Title: Classical Physics - Lecture 220907

Speakers:

Collection: Classical Physics (2022/2023)

Date: September 07, 2022 - 9:00 AM

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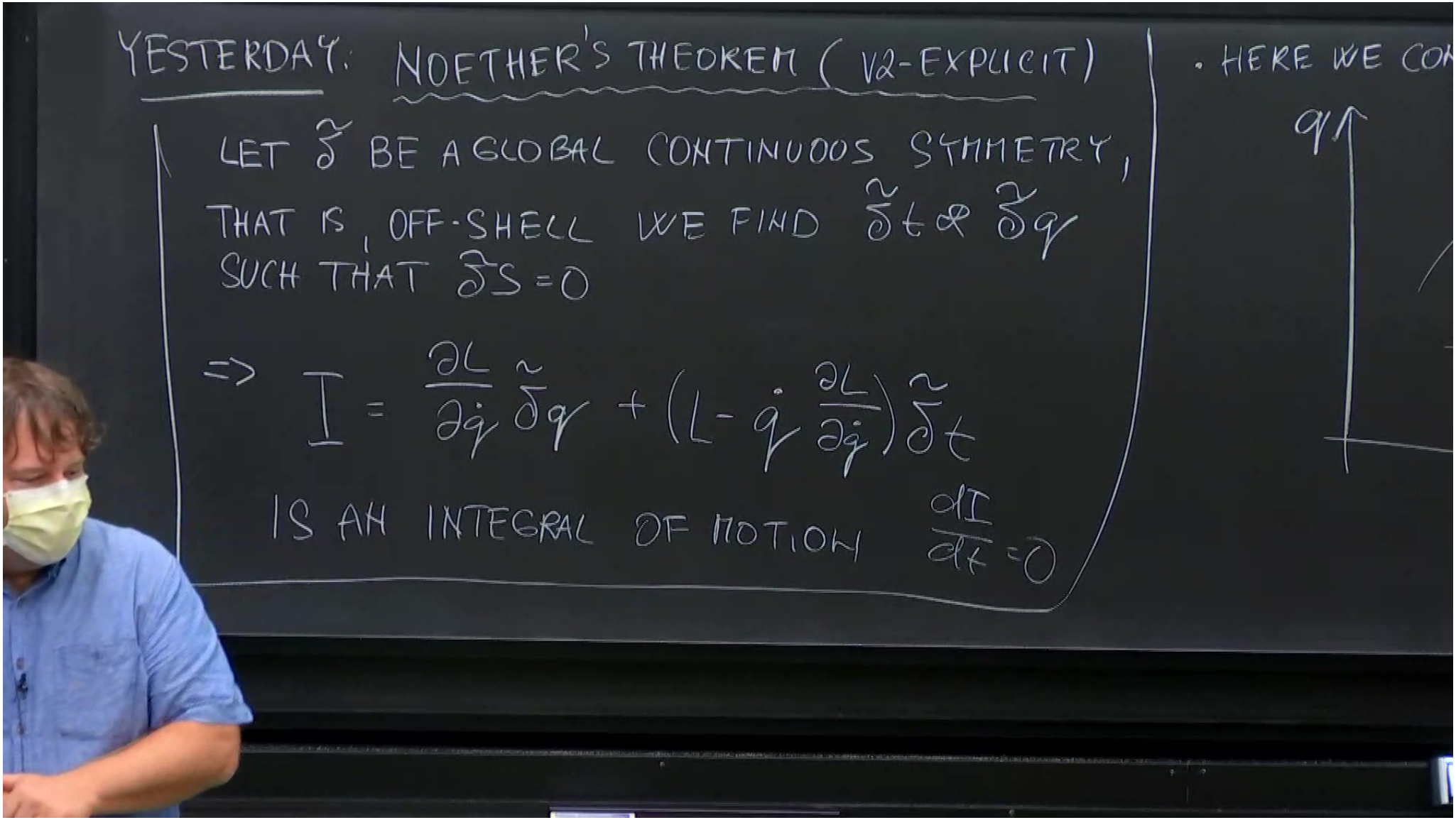
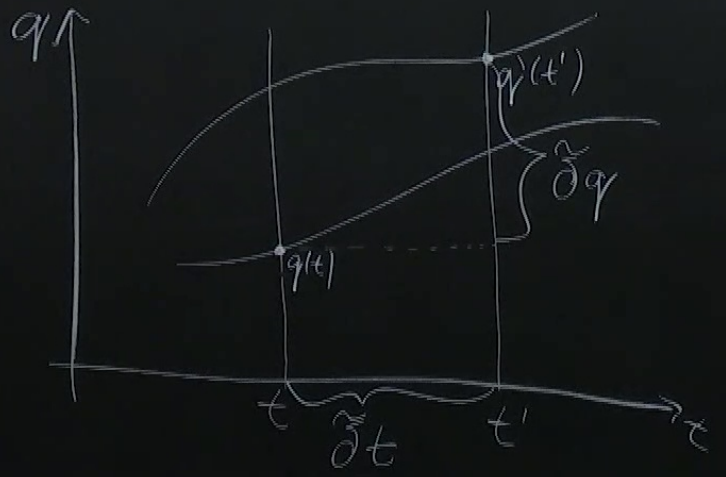

YESTERDAY: NOETHER'S THEOREM (V2-EXPLICIT)
LET $\tilde{\delta}$ BE A GLOBAL CONTINUOUS SYMMETRY,
THAT IS, OFF-SHELL WE FIND $\tilde{\delta}t$ & $\tilde{\delta}q$
SUCH THAT $\tilde{\delta}S = 0$
$\Rightarrow I = \frac{\partial L}{\partial \dot{q}} \tilde{\delta} q + \left(L - \dot{q} \frac{\partial L}{\partial \dot{q}}\right) \tilde{\delta} t$
IS AN INTEGRAL OF MOTION $\frac{dI}{dt} = 0$
. HERE WE CO
q

