Title: Quantum Theory - Lecture 220916

Speakers: Dan Wohns

Collection: Quantum Theory (2022-2023)

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# Lecture 6

Friday, 16 September 2022 5:42 PM

Announcements:

- H/W extension
- Open quantum systems
- Tutorial 4
- Lecture notes on portal

Tensor products

## Outline

- Density matrix continued...
  - $\rho$ for two,three,four,....n level system
  - | | | |
    |---|---|---|
    | Nomenclature | diagonal | off diagonal |
  - Coherences/dephasing/amplitude decay
  - $\rho$ for composite system
  - Reduced density matrix
  - Observables in density matrix representation

Tensor products

Outline

- Density matrix continued...
  - $\rho$ for two,three,four,....n level system
  - | Nomenclature | diagonal | off diagonal |
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  - Coherences/dephasing/amplitude decay
  - $\rho$ for composite system
  - Reduced density matrix
  - Observables in density matrix representation
  - Time evolution
- Touch open quantum systems
- Questions & discussions from feedback/office hours/tutorials

$$|\alpha\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \qquad \rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2} \qquad \text{qubit} \qquad s = 1/2$$

$$|\alpha\rangle = \frac{1}{\sqrt{3}}\left[|0\rangle + |1\rangle + |2\rangle\right] \qquad \rho = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3} \qquad S = 1 \qquad \text{qtrits}$$

$\sqrt{3}$ $\qquad$ $3 \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}_{3\times 3}$

qquad.

<u>n-level.</u>

$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \cdots \\ \vdots & \rho_{11} & & \\ \vdots & & & \\ \rho_{n-1\,0} & - & \cdots & \rho_{n-1\,n-1} \end{bmatrix}_{n\times n}$$

$$\rho_{ij} = \sum_{\alpha=1}^{m} W_\alpha C_i C_j^*$$

$$|\alpha\rangle = \frac{1}{\sqrt{2}}\left[|0\rangle + |1\rangle\right] \qquad \rho = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}_{2\times 2}$$

qubit $s = 1/2$

$$|\alpha\rangle = \frac{1}{\sqrt{3}}\left[|0\rangle + |1\rangle + |2\rangle\right] \qquad \rho = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}_{3\times 3}$$

$S = 1$

qtrits

qquad.

n- level.

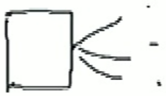
$$\rho = \begin{bmatrix} \rho_{00} & \rho_{01} & \rho_{02} & \cdots \\ 1 & \rho_{11} & & \\ \vdots & & & \\ \rho_{n-1\,0} & - & - & \rho_{n-1\,n-1} \end{bmatrix}_{n\times n}$$

$$\rho_{ij} = \sum_{\alpha=1}^{m} W_\alpha\, C_i\, C_j^*$$

$\Downarrow$

classical probabilities

$$\begin{bmatrix} \rho_{n-1\,0} & - & - & - & \rho_{n-1\,n-1} \end{bmatrix}_{n\times n}$$



$$|\alpha\rangle = c_0|0\rangle + c_1|1\rangle$$

$$|\alpha\rangle = c_0|0\rangle - c_1|1\rangle$$

$|c|^2 \to |0\rangle$

$|c|^2 \to |1\rangle$

$$\underline{|\alpha\rangle} = c_0|0\rangle + e^{i\phi}c_1|1\rangle$$

$$\rho = \begin{bmatrix} |c_0|^2 & c_0c_1^* \\ & \end{bmatrix}$$

$$|\alpha\rangle = c_0|0\rangle + e^{i\varphi}\, c_1|1\rangle$$

$$\rho = \begin{bmatrix} |c_0|^2 & c_0 c_1^* \\ c_1 c_0^* & |c_1|^2 \end{bmatrix}$$

$$|\alpha\rangle = c_0|0\rangle + e^{i\phi} c_1|1\rangle$$

$$\rho = \begin{bmatrix} |c_0|^2 & c_0^* c_1 \, e^{-i\phi} \\ c_1 \, {}_0 \, {}^{i\phi} & |c_1|^2 \end{bmatrix}$$

