Title: Quantum Theory - Lecture 220926

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LSZ Reduction Formula

$$\langle f|S|i\rangle_H \overset{\text{LSZ}}{\longleftrightarrow} \langle \Omega|T\,\varphi(x_1)\cdots\varphi(x_n)|\Omega\rangle \longleftrightarrow \text{Dyson}$$

- $\langle f|S|i\rangle_H$: scattering amplitude
- $|\Omega\rangle$: ground state of interacting theory
- T: time-ordering symbol

$$T\,\varphi(x_1)\,\varphi(x_2) = \begin{cases} \varphi(x_1)\,\varphi(x_2) \\ \varphi(x_2)\,\varphi(x_1) \end{cases}$$

latest to left

ion Formula

sz

$$\rightarrow \langle \Omega | T\,\varphi(x_1) \cdots \varphi(x_n) | \Omega \rangle \longleftrightarrow \langle 0 | T\,\varphi(x_1) \cdots \varphi(x_m) | 0$$

ground state of interacting theory

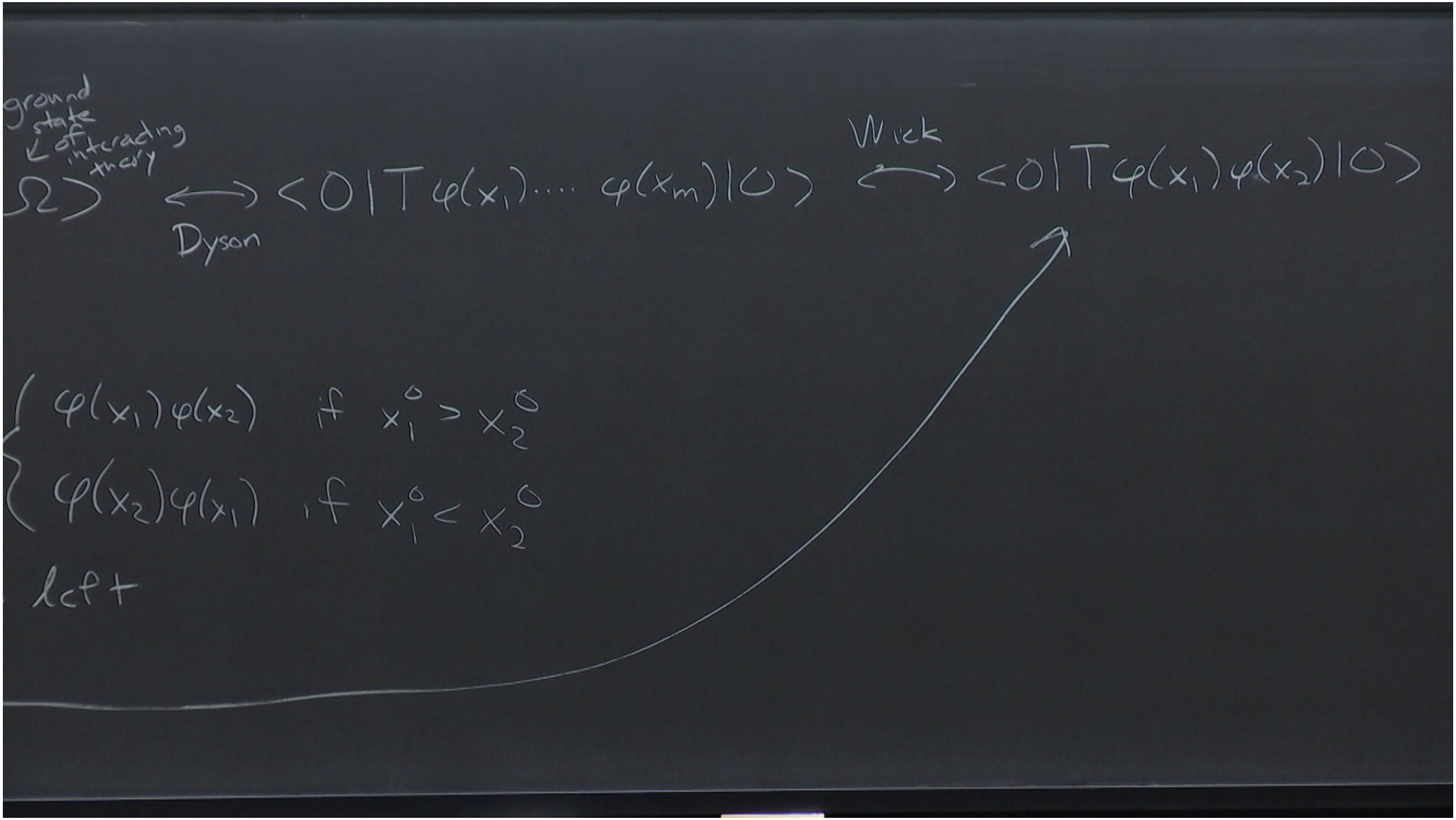
Dyson

time-ordering symbol

$$T\,\varphi(x_1)\varphi(x_2) = \begin{cases} \varphi(x_1)\varphi(x_2) & \text{if } x_1^0 > x_2^0 \\ \varphi(x_2)\varphi(x_1) & \text{if } x_1^0 < x_2^0 \end{cases}$$

latest to left

Feynman diagrams


ground state of interacting theory
$|\Omega\rangle$
$\longleftrightarrow$
Dyson
$\langle 0|T\varphi(x_1)\cdots\varphi(x_m)|0\rangle$
Wick
$\longleftrightarrow$
$\langle 0|T\varphi(x_1)\varphi(x_2)|0\rangle$
$\varphi(x_1)\varphi(x_2)$ if $x_1^0 > x_2^0$
$\varphi(x_2)\varphi(x_1)$ if $x_1^0 < x_2^0$
left

