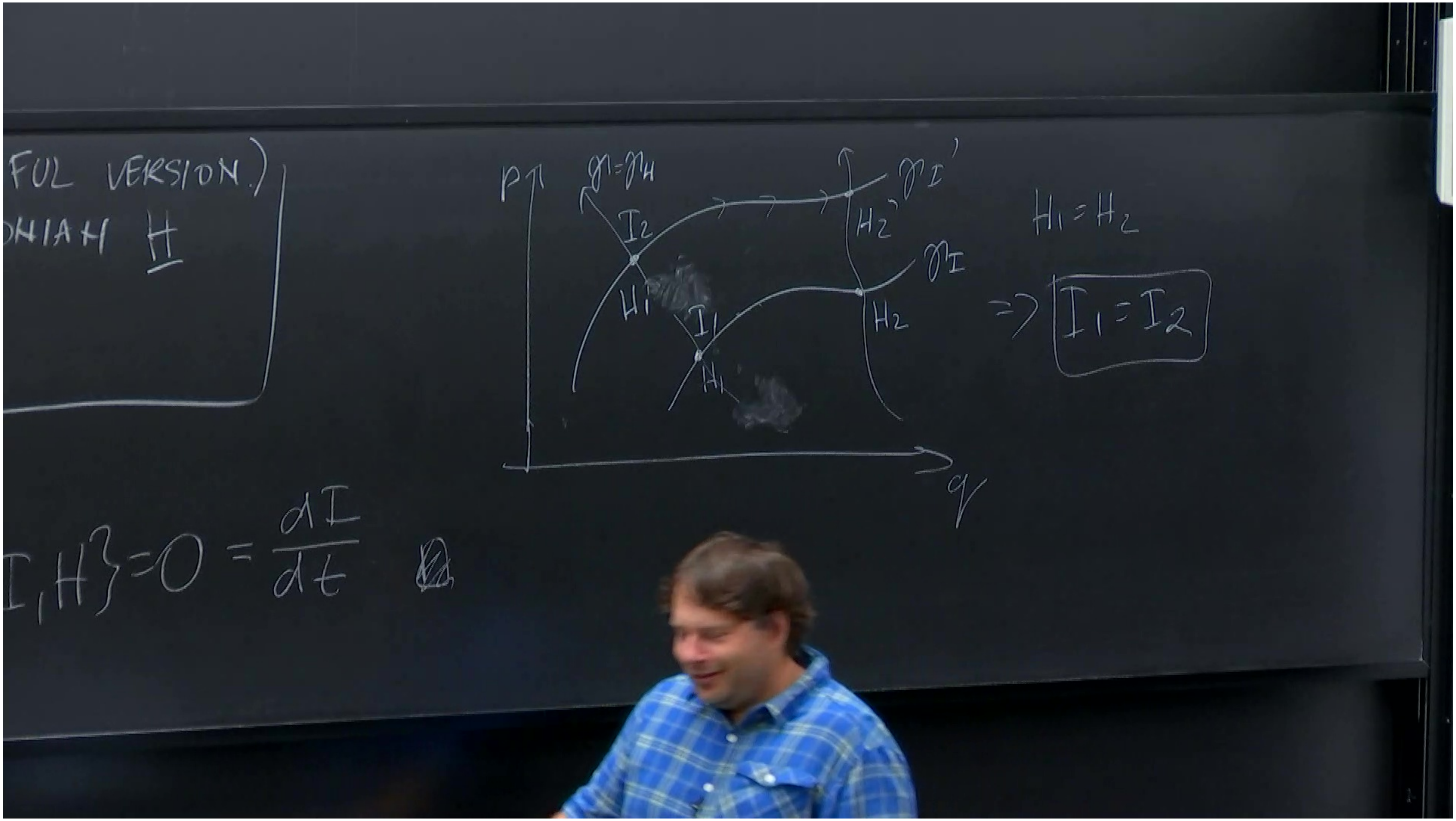
NOETHER'S THEOREM (V4 – HAMILTONIAN BEATIFUL VERSION)

> LET THERE BE A SYMMETRY OF HAMILTONIAN $\underline{H}$
> GENERATED BY FLOW OF $\underline{I}$ (γ_I)
> $\Rightarrow$ I IS AN INTEGRAL OF MOTION

PROOF: WE HAVE A SYMMETRY OF H.

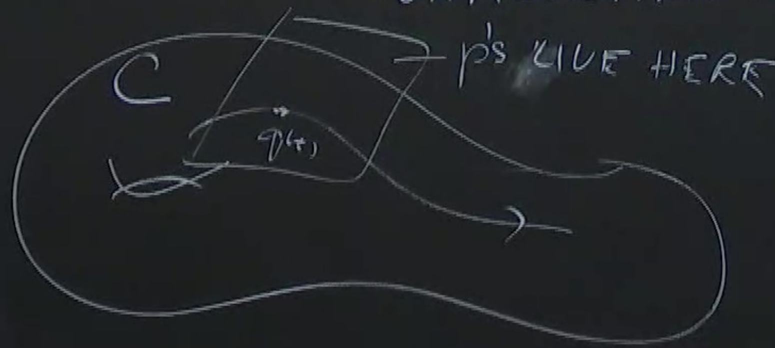
$$\{H, I\} = \frac{dH}{dt_I} = 0 \iff \{I, H\} = 0 = \frac{dI}{dt} \quad \blacksquare$$


