Title: Classical Physics - Lecture 220907

Speakers:

Collection: Classical Physics (2022/2023)

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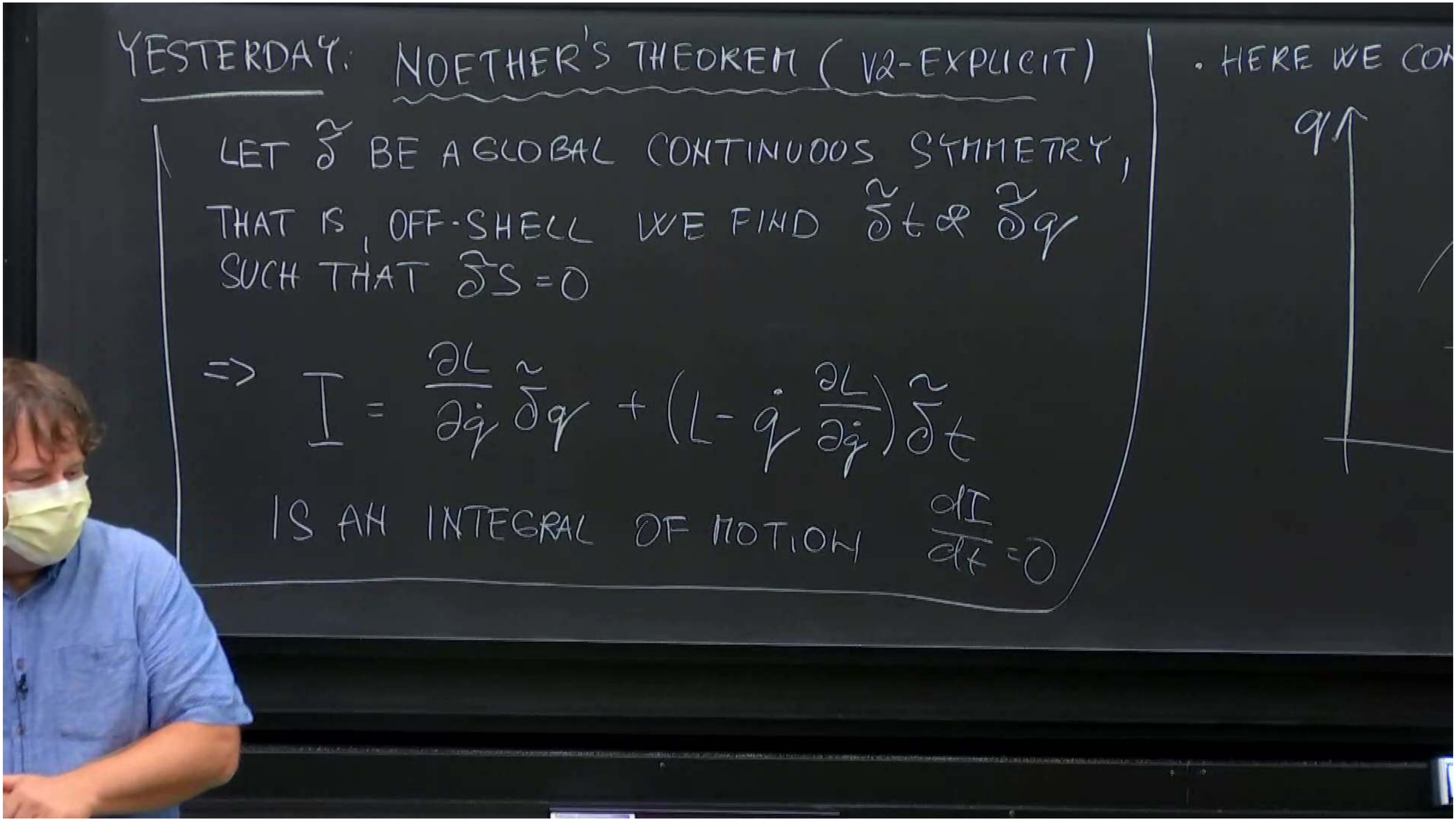
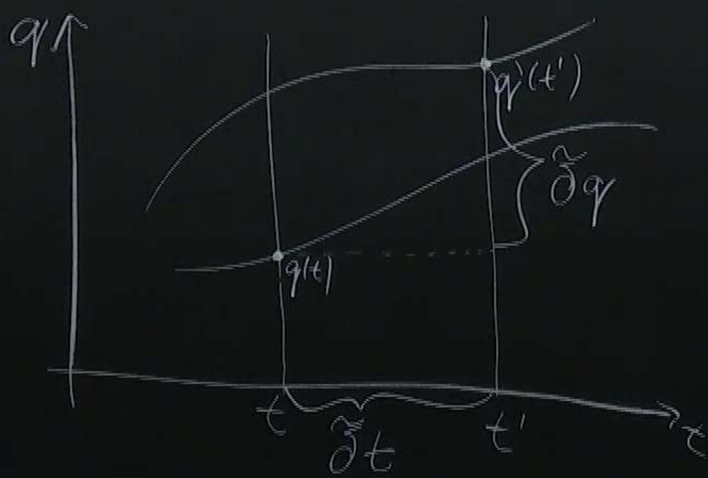

YESTERDAY: NOETHER'S THEOREM (V2-EXPLICIT)
LET $\tilde{\delta}$ BE A GLOBAL CONTINUOUS SYMMETRY,
THAT IS, OFF-SHELL WE FIND $\tilde{\delta} t$ & $\tilde{\delta} q$
SUCH THAT $\tilde{\delta} S = 0$
$\Rightarrow \quad I = \frac{\partial L}{\partial \dot{q}} \tilde{\delta} q + \left(L - \dot{q} \frac{\partial L}{\partial \dot{q}}\right) \tilde{\delta} t$
IS AN INTEGRAL OF MOTION $\frac{dI}{dt} = 0$
• HERE WE CON
q

