Title: Quantum Theory - Lecture 220923

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Collection: Quantum Theory (2022-2023)

Date: September 23, 2022 - 10:45 AM

URL: https://pirsa.org/22090040

Classical + Quantum Klein-Gordon Theory

Goal: Basis for perturbation theory

Measurements

Noether's Theorem

continuous symmetry $\rightarrow$ conserved current $\rightarrow$ conserved charge

classical result

$\boxed{\text{quantum anomalies!}}$

Klein-Gordon Theory

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

m: mass parameter

φ: real scalar fied

EoM: $(\partial_\mu \partial^\mu + m^2)\varphi = 0$ linear in φ

Klein-Gordon Theory

$$\mathcal{L}_{KG} = \frac{1}{2}\partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2$$

mass parameter

real scalar fied

$[S] = 0 = \int d^4x \, \mathcal{L}$

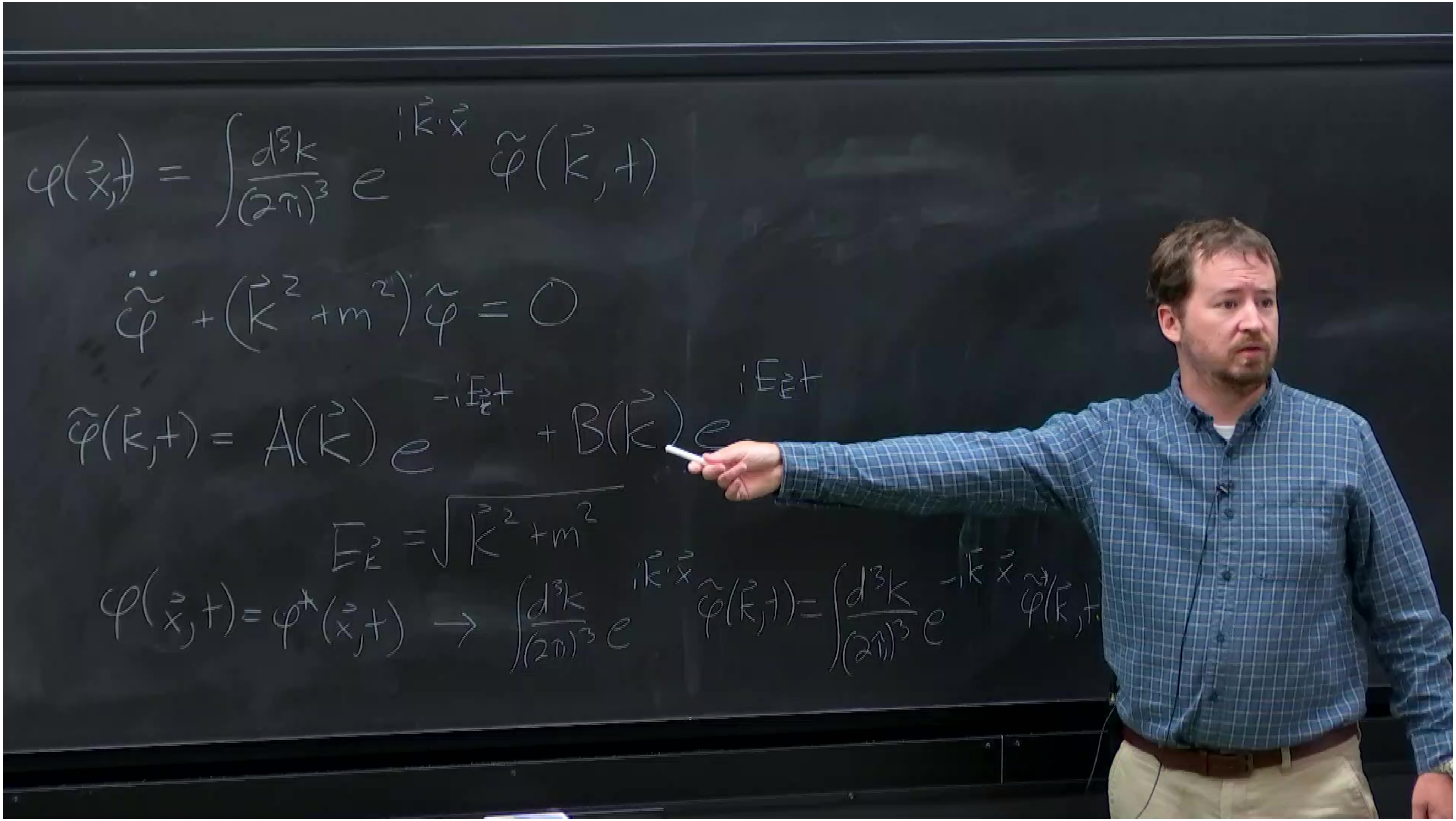
$[\mathcal{L}] = 4$

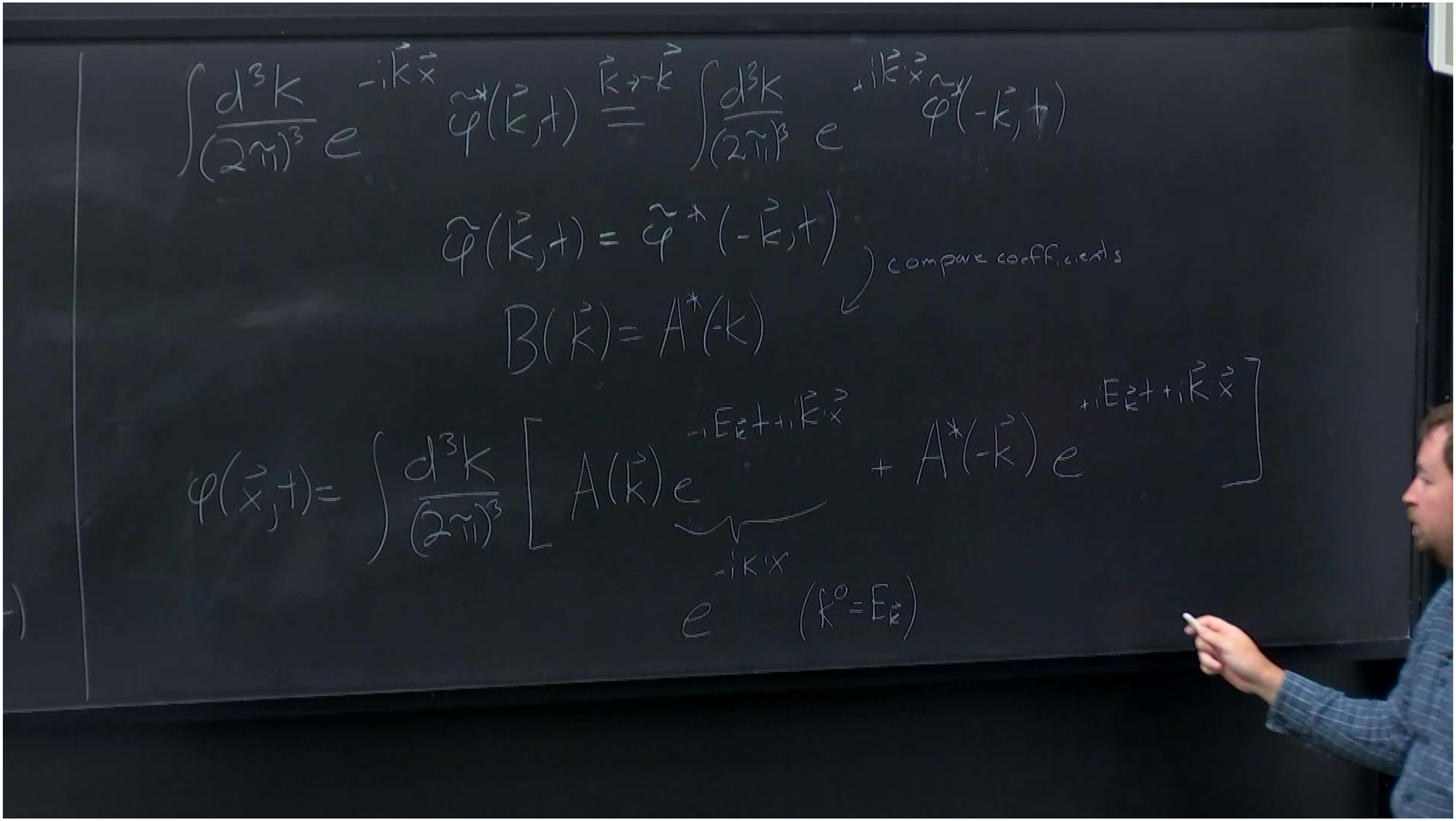
$[\partial^2 \varphi^2] = 2 + 2[\varphi]$

$[\varphi] = 1$

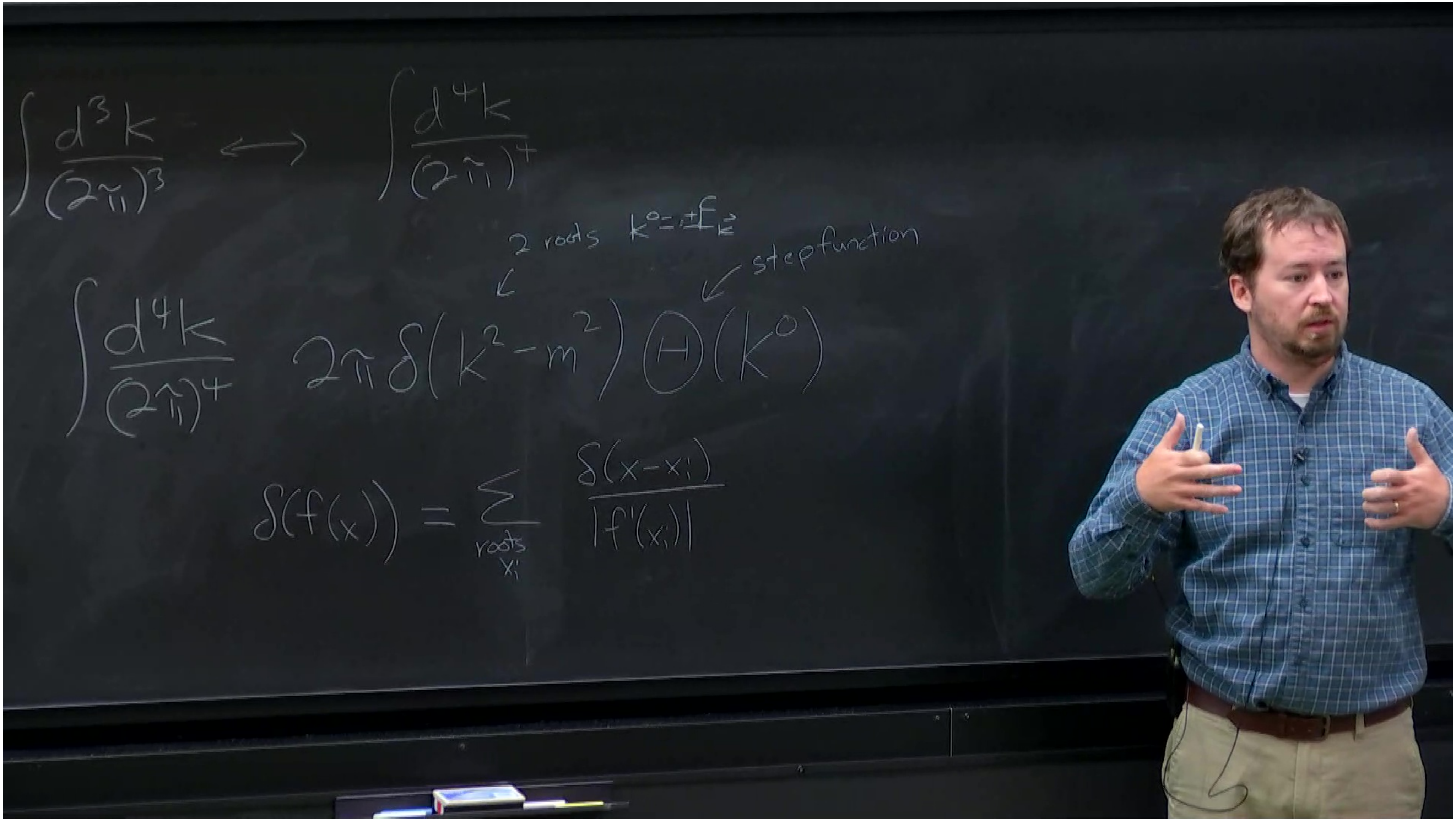
EoM: $(\partial_\mu \partial^\mu + m^2)\varphi = 0$ linear in φ