[redacted]

$$\checkmark\ \underline{|\alpha_1\rangle} = c_0|0\rangle + e^{i\phi}\, c_1|1\rangle \qquad W_{\alpha_1}$$

$$\rho = \begin{bmatrix} \sum_{i=1}^{m} w_{\alpha_i}|c_0|^2 & c_0c_1\left[w_{\alpha_1} e^{-i\phi_{\alpha_1}} + w_{\alpha_2} e^{-i\phi_2} \cdots\right] \\ c_1c_0\left[w e^{i\phi}\right] & \sum_{i=1}^{m} w_{\alpha_i}|c_1|^2 \end{bmatrix}$$

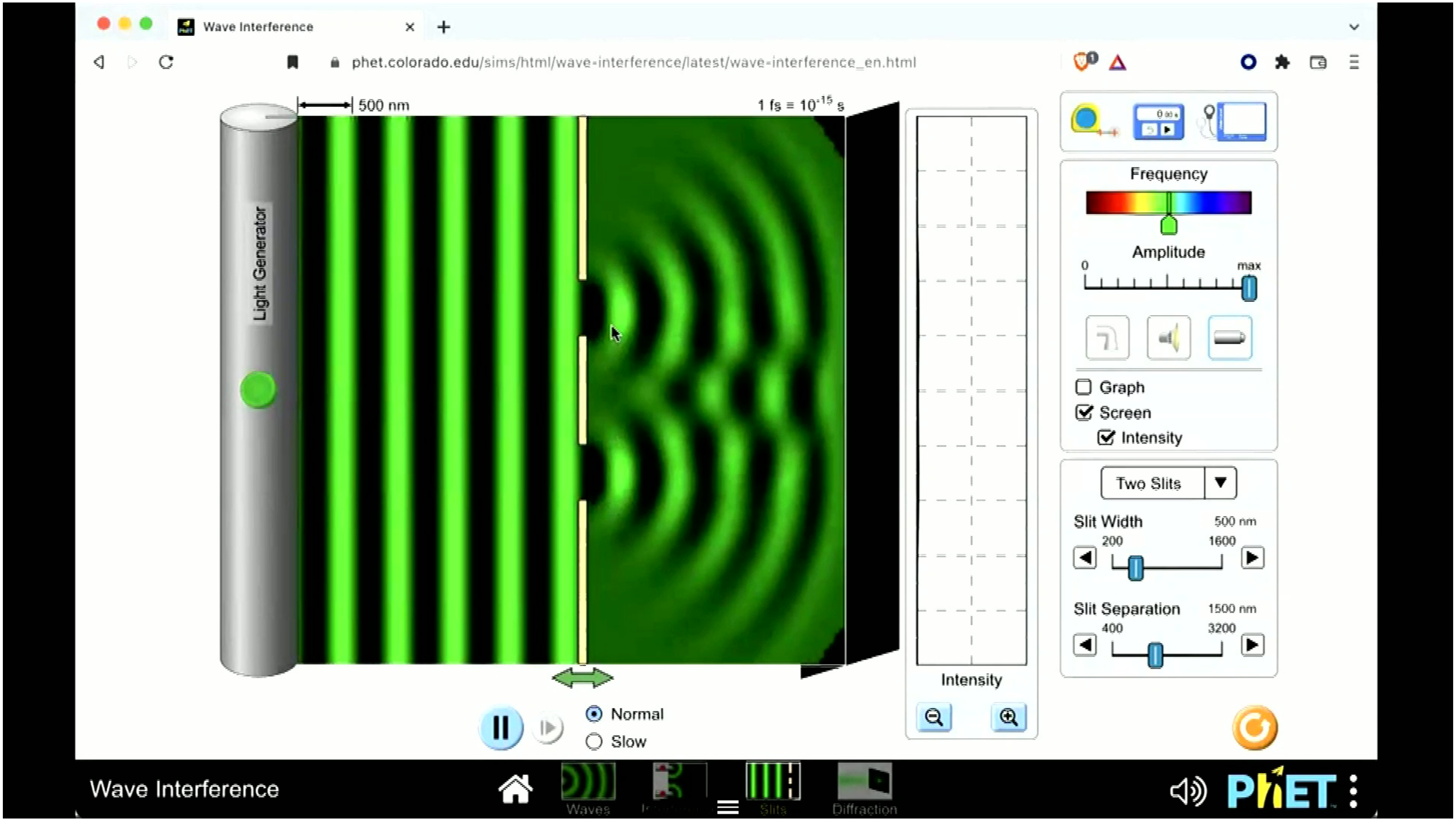
$$
\rho = \begin{bmatrix} \underbrace{\sum_{i=1}^{m} w_{\alpha_i} |c_0|^2} & c_0 c_1 \left[ w_{\alpha_1} e^{-i\phi_{\alpha_1}} + w_{\alpha_2} e^{-i\phi_2} \cdots \right] \\ c_1 c_0 \left[ w e^{i\phi} \right] & \underbrace{\sum_{i=1}^{m} w_{\alpha_i} |c_1|^2} \end{bmatrix}_{2\times 2}
$$