- HERE WE CONSIDER GENERAL TRANSFORMATIONS OF BOTH q & t

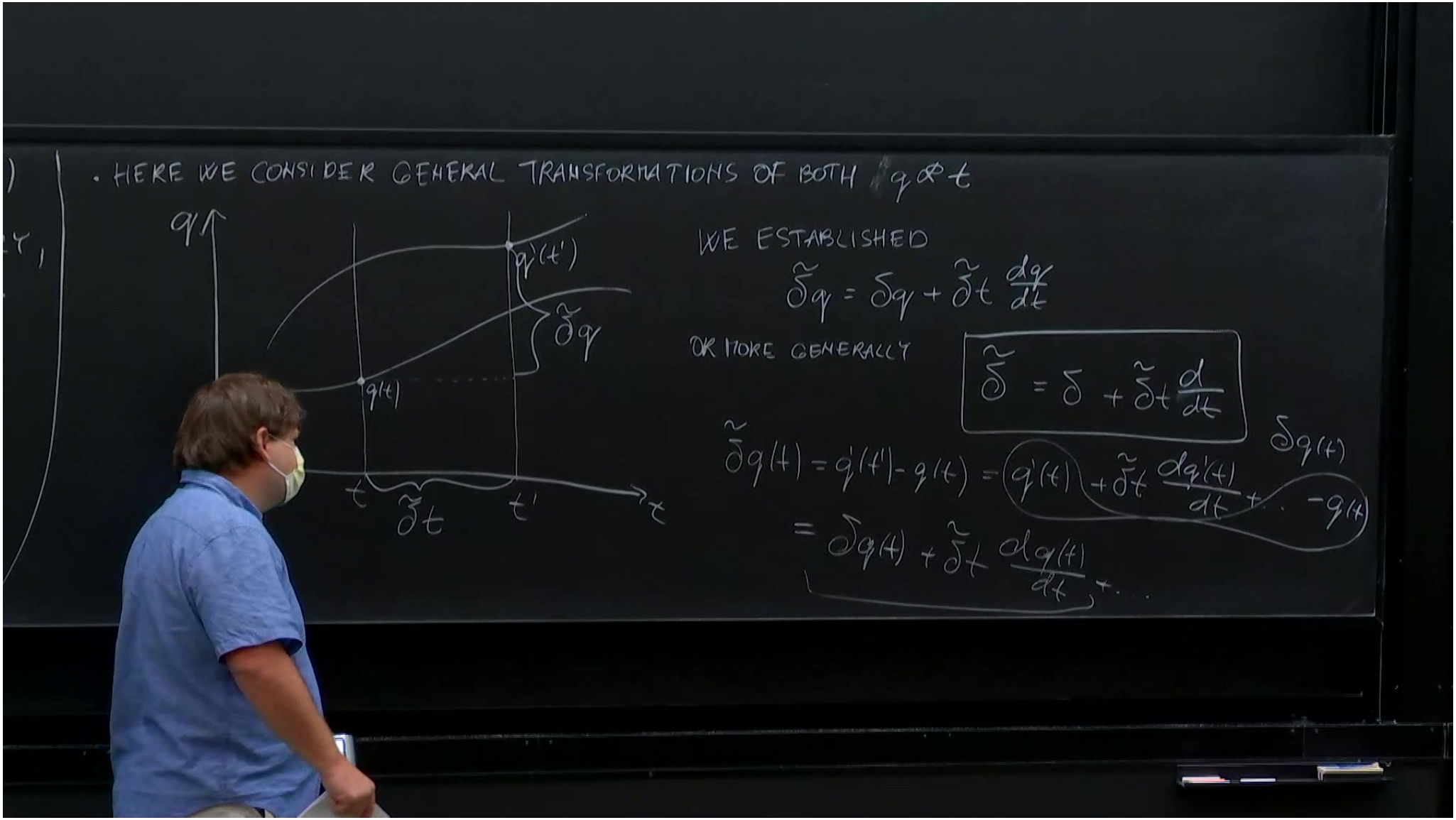


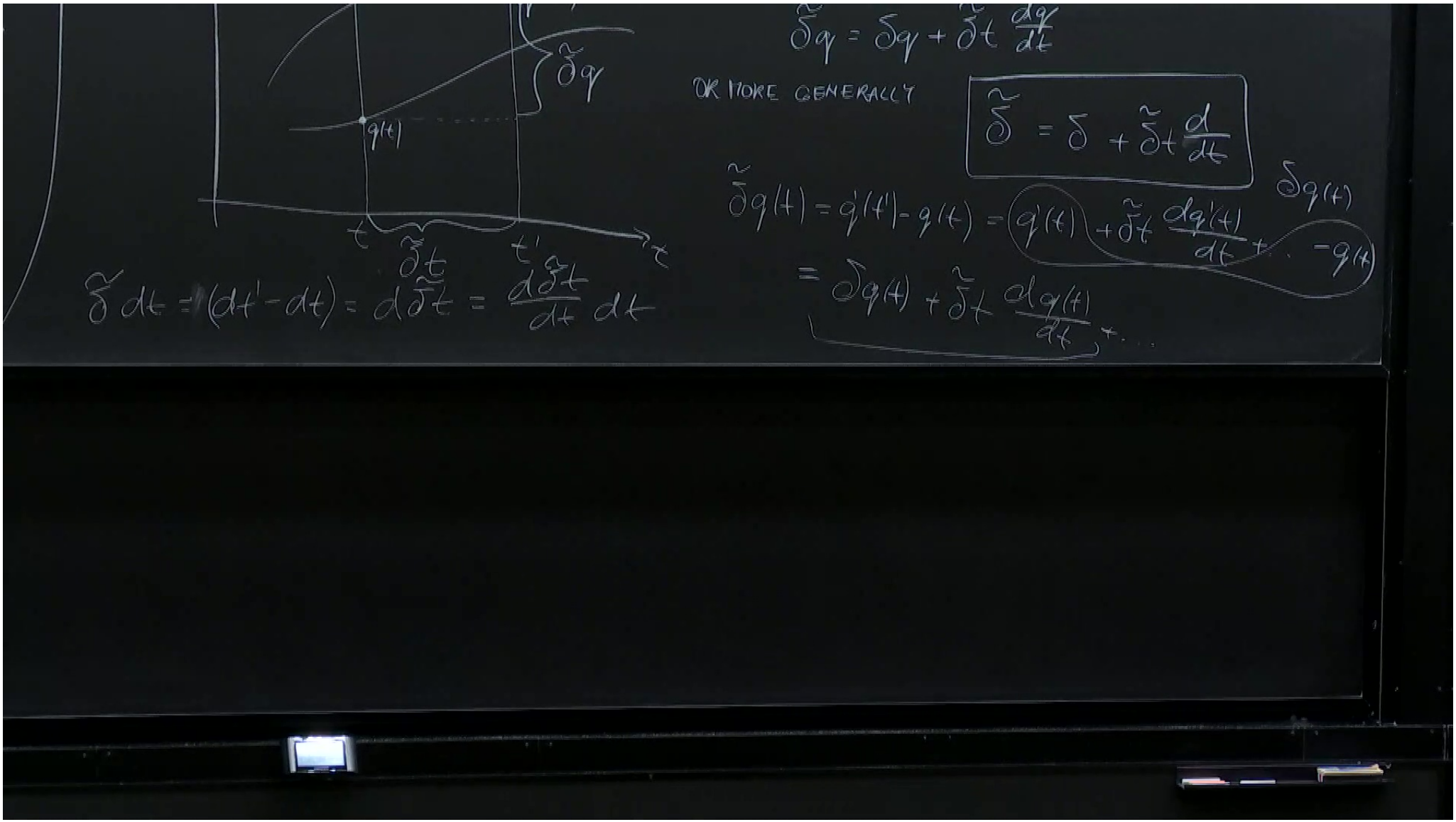
WE ESTABLISHED

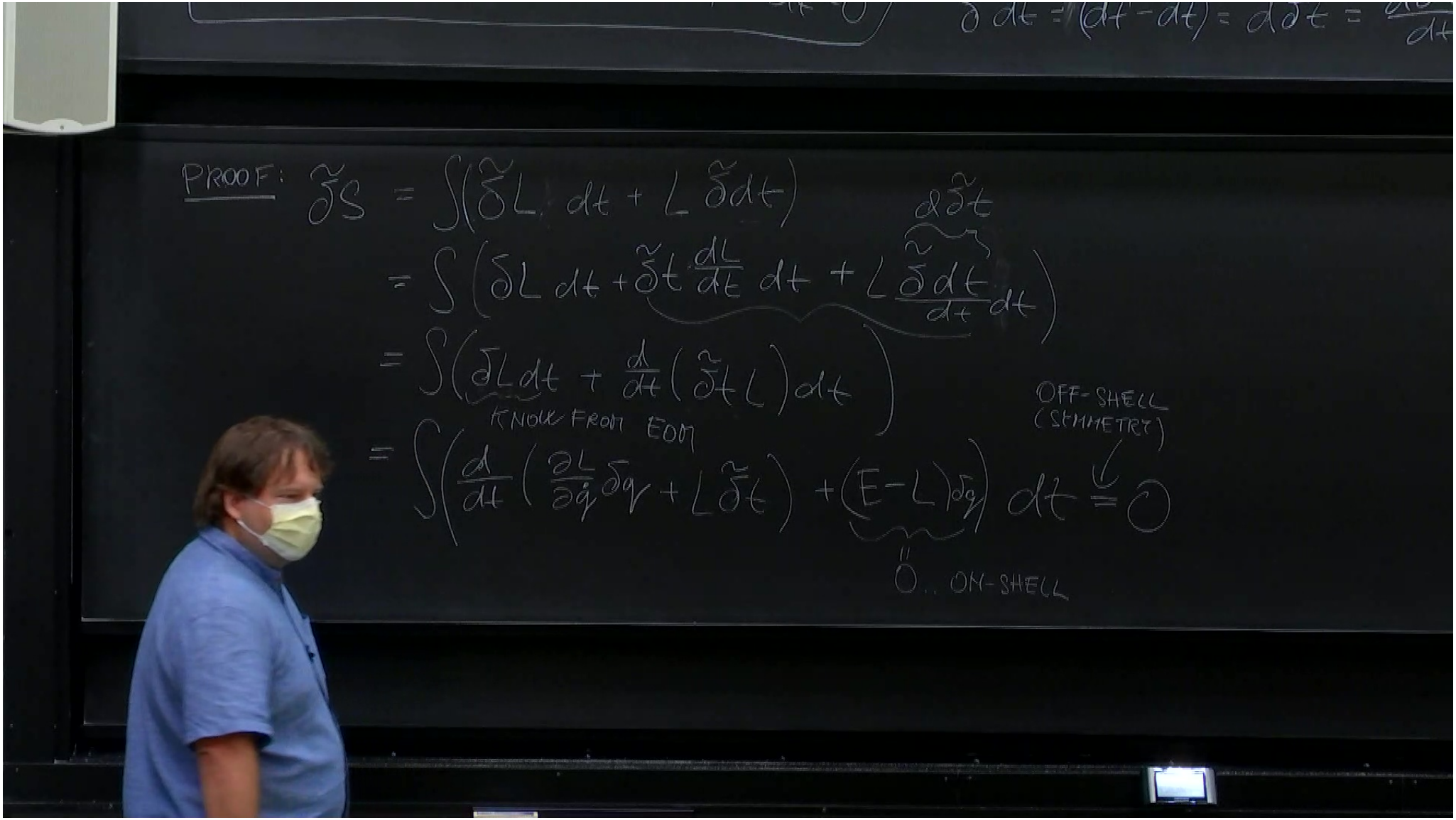
$$\tilde{\delta} q = \delta q + \tilde{\delta} t \frac{dq}{dt}$$

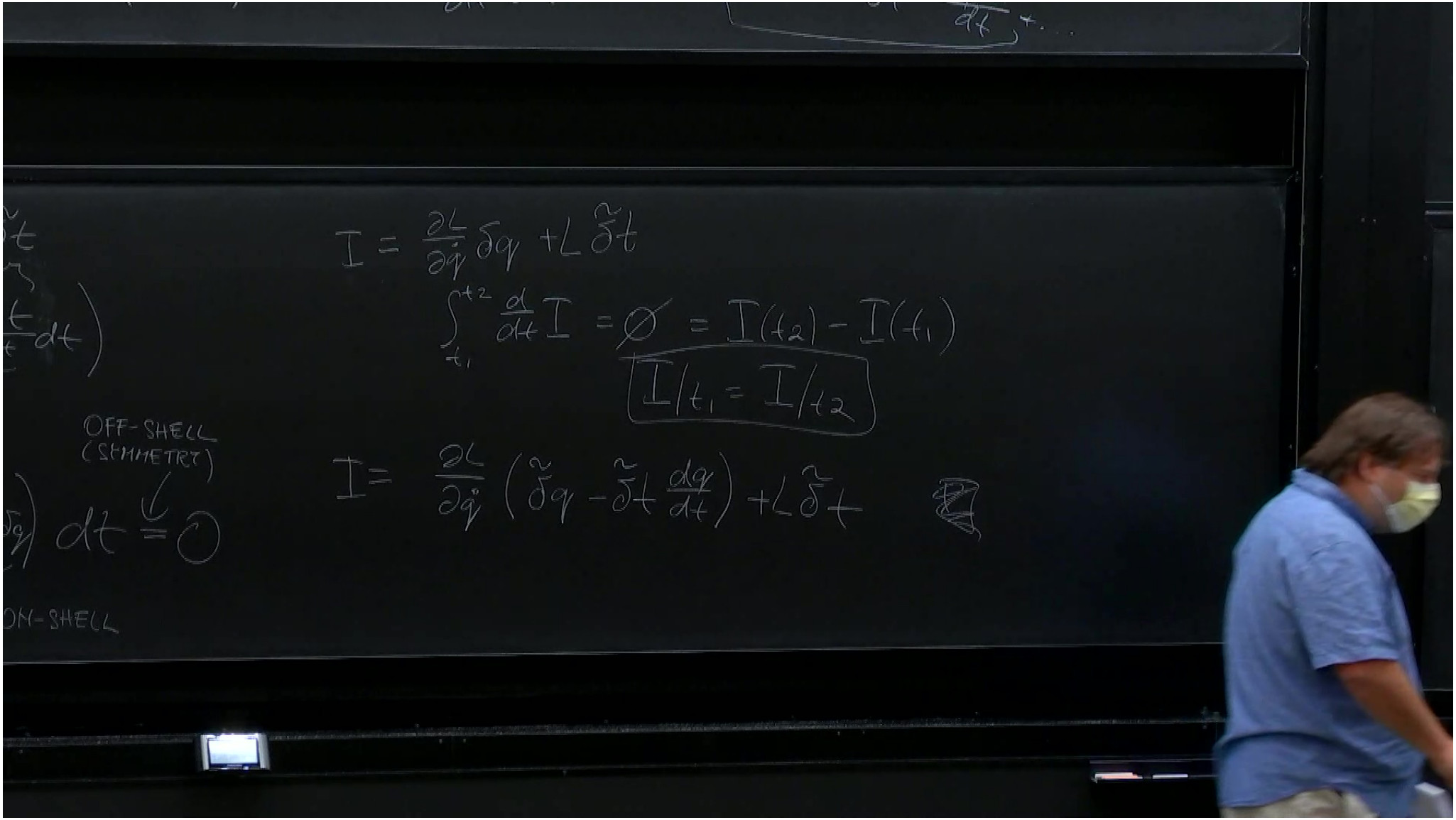
OR MORE GENERALLY

$$\tilde{\delta} = \delta + \tilde{\delta} t \frac{d}{dt}$$


• HERE WE CONSIDER GENERAL TRANSFORMATIONS OF BOTH q & t
q
q'(t')
δ̃q
q(t)
t
δ̃t
t'
t
WE ESTABLISHED
δ̃q = δq + δ̃t dq/dt
OR MORE GENERALLY
δ̃ = δ + δ̃t d/dt
δ̃q(t) = q'(t') - q(t) = q'(t) + δ̃t dq'(t)/dt + ... - q(t)
δq(t)
= δq(t) + δ̃t dq(t)/dt + ...