FUL VERSION)
ONIAN H
p
q
H1
H2
H1 = H2
I,H}=0 = dI/dt

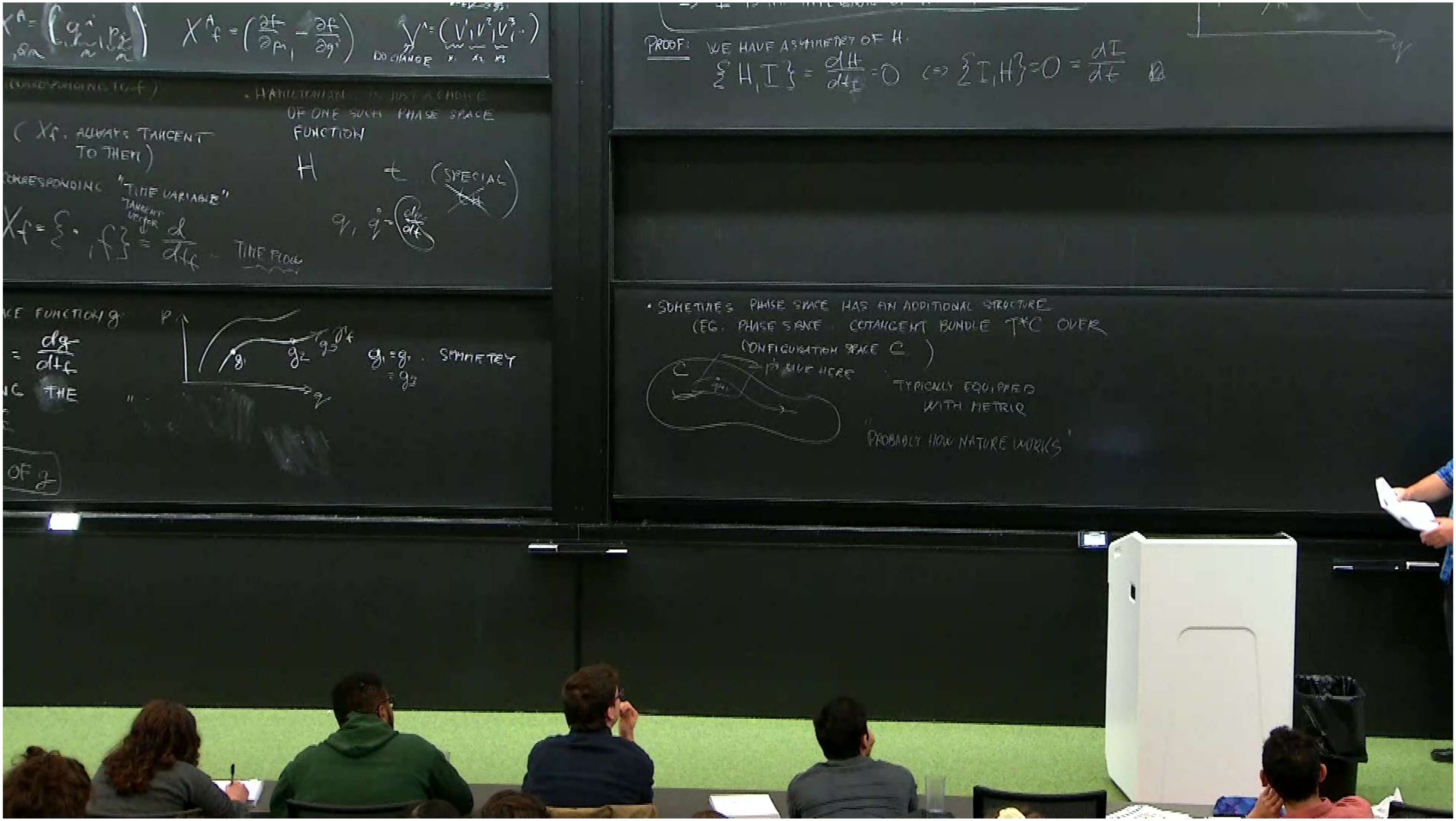

FUL VERSION.)
H1 = H2
=> I1 = I2
dI/dt

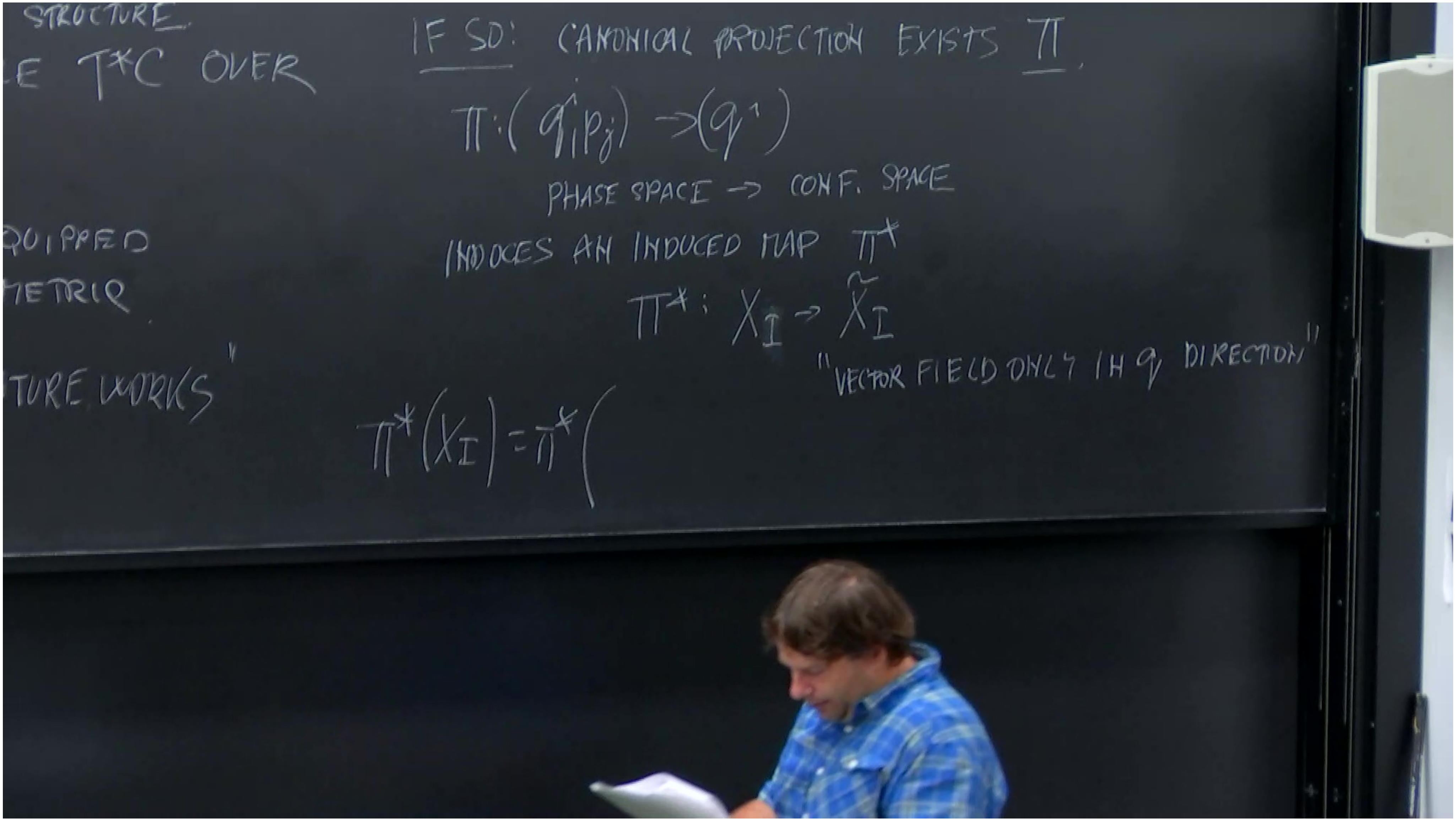
• SOMETIMES PHASE SPACE HAS AN ADDITIONAL STRUCTURE.

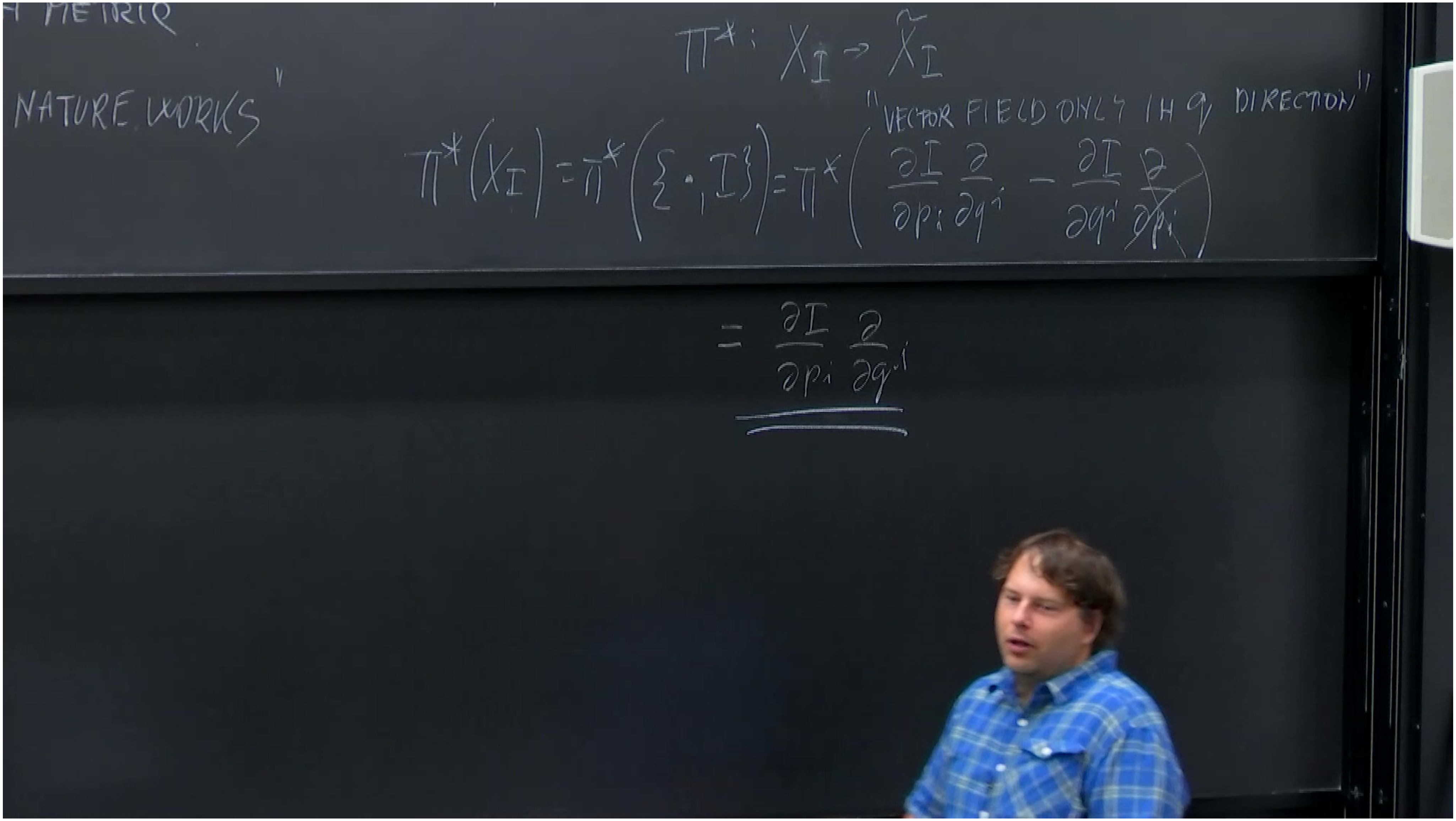
(EG, PHASE SPACE ... COTANGENT BUNDLE T^*C OVER CONFIGURATION SPACE C)