$$\langle 0 | \hat{V} | 1 \rangle$$

$$\rho = \begin{bmatrix} \underbrace{\sum_{i=1}^{m} w_{\alpha_i} |C_0|^2} & \underbrace{C_0 C_1 \left[ w_{\alpha_1} e^{-i\phi_{\alpha_1}} + w_{\alpha_2} e^{-i\phi_2} \cdots \right]} \\ C_1 C_0 \left[ w e^{i\phi} \right] & \underbrace{\sum_{i=1}^{m} w_{\alpha_i} |C_1|^2} \end{bmatrix}_{2\times 2} \rightarrow \text{weighted sum of quantum prob. of finding my system in 1}$$

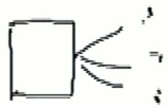
Diagonal matrix elements $\rightarrow$ populations

off-Diagonal matrix elements $\rightarrow$


Wave Interference
phet.colorado.edu/sims/html/wave-interference/latest/wave-interference_en.html
500 nm
1 fs = 10⁻¹⁵ s
Light Generator
Intensity
Frequency
Amplitude
0
max
Graph
Screen
Intensity
Two Slits
Slit Width
500 nm
200
1600
Slit Separation
1500 nm
400
3200
Normal
Slow
Wave Interference
Waves
Slits
Diffraction
PhET

$$\begin{bmatrix} \vdots & & \\ \rho_{n-1\,0} & - - - & \rho_{n-1\,n-1} \end{bmatrix}_{n\times n}$$

classical probabilities



$|\alpha\rangle = c_0|0\rangle + c_1|1\rangle$ $\qquad |c|^2 \to |0\rangle$

$|\alpha\rangle = c_0|0\rangle - c_1|1\rangle$ $\qquad |c|^2 \to |1\rangle$

$|\alpha_i\rangle = c_0|0\rangle + e^{i\phi} c_1|1\rangle$ $\qquad W_{\alpha_1}$

$$\rho = \begin{bmatrix} \underbrace{\sum_{i=1}^{m} w_{\alpha_i}|c_0|^2} & \underbrace{c_0c_1\left[w_{\alpha_1}e^{-i\phi_{\alpha_1}} + w_{\alpha_2}e^{-i\phi_2} - - -\right]}_{\to 0} \\ c_1c_0\left[w e^{i\phi}\right] & \underbrace{\sum_{i=1}^{m} w_{\alpha_i}|c_1|^2} \end{bmatrix}_{2\times 2}$$