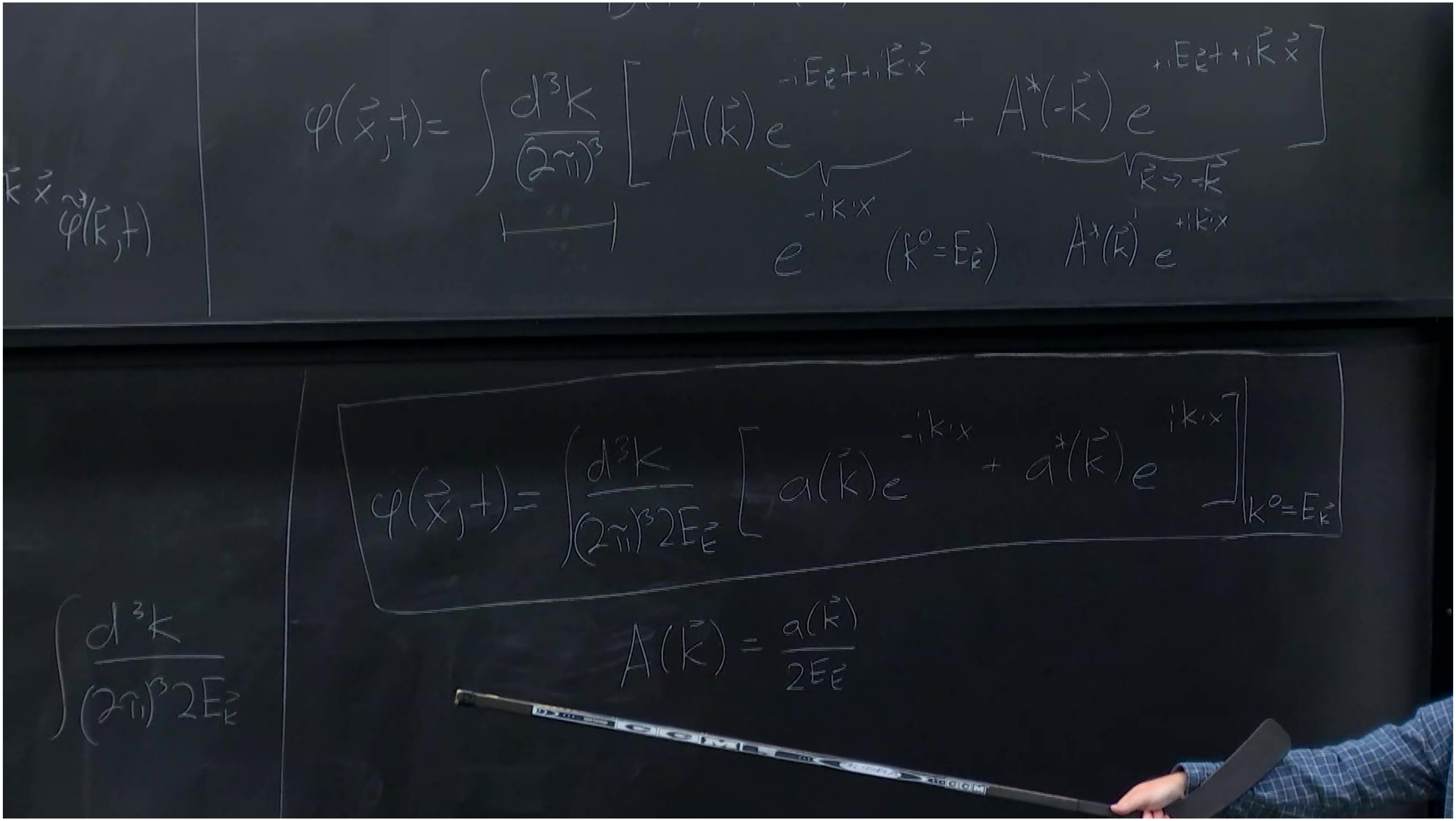
$$\partial_\mu \partial^\mu = \partial_\mu \partial_\nu \eta^{\mu\nu} = \partial_t^2 - \vec{\nabla}^2$$


$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\varphi}(\vec{k},t)$
$\ddot{\tilde{\varphi}} + (\vec{k}^2 + m^2)\tilde{\varphi} = 0$
$\tilde{\varphi}(\vec{k},t) = A(\vec{k}) e^{-iE_{\vec{k}}t} + B(\vec{k}) e^{iE_{\vec{k}}t}$
$E_{\vec{k}} = \sqrt{\vec{k}^2 + m^2}$
$\varphi(\vec{x},t) = \varphi^*(\vec{x},t) \rightarrow \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k}\cdot\vec{x}} \tilde{\varphi}(\vec{k},t) = \int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \tilde{\varphi}^*(\vec{k},t)$



$\int \frac{d^3k}{(2\pi)^3} e^{-i\vec{k}\cdot\vec{x}} \tilde{\varphi}^*(\vec{k},t) \overset{\vec{k}\to-\vec{k}}{=} \int \frac{d^3k}{(2\pi)^3} e^{+i\vec{k}\cdot\vec{x}} \tilde{\varphi}^*(-\vec{k},t)$
$\tilde{\varphi}(\vec{k},t) = \tilde{\varphi}^*(-\vec{k},t)$
compare coefficients
$B(\vec{k}) = A^*(-k)$
$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3} \left[A(\vec{k}) \underbrace{e^{-iE_{\vec{k}}t + i\vec{k}\cdot\vec{x}}}_{e^{-ik\cdot x}} + A^*(-\vec{k}) e^{+iE_{\vec{k}}t + i\vec{k}\cdot\vec{x}} \right]$
$(k^0 = E_{\vec{k}})$


$\int \frac{d^3k}{(2\pi)^3} \longleftrightarrow \int \frac{d^4k}{(2\pi)^4}$
2 roots $k^0 = \pm E_{\vec{k}}$
stepfunction
$\int \frac{d^4k}{(2\pi)^4} 2\pi\delta(k^2-m^2)\Theta(k^0)$
$\delta(f(x)) = \sum_{\text{roots } x_i} \frac{\delta(x-x_i)}{|f'(x_i)|}$


$\vec{k}\cdot\vec{x}\ \tilde{\varphi}(\vec{k},t)$
$\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3}\left[A(\vec{k})\underbrace{e^{-iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}}_{e^{-ik\cdot x}} + \underbrace{A^*(-\vec{k})\, e^{+iE_{\vec{k}}t+i\vec{k}\cdot\vec{x}}}_{\vec{k}\to-\vec{k}} \right]$
$(k^0 = E_{\vec{k}})$
$A^*(\vec{k})\, e^{+ik\cdot x}$
$\boxed{\varphi(\vec{x},t) = \int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}\left[a(\vec{k})\, e^{-ik\cdot x} + a^*(\vec{k})\, e^{ik\cdot x} \right]\Big|_{k^0=E_{\vec{k}}}}$
$A(\vec{k}) = \frac{a(\vec{k})}{2E_{\vec{k}}}$
$\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}}$