• HERE WE CONSIDER GENERAL TRANSFORMATIONS OF BOTH q & t

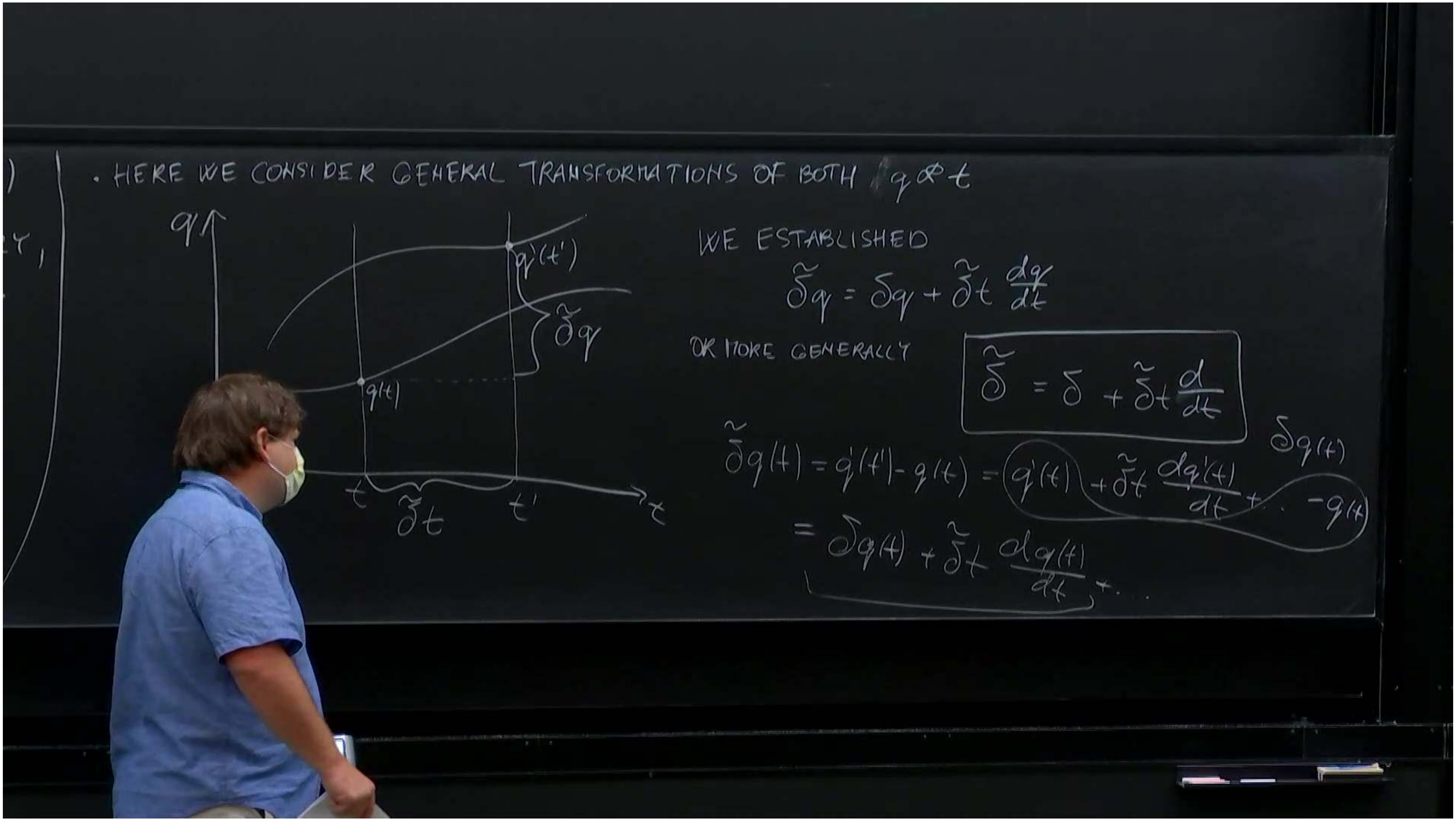


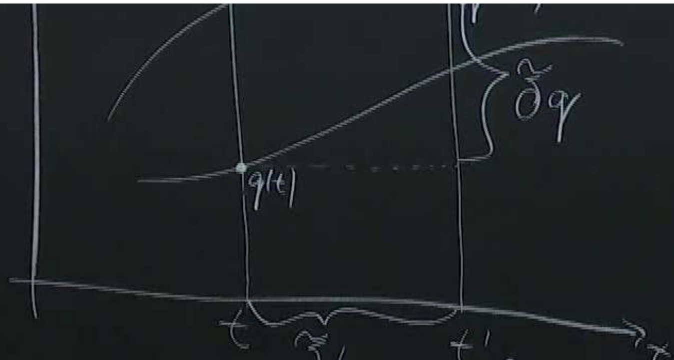
WE ESTABLISHED

$$\tilde{\delta} q = \delta q + \tilde{\delta} t \, \frac{dq}{dt}$$

OR MORE GENERALLY

$$\boxed{\tilde{\delta} = \delta + \tilde{\delta} t \, \frac{d}{dt}}$$


. HERE WE CONSIDER GENERAL TRANSFORMATIONS OF BOTH q & t
WE ESTABLISHED
$\tilde{\delta} q = \delta q + \tilde{\delta} t \frac{dq}{dt}$
OR MORE GENERALLY
$\boxed{\tilde{\delta} = \delta + \tilde{\delta} t \frac{d}{dt}}$
$\tilde{\delta} q(t) = q'(t') - q(t) = q'(t) + \tilde{\delta} t \frac{dq'(t)}{dt} + \dots - q(t)$
$= \delta q(t) + \tilde{\delta} t \frac{dq(t)}{dt} + \dots$
$\delta q(t)$
$q'(t')$
$q(t)$
$\tilde{\delta} q$
$\tilde{\delta} t$



$$\tilde{\delta} dt = (dt' - dt) = d\tilde{\delta}t = \frac{d\tilde{\delta}t}{dt}\, dt$$

$$\tilde{\delta} q = \delta q + \tilde{\delta} t \frac{dq}{dt}$$

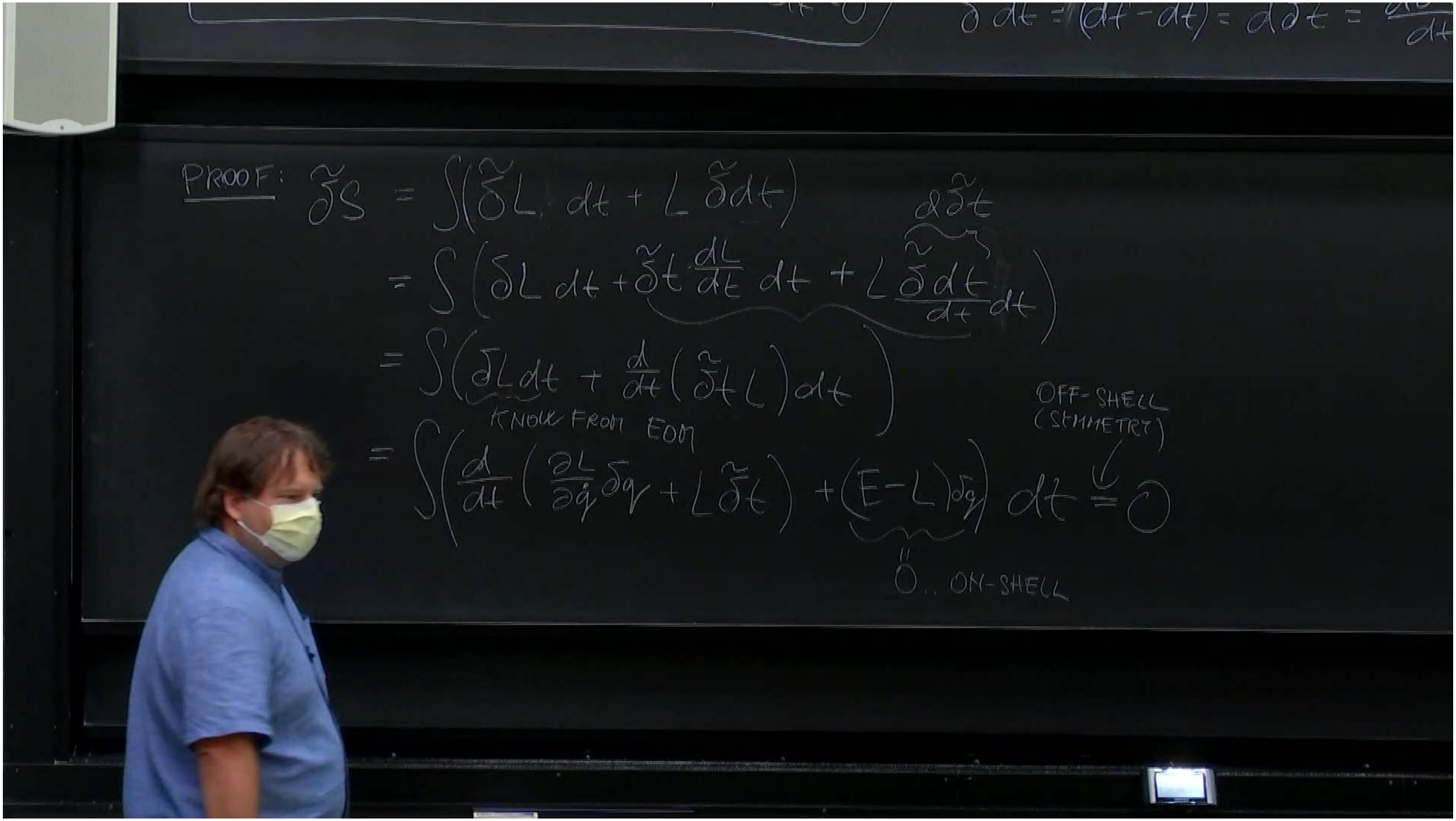
OR MORE GENERALLY

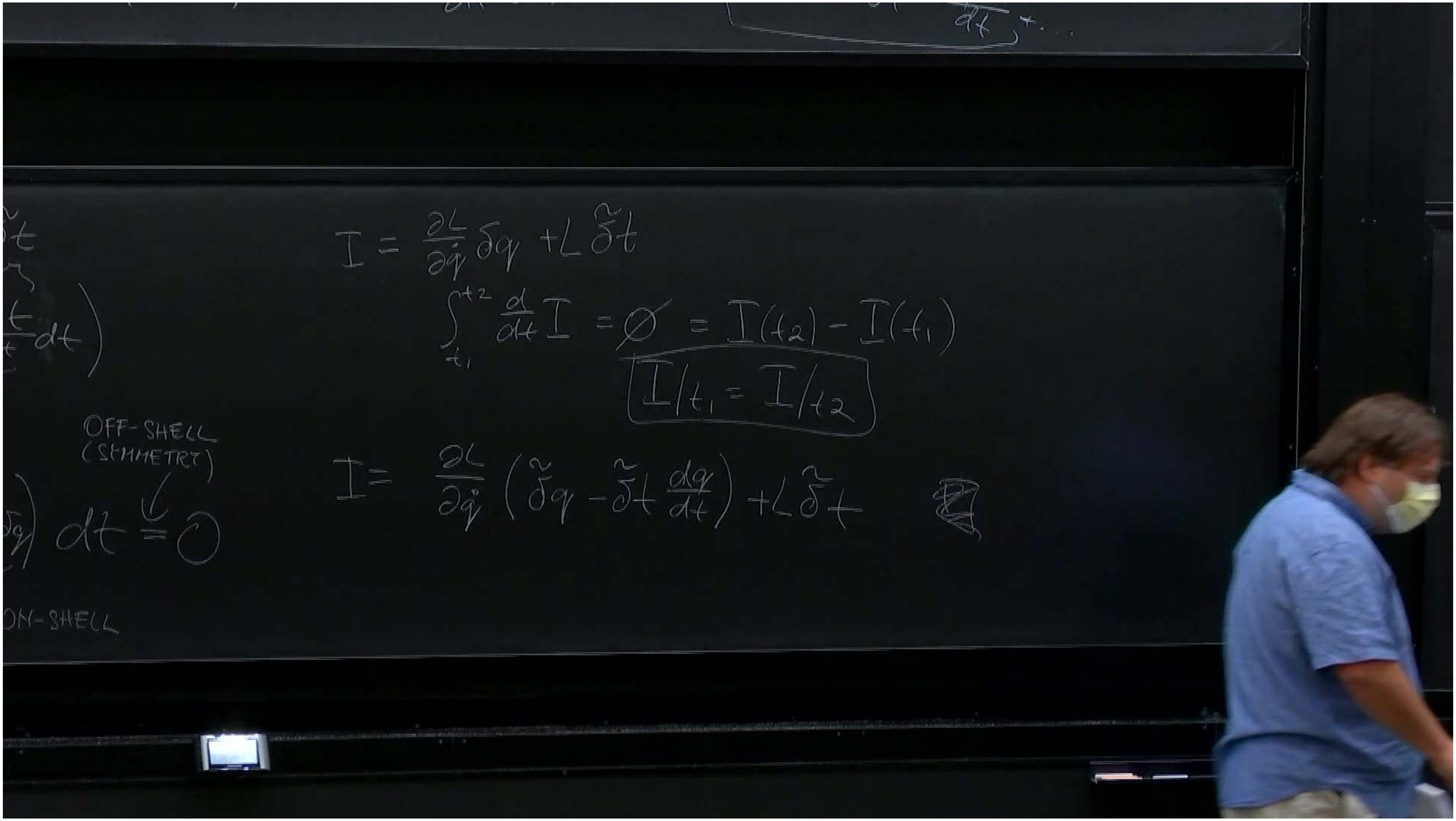
$$\boxed{\tilde{\delta} = \delta + \tilde{\delta} t \frac{d}{dt}}$$