Particles in Klein-Gordon theory

$$|\vec{k}\rangle = a^\dagger_{\vec{k}}|0\rangle$$

↑ one-particle state

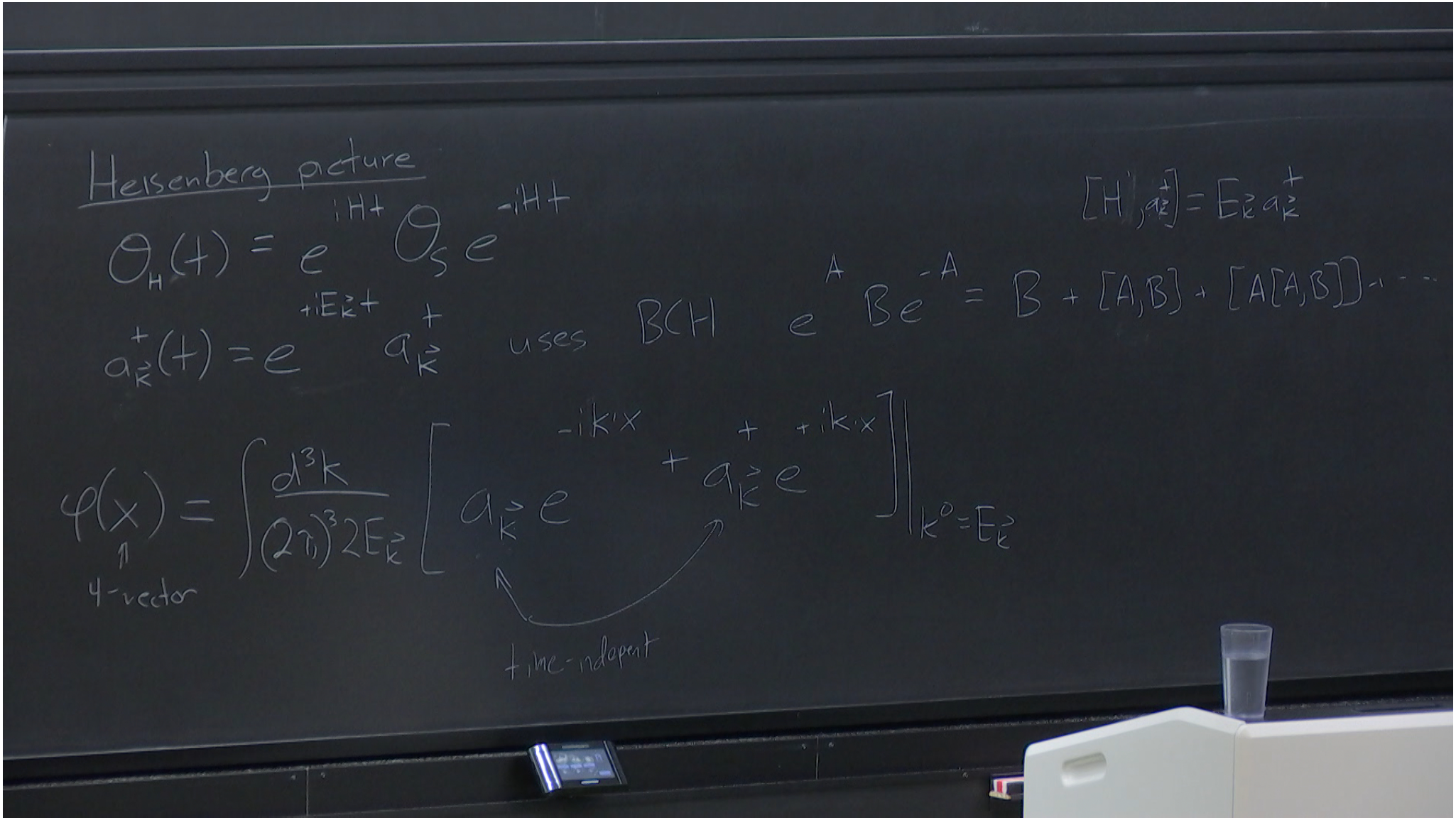
$$E^2_{\vec{k}} = \vec{k}^2 + m^2$$

$$\langle \vec{k}|\vec{p}\rangle = \langle 0| a_{\vec{k}} a^\dagger_{\vec{p}} |0\rangle$$

$$= (2\pi)^3 2E_{\vec{k}} \delta^{(3)}(\vec{k} - \vec{p})$$

$$|\vec{k}_1, \vec{k}_2, \cdots \vec{k}_n\rangle = a^\dagger_{\vec{k}_1} \cdots a^\dagger_{\vec{k}_n} |0\rangle$$

↑ commute → bosons


Heisenberg picture
$\mathcal{O}_H(t) = e^{iHt}\,\mathcal{O}_S\,e^{-iHt}$
$[H, a_{\vec{k}}^\dagger] = E_{\vec{k}}\, a_{\vec{k}}^\dagger$
$a_{\vec{k}}^\dagger(t) = e^{+iE_{\vec{k}}t}\, a_{\vec{k}}^\dagger$
uses BCH $e^{A} B e^{-A} = B + [A,B] + [A,[A,B]] + \cdots$
$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left[a_{\vec{k}} e^{-ik\cdot x} + a_{\vec{k}}^\dagger e^{+ik\cdot x} \right]\Big|_{k^0 = E_{\vec{k}}}$
4-vector
time-indepent

LSZ

Goal: $\langle f|S|i\rangle \Leftrightarrow \langle \Omega|T\,\varphi(x_1)\cdots\varphi(x_n)|\Omega\rangle$