$\tilde{\delta}q = \delta q + \tilde{\delta}t\,\frac{dq}{dt}$
OR MORE GENERALLY
$\tilde{\delta} = \delta + \tilde{\delta}t\,\frac{d}{dt}$
$\tilde{\delta}q(t) = q'(t') - q(t) = q'(t) + \tilde{\delta}t\,\frac{dq'(t)}{dt} + \ldots - q(t)$
$\delta q(t)$
$= \delta q(t) + \tilde{\delta}t\,\frac{dq(t)}{dt} + \ldots$
$\tilde{\delta}q(t)$
$q(t)$
t
$\tilde{\delta}t$
t'
$\tilde{\delta}\,dt = (dt' - dt) = d\tilde{\delta}t = \frac{d\tilde{\delta}t}{dt}\,dt$


PROOF:
$\tilde{\delta}S = \int(\tilde{\delta}L\,dt + L\,\tilde{\delta}dt)$
$= \int\left(\delta L\,dt + \tilde{\delta}t\frac{dL}{dt}dt + L\frac{\tilde{\delta}dt}{dt}dt\right)$
$= \int\left(\delta L\,dt + \frac{d}{dt}(\tilde{\delta}t\,L)dt\right)$
KNOWN FROM EOM
$= \int\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\delta q + L\tilde{\delta}t\right) + (E-L)\delta q\right)dt = 0$
OFF-SHELL (SYMMETRY)
0.. ON-SHELL


$I = \frac{\partial L}{\partial \dot{q}} \delta q + L \tilde{\delta} t$
$\int_{t_1}^{t_2} \frac{d}{dt} I = \emptyset = I(t_2) - I(t_1)$
$I|_{t_1} = I|_{t_2}$
OFF-SHELL
(SYMMETRY)
$I = \frac{\partial L}{\partial \dot{q}} \left(\tilde{\delta} q - \dot{q} \tilde{\delta} t \right) + L \tilde{\delta} t$
ON-SHELL

$= \int \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q + L \delta t \right) + (E - L) \delta q \right) dt \overset{!}{=} 0$