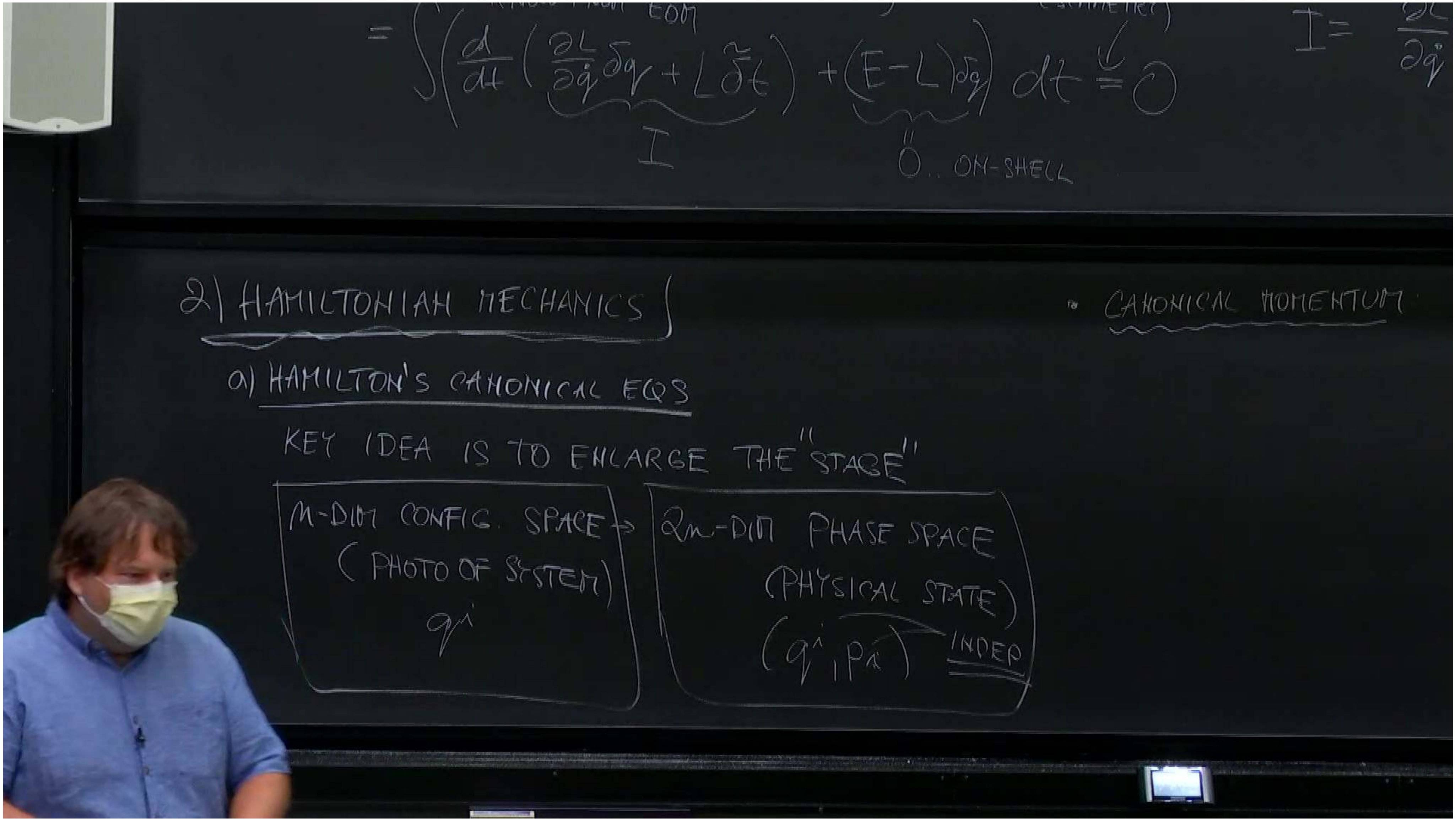
$$\tilde{\delta} q(t) = q'(t') - q(t) = q'(t) + \tilde{\delta} t \frac{dq'(t)}{dt} + \dots - q(t)$$

$\delta q(t)$

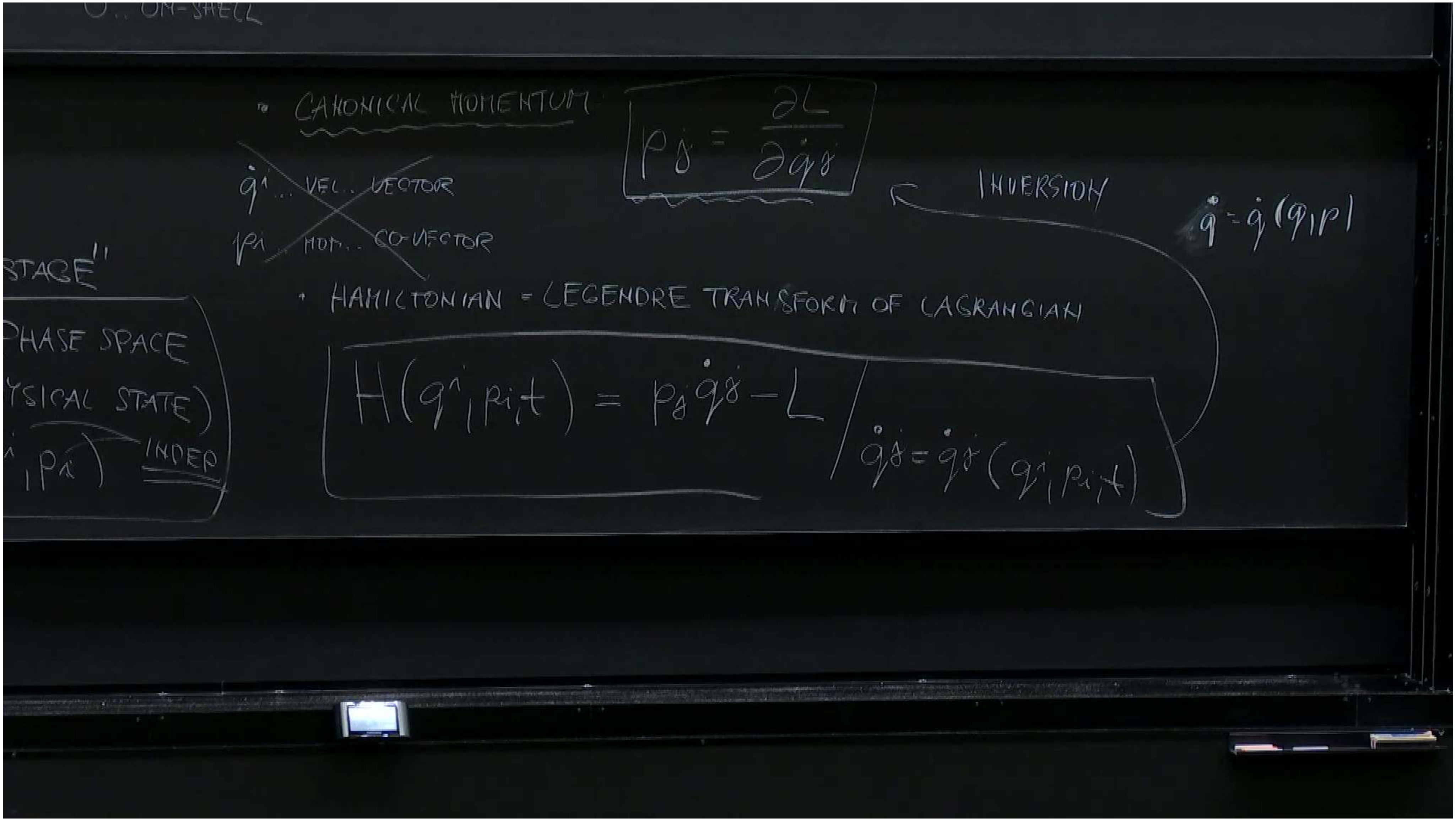
$$= \underbrace{\delta q(t) + \tilde{\delta} t \frac{dq(t)}{dt}} + \dots$$

