Title: Classical Physics - Lecture 220909

Speakers:

Collection: Classical Physics (2022/2023)

Date: September 09, 2022 - 9:00 AM

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YESTERDAY:

DEF: A CANONICAL TRANSFORMATION

$$Q^j = Q^j(q^i, p_i, t), \quad P_j = P_j(q^i, p_i, t)$$

PRESERVES THE FORM OF HAMILTON'S EQS.
THAT IS, WE CAN FIND NEW H':

$$\frac{\partial H'}{\partial P_j} = \dot{Q}^j, \quad \frac{\partial H'}{\partial Q^j} = -\dot{P}_j$$

- GENER

E.G.

$S =$

$=$

IF

- GENERATED BY GENERATING FUNCTIONS $F = F(\text{OLD}, \text{NEW})$

E.G. $F(q, Q, t)$

FREEDOM

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt}\right) dt = \int \left(\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right)\right) d$$

$$= \int \left(P\dot{Q} - H'\right) dt$$

IF $\boxed{p_i = \frac{\partial F}{\partial q^i}, \quad P_i = -\frac{\partial F}{\partial Q^i}, \quad H' = H + \frac{\partial F}{\partial t}}$ $\Rightarrow$ $\boxed{\frac{\partial H'}{\partial P_j} = \dot{Q}}$

GENERATING FUNCTIONS $F = F(\text{OLD}, \text{NEW})$

$t)$

FREEDOM

$$-H - \frac{dF}{dt}\Big)dt = \int\left(\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right)\right)dt$$

$$\dot{Q} - H'\Big)dt$$

$$\boxed{= \frac{\partial F}{\partial q^i}, \quad P_i = -\frac{\partial F}{\partial Q^i}, \quad H' = H + \frac{\partial F}{\partial t}} \Rightarrow \boxed{\frac{\partial H'}{\partial P_j} = \dot{Q}_j, \quad \frac{\partial H'}{\partial Q^j} = -\dot{P}_j}$$

- GENERATED BY GENERATING FUNCTIONS $F = F(\text{OLD}, \text{NEW})$

E.G. $F(q, Q, t)$

FREEDOM

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt = \int \left(\dot{q} \underbrace{\left(p - \frac{\partial F}{\partial q} \right)}_{\cancel{\partial}} \underbrace{- \frac{\partial F}{\partial Q}}_{P} \dot{Q} - \underbrace{\left(H + \frac{\partial F}{\partial t} \right)}_{H'} \right)$$

WE WANT

$$\overset{!}{=} \int \left(P\dot{Q} - H' \right) dt$$

$$\text{IF} \quad \boxed{p_i = \frac{\partial F}{\partial q^i}, \quad P_i = -\frac{\partial F}{\partial Q^i}, \quad H' = H + \frac{\partial F}{\partial t}} \Rightarrow \boxed{\frac{\partial H'}{\partial P_j} = \ldots}$$

REMARKS: i) $\{f, g\}_{q,p} = \{f, g\}_{Q,P}$ $\qquad \frac{df}{dt} = \{$

$g = H$

TIME EVOLUTION OF $\underline{f}$ AS GENERATED BY HAMILTONIAN $\underline{g}$.

IS INDEPENDENT OF CHOICE OF COORDS.

ii) (Q, P) ARE NEW CANONICAL COORDS.

$$f, H\} + \cancel{\frac{\partial f}{\partial t}}$$

$$\{f, g\} = \frac{\partial f}{\partial Q}\frac{\partial g}{\partial P} - \frac{\partial f}{\partial P}\frac{\partial g}{\partial Q}$$

$$\{Q^i, Q^j\}_{q,p} = 0 = \{P_i, P_j\}$$

$$\{Q^i, P_j\} = \delta^i_j$$

PROVIDES A CONVENIENT CHECK WHETHER TRANSF. CANONICAL.

iii) "TIME EVOLUTION = CHANGE OF COORDS"

IT IS A CANONICAL TRANSF. GENERATED BY $F = -S$

S... HAMILTON'S FUNCTION $\approx$ ACTION UNDERSTOOD AS A FUNCTION OF COORDINATES AND TIME (BOUNDARY DATA)

TO CONSTRUCT IT W/E

$$S = \int (p\dot{q} - H)\,dt$$

i) SOLVE EOM

$$q = q(\underbrace{t_1, q_1, t_2, q_2}_{\text{BOUNDARY DATA}}, t) \quad (\text{NON-TRIVIAL})$$

ii)

$$S(t_1, q_1, t_2, q_2) = \int_{t_1}^{t_2} L(t_1, q_1, t_2, q_2, t)\,dt$$

$\uparrow$ HAMILTON'S FUNCTION (CF. $S = S[q(t)]$)