$\langle f|S|i\rangle_H = \langle f;\, t=+\infty \,|\, i;\, t=-\infty\rangle_S$

S: time evol. op from $t=-\infty$ to $t=+\infty$

$|i\rangle = |\vec{k}_1, \vec{k}_2\rangle$

$|f\rangle = |\vec{p}_1, \vec{p}_2\rangle$

LSZ

Goal: $\langle f|S|i\rangle \Leftrightarrow \langle\Omega|T\,\varphi(x_1)\cdots\varphi(x_n)|\Omega\rangle$

$\langle f|S|i\rangle_H = \langle f;\, t=+\infty \,|\, i;\, t=-\infty\rangle_S$

S: time evol. op from $t=-\infty$ to $t=+\infty$

$a_i^+ \equiv a_{\vec{k}_i}^+$

$|i\rangle = |\vec{k}_1, \vec{k}_2\rangle = a_1^+(0)\, a_2^+(0)\,|\Omega\rangle$

$|f\rangle = |\vec{k}_3, \vec{k}_4\rangle = a_3^+(0)\, a_4^+(0)\,|\Omega\rangle$

Assumptions

- $[a_{\vec{k}}(t), a^{+}_{\vec{p}}(t)] = (2\pi)^3 2E_{\vec{k}}\,\delta^{(3)}(\vec{k}-\vec{p})$ (E: phys energy)

- $\varphi(x)|\Omega\rangle$ is a one-particle state $t \to \pm\infty$

- $\langle \vec{k}|\varphi(x)|\Omega\rangle = e^{ik\cdot x}$

$N =$

if real

$$\varphi(x) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \left(a_{\vec{k}}(t)\, e^{-ik\cdot x} + a^{+}_{\vec{k}}(t)\, e^{ik\cdot x} \right)$$

(a^{+}: linear in a^{+} at $t=\pm\infty$)

Assumptions

- $[a_{\vec{k}}(t), a^{+}_{\vec{p}}(t)] = (2\tilde{\pi})^3 \, 2E_{\vec{k}} \, \delta^{(3)}(\vec{k}-\vec{p})$ ← phys energy

- $\varphi(x)|\Omega\rangle$ is a one-particle state $t \to \pm\infty$

- $\langle \vec{k}|\varphi(x)|\Omega\rangle = e^{ik\cdot x}$

$$\varphi(x) = \int \frac{d^3k}{(2\tilde{\pi})^3 2E_{\vec{k}}} \left(a_{\vec{k}}(t)\, e^{-ik\cdot x} + a^{+}_{\vec{k}}(t)\, e^{ik\cdot x} \right)$$

if real ↓ ← linear in a^{+} at $t=\pm\infty$

$$\langle f|S|i\rangle = \langle \Omega | a_{4}^{+}(+\infty)\, a_{3}(+\infty)\, a_{1}^{\dagger}(-\infty)\, a_{2}^{\dagger}(-\infty) | \Omega \rangle$$

useful identity

$$\boxed{a_1^+(+\infty) - a_1^+(-\infty) = -i \int d^4x\, e^{-ik_1 \cdot x} (\partial^2 + m_{ph}^2)\, \varphi(x) = I_1^+}$$