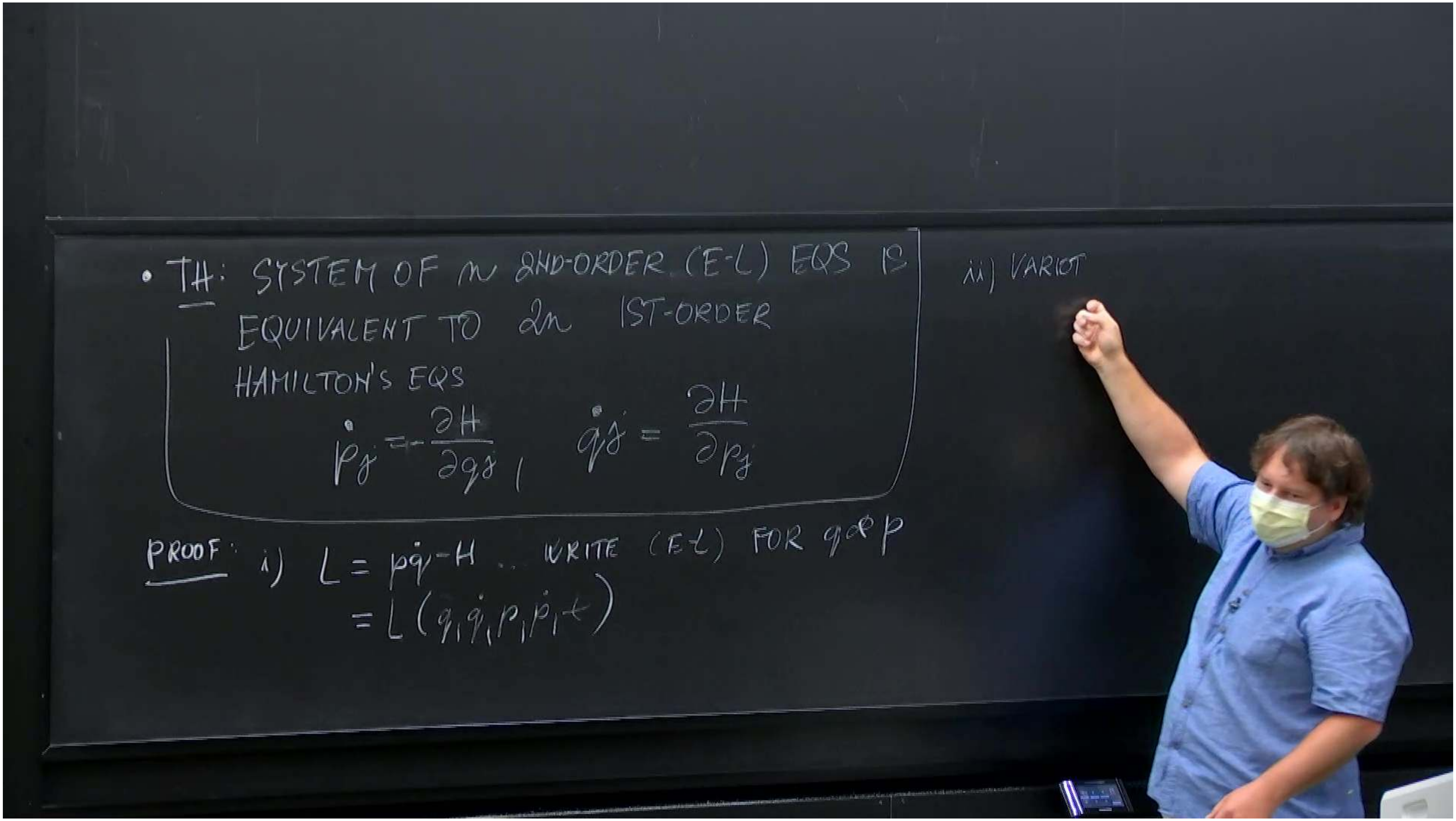

PROOF: $\tilde{\delta} S = \int (\tilde{\delta} L \, dt + L \, \tilde{\delta} dt)$
$= \int \left(\delta L \, dt + \tilde{\delta} t \frac{dL}{dt} dt + L \frac{\tilde{\delta} dt}{dt} dt \right)$
$= \int \left(\delta L \, dt + \frac{d}{dt} (\tilde{\delta} t \, L) \, dt \right)$
KNOW FROM EOM
$= \int \left(\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}} \delta q + L \tilde{\delta} t \right) + (E - L) \delta q \right) dt = 0$
OFF-SHELL
(SYMMETRY)
0 .. ON-SHELL


$I = \frac{\partial L}{\partial \dot q}\delta q + L\tilde\delta t$
$\int_{t_1}^{t_2} \frac{d}{dt} I = \emptyset = I(t_2) - I(t_1)$
$\boxed{I|_{t_1} = I|_{t_2}}$
OFF-SHELL
(SYMMETRY)
$I = \frac{\partial L}{\partial \dot q}\left(\tilde\delta q - \tilde\delta t \frac{dq}{dt}\right) + L\tilde\delta t$
$dt = 0$
ON-SHELL



EOM
I
O.. ON-SHELL
2) HAMILTONIAN MECHANICS
• CANONICAL MOMENTUM
a) HAMILTON'S CANONICAL EQS
KEY IDEA IS TO ENLARGE THE "STAGE"
n-DIM CONFIG. SPACE → 2n-DIM PHASE SPACE
(PHOTO OF SYSTEM)
(PHYSICAL STATE)
INDEP


O.. ON-SHELL
• CANONICAL MOMENTUM
$p_j = \frac{\partial L}{\partial \dot{q}^j}$
$\dot{q}^i$ .. VEL.. VECTOR
p_i .. MOM.. CO-VECTOR
INVERSION
$\dot{q} = \dot{q}(q, p)$
STAGE"
PHASE SPACE
YSICAL STATE)
$, p_i)$ INDEP
• HAMILTONIAN = LEGENDRE TRANSFORM OF LAGRANGIAN
$H(q^i, p_i, t) = p_j \dot{q}^j - L$
$\dot{q}^j = \dot{q}^j(q^i, p_i, t)$


• TH: SYSTEM OF n 2ND-ORDER (E-L) EQS IS EQUIVALENT TO $2n$ 1ST-ORDER HAMILTON'S EQS
$\dot{p}_j = -\frac{\partial H}{\partial q_j}, \quad \dot{q}_j = \frac{\partial H}{\partial p_j}$
PROOF: i) $L = p\dot{q} - H$ .. WRITE (E-L) FOR q & p
$= L(q, \dot{q}, p, \dot{p}, t)$
ii) VARIAT