Canonical Quantization

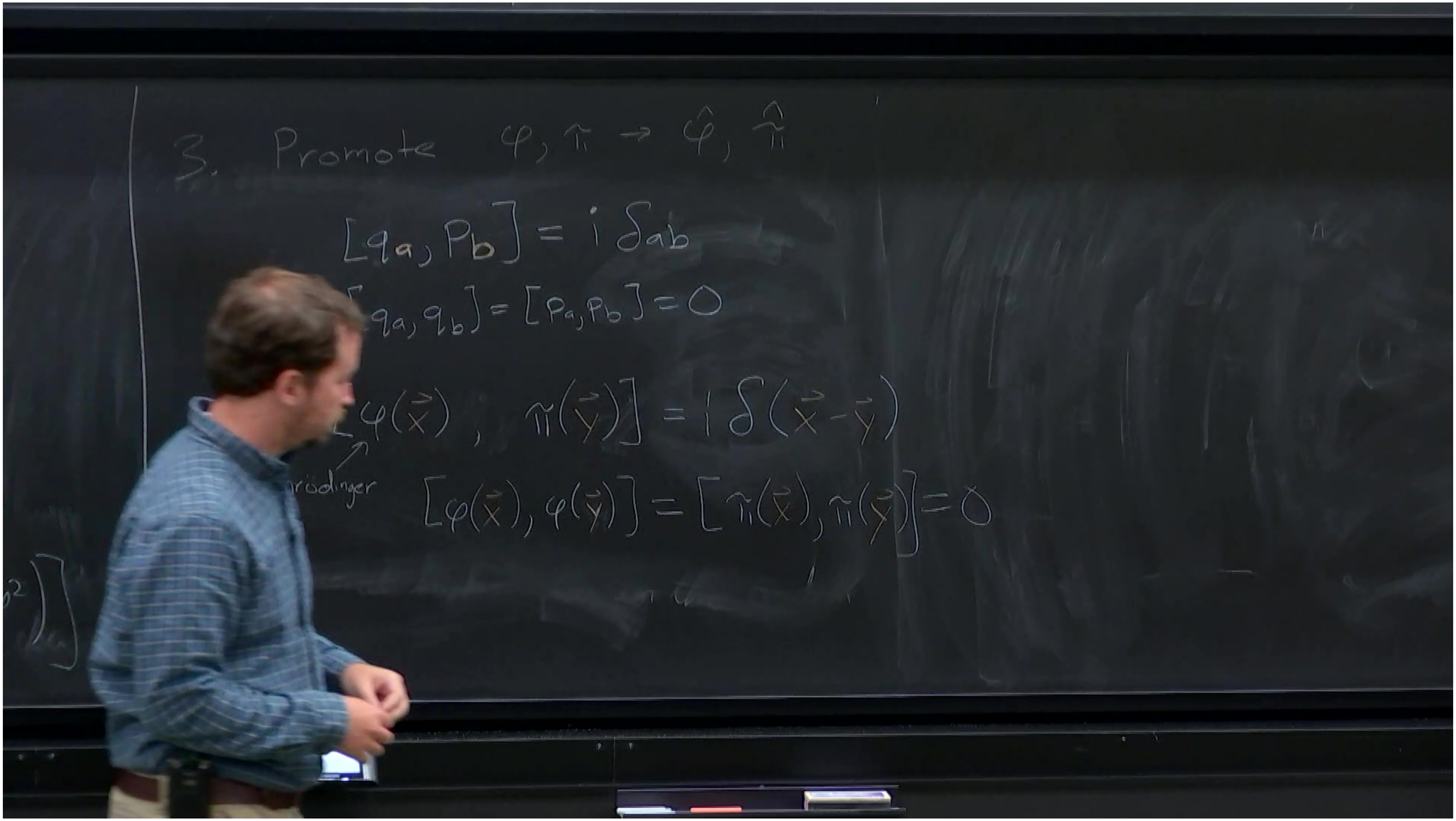
1. Choose $\mathcal{L}$ (Lorentz-invariant, local)

$$\mathcal{L} = \mathcal{L}_{KG}$$

2. Compute π, H

$$\pi = \frac{\partial \mathcal{L}_{KG}}{\partial \dot{\varphi}} = \dot{\varphi}$$

$$\left(\mathcal{H} = \dot{\varphi}\pi - \mathcal{L} \right) \left[H = \int d^3x \left[\frac{1}{2}\pi^2 + \frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 \right] \right]$$


3. Promote $\varphi, \pi \to \hat{\varphi}, \hat{\pi}$
$[q_a, p_b] = i\,\delta_{ab}$
$[q_a, q_b] = [p_a, p_b] = 0$
$[\varphi(\vec{x}), \pi(\vec{y})] = i\,\delta(\vec{x}-\vec{y})$
rödinger
$[\varphi(\vec{x}), \varphi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$