physical mass

$\neq 0$ for interacting theory

$(\vec{k}-\vec{p})$

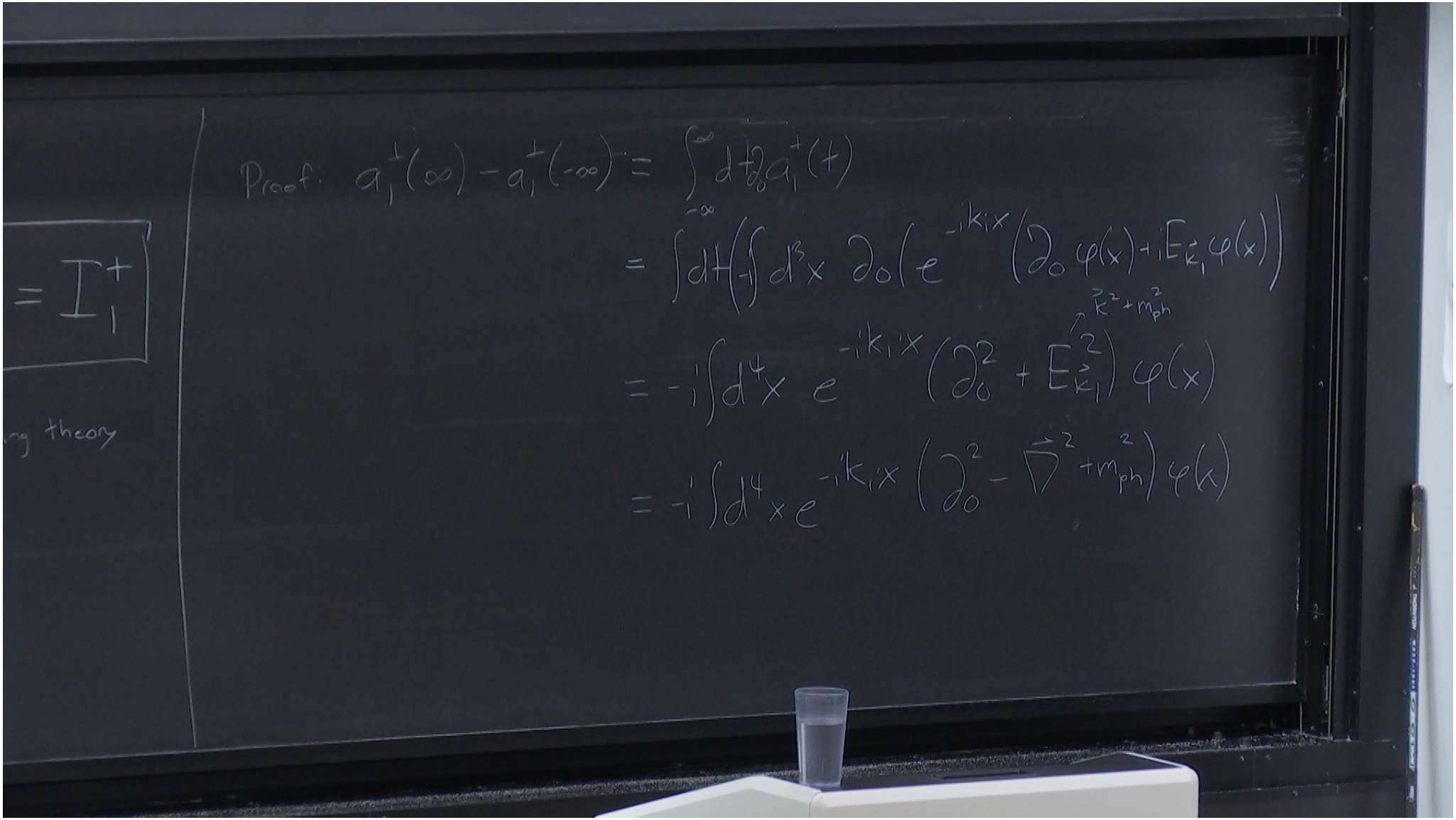
$t \to \pm\infty$

$$N = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} \, a^{+}_{\vec{k}} a_{\vec{k}}$$

linear in a^{+} at $t=\pm\infty$

$$a^{+}_{\vec{k}}(t)\, e^{ik\cdot x}$$

$$\partial_0 a_1^{+}(t)\Big|_{t=\pm\infty} = 0$$


$= I_1^+$
g theory
Proof: $a_1^+(\infty) - a_1^+(-\infty) = \int_{-\infty}^{\infty} dt\, \partial_0 a_1^+(t)$
$= \int dt \left(\int d^3x\, \partial_0 \left(e^{-ik_1 x} \left(\partial_0 \varphi(x) + i E_{\vec{k}_1} \varphi(x) \right) \right) \right)$
$E_{\vec{k}_1}^2 = \vec{k}^2 + m_{ph}^2$
$= -i \int d^4x\, e^{-ik_1 x} \left(\partial_0^2 + E_{\vec{k}_1}^2 \right) \varphi(x)$
$= -i \int d^4x\, e^{-ik_1 x} \left(\partial_0^2 - \vec{\nabla}^2 + m_{ph}^2 \right) \varphi(x)$