DER (E-L) EQS IS

1ST-ORDER

$$\dot{q}^j = \frac{\partial H}{\partial p_j}$$

E (E-L) FOR q & p

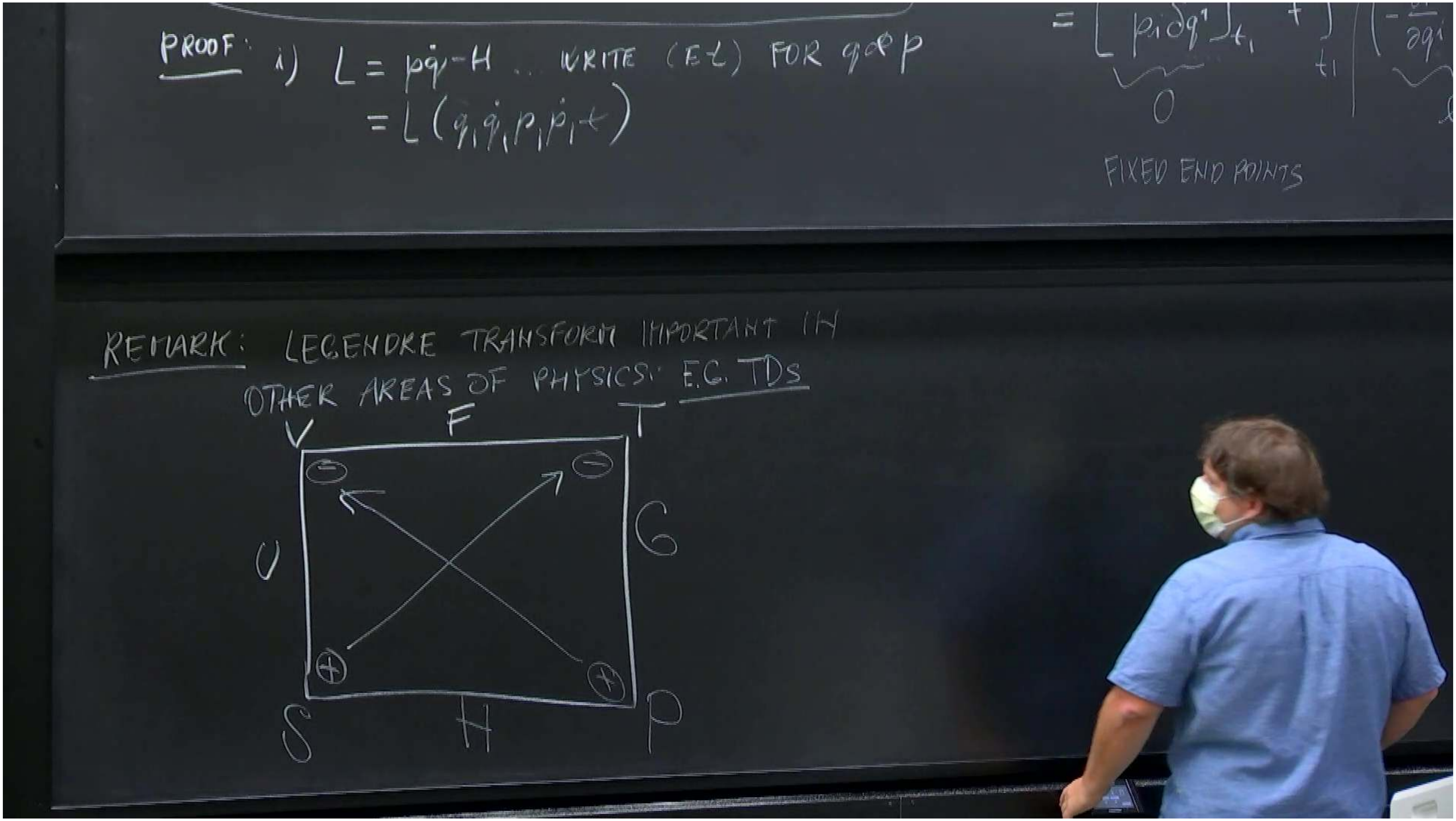
ii) VARIATIONAL PRINCIPLE

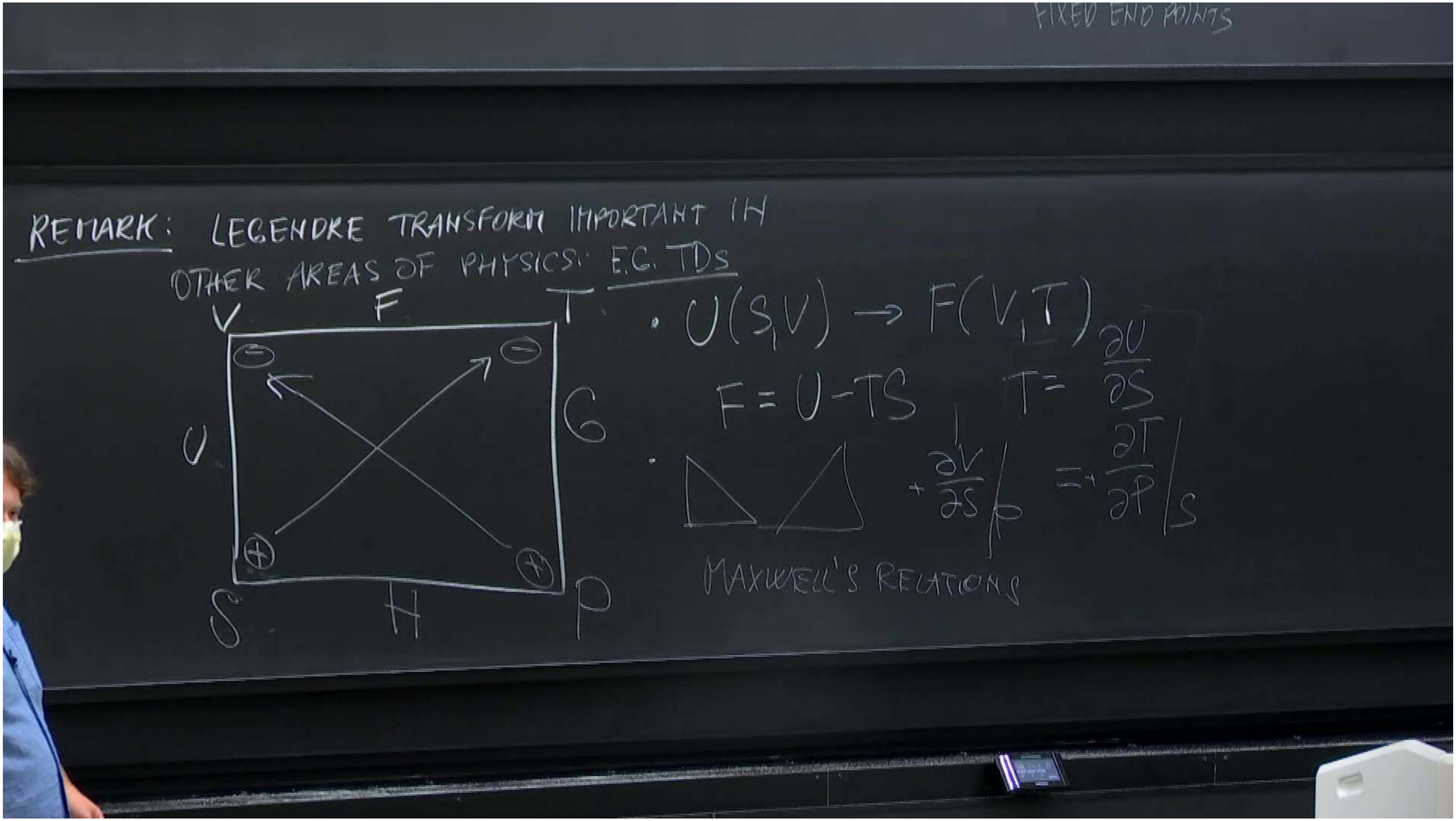
$$0 = \delta S = \delta \int_{t_1}^{t_2} \left(p_i \dot{q}^i - H(q,p) \right) dt$$

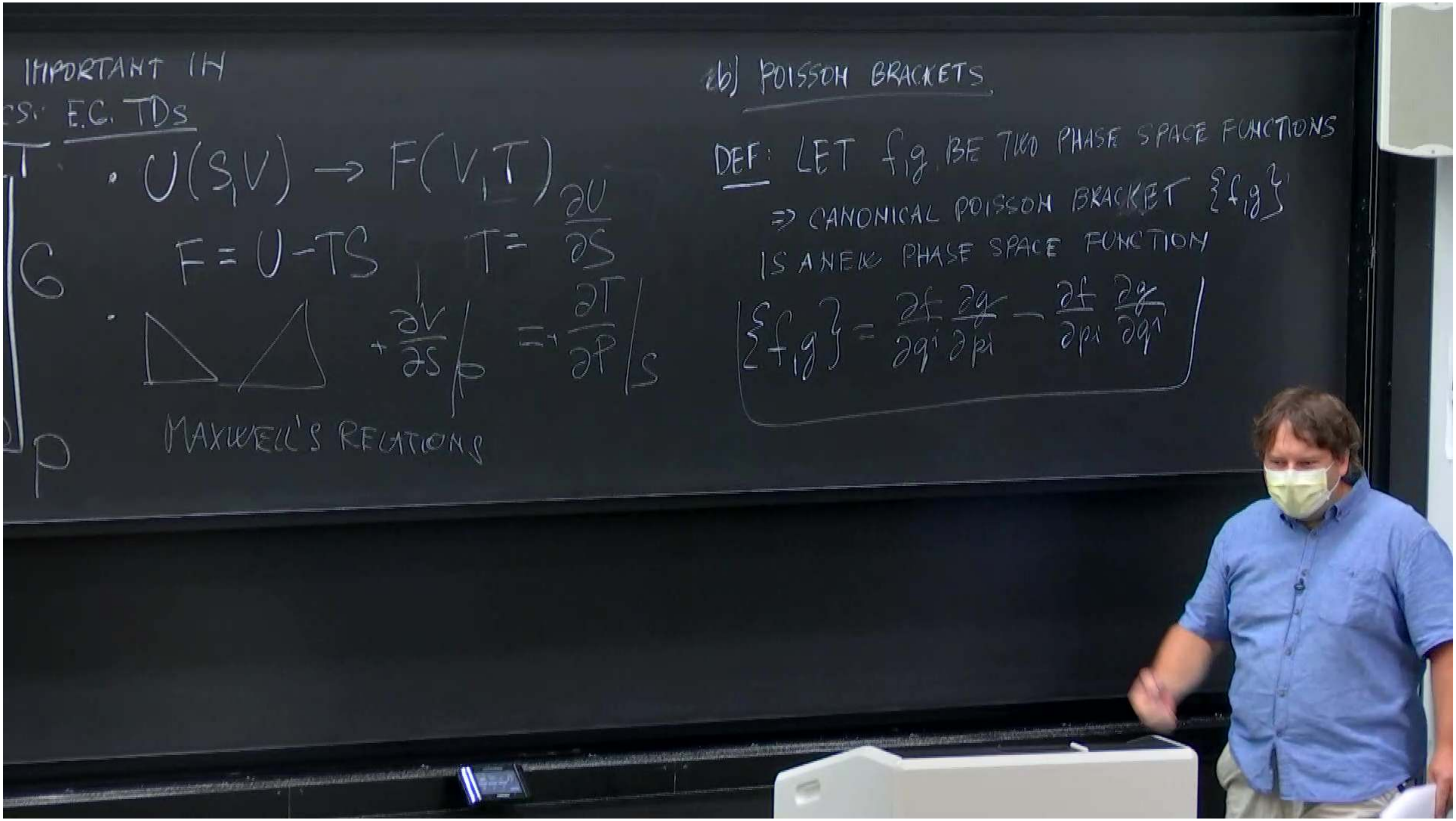
$$= \int_{t_1}^{t_2} \left((\delta p_i)\dot{q}^i + p_i \delta \dot{q}^i - \frac{\partial H}{\partial q^i}\delta q^i - \frac{\partial H}{\partial p_i}\delta p_i \right) dt$$

$$= \underbrace{\Big[p_i \delta q^i \Big]_{t_1}^{t_2}}_{0} + \int_{t_1}^{t_2} \left(\underbrace{\left(-\frac{\partial H}{\partial q^i} - \dot{p}_i \right)}_{\emptyset} \delta q^i + \underbrace{\left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right)}_{\emptyset} \delta p_i \right) dt$$

FIXED END POINTS


PROOF: i) $L = p\dot{q} - H$ ... WRITE (E-L) FOR q & p
$= L(q, \dot{q}, p, \dot{p}, t)$
0
FIXED END POINTS
REMARK: LEGENDRE TRANSFORM IMPORTANT IN
OTHER AREAS OF PHYSICS: E.G. TDs
V
F
T
U
G
S
H
P


FIXED END POINTS
REMARK: LEGENDRE TRANSFORM IMPORTANT IN
OTHER AREAS OF PHYSICS: E.G. TDs
V F T
U G
S H P
U(S,V) → F(V,T)
F = U − TS
T = ∂U/∂S
+∂V/∂S|P = +∂T/∂P|S
MAXWELL'S RELATIONS