I = 0 .. ON-SHELL

$I = \frac{\partial L}{\partial \dot{q}}$

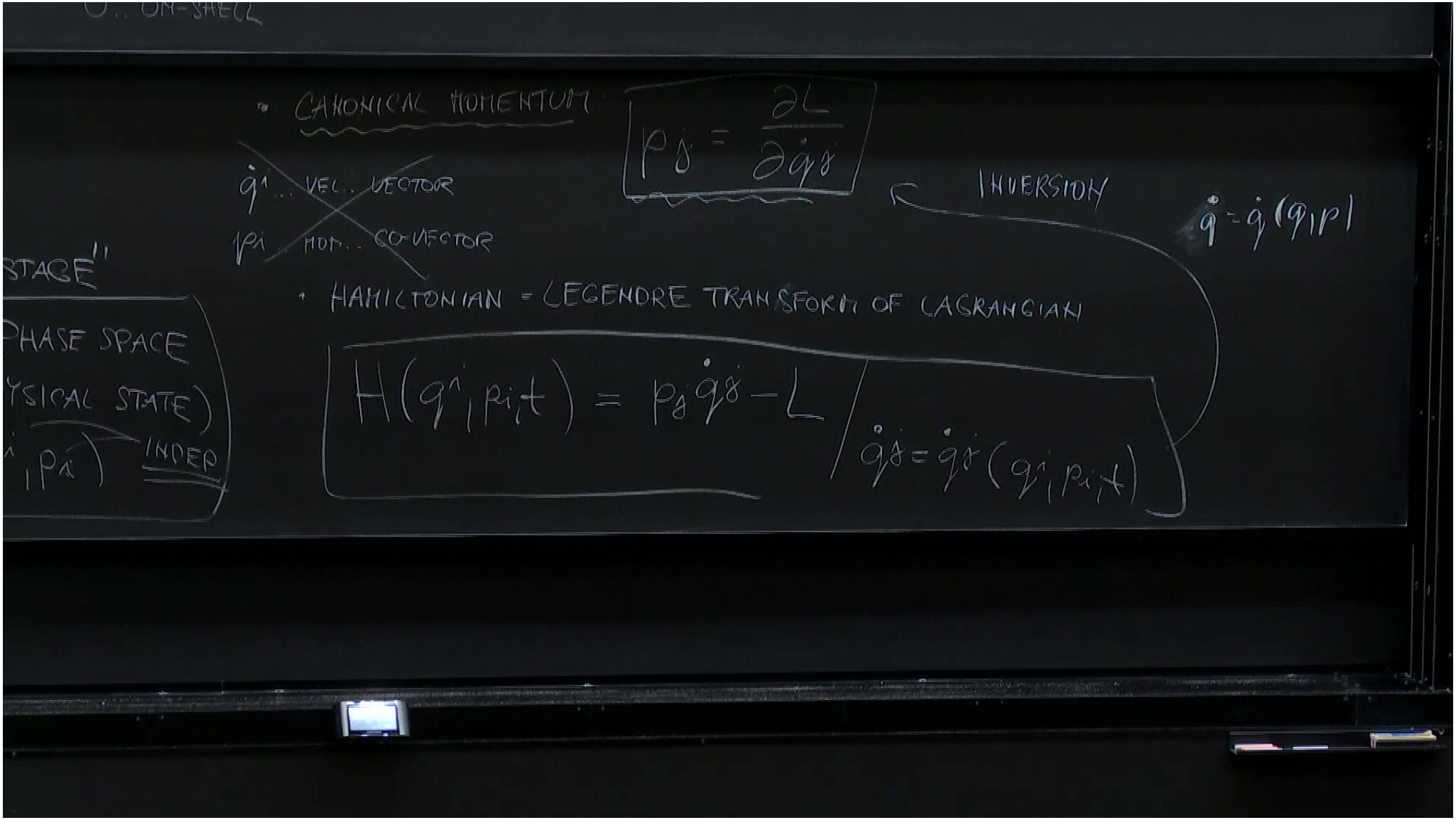
2) HAMILTONIAN MECHANICS

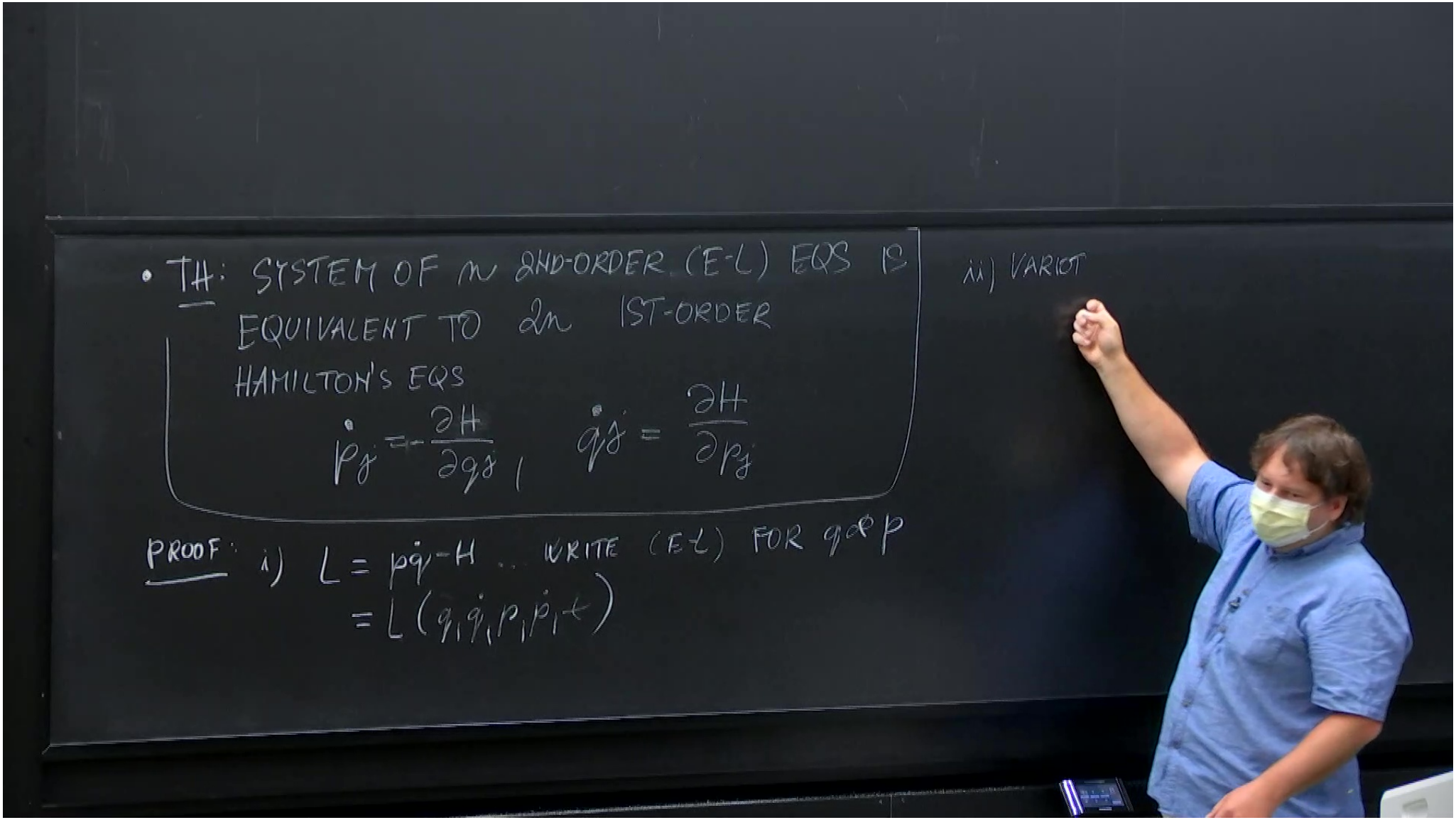
- CANONICAL MOMENTUM

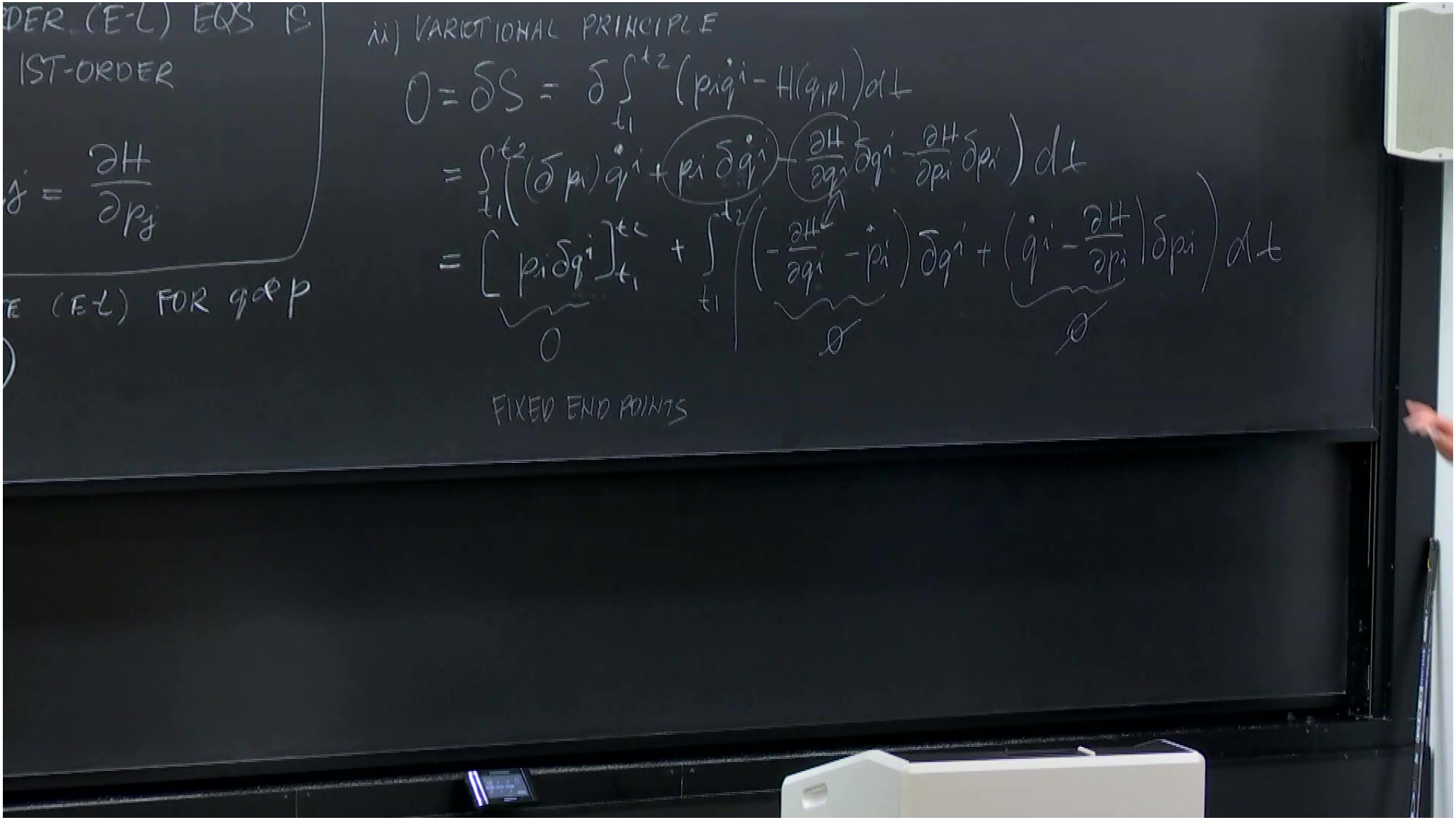
a) HAMILTON'S CANONICAL EQS

KEY IDEA IS TO ENLARGE THE "STAGE"

n-DIM CONFIG. SPACE ⇒	2n-DIM PHASE SPACE
(PHOTO OF SYSTEM)	(PHYSICAL STATE)
q^i	(q^i, p_i) INDEP


ON-SHELL
CANONICAL MOMENTUM
$p_j = \frac{\partial L}{\partial \dot{q}^j}$
$\dot{q}^i$ ... VEL. VECTOR
p_i ... MOM. CO-VECTOR
INVERSION
$\dot{q} = \dot{q}(q,p)$
STAGE"
HAMILTONIAN = LEGENDRE TRANSFORM OF LAGRANGIAN
$H(q^i, p_i, t) = p_j \dot{q}^j - L$
$\dot{q}^j = \dot{q}^j(q^i, p_i, t)$
PHASE SPACE
YSICAL STATE)
p_i)
INDEP


• TH: SYSTEM OF n 2ND-ORDER (E-L) EQS IS EQUIVALENT TO 2n 1ST-ORDER HAMILTON'S EQS
$\dot{p}_j = -\frac{\partial H}{\partial q_j}$, $\dot{q}_j = \frac{\partial H}{\partial p_j}$
PROOF: i) $L = p\dot{q} - H$ ... WRITE (E-L) FOR $q \& p$
$= L(q, \dot{q}, p, \dot{p}, t)$
ii) VARIOT


DER (E-L) EQS IS
IST-ORDER
$\dot{q}^i = \frac{\partial H}{\partial p_j}$
TE (E-L) FOR q & p
ii) VARIATIONAL PRINCIPLE
$0 = \delta S = \delta \int_{t_1}^{t_2} (p_i \dot{q}^i - H(q,p))\, dt$
$= \int_{t_1}^{t_2} \left((\delta p_i)\dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \right) dt$
$= \left[p_i \delta q^i \right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \left(\left(-\frac{\partial H}{\partial q^i} - \dot{p}_i \right)\delta q^i + \left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right)\delta p_i \right) dt$
0
0
0
FIXED END POINTS

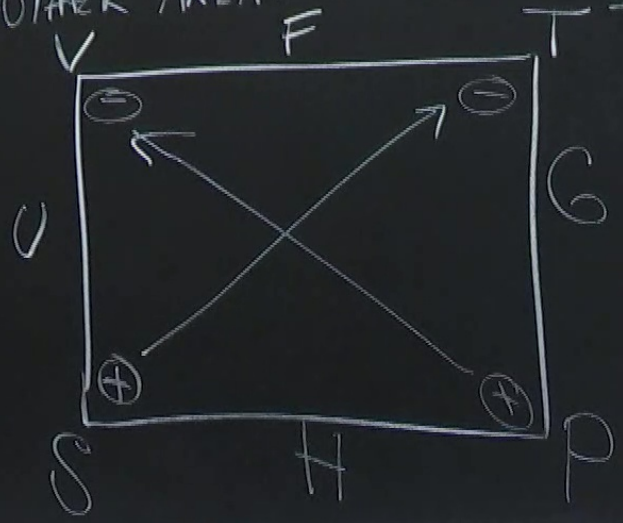
PROOF: i) $L = p\dot{q} - H$ ... WRITE (E-L) FOR $q \& p$