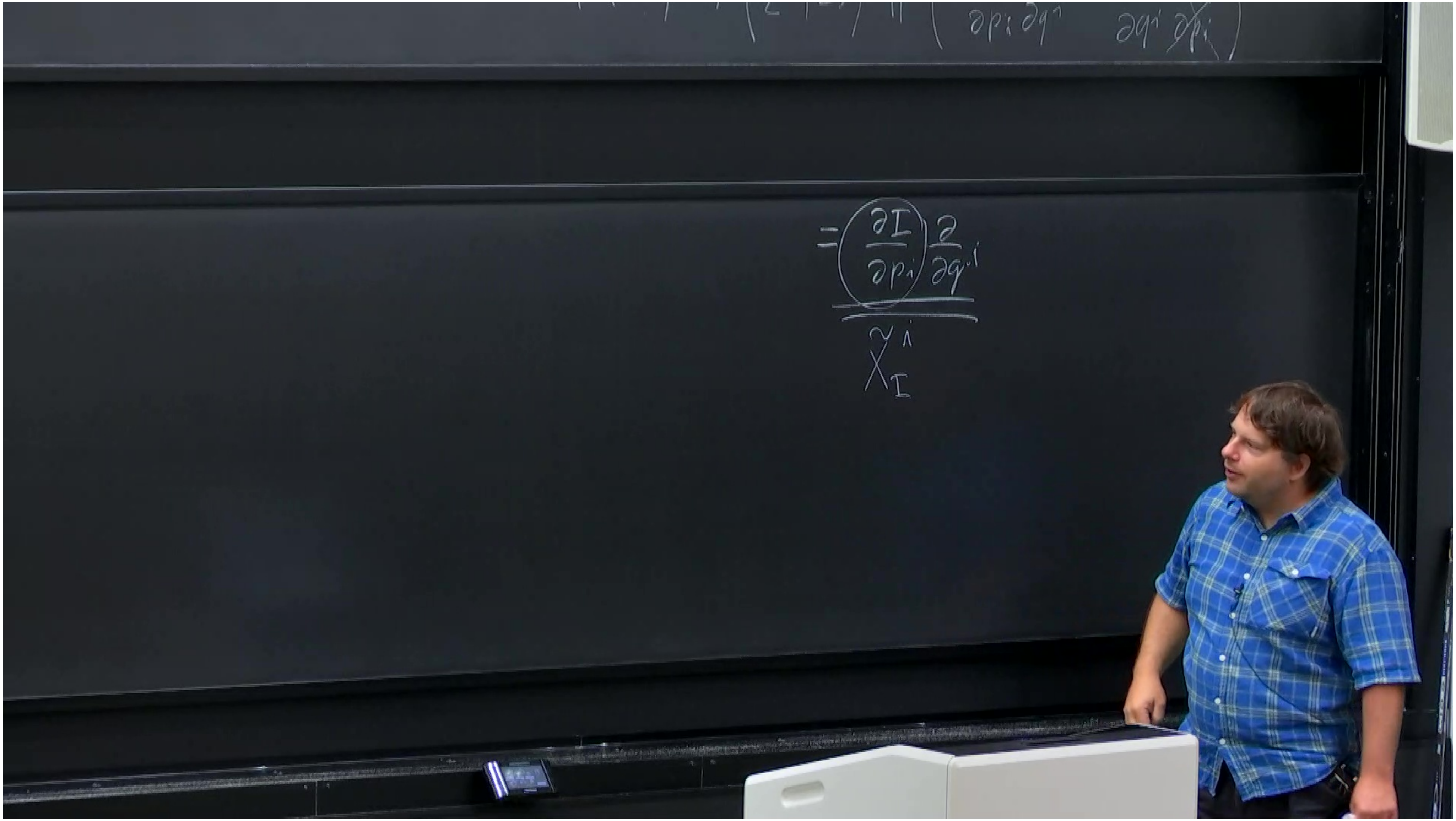


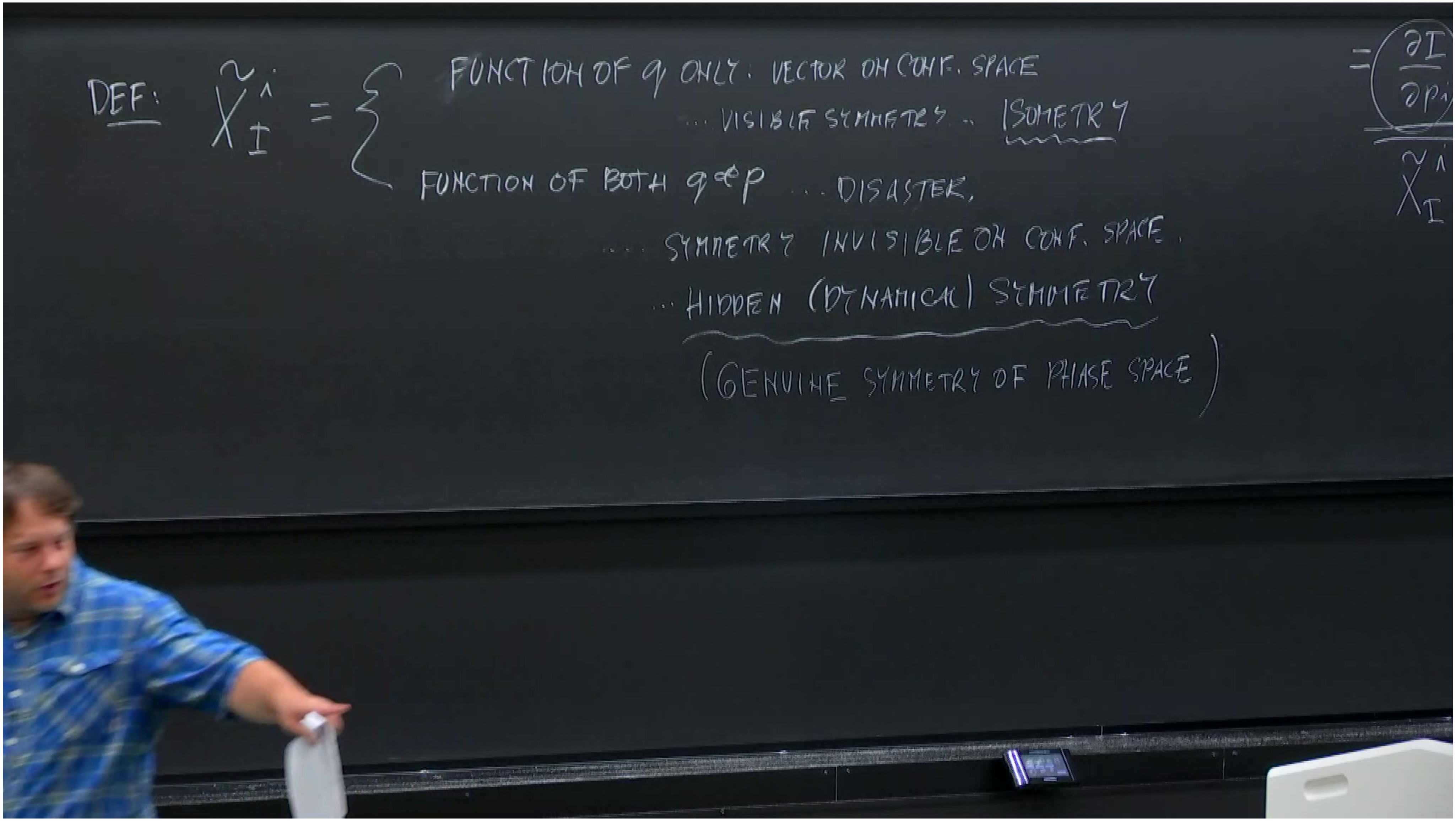
TYPICALLY EQUIPPED WITH METRIC

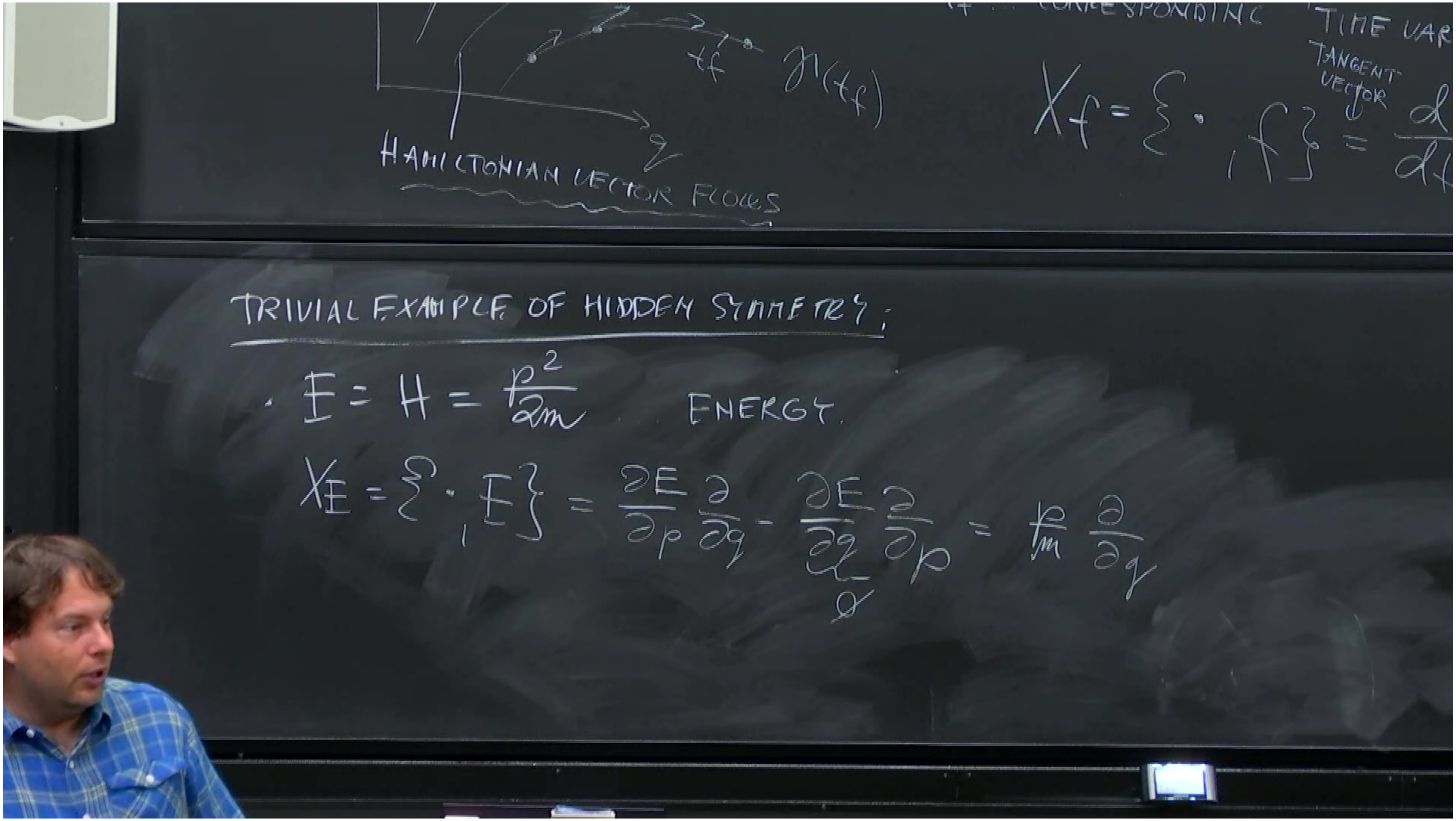

PROOF: WE HAVE A SYMMETRY OF H.
{H, I} = dH/dt_I = 0 ⟺ {I, H} = 0 = dI/dt
OF ONE SUCH PHASE SPACE FUNCTION
(X_f ALWAYS TANGENT TO THEM)
CORRESPONDING "TIME VARIABLE"
TANGENT VECTOR
TIME FLOW
(SPECIAL
SYMMETRY
• SOMETIMES PHASE SPACE HAS AN ADDITIONAL STRUCTURE
(EG. PHASE SPACE COTANGENT BUNDLE T*C OVER
CONFIGURATION SPACE C)
TYPICALLY EQUIPPED WITH METRIC
"PROBABLY HOW NATURE WORKS"


STRUCTURE.
E T*C OVER
QUIPPED
METRIC
TURE WORKS"
IF SO: CANONICAL PROJECTION EXISTS π.
π : (q^i, p_j) → (q^i)
PHASE SPACE → CONF. SPACE
INDUCES AN INDUCED MAP π*
π* : X_I → X̃_I
"VECTOR FIELD ONLY IN q DIRECTION"
π*(X_I) = π*(