PROOF : SEE L&L

CALCULATE DIFFERENTIAL OF S

$$dS = p\,dq - H\,dt\Big|_t - \left(p\,dq - H\,dt\right)\Big|_0$$

"OBVIOUS" (OR MAYBE NOT)

IMMEDIATELY CAN READ RELATIONS LIKE

$$\frac{\partial S}{\partial q} = p, \quad \frac{\partial S}{\partial t} = -H, \quad \frac{\partial S}{\partial q_0} = -p_0, \quad \frac{\partial S}{\partial t_0} = H_0$$

TIME EVOLUTION

$$q = q(q_0, p_0, t_0, t)$$
$$p = p(q_0, p_0, t_0, t)$$

TRANSF $\underbrace{(q_0, p_0)}_{\text{OLD}} \rightarrow \underbrace{(q, p)}_{\text{NEW}}$

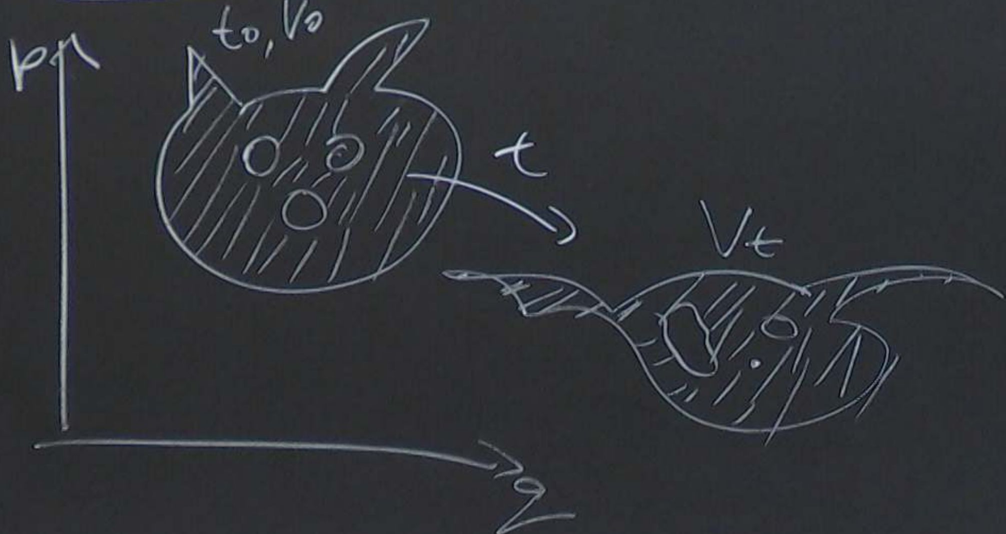
CHECKS WITH CONDITIONS

FOR $\underline{F = -S}$.

H_0

$$\int p\,dq - H\,dt = \int \left(\frac{dS}{dt_0} + p_0 \frac{dq_0}{dt_0} - H_0 \right) dt_0$$

iv) LIOUVILLE'S THEOREM: TIME EVOLUTION PRESERVES PHASE SPACE VOLUME



PROOF ($n=1$)

$$V_t = \int dP\,dQ = \int dq\,dp\,|JAC|$$

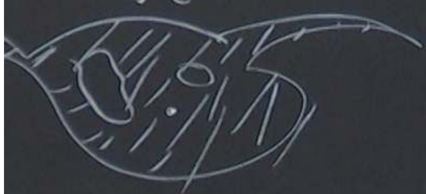
REM : TIME EVOLUTION

ASE SPACE VOLUME

PROOF ($n=1$)

$$V_t = \int dP\,dQ = \int dq\,dp\,|JAC|$$

V_t



$$|JAC| = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix}$$

EVOLUTION

VOLUME

PROOF $(n=1)$

$$V_t = \int dP\,dQ = \int dq\,dp\,|JAC| = \int dq\,dp = V_0$$

$$|JAC| = \begin{vmatrix} \frac{\partial Q}{\partial q} & \frac{\partial Q}{\partial p} \\ \frac{\partial P}{\partial q} & \frac{\partial P}{\partial p} \end{vmatrix} = \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial P}{\partial q}\frac{\partial Q}{\partial p} = \{Q,P\}_{q,p} \overset{\text{CAN. TR.}}{=} 1$$

d) HAMILTON - JACOBI THEORY

INGENIOUS IDEA: FIND A CANONICAL TRANSF.