$= L(q, \dot{q}, p, \dot{p}, t)$

$= \left[p_i \delta q^i \right]_{t_1}^{t} = 0$

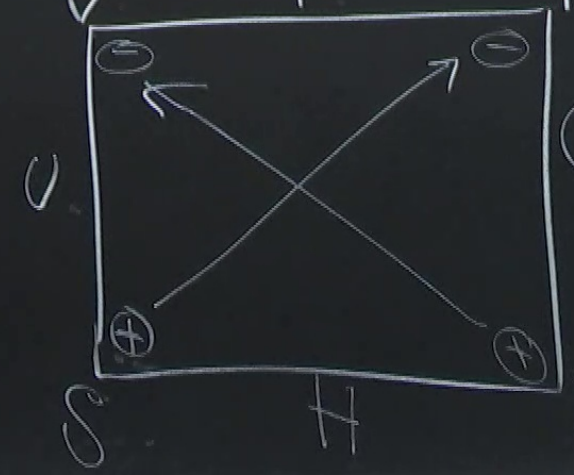
FIXED END POINTS

REMARK: LEGENDRE TRANSFORM IMPORTANT IN OTHER AREAS OF PHYSICS: E.G. TDs



FIXED END POINTS

REMARK: LEGENDRE TRANSFORM IMPORTANT IN OTHER AREAS OF PHYSICS: E.G. TDs

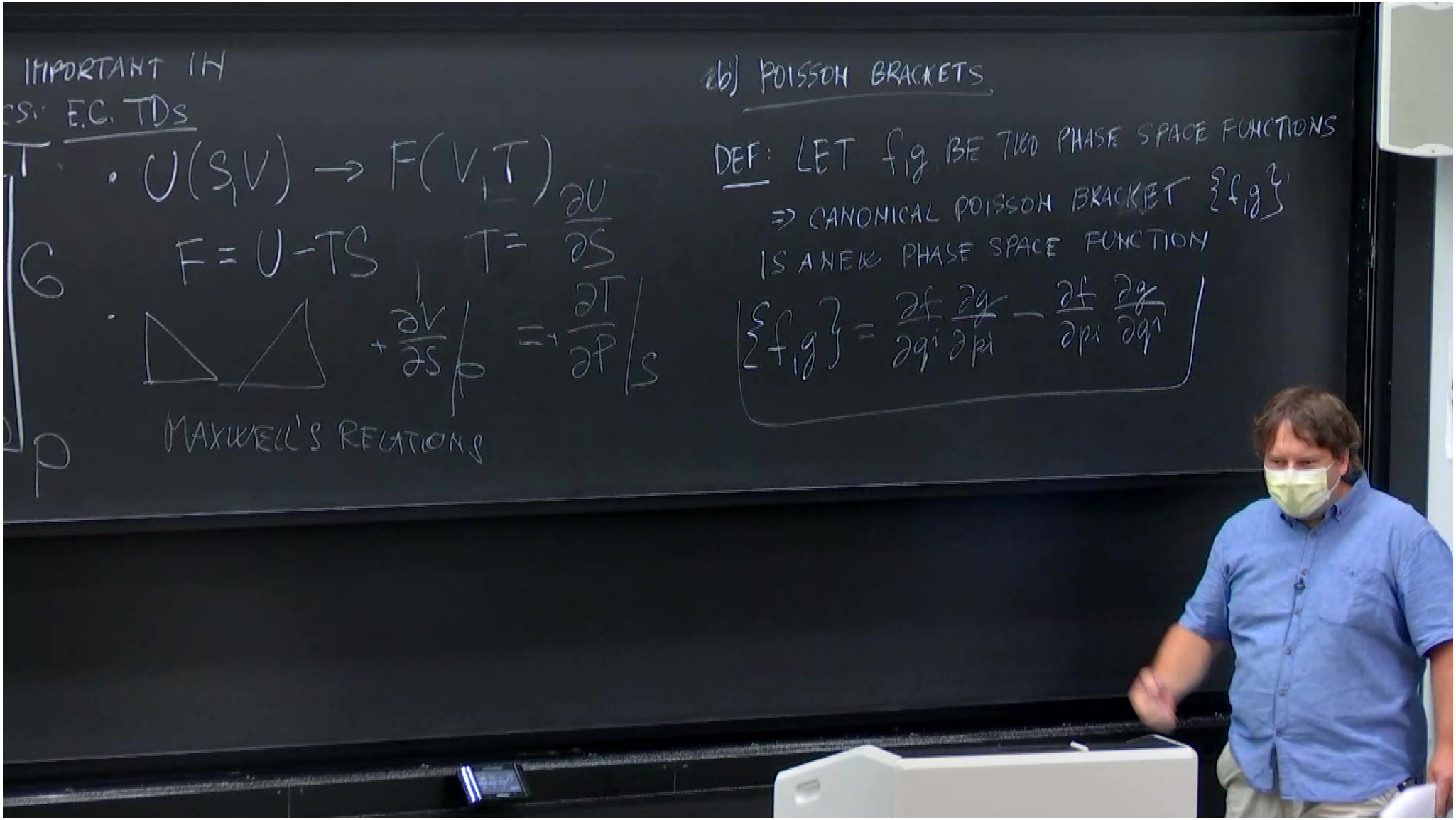


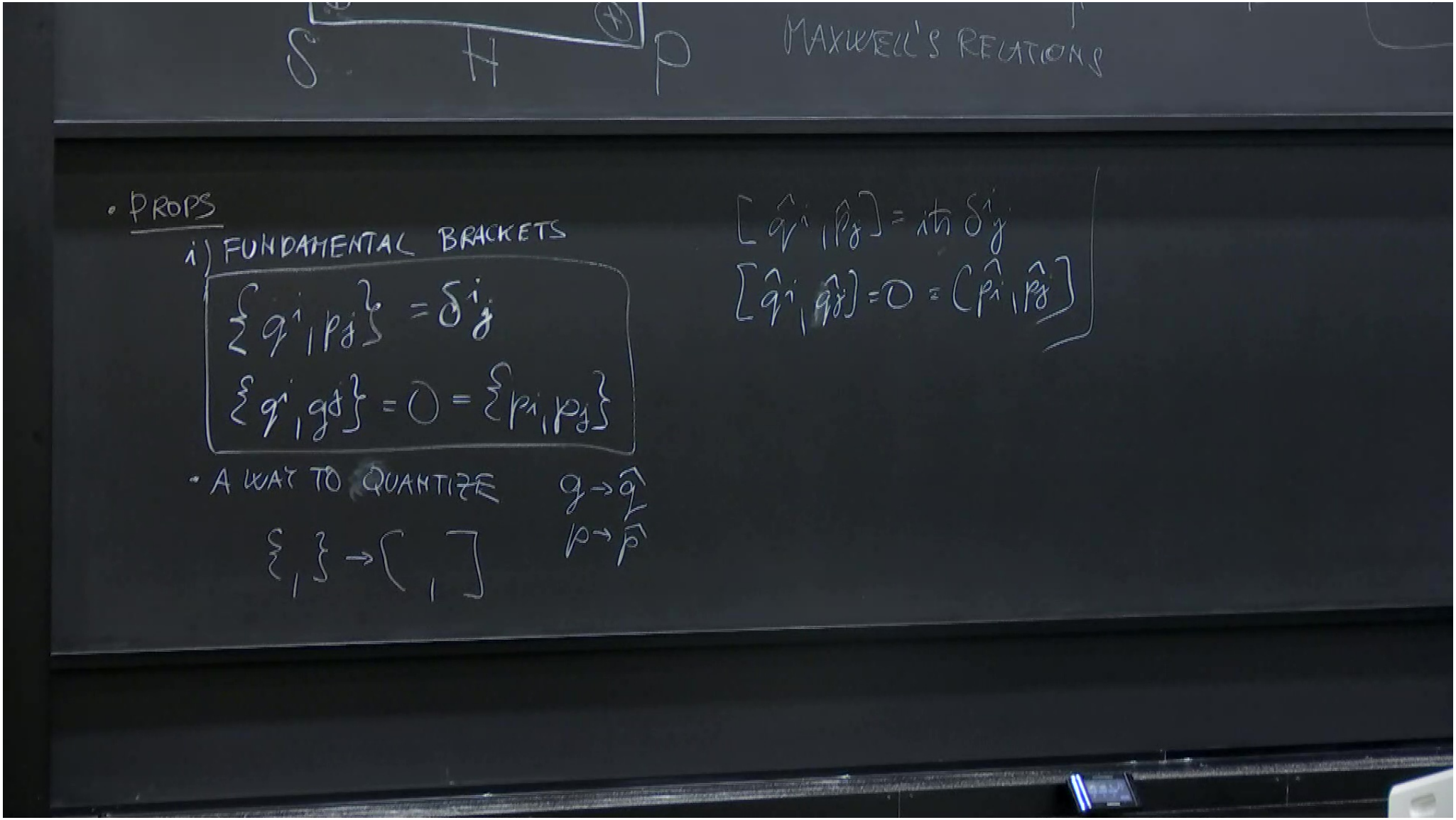
- $U(S,V) \rightarrow F(V,T)$

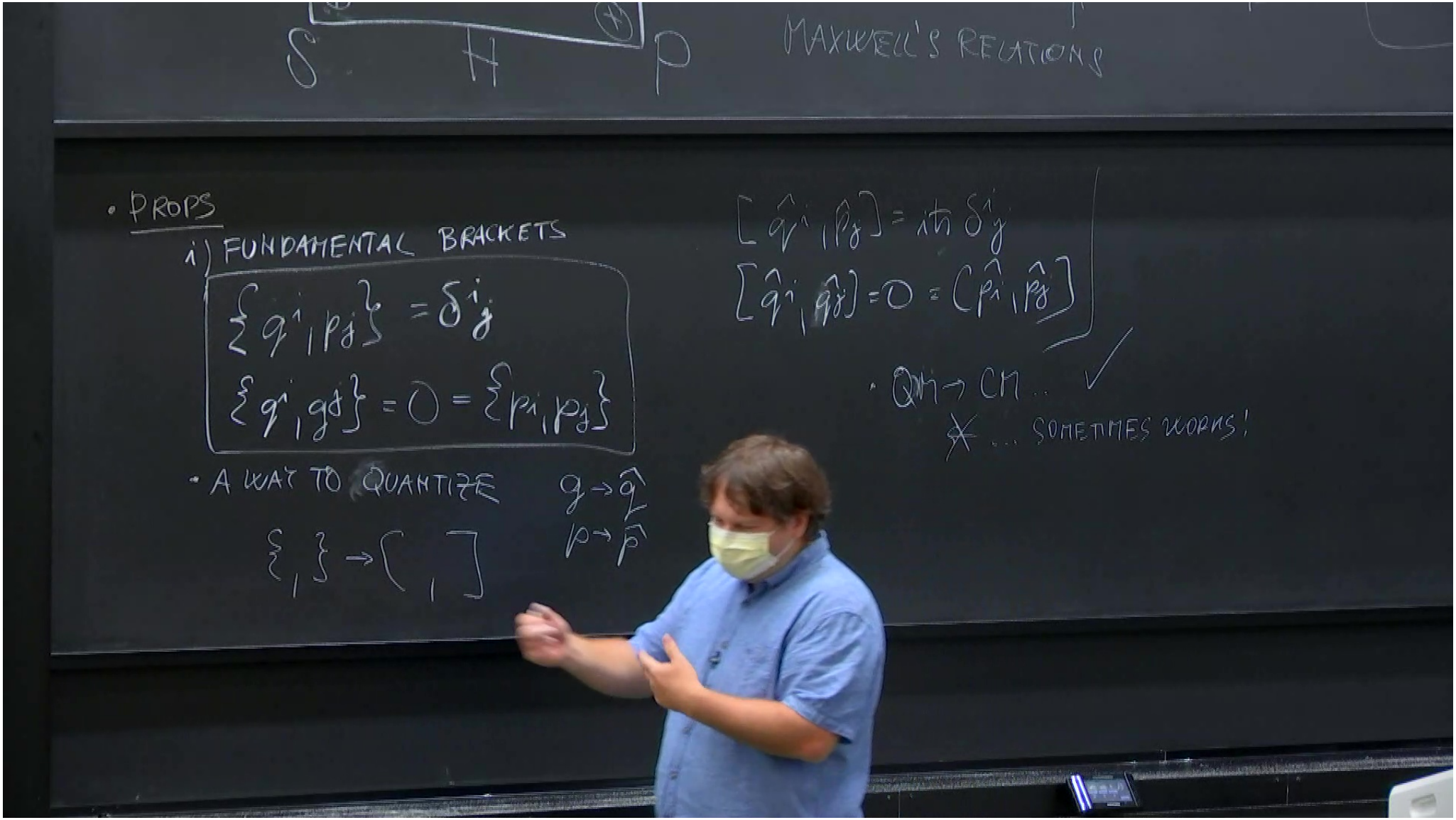
$F = U - TS$ $\quad T = \frac{\partial U}{\partial S}$

- $+\frac{\partial V}{\partial S}\Big|_p = +\frac{\partial T}{\partial p}\Big|_S$

MAXWELL'S RELATIONS


IMPORTANT IN
E.G. TDs
$U(S,V) \rightarrow F(V,T)$
$F = U - TS$
$T = \frac{\partial U}{\partial S}$
$+\frac{\partial V}{\partial S}\Big|_P = +\frac{\partial T}{\partial P}\Big|_S$
MAXWELL'S RELATIONS
b) POISSON BRACKETS
DEF: LET f, g BE TWO PHASE SPACE FUNCTIONS
$\Rightarrow$ CANONICAL POISSON BRACKET $\{f,g\}$
IS A NEW PHASE SPACE FUNCTION
$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$


S
H
p
MAXWELL'S RELATIONS
• PROPS
i) FUNDAMENTAL BRACKETS
$\{q^i, p_j\} = \delta^i_j$
$\{q^i, q^j\} = 0 = \{p_i, p_j\}$
$[\hat{q}^i, \hat{p}_j] = i\hbar\,\delta^i_j$
$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j]$
• A WAY TO QUANTIZE
$\{,\} \to [,]$
$q \to \hat{q}$
$p \to \hat{p}$