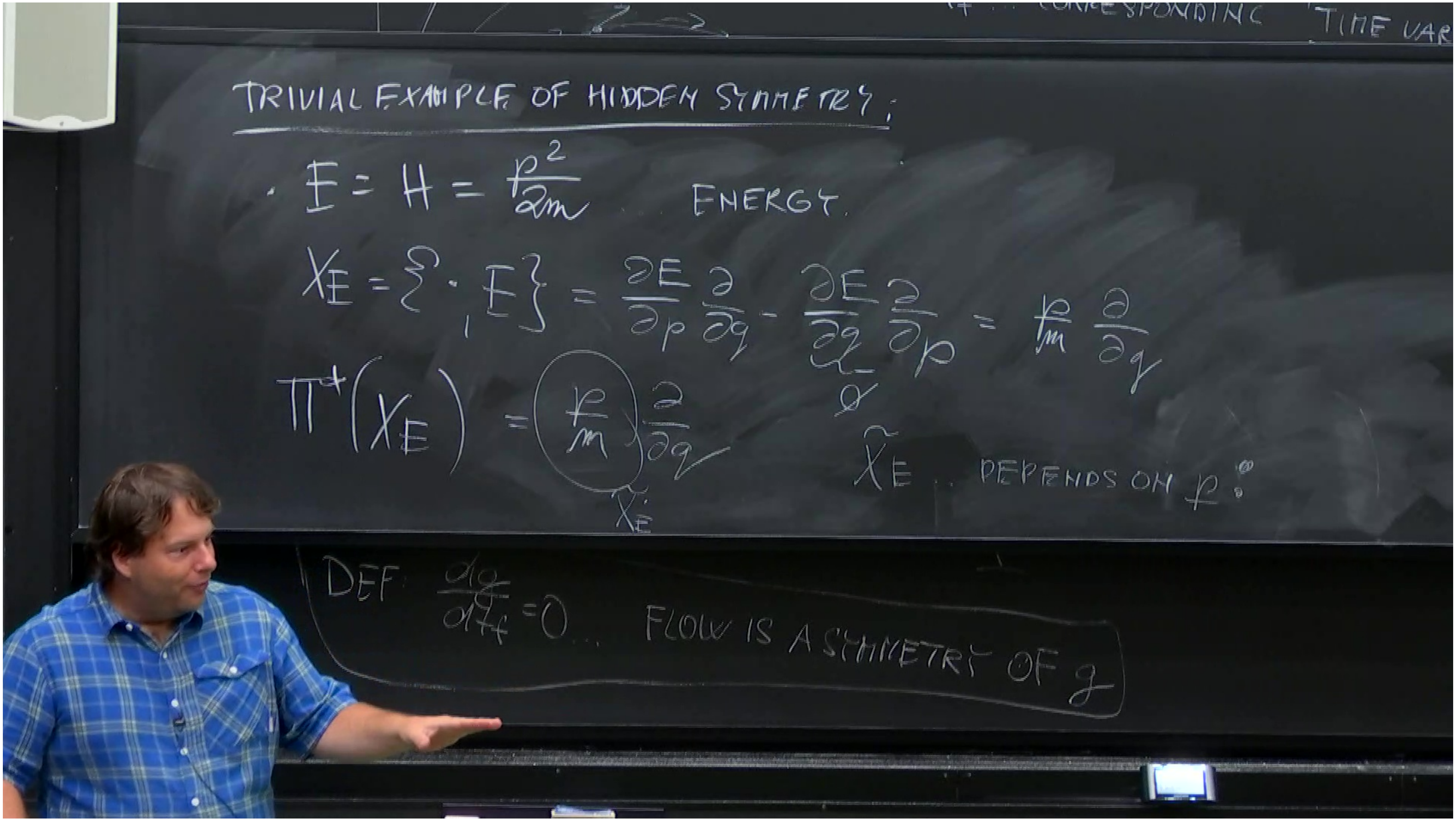

METRIC
"NATURE WORKS"
$\pi^*: X_I \to \tilde{X}_I$
"VECTOR FIELD ONLY IN q DIRECTION"
$\pi^*(X_I) = \pi^*(\{\cdot, I\}) = \pi^*\left(\frac{\partial I}{\partial p_i}\frac{\partial}{\partial q^i} - \cancel{\frac{\partial I}{\partial q^i}\frac{\partial}{\partial p_i}}\right)$
$= \frac{\partial I}{\partial p_i}\frac{\partial}{\partial q^i}$

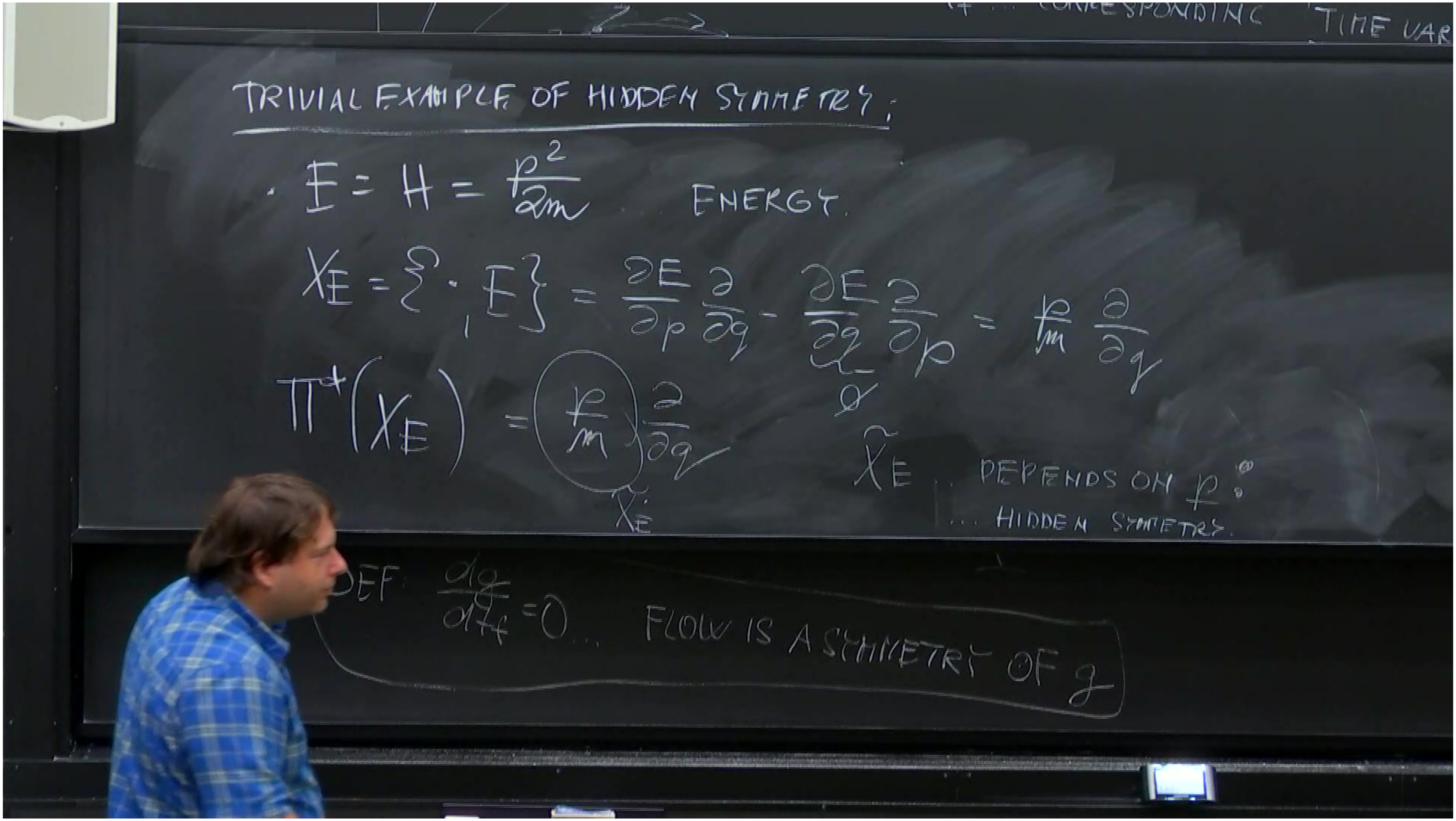

$= \frac{\partial I}{\partial p_i} \frac{\partial}{\partial q^i}$
$\tilde{X}_I^{\,i}$