$$\langle f|S|i\rangle = \langle\Omega|T\, a_4(+\infty)\, a_3(+\infty)\, a_1^\dagger(-\infty)\, a_2^\dagger(-\infty)|\Omega\rangle$$

$$= \langle\Omega|T\, a_4(-\infty) a_3(-\infty) a_1^\dagger(+\infty) a_2^\dagger(+\infty)|\Omega\rangle + \cdots^0$$

$$+ \langle\Omega|T\, I_4 I_3 I_1^\dagger I_2^\dagger|\Omega\rangle$$

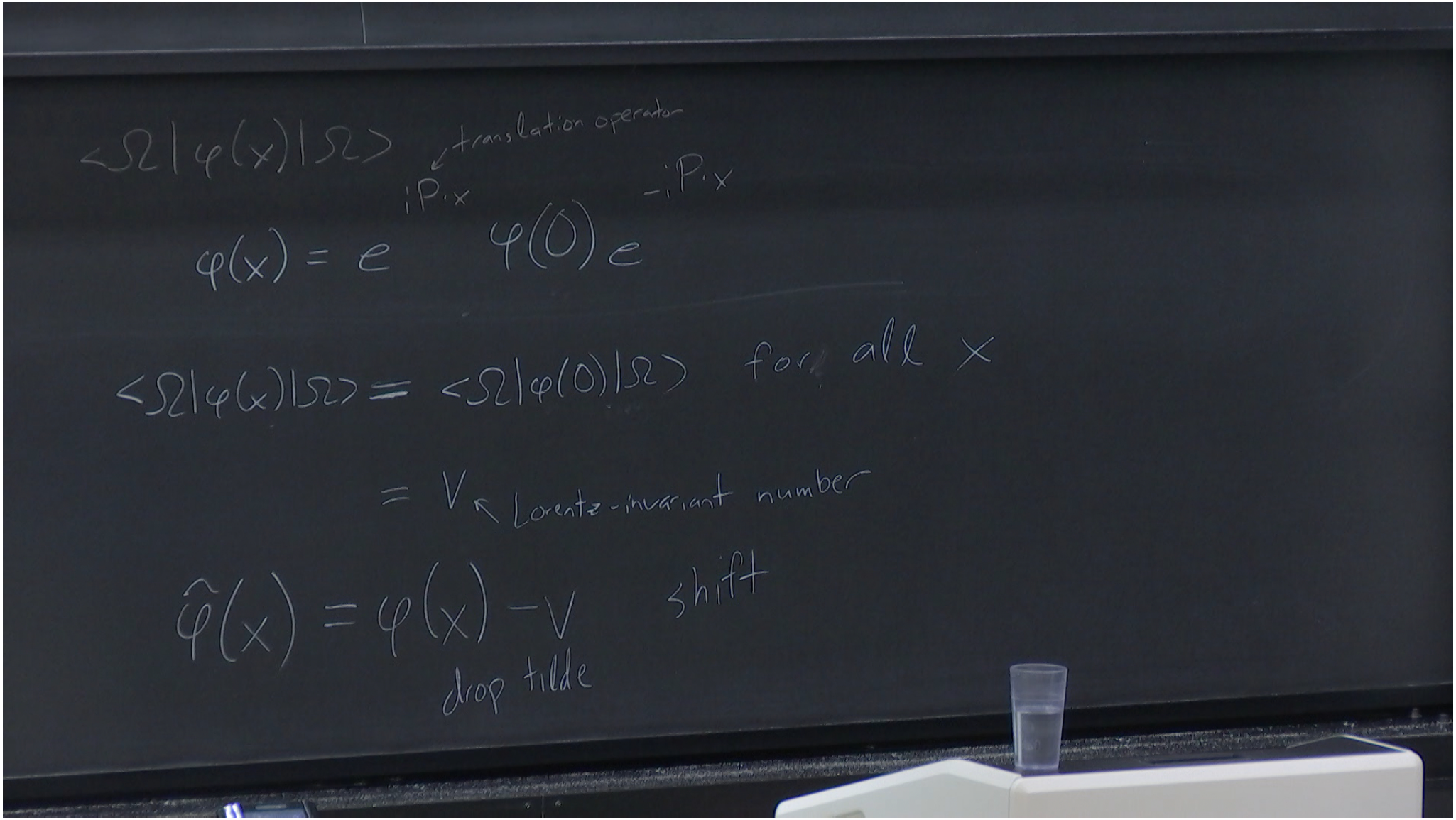
$\varphi_i = \varphi(x_i)$

$$\boxed{\langle f|S|i\rangle = i^4 \prod_{j=1}^{4} \int d^4x_j\, e^{-i\lambda_j k_j \cdot x_j} (\partial_j^2 + m^2) \langle\Omega|T\, \varphi_1 \varphi_2 \varphi_3 \varphi_4|\Omega\rangle}$$

$\lambda_j = +1$ initial state

-1 final states

$\partial_j^2 = \eta^{\mu\nu} \frac{\partial^2}{\partial x_j^\mu \partial x_j^\nu}$


$\langle\Omega|\varphi(x)|\Omega\rangle$
translation operator
$\varphi(x) = e^{iP\cdot x}\,\varphi(0)\,e^{-iP\cdot x}$
$\langle\Omega|\varphi(x)|\Omega\rangle = \langle\Omega|\varphi(0)|\Omega\rangle$ for all x
$= V$ Lorentz-invariant number
$\hat{\varphi}(x) = \varphi(x) - V$ shift
drop tilde

$$q_{\vec{k}}(t)\, e^{\quad} + q^{\dagger}_{\vec{k}}(t)\, e^{\quad}$$

$$\partial_0 q_1(t)\Big|_{t=\pm\infty} = 0$$

$$\langle \vec{k}|\varphi(x)|\Omega\rangle = \langle \vec{k}|\, e^{iP\cdot x}\, \varphi(0)\, e^{-iP\cdot x}\, |\Omega\rangle$$