3. Promote $\varphi, \pi \rightarrow \hat{\varphi}, \hat{\pi}$

$$[q_a, p_b] = i\delta_{ab}$$

$$[q_a, q_b] = [p_a, p_b] = 0$$

at equal times $(t=0)$

$$[\varphi(\vec{x}), \pi(\vec{y})] = i\delta(\vec{x}-\vec{y})$$

Schrödinger

$$[\varphi(\vec{x}), \varphi(\vec{y})] = [\pi(\vec{x}), \pi(\vec{y})] = 0$$

$\hat{\varphi}, \hat{\pi}$

ab

$= 0$

at equal times $(t=0)$

$= i\,\delta(\vec{x}-\vec{y})$

$= [\pi(\vec{x}), \pi(\vec{y})] = 0$

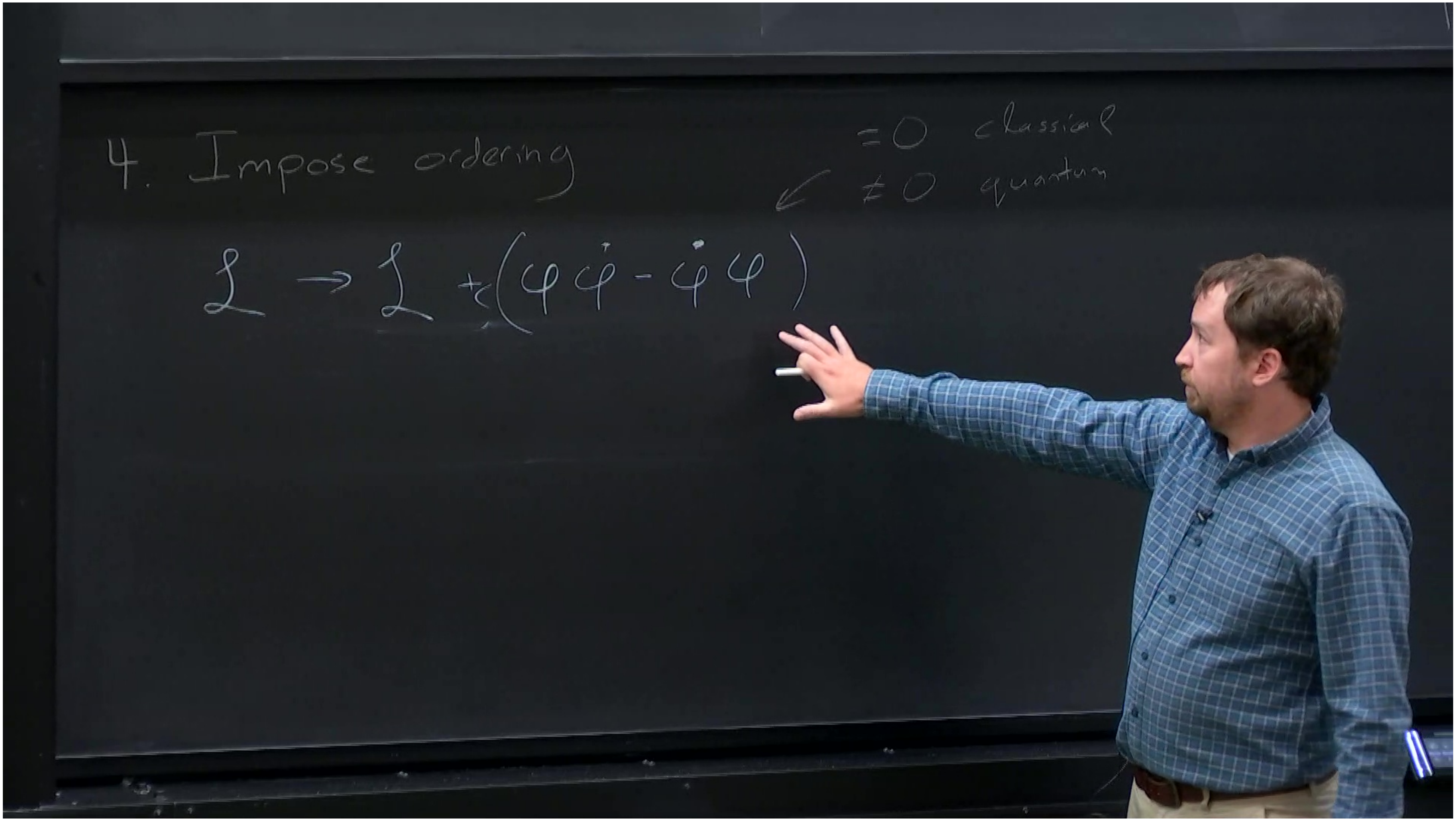
$a(\vec{k}) \to a_{\vec{k}}$ anihilation op

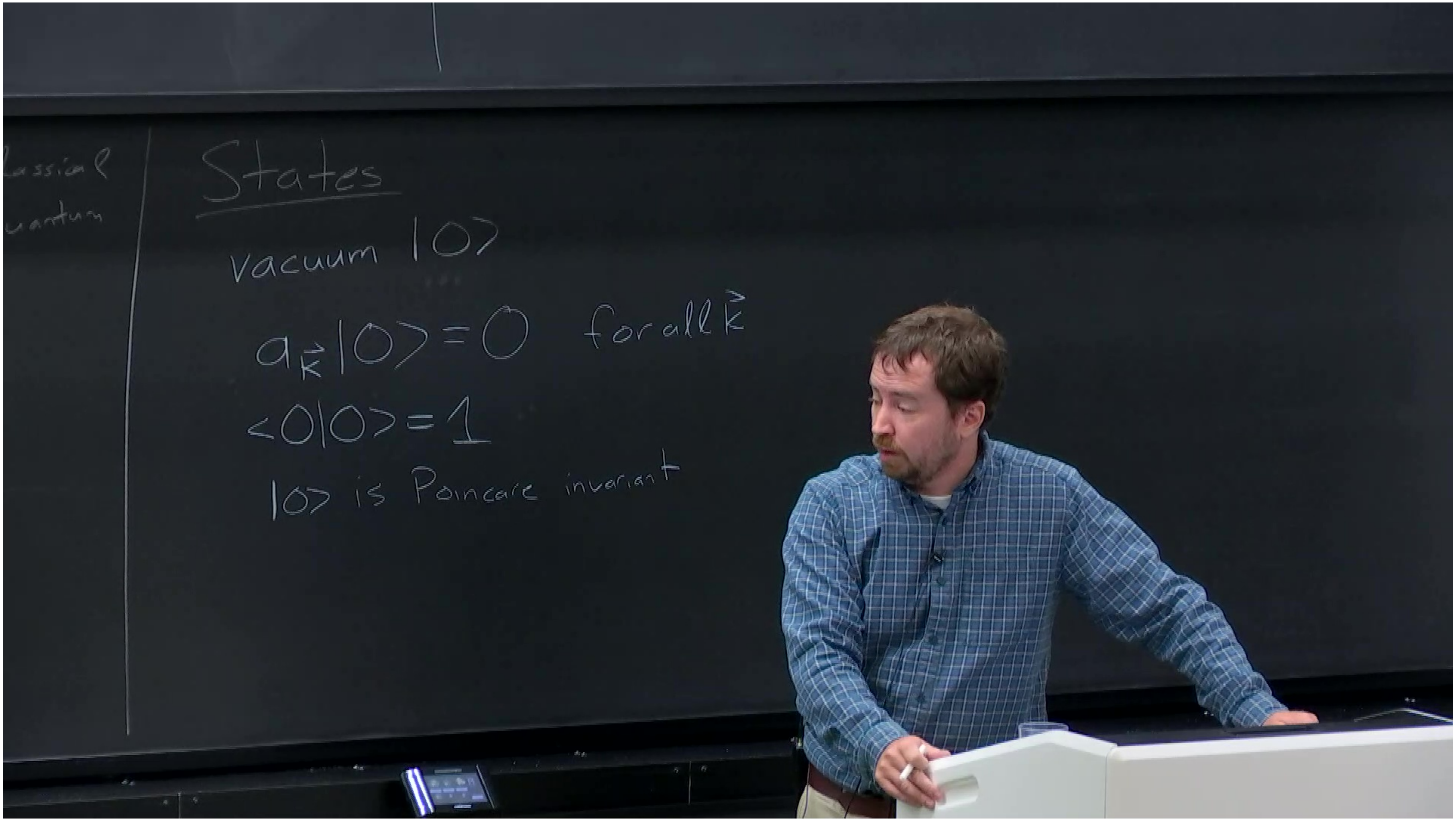
$a^*(\vec{k}) \to a^\dagger_{\vec{k}}$ creation op