S
H
p
MAXWELL'S RELATIONS
• PROPS
i) FUNDAMENTAL BRACKETS
$\{q^i, p_j\} = \delta^i_j$
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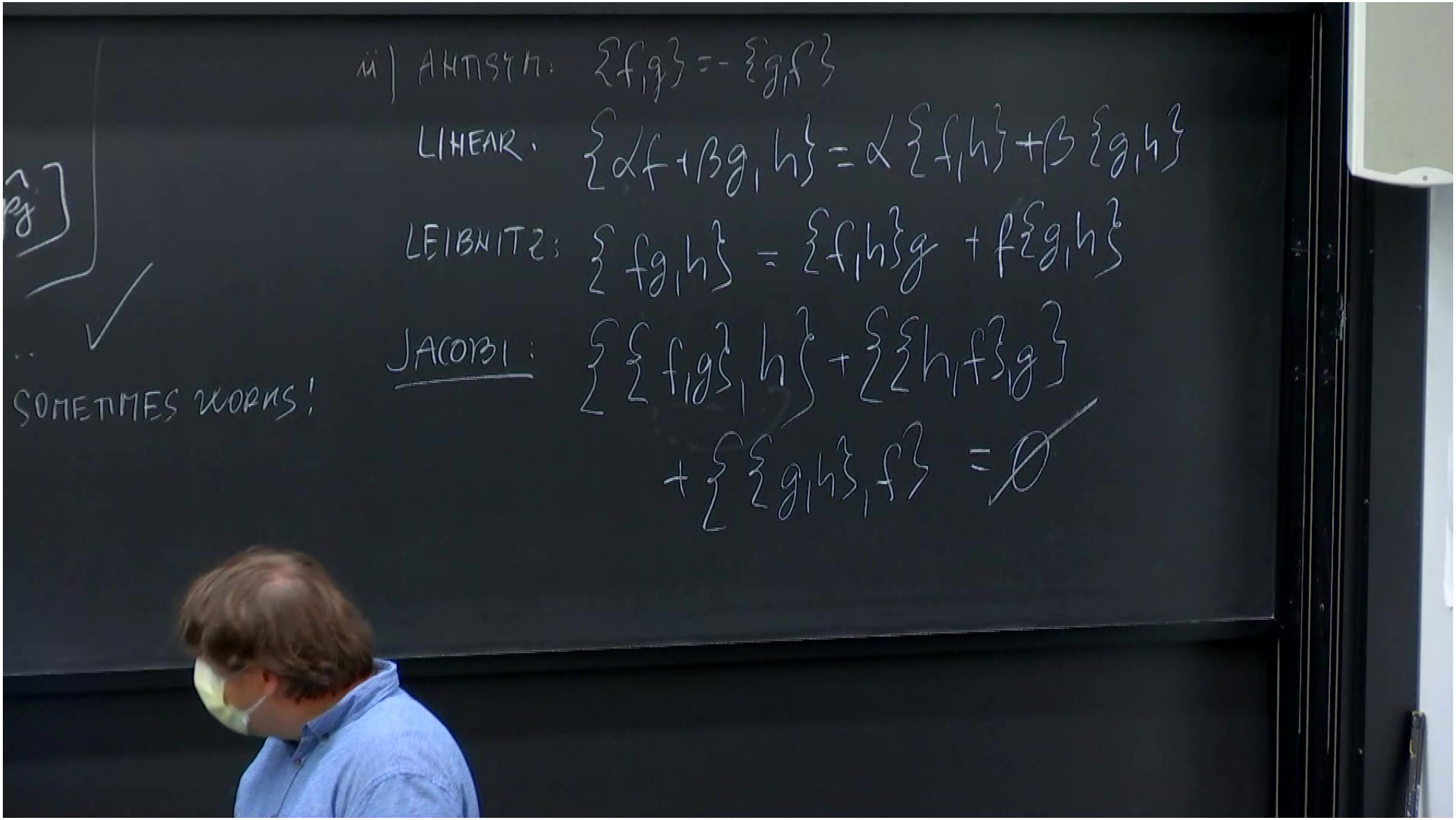
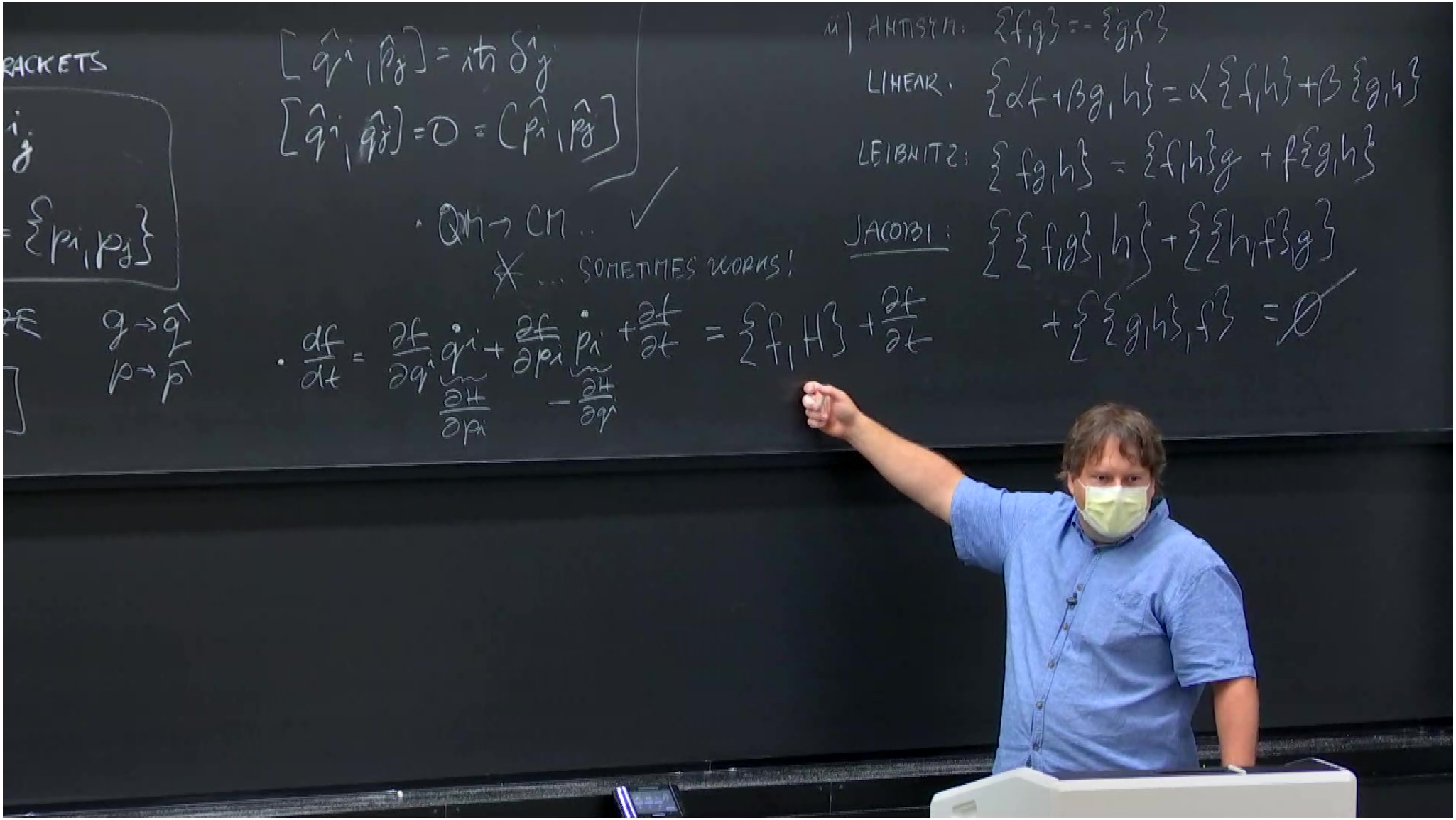
• A WAY TO QUANTIZE
$\{,\} \to [,]$
$q \to \hat{q}$
$p \to \hat{p}$
$[\hat{q}^i, \hat{p}_j] = i\hbar\,\delta^i_j$
$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j]$
• QM → CM ✓
✗ ... SOMETIMES WORKS!


ii) ANTISYM: $\{f,g\} = -\{g,f\}$
LINEAR: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$
LEIBNITZ: $\{fg,h\} = \{f,h\}g + f\{g,h\}$
JACOBI: $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$
SOMETIMES WORKS!


RACKETS
$[\hat{q}^i, \hat{p}_j] = i\hbar\,\delta^i_j$
$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j]$
$= \{p_i, p_j\}$
$q \to \hat{q}$
$p \to \hat{p}$
· QM → CM .. ✓
✗ ... SOMETIMES WORKS!
$\frac{df}{dt} = \frac{\partial f}{\partial q^i}\dot{q}^i + \frac{\partial f}{\partial p_i}\dot{p}_i + \frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial f}{\partial t}$
$\frac{\partial H}{\partial p_i}$
$-\frac{\partial H}{\partial q^i}$
ii) ANTISYM: $\{f,g\} = -\{g,f\}$
LINEAR: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$
LEIBNITZ: $\{fg,h\} = \{f,h\}g + f\{g,h\}$
JACOBI: $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$