SO THAT $\boxed{H' = 0}$

$\Rightarrow$ (4) & (5):

$$Q^j = \alpha^j = \text{CONST}$$

$$P_j = -\beta_j = \text{CONST}$$

a) HAMILTON – JACOBI THEORY

INGENIOUS IDEA: FIND A CANONICAL TRANSF. SO THAT $\boxed{H'=0}$

$\Rightarrow$ (4) & (5): $Q^j = \alpha^j = \text{CONST}$

$P_j = -\beta_j = \text{CONST}$

(2): $P_j = -\dfrac{\partial S}{\partial Q^j}$ ← GEN. F.

• WHAT IS THIS MIRACULOUS S ?

...NSF.

$$\beta_j = \frac{\partial S(q^i, \alpha^i, t)}{\partial \alpha^j}$$

BY INVERSION:

$$\boxed{q^i = q^i(\alpha, \beta, t)}$$

<u>SOLUTION !</u>

$$\beta_j = \frac{\partial S(q^i, \alpha^i, t)}{\partial \alpha^j}$$

ERSION:

$$\underline{q^i = q^i(\alpha, \beta, t)}$$

SOLUTION !

• WHAT IS THIS MIRACULOUS S ?

$$p_i = \frac{\partial S}{\partial q^i} \quad (1)$$

$$H' = 0 = H + \frac{\partial S}{\partial t} \quad (3)$$

$$\boxed{H\Big(q^i, \underbrace{\frac{\partial S}{\partial q^i}}_{= p_i}, t\Big) + \frac{\partial S}{\partial t} = 0}$$

HAMILTON-JACOBI EQ.

SINGLE PDE FOR S

PDE < COMPLETE INTEGRAL $S = S(q^i, \alpha^i, t)$
GENERAL INTEGRAL (WAVE EQ: $f(x - vt)$)

TO FIND IT: EMPLOY METHOD OF SEPARATION OF VARIABLES

$H \neq H(q_c)$ $\qquad$ $H \neq H(t)$

$$S(q^i, t) = \underbrace{S_1(q_1) + S_2(q_2)}_{\text{ANSATZ}} + \dots + \underset{\text{CONST.}}{\alpha_c}\, q_c + \dots - \underset{\text{CONST.}}{E}\, t$$

(COORDINATE DEPENDENT)

$\alpha_i, t)$

$f(x - vt))$

...TION OF VARIABLES

$H \neq H(q_c)$ $\quad H \neq H(t)$

$+ \alpha_c q_c + \ldots - E t$

CONST. CONST.

T)

EXAMPLE: FREE FALL IN HOMOGENEOUS GRAV. F

$$H = \frac{p^2}{2m} + V \qquad V = -mgX$$

H-J: $\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V + \frac{\partial S}{\partial t} = 0$ $\qquad S = $

$$S = S_0(x) - Et$$

$$\frac{1}{2m}\left(\frac{dS_0}{dx}\right)^2 + V - E = \emptyset$$

HAMILTON'S FUNCTION (CF. $S = S$

• H-J: $\frac{1}{2m}\left(\frac{\partial S}{\partial x}\right)^2 + V + \frac{\partial S}{\partial t} = 0$, $S = S(x,t)$

$$S = S_0(x) - Et$$

$$\frac{1}{2m}\left(\frac{dS_0}{dx}\right)^2 + V - E = 0 \Rightarrow S_0 = \int \sqrt{2m(E-V)}\, dx$$

$$\underline{S = \frac{1}{3gm^2}\left(2m(E+mgx)\right)^{3/2} - Et}$$

BOUNDARY DATA

1.1) $S(t_1, q_1, t_2, q_2) = \int_{t_1}^{t_2} L(t_1, q_1, t_2, q_2, t)\, dt$

↑ HAMILTON'S FUNCTION (CF. $S = S[q(t)]$)

$$\frac{\partial S}{\partial E} = \beta = \frac{1}{mg}\sqrt{2m(E-V)} - t$$

$$\Rightarrow \quad x = \frac{g}{2}(t+\beta)^2 - \frac{E}{mg}$$

BY INVERSION.