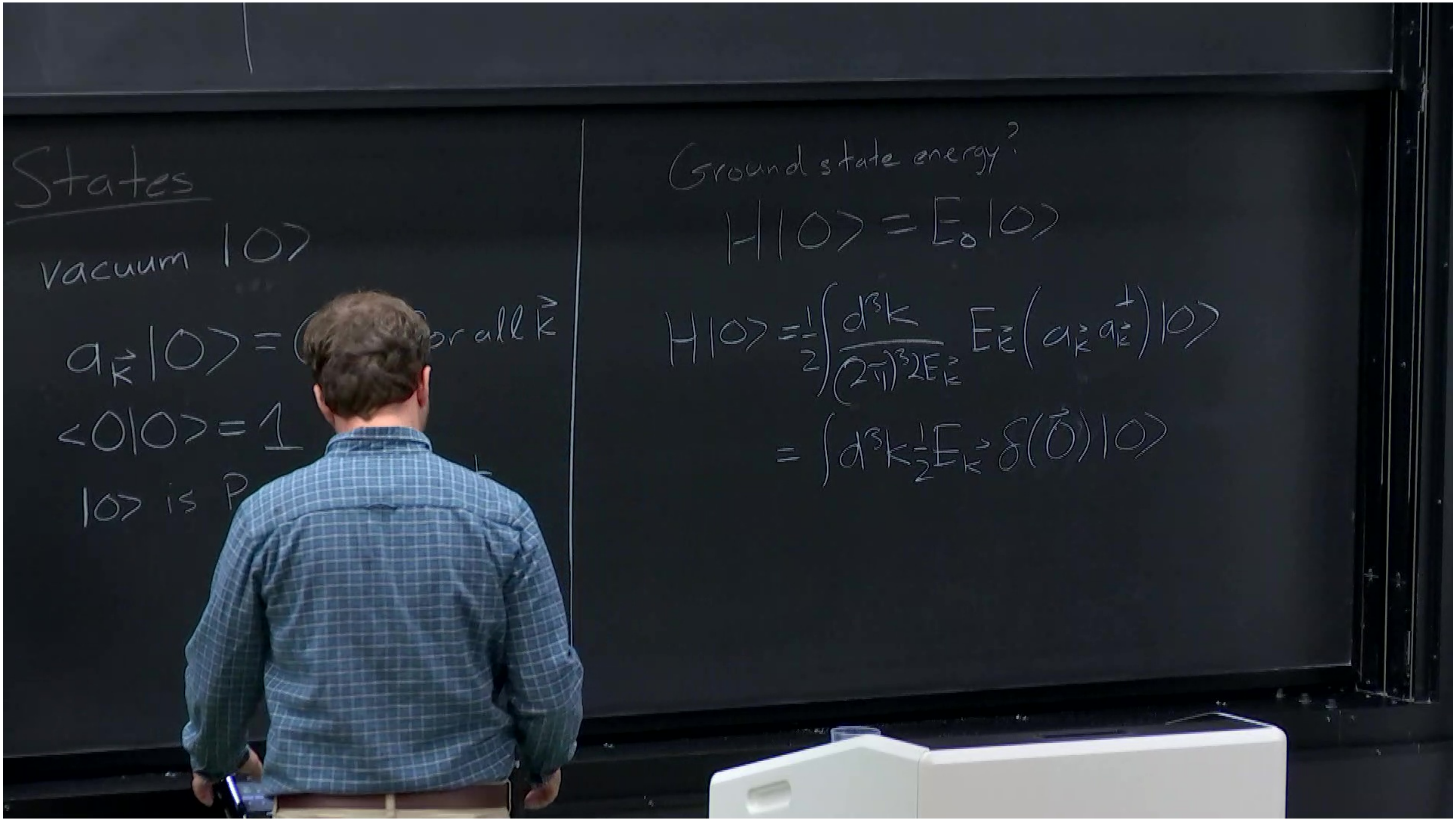
quiz

$$[a_{\vec{k}}, a^\dagger_{\vec{p}}] = (2\pi)^3\, 2E_{\vec{k}}\, \delta(\vec{p}-\vec{k})$$

$$[a_{\vec{k}}, a_{\vec{p}}] = 0 = [a^\dagger_{\vec{k}}, a^\dagger_{\vec{p}}]$$


4. Impose ordering
= 0 classical
≠ 0 quantum
$\mathcal{L} \to \mathcal{L} + c(\varphi\dot{\varphi} - \dot{\varphi}\varphi)$


States
vacuum |0>
$a_{\vec{k}}|0\rangle = 0$ for all $\vec{k}$
$\langle 0|0\rangle = 1$
|0> is Poincare invariant


States
vacuum $|0\rangle$
$a_{\vec{k}}|0\rangle =$
or all $\vec{k}$
$\langle 0|0\rangle = 1$
$|0\rangle$ is P
Ground state energy?
$H|0\rangle = E_0|0\rangle$
$H|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} \left(a_{\vec{k}} a_{\vec{k}}^{\dagger}\right)|0\rangle$
$= \int d^3k \frac{1}{2} E_{\vec{k}} \delta(\vec{0})|0\rangle$

States

vacuum $|0\rangle$

$$a_{\vec{k}}|0\rangle = 0 \quad \text{for all } \vec{k}$$

$$\langle 0|0\rangle = 1$$

$|0\rangle$ is Poincare invariant

Ground state energy?