I

0.. ON-SHELL

2| HAMILTONIAN MECHANICS

- CANONICAL MOMENTUM

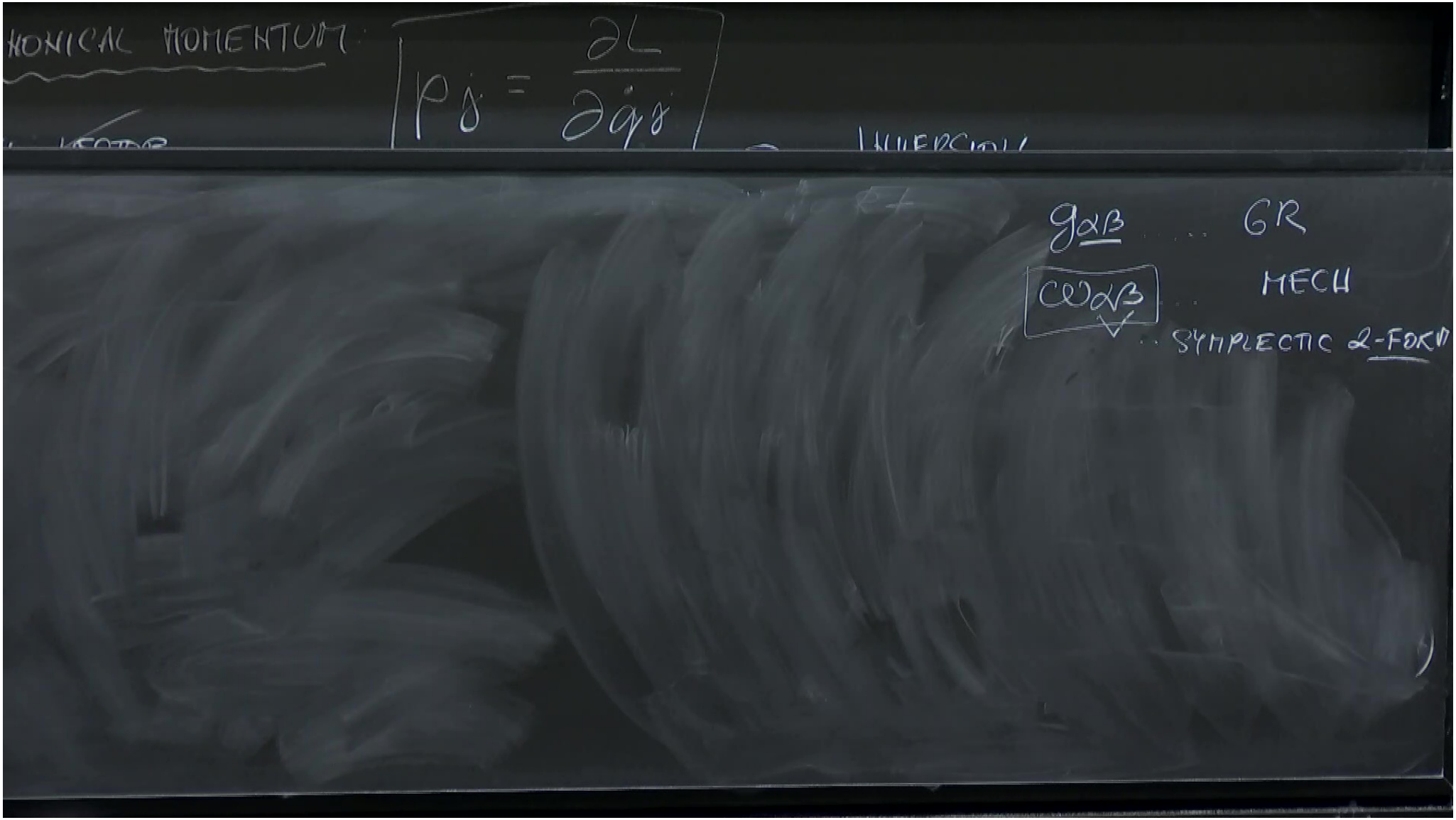
- FOR INTEGRAL OF MOTION

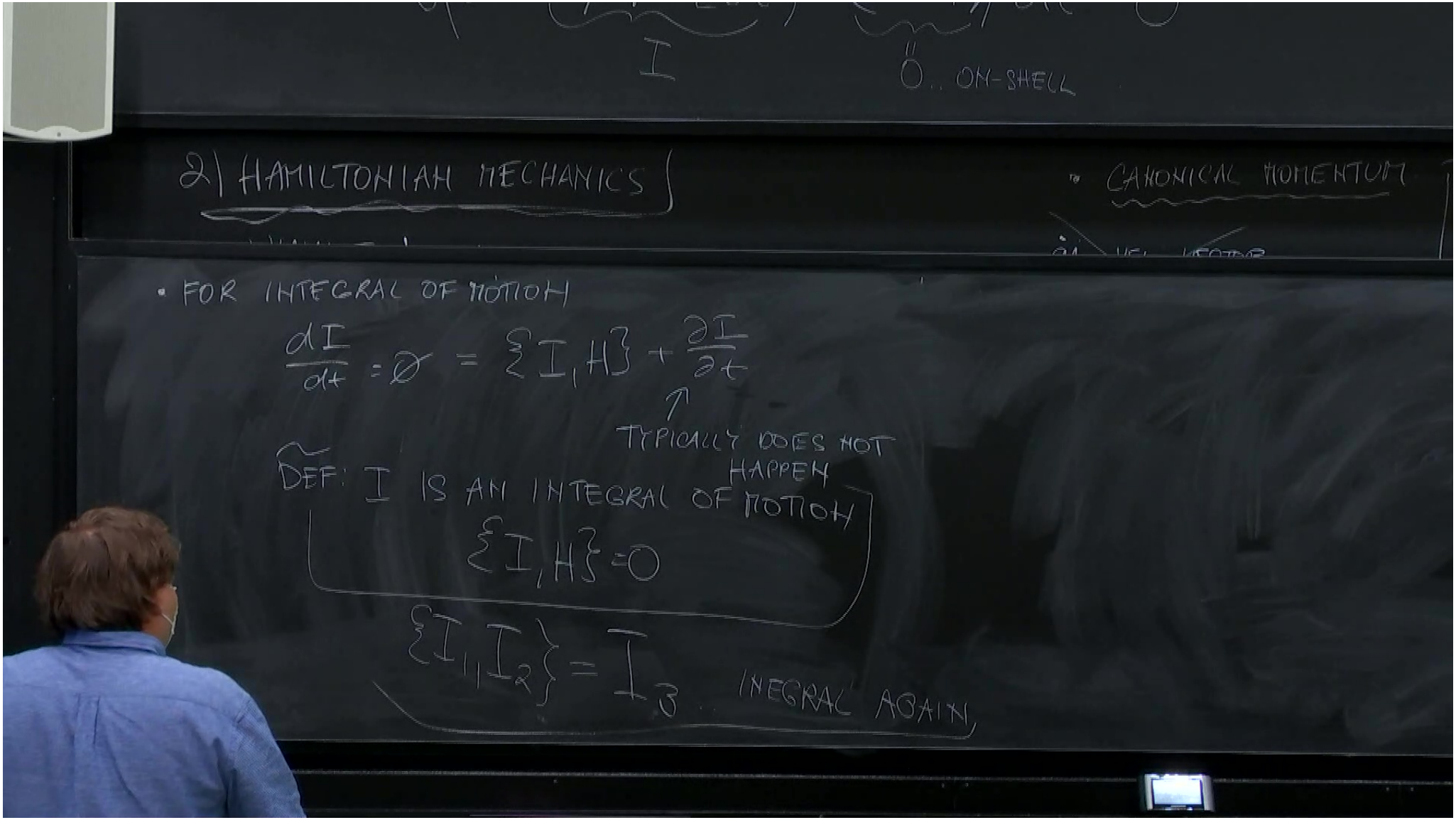
$$\frac{dI}{dt} = 0 = \{I, H\} + \frac{\partial I}{\partial t}$$

TYPICALLY DOES NOT HAPPEN

DEF: I IS AN INTEGRAL OF MOTION

$$\{I, H\} = 0$$


HONICAL MOMENTUM
$p_i = \frac{\partial L}{\partial \dot{q}_i}$
$g_{\alpha\beta}$ ... GR
$\omega_{\alpha\beta}$ ... MECH
... SYMPLECTIC 2-FORM


I
0.. ON-SHELL
2| HAMILTONIAN MECHANICS
• CANONICAL MOMENTUM
• FOR INTEGRAL OF MOTION
$\frac{dI}{dt} = 0 = \{I, H\} + \frac{\partial I}{\partial t}$
TYPICALLY DOES NOT HAPPEN
DEF: I IS AN INTEGRAL OF MOTION
$\{I, H\} = 0$
$\{I_1, I_2\} = I_3$ INTEGRAL AGAIN,

I

0 .. ON-SHELL

2| HAMILTONIAN MECHANICS

- CANONICAL MOMENTUM

- FOR INTEGRAL OF MOTION

$$\frac{dI}{dt} = 0 = \{I, H\} + \frac{\partial I}{\partial t}$$

TYPICALLY DOES NOT HAPPEN

DEF: I IS AN INTEGRAL OF MOTION

$$\{I, H\} = 0$$

$$\{I_1, I_2\} = I_3$$ INTEGRAL AGAIN, (UNLESS ZERO)

CANONICAL MOMENTUM

$$p_j = \frac{\partial L}{\partial \dot{q}_j}$$

c) CANONICAL TRANSFORMATIONS

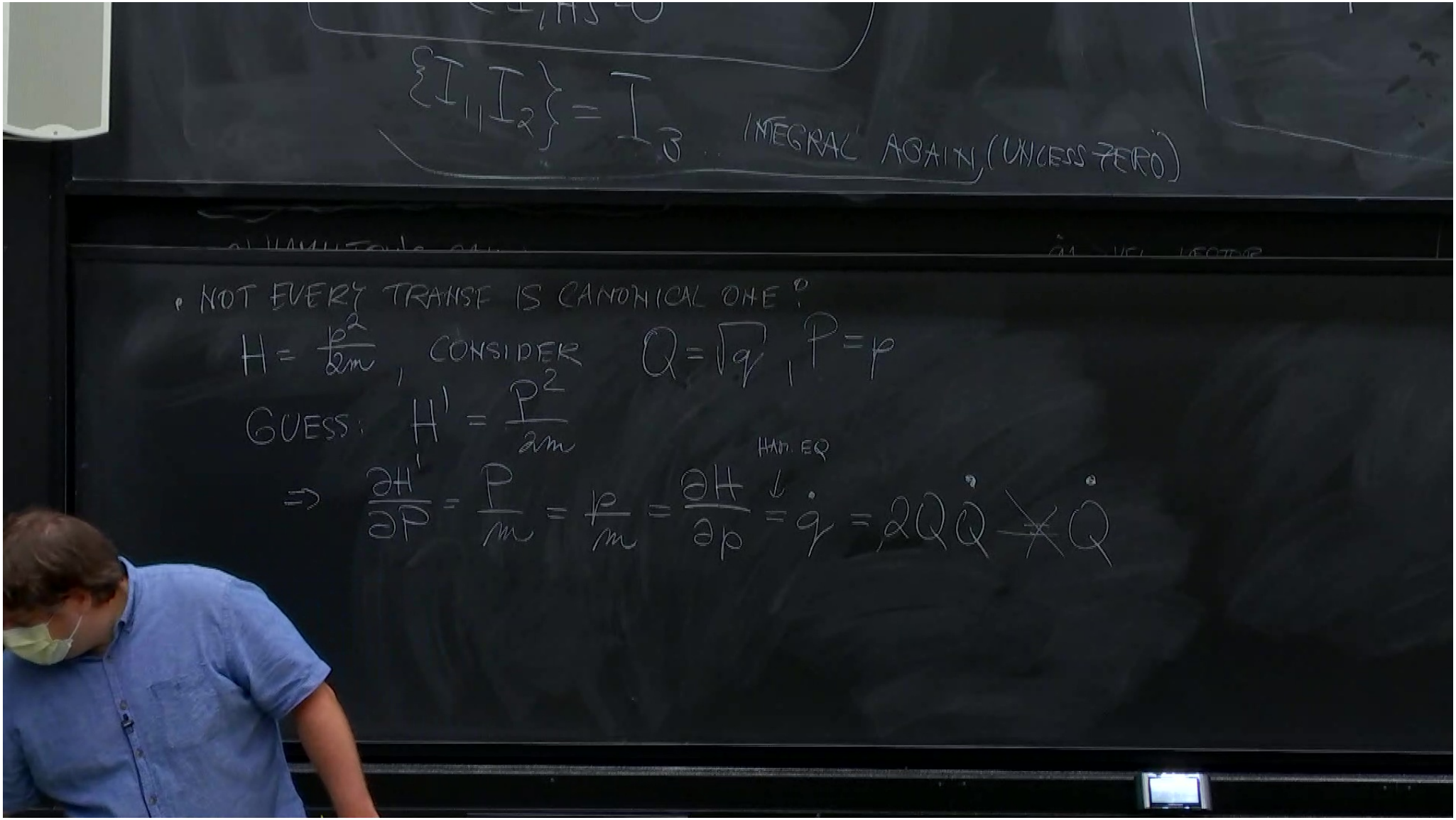
DEF. A CANONICAL TRANSF. IS A TRANSF OF PHASE SPACE COORDS.