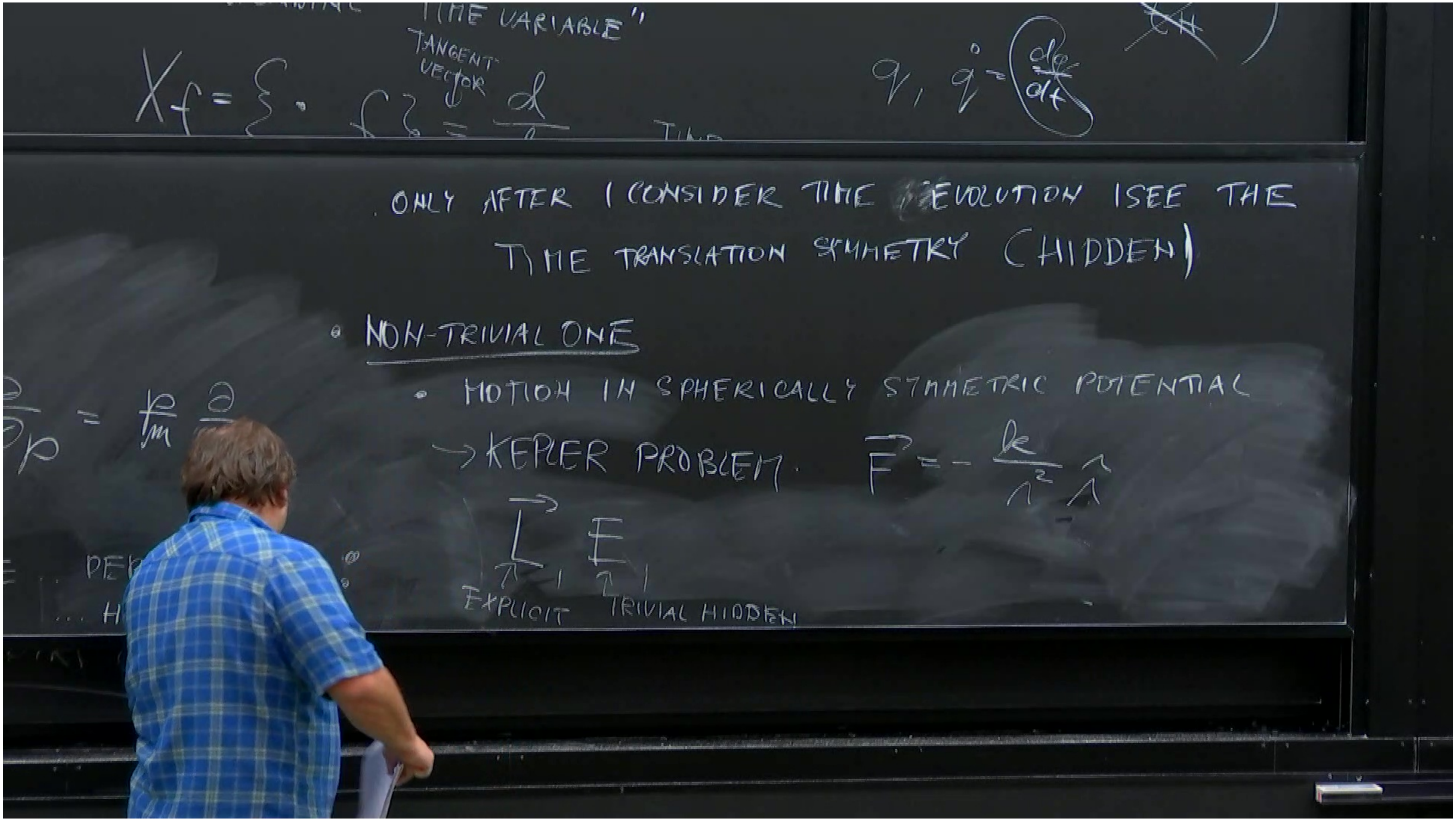

DEF: $\tilde{X}_{I}^{\hat{i}}$ = FUNCTION OF q ONLY: VECTOR ON CONF. SPACE
... VISIBLE SYMMETRY .. ISOMETRY
FUNCTION OF BOTH q & p ... DISASTER.
.... SYMMETRY INVISIBLE ON CONF. SPACE.
... HIDDEN (DYNAMICAL) SYMMETRY
(GENUINE SYMMETRY OF PHASE SPACE)