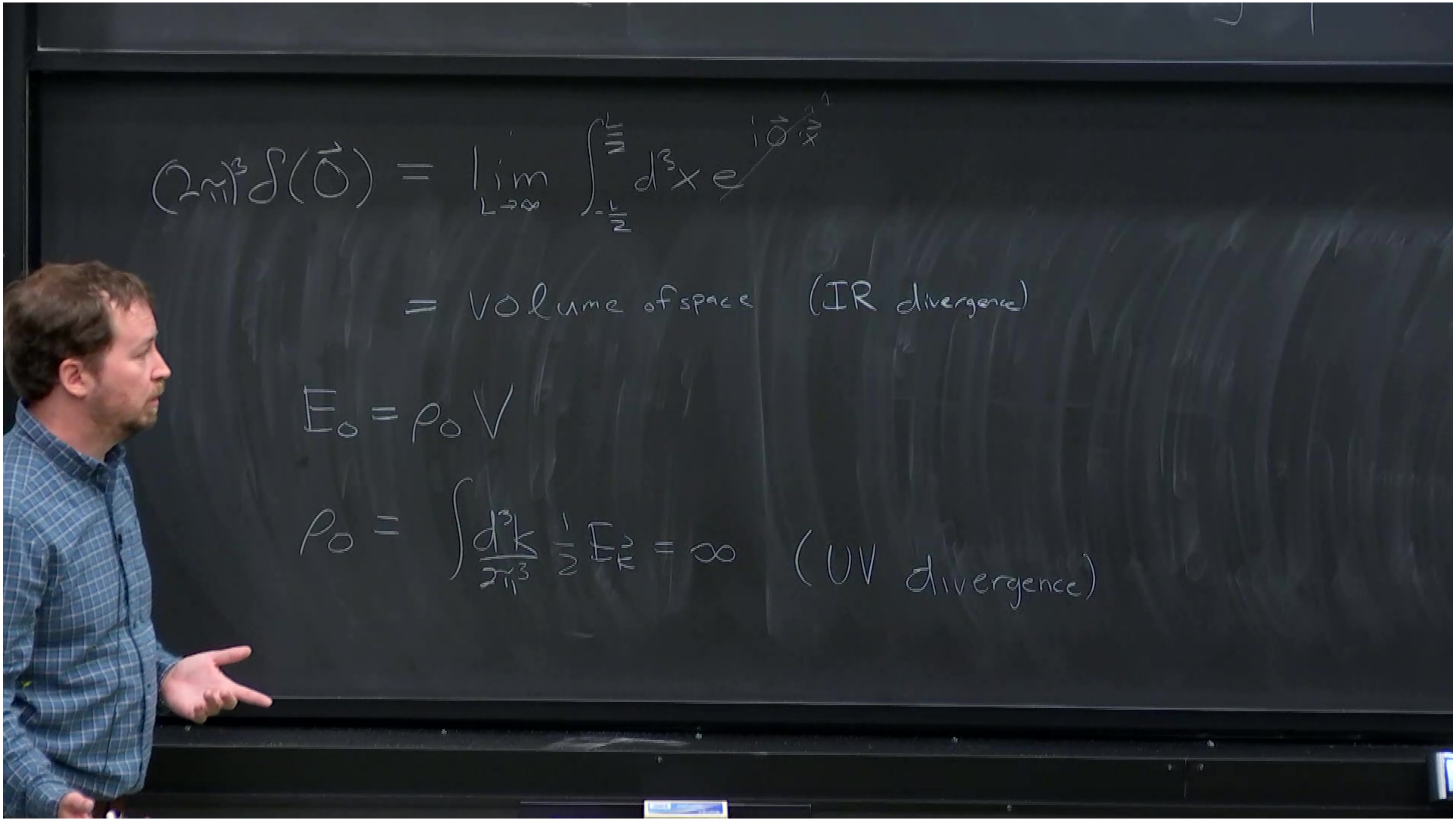
$$H|0\rangle = E_0|0\rangle$$

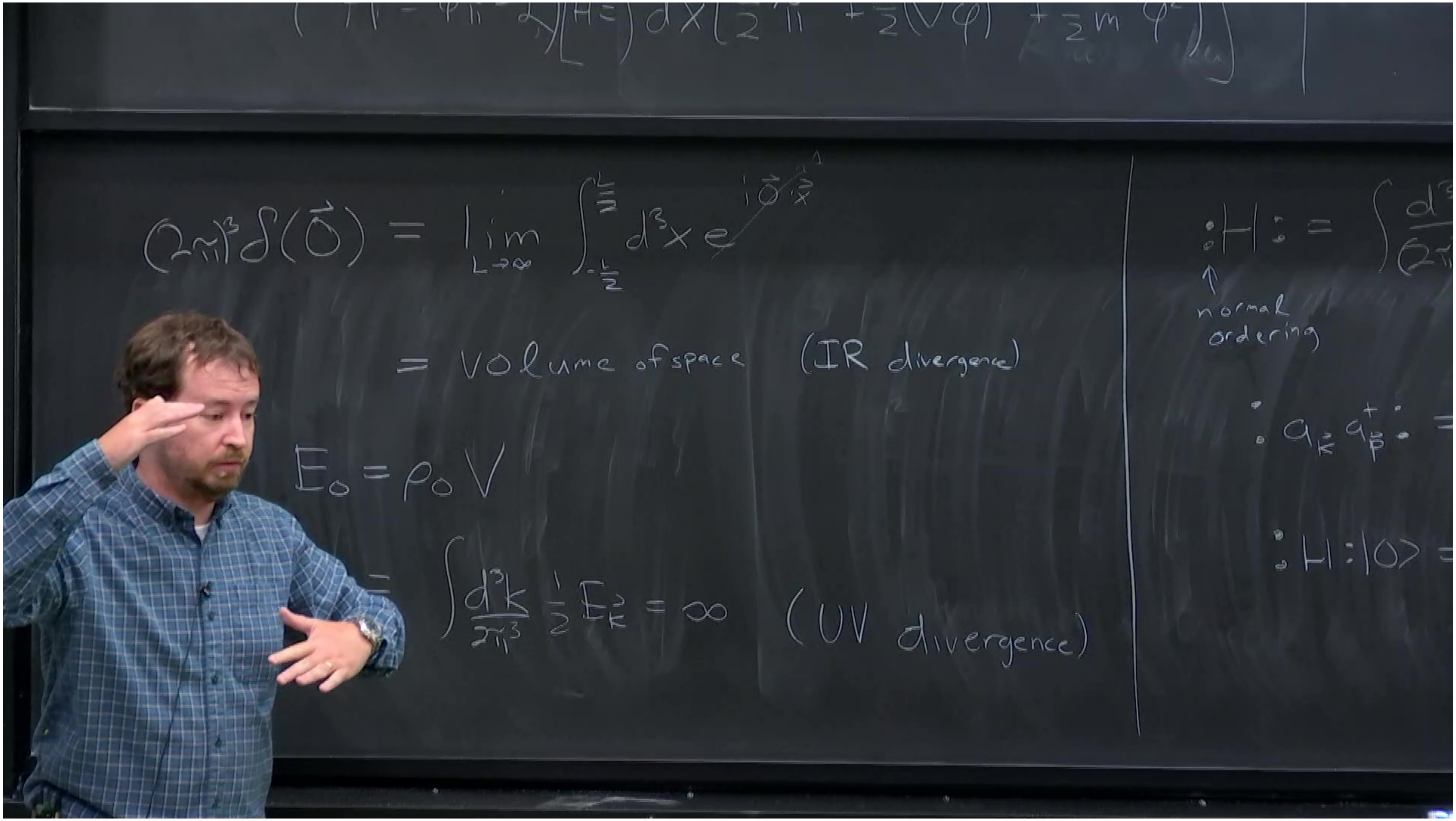
$$H|0\rangle = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3 2E_{\vec{k}}} E_{\vec{k}} \left(a_{\vec{k}} a_{\vec{k}}^{\dagger}\right)|0\rangle$$