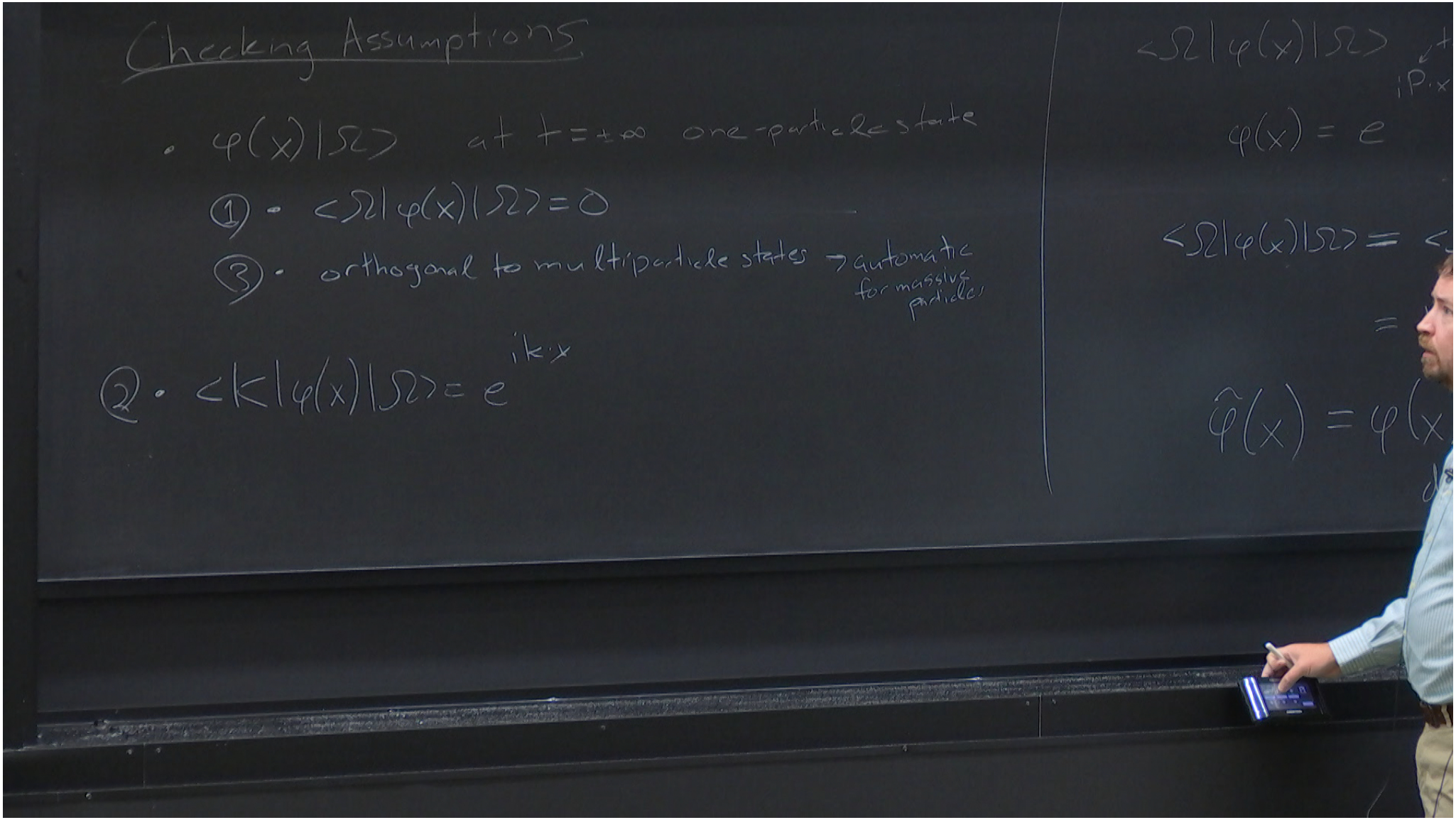
$$= e^{ik\cdot x}\langle \vec{k}|\, \varphi(0)\, |\Omega\rangle$$

$$f(k^2) = f(m^2) = \text{Lorentz invariant number}$$

$$= e^{ik\cdot x}\,\frac{1}{N}$$

$$\tilde{\varphi}(x) = N\varphi(x) \quad \text{rescale}$$

drop tilde


Checking Assumptions
• φ(x)|Ω⟩ at t = ±∞ one-particle state
① • ⟨Ω|φ(x)|Ω⟩ = 0
③ • orthogonal to multiparticle states → automatic for massive particles
② • ⟨k|φ(x)|Ω⟩ = e^{ik·x}
⟨Ω|φ(x)|Ω⟩
φ(x) = e
φ̂(x) = φ(x

$\hat{\varphi}(x) = \varphi(x)$

drop

$$\mathcal{L} = \frac{1}{2}(\partial\varphi)^2 - \frac{1}{2}m^2\varphi^2 + \frac{g}{3!}\varphi^3$$

Shift + rescale + rename

new term

$$\mathcal{L} = \frac{1}{2}Z_\varphi(\partial\varphi)^2 - \frac{1}{2}Z_m m^2\varphi^2 + \frac{gZ_g}{3!}\varphi^3 + Y\varphi$$

Renormalization is required!

for now: $Z_i = 1$, $Y = 0$ } true at leading order in perturbation theory