$\rightarrow$ weighted sum of quantum prob. of finding my system in 1

Diagonal matrix elements $\rightarrow$ populations

off-Diagonal matrix elements $\rightarrow$

$\Big]_{2\times 2}$ of

Diagonal matrix elements $\rightarrow$ populations

off-Diagonal matrix elements $\rightarrow$

They tell me how well I am preparing system states

control the phase difference b/w

## off-Diagonal matrix elements →

They tell me how well I am preparing system states

control the phase difference b/w

| | |
|---|---|
| Coherent system $\rho_{ij} \neq 0$ $i \neq j$ | incoherent system $\rho_{ij} = 0$ $i \neq j$ |

off- Diagonal matrix elements $\rightarrow$

They tell me how well I am preparing system states

control the phase difference b/w

Coherent system $\rho_{ij} \neq 0 \quad i \neq j$

$$|\alpha\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right)$$

$$\rho_c = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

incoherent system $\rho_{ij} = 0 \quad i \neq j$

$$\rho_{in} = |1\rangle\langle 1| + |0\rangle\langle 0|$$

$$\rho_{in} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

off-Diagonal matrix elements →

They tell me how well I am preparing system states

control the phase difference b/w

Coherent system $\rho_{ij} \neq 0 \quad i \neq j$

$$|\alpha\rangle = \frac{1}{\sqrt{2}}\left(|1\rangle + |0\rangle\right)$$

$$\rho = \left(|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1| + |1\rangle\langle 0|\right)$$

$$\rho_c = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

incoherent system $\rho_{ij} = 0 \quad i \neq j$

$$\rho_{in} = \left(|1\rangle\langle 1| + |0\rangle\langle 0|\right)$$

$$\rho_{in} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

1

$$\rho_c = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \quad + |0\rangle\langle 1| + |1\rangle\langle 0|$$

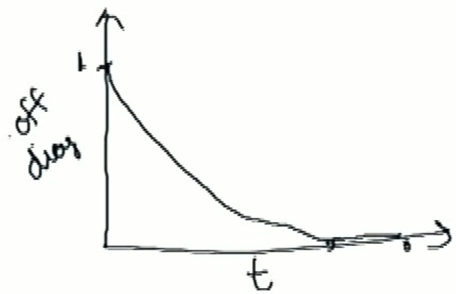
$$1 < \rho_{ij} < 0$$

$$\rho_{in} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} |c_0|^2 & 0.5 \\ 0.5 & |c_1|^2 \end{bmatrix}$$

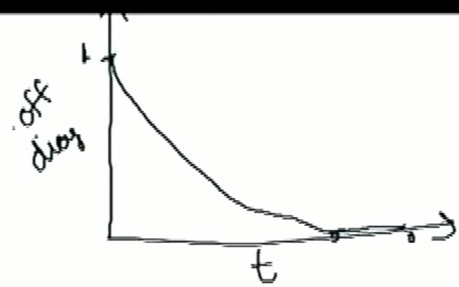
$$\begin{bmatrix} |c_0|^2 & 0.5 \\ 0.5 & |c_1|^2 \end{bmatrix}$$

$$1 < \rho_{ij} < 0$$




$$\rho_c \rightarrow \rho_{in} \quad \{\text{dephasing}\}$$

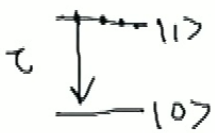
$$\text{time} \rightarrow \underline{T_2}$$




$\rho_c \rightarrow \rho_{in}$ $\{$ dephasing $\}$

time $\rightarrow T_2$

The other way off-diagonal terms $\rightarrow 0$




$c_1 = 0$ or $c_0 = 0$

amplitude damping $\rightarrow$ prob. of the state being in $|0\rangle$ or $|1\rangle$

$$\rho = \begin{bmatrix} \underbrace{\sum_{i=1}^{m} w_{\alpha_i} |c_0|^2} & \underbrace{c_0 c_1 [w_{\alpha_1} e^{-i\phi_{\alpha_1}} + w_{\alpha_2} e^{-i\phi_2} \cdots]} \\ c_1 c_0 [w e^{i\phi}] & \underbrace{\sum_{i=1}^{m} w_{\alpha_i} |c_1|^2} \end{bmatrix}_{2\times 2}$$

$\rightarrow$ weighted sum of quantum prob. of finding my system in 1

Diagonal matrix elements $\rightarrow$ populations

off-Diagonal matrix elements $\rightarrow$

They tell me how well I am preparing system states

control the phase difference b/w

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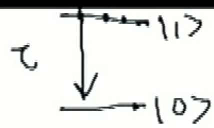
Coherent system $\rho_{ij} \neq 0$ $i \neq j$