$$= \int d^3k \, \frac{1}{2} E_{\vec{k}} \, \delta(\vec{0})|0\rangle$$

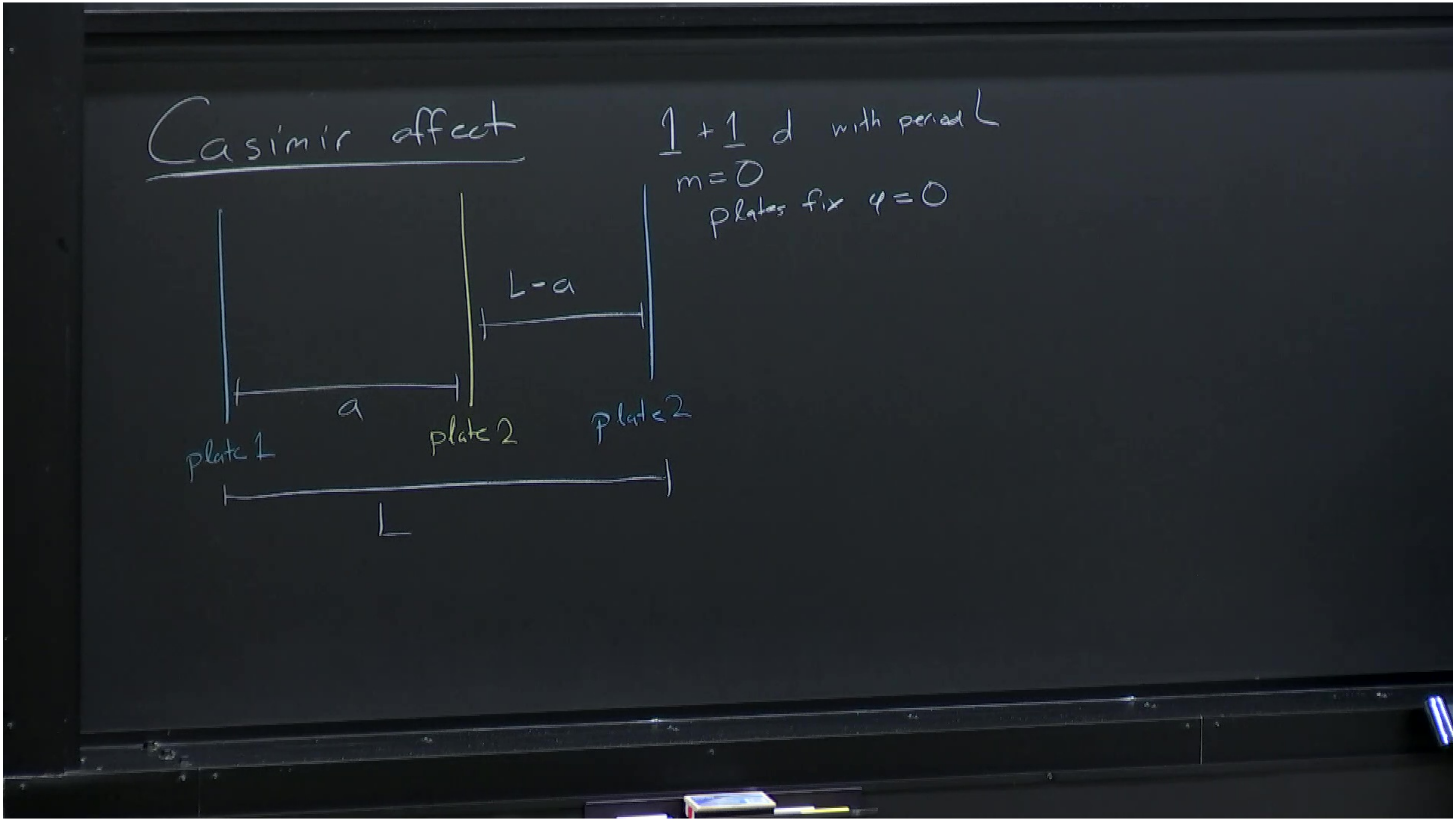
$$= \infty \, |0\rangle$$

not observable $\rightarrow$ not finite

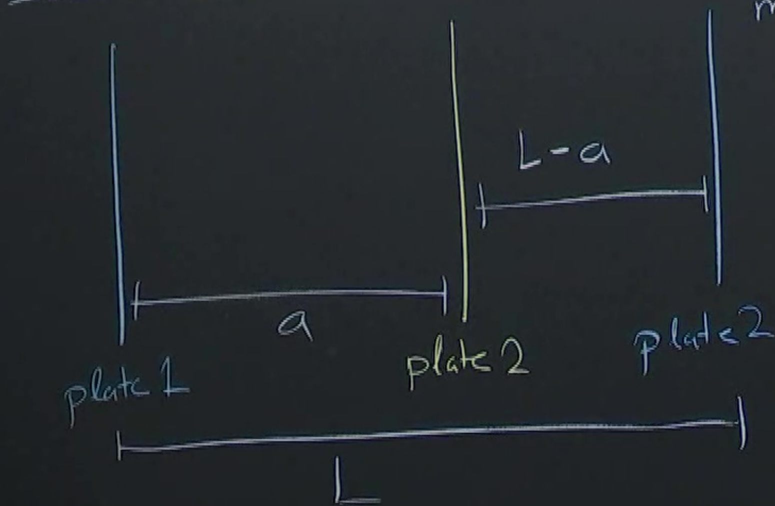

$(2\pi)^3\delta(\vec{0}) = \lim_{L\to\infty}\int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x\, \cancel{e^{i\vec{0}\cdot\vec{x}}}$
= volume of space (IR divergence)
$E_0 = \rho_0 V$
$\rho_0 = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2}E_k = \infty$ (UV divergence)



$(2\pi)^3 \delta(\vec{0}) = \lim_{L\to\infty} \int_{-\frac{L}{2}}^{\frac{L}{2}} d^3x\, e^{i\vec{0}\cdot\vec{x}}$
$=$ volume of space (IR divergence)
$E_0 = \rho_0 V$
$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} E_{\vec{k}} = \infty$ (UV divergence)
$:H: = \int \frac{d^3}{(2\pi}$
normal ordering
$:a_{\vec{k}} a^{\dagger}_{\vec{p}}: =$
$:H:|0\rangle =$


Casimir affect
1 + 1 d with period L
m = 0
plates fix φ = 0
L - a
a
plate 1
plate 2
plate 2
L

Casimir affect



$1+1$ d with period L

$m=0$

plates fix $\varphi=0$

$E_{\vec{k}} = k = \omega$

$$E(r) = \sum_n \frac{1}{2}\omega_n$$

energy inside box of size r

$$\omega_n = \frac{n\pi}{L}$$

E_0

not observable → not finite

$$E_0(a) = E(a) + E(L-a)$$

$$= \frac{\pi}{2}\left(\frac{1}{L-a} + \frac{1}{a}\right)\sum_{n=0}^{\infty} n \to \infty$$

$$F = -\frac{dE_0}{da}$$

$$= \frac{\pi}{2}\left(\frac{1}{a^2} - \frac{1}{(L-a)^2}\right)\sum_{n=0}^{\infty} n \to \infty \, !$$

Mistake!

High-frequency modes unaffected by plates