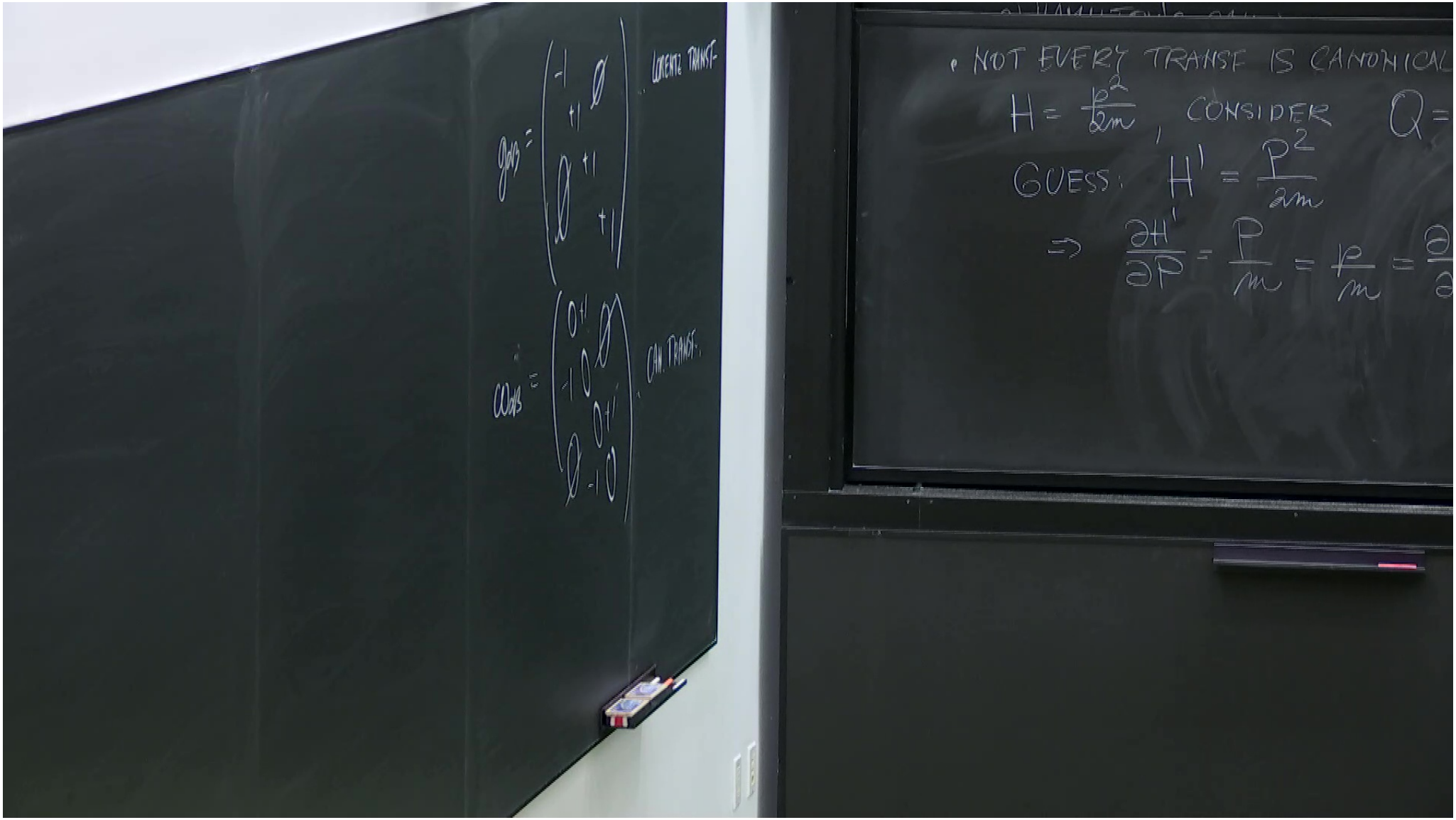
$$Q^j = Q^j(q^i, p_i, t), \quad P_j = P_j(q^i, p_i, t)$$

SUCH THAT IT PRESERVES THE FORM OF HAMILTON'S EQS.

THAT IS, THERE EXISTS NEW $H'(Q, P, t)$

$$\frac{\partial H'}{\partial P_j} = \dot{Q}^j, \quad \frac{\partial H'}{\partial Q_j} = -\dot{P}_j$$


$\{I_1, I_2\} = I_3$ INTEGRAL AGAIN, (UNLESS ZERO)
• NOT EVERY TRANSF IS CANONICAL ONE?
$H = \frac{p^2}{2m}$, CONSIDER $Q = \sqrt{q}$, $P = p$
GUESS: $H' = \frac{P^2}{2m}$
HAM EQ
$\Rightarrow \frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} = 2Q\dot{Q} \neq \dot{Q}$

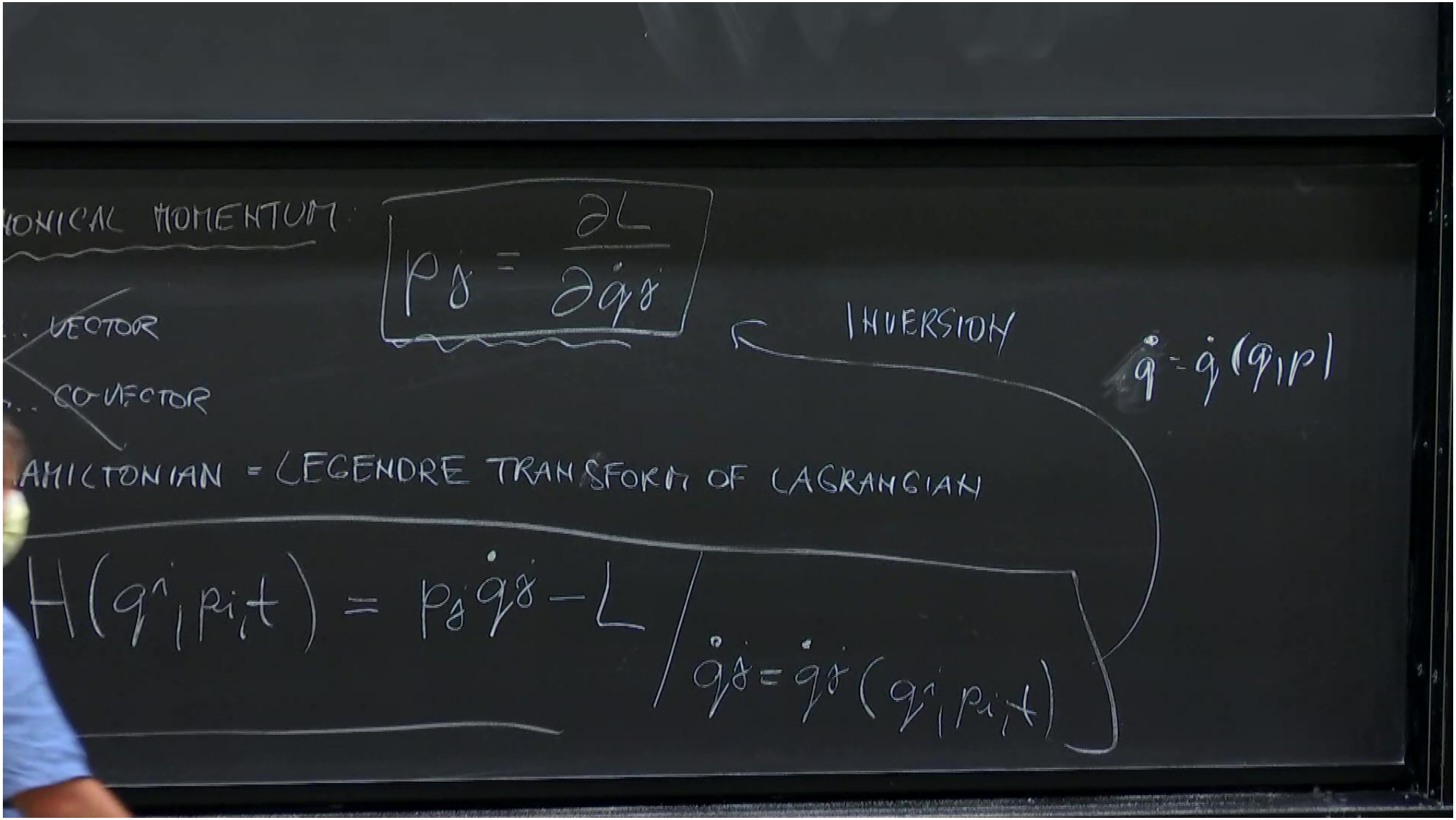

LORENTZ TRANSF.
CAN. TRANSF.
• NOT EVERY TRANSF IS CANONICAL
H = p²/2m , CONSIDER Q =
GUESS: H' = P²/2m
⇒ ∂H'/∂P = P/m = p/m = ∂

- CANONICAL TRANST. GENERATED BY GENERATING FUNCTIONS,

$$F = F(\text{OLD}, \text{NEW})$$

$\text{OLD} \in (q, p)$ $\quad$ $\text{NEW} \in (Q, P)$

... 4 POSSIBILITIES ... 4 KINDS OF GEN. F.


HONICAL MOMENTUM
$p_\delta = \frac{\partial L}{\partial \dot{q}^\delta}$
.. VECTOR
.. COVECTOR
INVERSION
$\dot{q}^i = \dot{q}^i(q, p)$
AMILTONIAN = LEGENDRE TRANSFORM OF LAGRANGIAN
$H(q^i, p_i, t) = p_\delta \dot{q}^\delta - L$
$\dot{q}^\delta = \dot{q}^\delta(q^i, p_i, t)$

$\Rightarrow \frac{\partial H}{\partial P} = \frac{1}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} = 2QQ \neq Q$

FOR EXAMPLE

$F(q, Q, t)$

... 4 POSSIBILITIES ... 4 KIN

• USING THE FREEDOM IN DEF OF L (BUT NOW IN HAM. PICTURE)

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt = \int \left(p\dot{q} - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \right) dt$$

$$= \int \left(\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right) \right) dt$$

WHAT I WANT ↓

$$= \int \left(P\dot{Q} - H' \right) dt$$

$$P = -\frac{\partial F}{\partial Q}, \quad p = \frac{\partial F}{\partial q}$$

$$H' = H + \frac{\partial F}{\partial t}$$