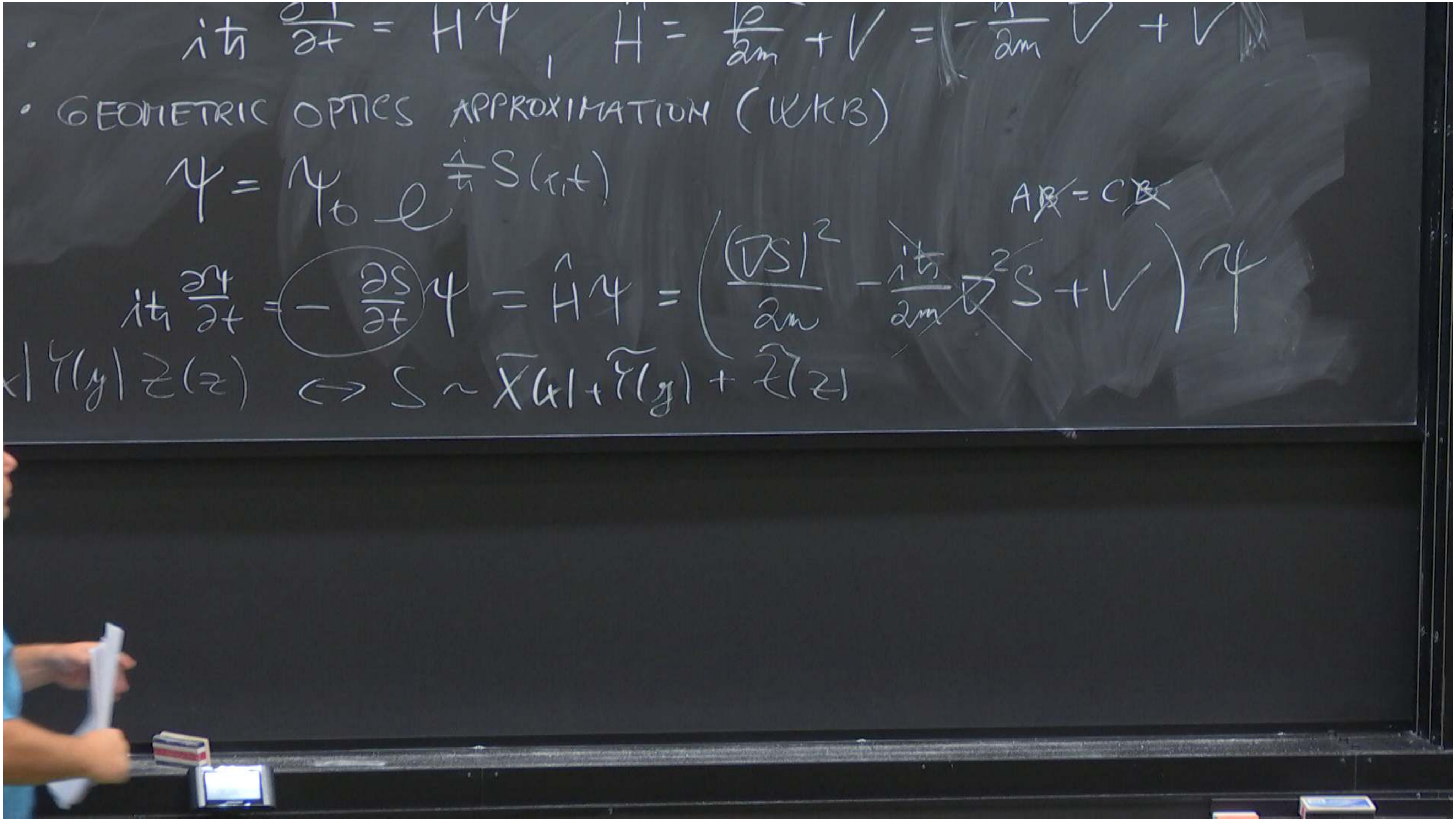
CONNECTION TO QM: SCHRÖDINGER EQ. $\hat{p} = -i\hbar\nabla$

- $i\hbar \frac{\partial \psi}{\partial t} = \hat{H}\psi$, $\hat{H} = \frac{\hat{p}^2}{2m} + V = -\frac{\hbar^2}{2m}\nabla^2 + V$