$|\alpha\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle)$

$\rho = (|0\rangle\langle 0| + |1\rangle\langle 1| + |0\rangle\langle 1|$

incoherent system $\rho_{ij} = 0$ $i \neq j$

$\rho_{in} = (|1\rangle\langle 1| + |0\rangle\langle 0|)$




$C_1 = 0 \quad \text{or} \quad C_0 = 0$

✓amplitude damping $\rightarrow$ prob. of the state being in $|0\rangle$ or $|1\rangle$

$T_1$ ✓

Density matrix for composite system

Single qubit basis $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

two qubit system $|0\rangle^A \otimes |0\rangle^B$ $|0\rangle^A \otimes |1\rangle^B$ $|1\rangle^A \otimes |0\rangle^B$ $|1\rangle^A \times |1\rangle^B$

$$|0^A 0^B\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad |01\rangle$$

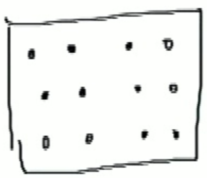
Single qubit basis $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

✓ two qubit system $|0\rangle^A \otimes |0\rangle^B$ $|0\rangle^A \otimes |1\rangle^B$ $|1\rangle^A \otimes |0\rangle^B$ $|1\rangle^A \times |1\rangle^B$

$|0^A 0^B\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $|0^A 1^B\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$ $|1^A 0^B\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $|1^A 1^B\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Single qubit basis $|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $|1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

✓ two qubit system $|0\rangle^A \otimes |0\rangle^B$ $|0\rangle^A \otimes |1\rangle^B$ $|1\rangle^A \otimes |0\rangle^B$ $|1\rangle^A \times |1\rangle^B$



12 single ~~atom~~ qubit

6 composite ~~atom~~ qubit

$|0^A 0^B\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ $|0^A 1^B\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ $|1^A 0^B\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $|1^A 1^B\rangle = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

✓ 2 single qubit Vs. ✓ 1 two qubit

Pure

$$\rho^{AB} = \frac{1}{?} |0^A 0^B\rangle\langle 0^A 0^B| = \begin{bmatrix} 1 & & & \phi \\ & 0 & & \\ & & 0 & \\ \phi & & & 0 \end{bmatrix}$$

Mixture

$\rho^{AB}$ = 25% $|00\rangle$ ; 25% $|01\rangle$ ; 25% $|10\rangle$ ; 25% $|11\rangle$

=

Mixture

$\rho^{AB} = 25\% \ |00\rangle ; 25\% \ |01\rangle ; 25\% \ |10\rangle ; 25\% \ |11\rangle$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$\rho^{AB} = \frac{1}{4} \ |0$

$$= \frac{1}{4}\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\rho^{AB} = \frac{1}{4}|0^A 0^B\rangle\langle 0^A 0^B| + \frac{1}{4}|0^A 1^B\rangle\langle 0^A 1^B| + \frac{1}{4}|1^A 0^B\rangle\langle 1^A 0^B| + \frac{1}{4}|1^A 1^B\rangle\langle 1^A 1^B|$$

Extract the information

[redacted]

$$\frac{1}{4}\,|1^A 1^B\rangle\langle 1^A 1^B|$$

Extract the information for just the system I am interested in

$\rho^{A}$

Reduced density matrix $\rightarrow$ partial trace over the unwanted sys.

$$\rho^{AB} = \frac{1}{4}\,|0^A 0^B\rangle\langle 0^A 0^B| + \frac{1}{4}\,|0^A 1^B\rangle\langle 0^A 1^B| + \frac{1}{4}\,|1^A 0^B\rangle\langle 1^A 0^B| + \frac{1}{4}\,|1^A 1^B\rangle\langle 1^A 1^B|$$

Extract the information for just the system I am interested in

$\rho^{A}$

Reduced density matrix $\rightarrow$ partial trace over the unwanted sys.