IMPORTANT IN
S: E.C. TDs
$U(S,V) \to F(V,T)$
$F = U - TS$
$T = \frac{\partial U}{\partial S}$
$\frac{\partial V}{\partial S}\Big|_P = - \frac{\partial T}{\partial P}\Big|_S$
MAXWELL'S RELATIONS
b) POISSON BRACKETS
DEF: LET f,g BE TWO PHASE SPACE FUNCTIONS
⇒ CANONICAL POISSON BRACKET $\{f,g\}$
IS A NEW PHASE SPACE FUNCTION
$\{f,g\} = \frac{\partial f}{\partial q^i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q^i}$

$S \quad H \quad P$ MAXWELL'S RELATIONS

- PROPS

i) FUNDAMENTAL BRACKETS

$$\boxed{\begin{aligned} \{q^i, p_j\} &= \delta^i_j \\ \{q^i, q^j\} &= 0 = \{p_i, p_j\} \end{aligned}}$$

- A WAY TO QUANTIZE

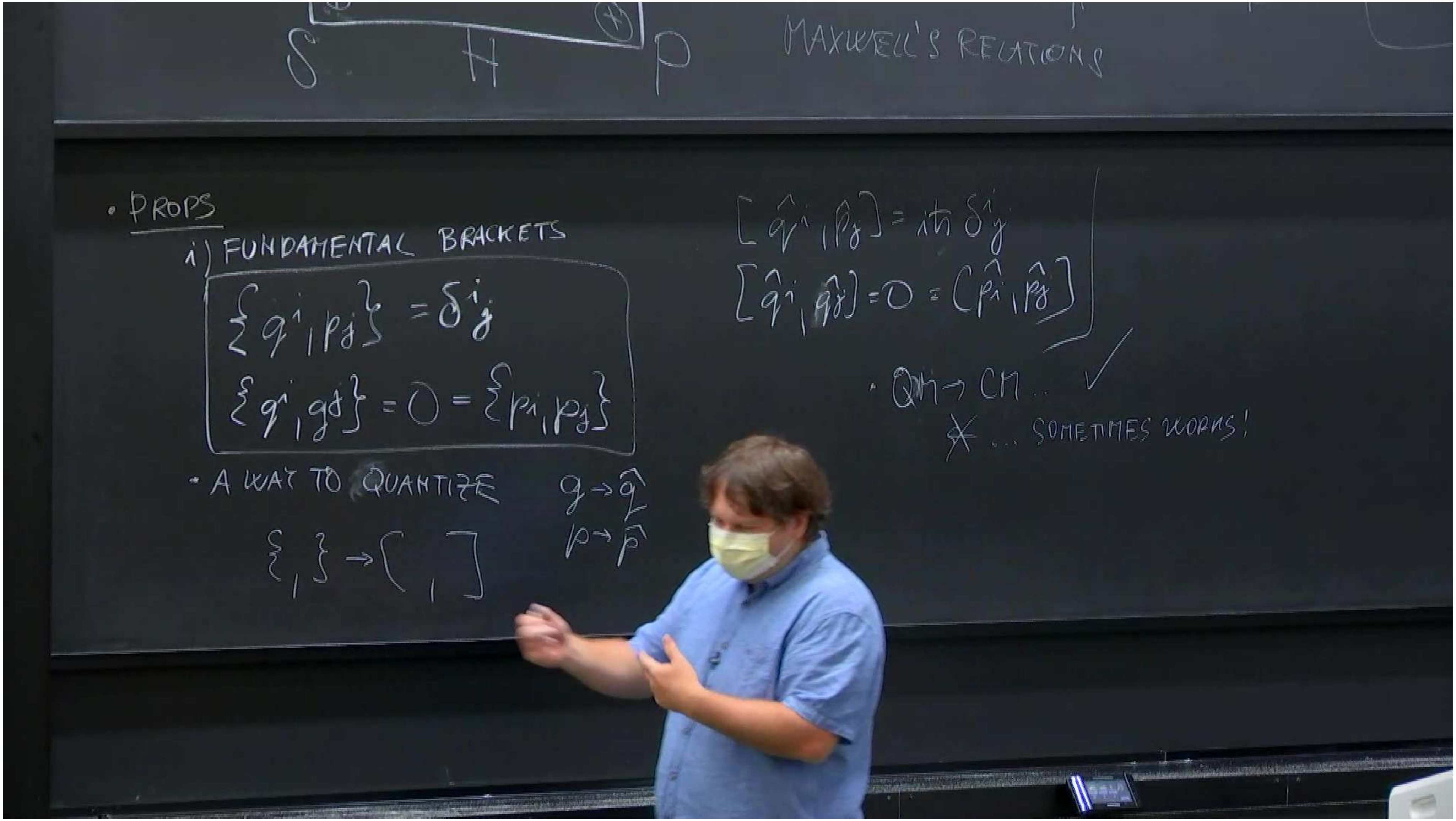
$q \to \hat{q}$

$p \to \hat{p}$

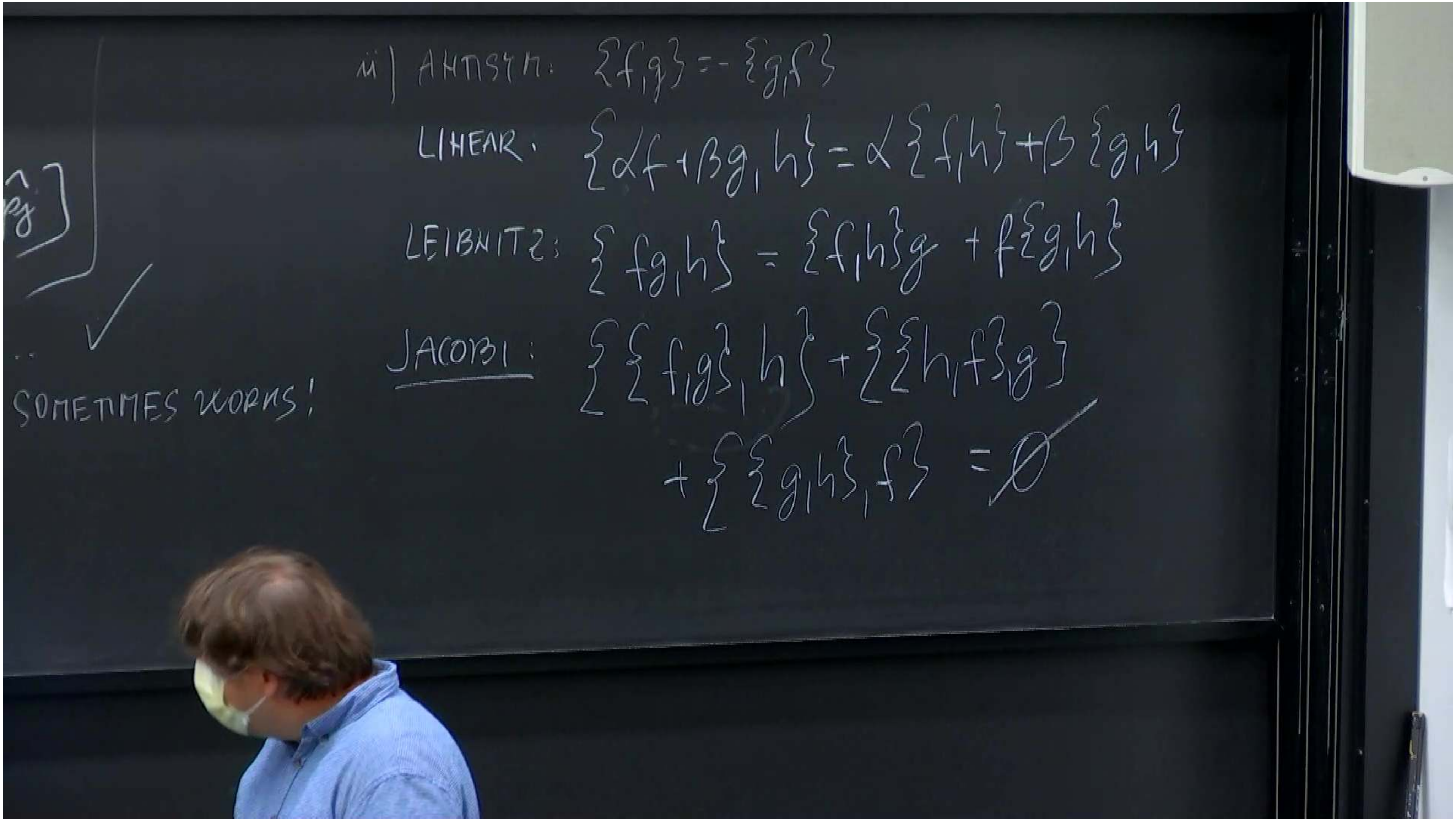
$\{\,,\,\} \to [\,,\,]$

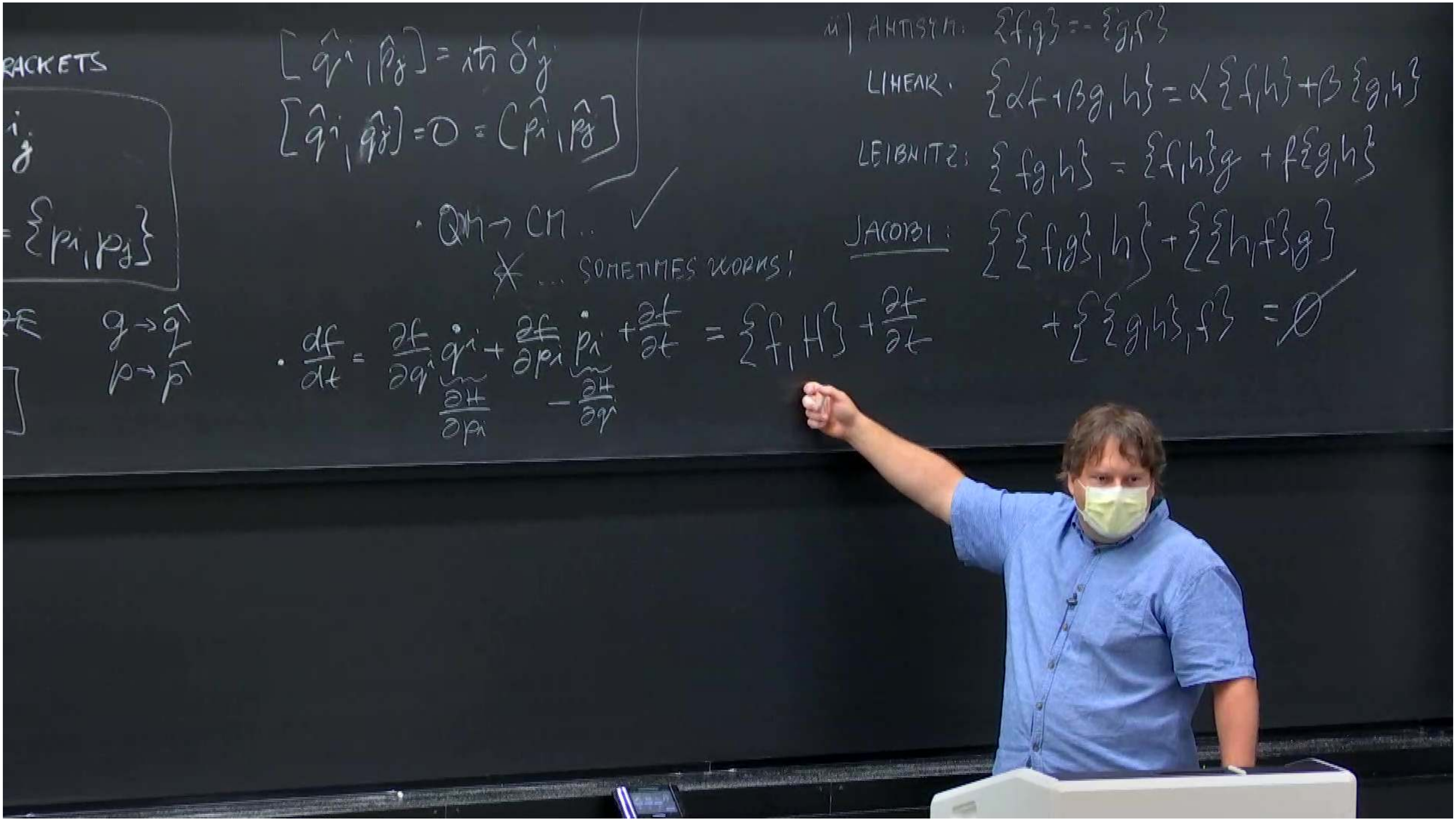
$$[\hat{q}^i, \hat{p}_j] = i\hbar\,\delta^i_j$$

$$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j]$$



S H P
MAXWELL'S RELATIONS
• PROPS
i) FUNDAMENTAL BRACKETS
$\{q^i, p_j\} = \delta^i_j$
$\{q^i, q^j\} = 0 = \{p_i, p_j\}$
• A WAY TO QUANTIZE
$q \to \hat{q}$
$p \to \hat{p}$
$\{\,,\,\} \to [\,,\,]$
$[\hat{q}^i, \hat{p}_j] = i\hbar\,\delta^i_j$
$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j]$
• QM → CM ... ✓
✗ ... SOMETIMES WORKS!


iii) ANTISYM: {f,g} = -{g,f}
LINEAR: {αf+βg, h} = α{f,h}+β{g,h}
LEIBNITZ: {fg,h} = {f,h}g + f{g,h}
JACOBI: {{f,g},h} + {{h,f},g} + {{g,h},f} = 0
SOMETIMES WORKS!