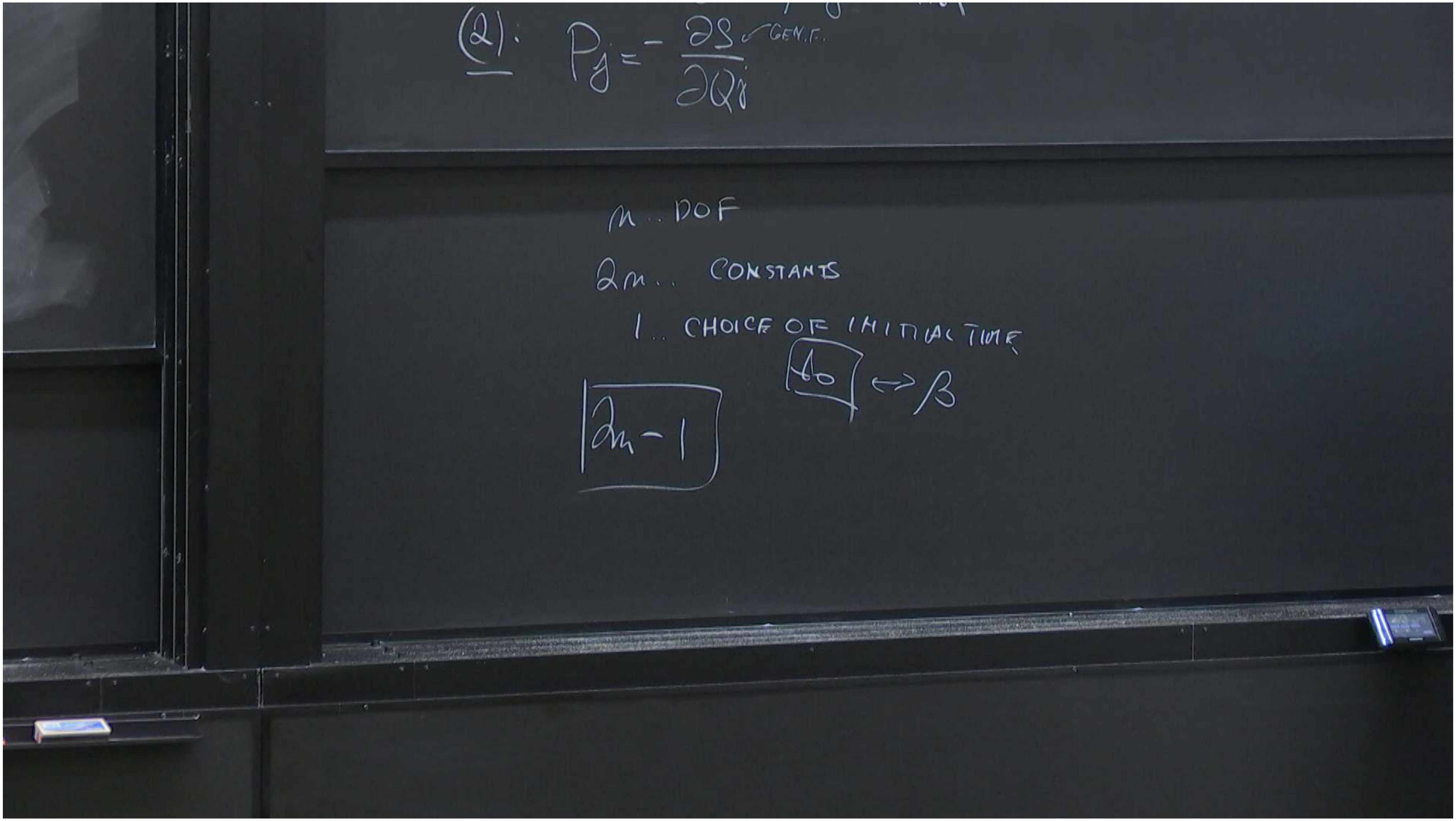
- GEOMETRIC OPTICS APPROXIMATION (WKB)

$$\psi = \psi_0 \, e^{\frac{i}{\hbar} S(x,t)}$$

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \hat{H}\psi = \left(\frac{(\nabla S)^2}{2m} - \frac{i\hbar}{2m}\nabla^2 S + V \right)\psi$$



• $i\hbar \frac{\partial \psi}{\partial t} = H\psi$, $H = \frac{p^2}{2m} + V = -\frac{\hbar^2}{2m}\nabla^2 + V$
• GEOMETRIC OPTICS APPROXIMATION (WKB)
$\psi = \psi_0 e^{\frac{i}{\hbar}S(x,t)}$
$A\cancel{B} = C\cancel{B}$
$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\partial S}{\partial t}\psi = \hat{H}\psi = \left(\frac{(\nabla S)^2}{2m} \cancel{- \frac{i\hbar}{2m}\nabla^2 S} + V\right)\psi$
$X(x)\,Y(y)\,Z(z) \Leftrightarrow S \sim \tilde{X}(x) + \tilde{Y}(y) + \tilde{Z}(z)$


(2): $P_j = -\frac{\partial S}{\partial Q_j}$ GEN.F.
n .. DOF
2n .. CONSTANTS
1 .. CHOICE OF INITIAL TIME
$\boxed{t_0} \leftrightarrow \beta$
$\boxed{2n-1}$