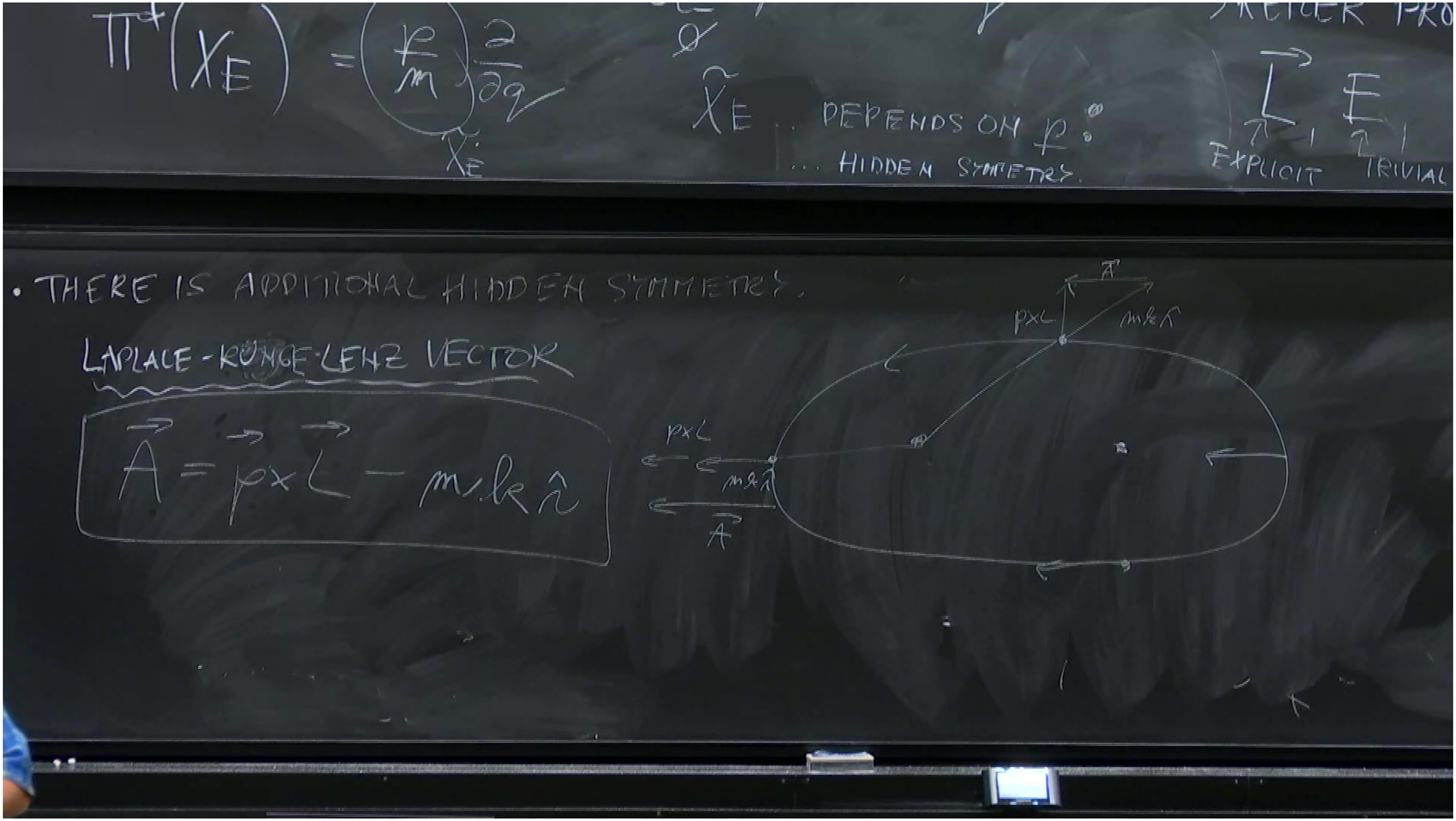


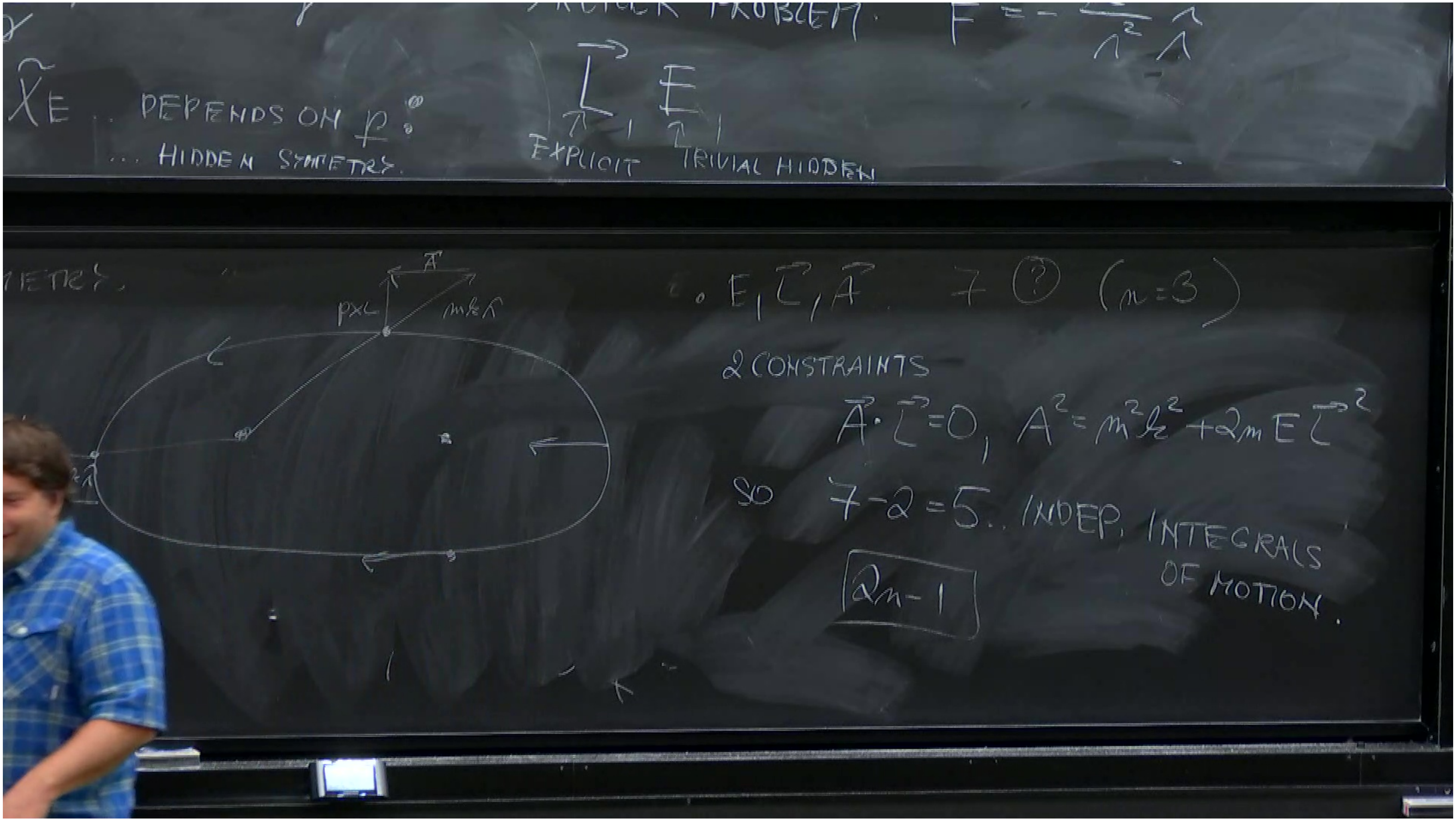
CORRESPONDING
TIME VAR
TANGENT VECTOR
$\gamma(t_f)$
t_f
q
HAMILTONIAN VECTOR FLOWS
$X_f = \{\cdot, f\} = \frac{d}{dt}$
TRIVIAL EXAMPLE OF HIDDEN SYMMETRY:
$E = H = \frac{p^2}{2m}$
ENERGY
$X_E = \{\cdot, E\} = \frac{\partial E}{\partial p}\frac{\partial}{\partial q} - \frac{\partial E}{\partial q}\frac{\partial}{\partial p} = \frac{p}{m}\frac{\partial}{\partial q}$