$$tr_B(\rho^{AB}) = \rho^A = \frac{1}{4}\Big[\,|0\rangle^A\langle 0|\,\langle 0^B|0^B\rangle + |0^A\rangle\langle 0^A|\,\langle 1^B|1^B\rangle + |1\rangle^A\langle 1|$$

$$\frac{1}{4}\,|1^A 1^B\rangle\langle 1^A 1^B|$$

Extract the information for just the system I am interested in

$\rho^{A}$

Reduced density matrix → partial trace over the unwanted sys.

$$tr_B(\rho^{AB}) = \rho^A = \frac{1}{4}\Big[\,|0\rangle^A\,{}^A\langle 0|\,\underline{\langle 0^B|0^B\rangle} + |0^A\rangle\langle 0^A|\,\underline{\langle 1^B|1^B\rangle} + |1\rangle^A\,{}^A\langle 1|\,\underline{\langle 0^B|0^B\rangle}$$

$$+\;|1^A\rangle\langle 1^A|\,\underline{\langle 1^B|1^B\rangle}\,\Big]$$

$$\rho^A = \frac{1}{2}\Big[\,|0\rangle^A\,{}^A\langle 0| + |1\rangle^A\,{}^B\langle 1|\,\Big] = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\rho^B =$$

$$\rho^A = \frac{1}{2}\left[ |0\rangle^A {}^A\langle 0| + |1\rangle^A {}^B\langle 1| \right] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho^B = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho^A \otimes \rho^B = \frac{1}{4}\begin{bmatrix} 1 & & & \emptyset \\ & 1 & & \\ & & 1 & \\ \emptyset & & & 1 \end{bmatrix} = \rho^{AB}$$

$$\rho^{B} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\sqrt{\rho^{A}} \otimes \rho^{B} = \frac{1}{4}\begin{bmatrix} 1 & & & \phi \\ & 1 & & \\ \phi & & 1 & \\ & & & 1 \end{bmatrix} = \sqrt{\rho^{AB}} \Rightarrow \text{Separable}$$

---

$\rho^{A} \otimes \rho^{B} = \rho^{AB}$ not always true

$$\phi \quad 1 \quad 1$$

$$\rho^A \otimes \rho^B = \underline{\underline{\rho}}^{AB} \quad \text{not always true}$$

$$|\alpha\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \text{Bell state}$$

$$\rho^{AB} = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$tr_B(\rho^{AB}) = \begin{bmatrix} \\ \\ \end{bmatrix}$$

$$|\alpha\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \text{Bell State}$$

$$\rho^{AB} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$$\rho^{A} = tr_B\left(\rho^{AB}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho^{A} \otimes \rho^{B} = \rho^{AB} \quad ???$$

Reduced density matrix

$$\mathrm{tr}_B(\rho^{AB}) = \rho^A = \frac{1}{2}\Big[ |0\rangle^A {}^A\langle 0| \langle 0^B|0^B\rangle + |0^A\rangle\langle 0^A| \langle 1^B|1^B\rangle + |1\rangle^A {}^A\langle 1| \langle 0^B|0^B\rangle$$

$$+ |1^A\rangle\langle 1^A| \langle 1^B|1^B\rangle \Big]$$

$$\rho^A = \frac{1}{2}\left[ |0\rangle^A {}^A\langle 0| + |1\rangle^A {}^B\langle 1| \right] = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho^B = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\rho^{AB} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$$\rho^{A} = tr_{B}\left(\rho^{AB}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$\rho^{A} \otimes \rho^{B} = \rho^{AB} \quad ???$$

$$\rho^{B} = tr_{A}\left(\rho^{AB}\right) = \begin{bmatrix} 1 & 0 \\ \ldots \end{bmatrix}$$

$$|\alpha\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}} \rightarrow \text{Bell state}$$

$$\rho^{AB} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix}_{4\times 4}$$

$$\rho^{A} = tr_B\left(\rho^{AB}\right) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

$$\underline{\rho}^{A} \otimes \underline{\rho}^{B} = \rho^{AB} \quad ???$$

$$\rho^{B} = tr_A\left(\rho^{AB}\right) = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times 2}$$