RACKETS
$[\hat{q}^i, \hat{p}_j] = i\hbar\,\delta^i_j$
$[\hat{q}^i, \hat{q}^j] = 0 = [\hat{p}_i, \hat{p}_j]$
$= \{p_i, p_j\}$
· QM → CM .. ✓
✗ ... SOMETIMES WORKS!
$q \to \hat{q}$
$p \to \hat{p}$
$\cdot\ \frac{df}{dt} = \frac{\partial f}{\partial q^i}\underbrace{\dot{q}^i}_{\frac{\partial H}{\partial p_i}} + \frac{\partial f}{\partial p_i}\underbrace{\dot{p}_i}_{-\frac{\partial H}{\partial q^i}} + \frac{\partial f}{\partial t} = \{f, H\} + \frac{\partial f}{\partial t}$
ii) ANTISYM: $\{f,g\} = -\{g,f\}$
LINEAR: $\{\alpha f + \beta g, h\} = \alpha\{f,h\} + \beta\{g,h\}$
LEIBNITZ: $\{fg, h\} = \{f,h\}g + f\{g,h\}$
JACOBI: $\{\{f,g\},h\} + \{\{h,f\},g\} + \{\{g,h\},f\} = 0$

I

Ö.. ON-SHELL

2) HAMILTONIAN MECHANICS

• CANONICAL MOMENTUM

• FOR INTEGRAL OF MOTION

$$\frac{dI}{dt} = \emptyset = \{I, H\} + \frac{\partial I}{\partial t}$$

TYPICALLY DOES NOT HAPPEN

DEF: I IS AN INTEGRAL OF MOTION

$$\boxed{\{I, H\} = 0}$$

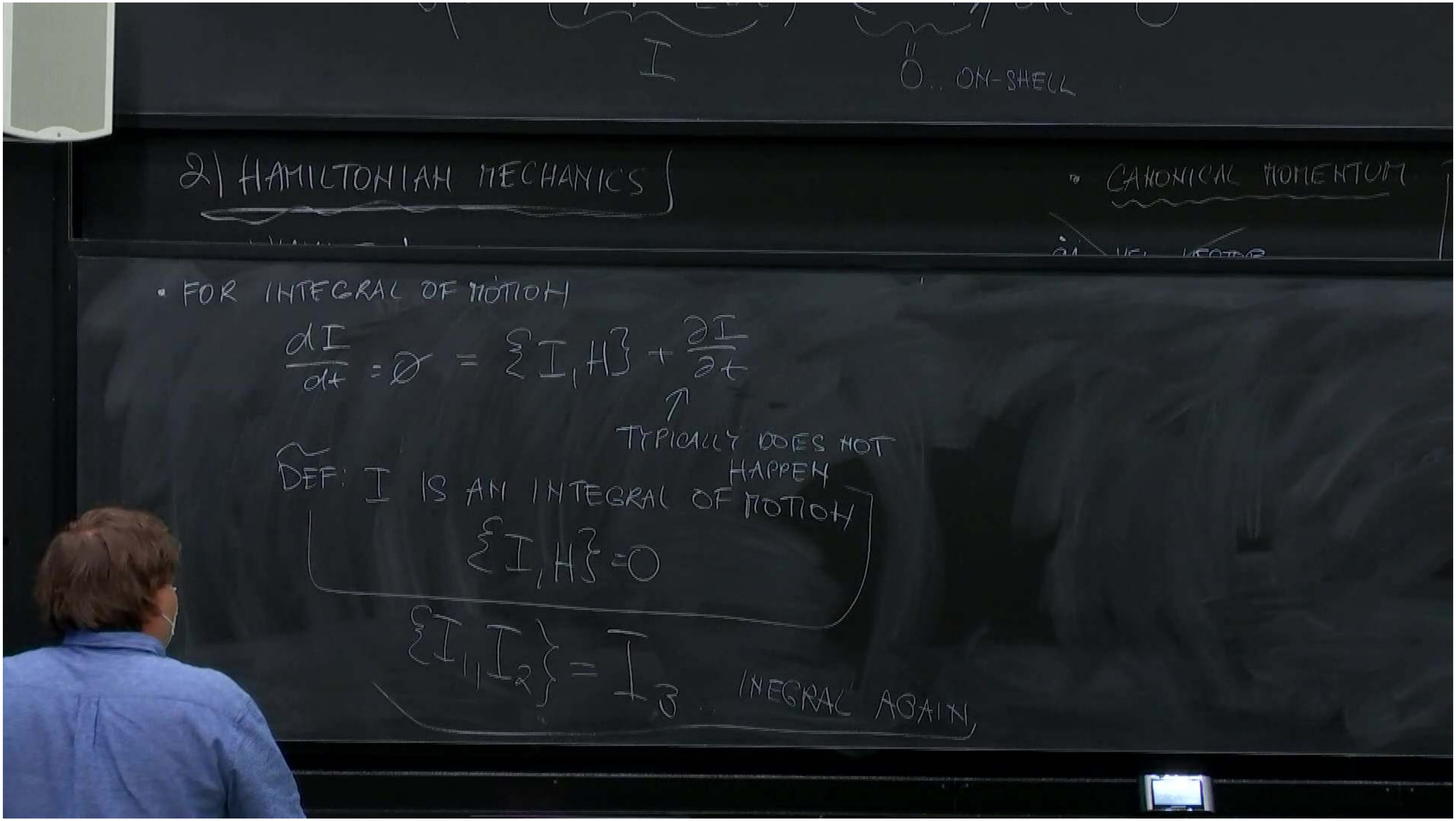
NONICAL MOMENTUM:

$$\boxed{p_j = \frac{\partial L}{\partial \dot{q}_j}}$$

$\underline{g_{\alpha\beta}}$ ... GR

$\boxed{\omega_{\alpha\beta}}$ ... MECH

... SYMPLECTIC $\underline{2\text{-FORM}}$


I
0 .. ON-SHELL
2) HAMILTONIAN MECHANICS
• CANONICAL MOMENTUM
• FOR INTEGRAL OF MOTION
$\frac{dI}{dt} = 0 = \{I, H\} + \frac{\partial I}{\partial t}$
TYPICALLY DOES NOT HAPPEN
DEF: I IS AN INTEGRAL OF MOTION
$\{I, H\} = 0$
$\{I_1, I_2\} = I_3$
INTEGRAL AGAIN