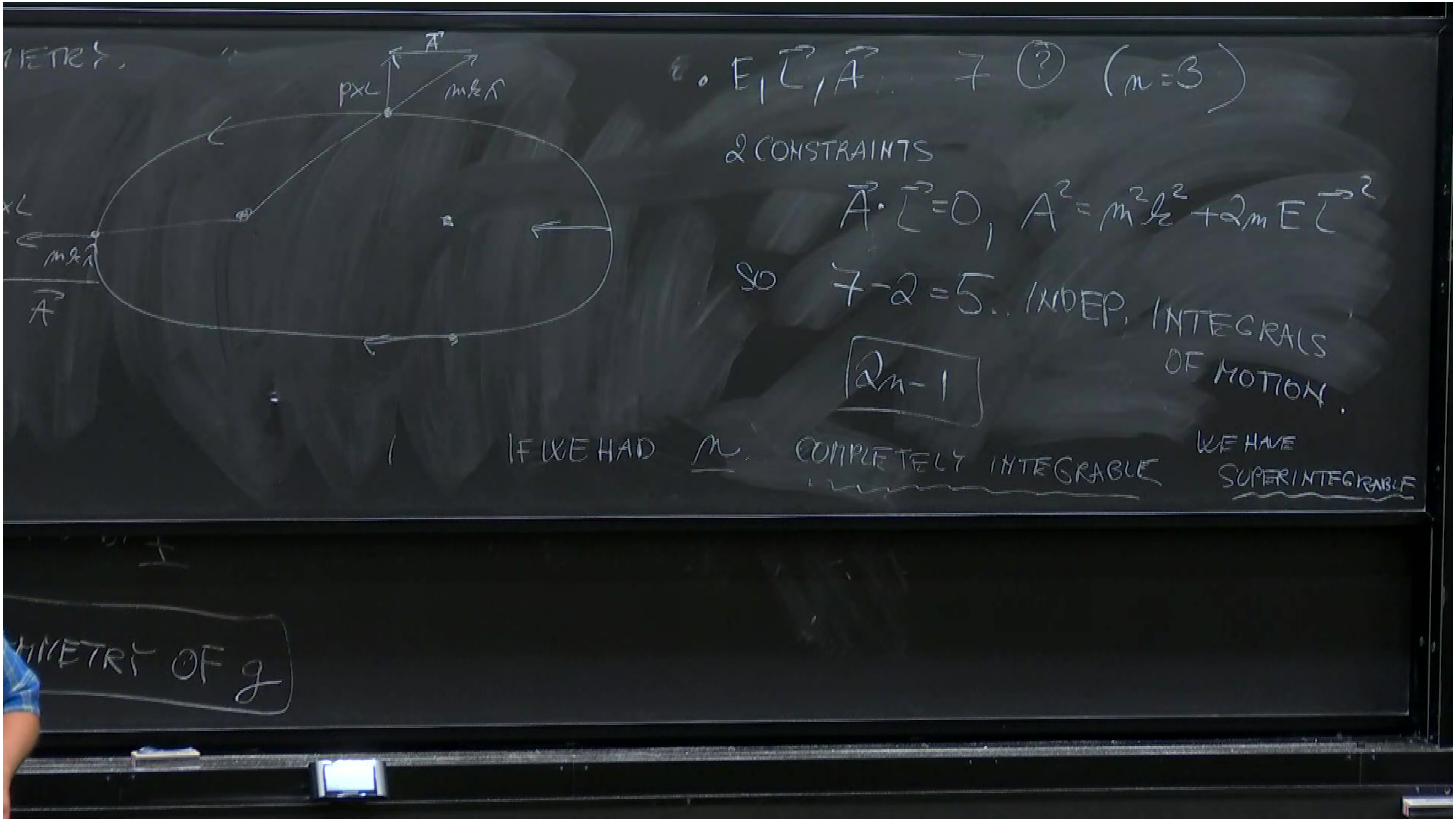

TRIVIAL EXAMPLE OF HIDDEN SYMMETRY:
$E = H = \frac{p^2}{2m}$ ENERGY
$X_E = \{\cdot, E\} = \frac{\partial E}{\partial p}\frac{\partial}{\partial q} - \frac{\partial E}{\partial q}\frac{\partial}{\partial p} = \frac{p}{m}\frac{\partial}{\partial q}$
$\pi^*(X_E) = \frac{p}{m}\frac{\partial}{\partial q}$
$\tilde{X}_E$
$\tilde{X}_E$ DEPENDS ON p!
DEF: $\frac{dg}{d\tau_f} = 0$ ... FLOW IS A SYMMETRY OF g