✓ Observables using density matrix $\qquad |\alpha\rangle = \sum_n c_n |n\rangle$

$$\langle A\rangle_\alpha = \langle\alpha| A|\alpha\rangle$$

$$= \sum_{nn'} \underbrace{\langle n|\alpha\rangle}_{c_n} \underbrace{\langle\alpha|n'\rangle}_{c_{n'}^*} \langle n'|A|n\rangle$$

Average of $\langle A\rangle_\alpha$ weighted over $W_\alpha$

$$\langle A\rangle = \sum_{\alpha=1}^{m} W_\alpha \langle A\rangle_\alpha = \underbrace{Tr(\underline{\rho}\,\underline{A})}$$

$\langle S$

Observables using density matrix $\quad |\alpha\rangle = \sum_n c_n |n\rangle$

$$\langle A\rangle_\alpha = \langle\alpha|A|\alpha\rangle$$

$$= \sum_{nn'} \overset{c_n}{\langle n|\alpha\rangle}\, \overset{c_{n'}^*}{\langle\alpha|n'\rangle}\, \langle n'|A|n\rangle$$

Average of $\langle A\rangle_\alpha$ weighted over $W_\alpha$

$$\langle A\rangle = \sum_{\alpha=1}^{m} W_\alpha \langle A\rangle_\alpha = \underbrace{Tr(\rho A)}$$

$|\uparrow\rangle_z \quad \langle S_z\rangle = Tr[\rho S_z] = 1$

$\rho = |\uparrow\rangle\langle\uparrow|$

$$\langle A \rangle = \sum_{\alpha=1}^{m} W_\alpha \langle A \rangle_\alpha = \underbrace{Tr(\rho A)}$$ ✓

$|\uparrow\rangle_z$ $\langle S_z \rangle = Tr[\rho S_z] = 1$

$\rho = |\uparrow\rangle\langle\uparrow|$

Time evolution of density matrix

$|\alpha(t)\rangle = U^+ |\alpha(t_0)\rangle$

$$\rho(t_0) = \sum_{\alpha=1}^{m} |\underbrace{\alpha(t_0)}\rangle W_\alpha \langle \underbrace{\alpha(t_0)} |$$

$$\Rightarrow \underline{\underline{\rho(t)}} = \sum$$

$\rho = |\uparrow\rangle\langle\uparrow|$

Time evolution of density matrix

$|\alpha(t)\rangle = U^{+} |\alpha(t_0)\rangle$

$$\rho(t_0) = \sum_{\alpha=1}^{m} |\underbrace{\alpha(t_0)}\rangle W_\alpha \langle\underbrace{\alpha(t_0)}|$$

$$\Rightarrow \underline{\underline{\rho(t)}} = U(t,t_0)\, \rho(t_0)\, U^{\dagger}(t,t_0)$$

derivative

## Time evolution of density matrix

$$|\alpha(t)\rangle = U \; |\alpha(t_0)\rangle$$

$$\rho(t_0) = \sum_{\alpha=1}^{m} |\alpha(t_0)\rangle W_\alpha \langle\alpha(t_0)|$$

$$\Rightarrow \rho(t) = U(t,t_0)\, \rho(t_0)\, U^\dagger(t,t_0)$$

derivative

$$\boxed{\frac{\partial U}{\partial t} = \frac{-i}{\hbar} H U}$$

$$\dot{\rho}(t) =$$

Time evolution of density matrix

$|\alpha(t)\rangle = U\,|\alpha(t_0)\rangle$

$$\rho(t_0) = \sum_{\alpha=1}^{m} |\alpha(t_0)\rangle\, W_\alpha\, \langle\alpha(t_0)|$$

$$\Rightarrow \rho(t) = U(t,t_0)\,\rho(t_0)\,U^\dagger(t,t_0)$$

derivative

$$\boxed{\frac{\partial U}{\partial t} = \frac{-i}{\hbar} H U}$$

$$\boxed{\dot{\rho}(t) = \frac{i}{\hbar}\left[\rho(t), H\right]}$$