I $\qquad 0$ .. ON-SHELL

2) HAMILTONIAN MECHANICS

• CANONICAL MOMENTUM

• FOR INTEGRAL OF MOTION

$$\frac{dI}{dt} = \emptyset = \{I, H\} + \frac{\partial I}{\partial t}$$

TYPICALLY DOES NOT HAPPEN

DEF: I IS AN INTEGRAL OF MOTION

$$\boxed{\{I, H\} = 0}$$

$$\{I_1, I_2\} = I_3 \text{ .. INTEGRAL AGAIN, (UNLESS ZERO)}$$

CANONICAL MOMENTUM: $\boxed{p_j = \frac{\partial L}{\partial \dot{q}^j}}$

c) CANONICAL TRANSFORMATIONS

DEF. A CANONICAL TRANSF. IS A TRANSF OF PHASE SPACE COORDS.

$$Q^j = Q^j(q^i, p_i, t), \quad P_j = P_j(q^i, p_i, t)$$

SUCH THAT IT PRESERVES THE FORM OF HAMILTON'S EQS.

THAT IS, THERE EXISTS NEW $H'(Q, P, t)$

$$\frac{\partial H'}{\partial P_j} = \dot{Q}^j, \quad \frac{\partial H'}{\partial Q^j} = -\dot{P}_j$$

...S ZERO)

$$\{I_1, I_2\} = I_3$$ INTEGRAL AGAIN, (UNLESS ZERO)

• NOT EVERY TRANSF IS CANONICAL ONE:

$$H = \frac{p^2}{2m}, \quad \text{CONSIDER} \quad Q = \sqrt{q}, \quad P = p$$

GUESS: $$H' = \frac{P^2}{2m}$$

$$\Rightarrow \frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} \overset{\text{HAM. EQ}}{=} \dot{q} = 2Q\dot{Q} \neq \dot{Q}$$

$$g_{\alpha\beta} = \begin{pmatrix} -1 & & & \emptyset \\ & +1 & & \\ & & +1 & \\ \emptyset & & & +1 \end{pmatrix} \; \text{LORENTZ TRANSF.}$$

$$\omega_{\alpha\beta} = \begin{pmatrix} 0 & +1 & & \emptyset \\ -1 & 0 & & \\ & & 0 & +1 \\ \emptyset & & -1 & 0 \end{pmatrix} \; \text{CAN. TRANSF.}$$

• NOT EVERY TRANSF IS CANONICAL

$H = \frac{p^2}{2m}$, CONSIDER $Q =$

GUESS: $H' = \frac{P^2}{2m}$

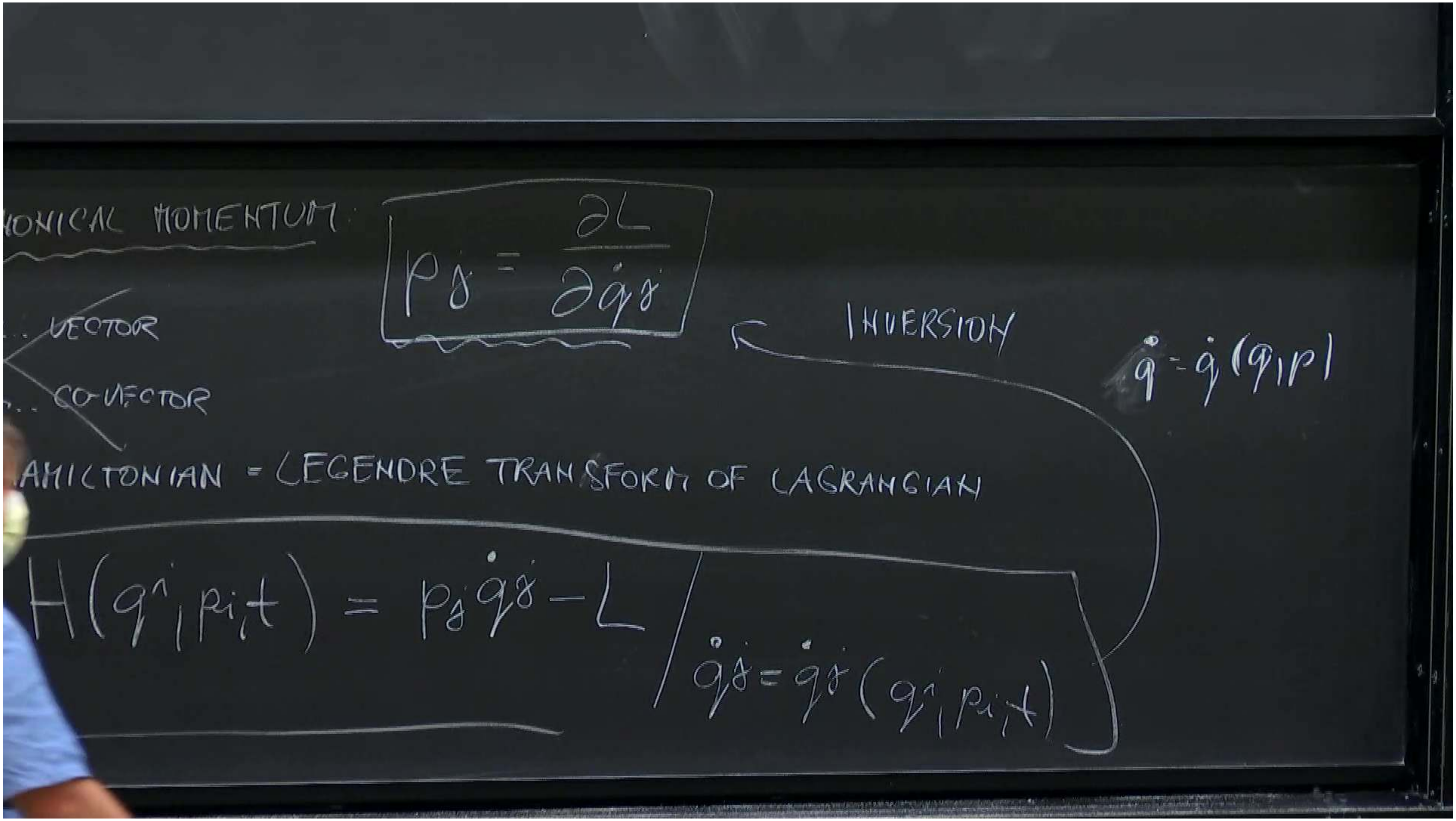
$$\Rightarrow \frac{\partial H'}{\partial P} = \frac{P}{m} = \frac{p}{m} = \frac{\partial}{\partial}$$

- CANONICAL TRANSF. GENERATED BY GENERATING FUNCTIONS,

$$F = F(\text{OLD}, \text{NEW})$$

$$\text{OLD} \in (q, p) \qquad \text{NEW} \in (Q, P)$$

... 4 POSSIBILITIES ... 4 KINDS OF GEN. F.


ONICAL MOMENTUM
$\boxed{p_j = \frac{\partial L}{\partial \dot{q}^j}}$
VECTOR
CO-VECTOR
INVERSION
$\dot{q} = \dot{q}(q,p)$
AMILTONIAN = LEGENDRE TRANSFORM OF LAGRANGIAN
$H(q^i, p_i, t) = p_j \dot{q}^j - L \Big|_{\dot{q}^j = \dot{q}^j(q^i, p_i, t)}$

$$\Rightarrow \frac{\partial H}{\partial P} = \frac{1}{m} = \frac{p}{m} = \frac{\partial H}{\partial p} = \dot{q} = 2Q\dot{Q} \neq Q$$

FOR EXAMPLE

$F(q, Q, t)$

... 4 POSSIBILITIES ... 4 KIN

• USING THE FREEDOM IN DEF OF L (BUT NOW IN HAM. PICTURE)

$$S = \int \left(p\dot{q} - H - \frac{dF}{dt} \right) dt = \int \left(p\dot{q} - H - \frac{\partial F}{\partial q}\dot{q} - \frac{\partial F}{\partial Q}\dot{Q} - \frac{\partial F}{\partial t} \right) dt$$

$$= \int \left(\dot{q}\left(p - \frac{\partial F}{\partial q}\right) - \frac{\partial F}{\partial Q}\dot{Q} - \left(H + \frac{\partial F}{\partial t}\right) \right) dt$$

WHAT I WANT ↓

$$= \int \left(P\dot{Q} - H' \right) dt$$

$$\boxed{P = -\frac{\partial F}{\partial Q}, \quad p = \frac{\partial F}{\partial q}, \quad H' = H + \frac{\partial F}{\partial t}}$$