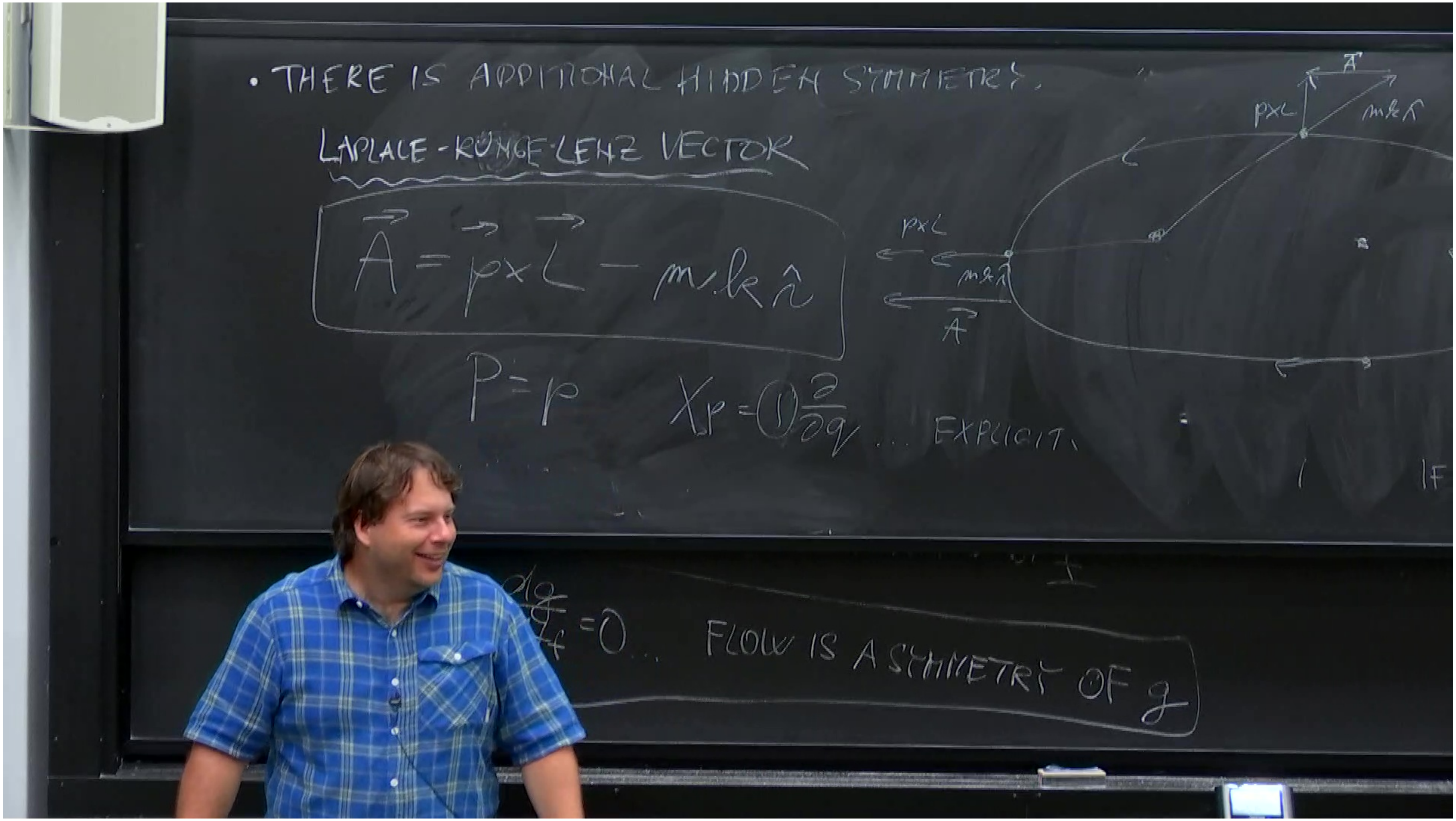

TRIVIAL EXAMPLE OF HIDDEN SYMMETRY:
$E = H = \frac{p^2}{2m}$ ENERGY
$X_E = \{\cdot, E\} = \frac{\partial E}{\partial p}\frac{\partial}{\partial q} - \frac{\partial E}{\partial q}\frac{\partial}{\partial p} = \frac{p}{m}\frac{\partial}{\partial q}$
$\pi^*(X_E) = \frac{p}{m}\frac{\partial}{\partial q}$
$\tilde{X}_E$ ... DEPENDS ON p!
... HIDDEN SYMMETRY.
DEF: $\frac{dg}{dt_f} = 0$ ... FLOW IS A SYMMETRY OF g


"TIME VARIABLE"
TANGENT VECTOR
$q, \dot{q} = \frac{dq}{dt}$
ONLY AFTER I CONSIDER TIME EVOLUTION I SEE THE
TIME TRANSLATION SYMMETRY (HIDDEN)
NON-TRIVIAL ONE
MOTION IN SPHERICALLY SYMMETRIC POTENTIAL
→ KEPLER PROBLEM. $\vec{F} = -\frac{k}{r^2}\hat{r}$
$\vec{L}$, E,
EXPLICIT TRIVIAL HIDDEN


$\Pi^{*}(X_E) = \frac{p}{m}\frac{\partial}{\partial q}$
$\tilde{X}_E$
$\tilde{X}_E$ .. DEPENDS ON p!
... HIDDEN SYMMETRY.
$\vec{L}$, E
EXPLICIT
TRIVIAL
• THERE IS ADDITIONAL HIDDEN SYMMETRY.
LAPLACE-RUNGE-LENZ VECTOR
$\boxed{\vec{A} = \vec{p}\times\vec{L} - mk\hat{r}}$
$\vec{A}$
$p\times L$
$mk\hat{r}$
$p\times L$
$mk\hat{r}$
$\vec{A}$


PROBLEM. F = -
DEPENDS ON p
... HIDDEN SYMMETRY.
L, E,
EXPLICIT TRIVIAL HIDDEN
pxL
A
mk r
E, L, A 7 ? (n=3)
2 CONSTRAINTS
A · L = 0, A² = m²k² + 2mEL²
SO 7-2=5 .. INDEP. INTEGRALS OF MOTION.
2n-1


E, L, A 7 ? (n=3)
2 CONSTRAINTS
A·L=0, A^2=m^2k^2+2mEL^2
SO 7-2=5 INDEP. INTEGRALS OF MOTION.
2n-1
IF WE HAD n COMPLETELY INTEGRABLE
WE HAVE SUPERINTEGRABLE
pxL
mkr
OF g


• THERE IS ADDITIONAL HIDDEN SYMMETRY.
LAPLACE-RUNGE-LENZ VECTOR
A = p × L − mk r̂
P = p
X_P = (1) ∂/∂q ... EXPLICIT.
FLOW IS A SYMMETRY OF g
p×L
mkr̂
A