

Title: Quantum Theory - Lecture 220912

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Collection: Quantum Theory (2022-2023)

Date: September 12, 2022 - 10:45 AM

URL: <https://pirsa.org/22090035>

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Quantum Theory(2022) Tutorial1Q1 Lecture-4 Time Dyna... Lectures

## Lecture-4 Time Dynamics

### Announcements

Our learning goals:

- ▶ Time evolution operator
  - ◇ Properties
  - ◇ Evolution equation
  - ◇ Solution
    - Time independent H
    - Time dependent H (commuting/non commuting)
- ▶ Time evolution of statekets
  - ◇ Stationary states
- ▶ Time evolution of observables

BA Add section Add page

The image shows a OneNote application window with a purple title bar. The title bar contains the text "OneNote" and a share icon. Below the title bar is a ribbon with tabs for "Home", "Insert", "Draw", "View", and "Tell me". The "Draw" tab is active, showing a toolbar with icons for "Text Mode", "Lasso Select", "Insert Space", "Eraser", "Pen", "Marker", "Highlighter", "Ink Colour", and line thickness options (0.25 mm, 0.35 mm, 0.5 mm, 0.7 mm, 1 mm). Below the toolbar, the text "Announcements" is underlined. Below that, the text "Our learning goals:" is followed by a list of items:

- ▶ Time evolution operator
  - ◇ Properties
  - ◇ Evolution equation
  - ◇ **Solution**
    - Time independent  $H$
    - Time dependent  $H$  (commuting/non commuting)
- ▶ Time evolution of statekets
  - ◇ Stationary states
- ▶ Time evolution of observables

In the bottom left corner, there is a circular icon with the letters "BA".



# Time Evolution operator.

Monday, 12 September 2022 8:27 PM

$$i\hbar \frac{\partial}{\partial t} |\alpha\rangle = H|\alpha\rangle$$
$$|\alpha(t)\rangle = |\alpha(t_0)\rangle$$

$|\alpha\rangle \rightarrow$  state ket  
 $|\alpha'\rangle \rightarrow$  basis ket  
 $|\alpha\rangle = \sum_{\alpha'} c_{\alpha'} |\alpha'\rangle$

1) Unitary

$$\begin{aligned} \langle \alpha(t_0) | \alpha(t_0) \rangle &= 1 = \langle \alpha(t) | \alpha(t) \rangle \\ &= \langle \alpha(t) | \underbrace{U^\dagger U}_{\mathbb{I}} | \alpha(t) \rangle \end{aligned}$$

2) Composite

$$| \alpha(t_2) \rangle = U(t_2, t_1) | \alpha(t_1) \rangle$$



COMPOSITIVE

$$\begin{aligned} |\alpha(t_2)\rangle &= U(t_2, t_1) |\alpha(t_1)\rangle \\ &= U(t_2, t_1) U(t_2, t_0) |\alpha(t_0)\rangle \\ |\alpha(t_2)\rangle &= U(t_2, t_0) |\alpha(t_0)\rangle \end{aligned}$$

Time-reversal

$$U(t_2, t_1) U(t_1, t_2)$$

Time reversal

$$U(t, t_0) = U^{-1}(t_0, t)$$

$$| U(t, t_0) = U^+(t_0, t)$$

$$U^+ U = \underline{I}$$

$$U^+(t_0, t) U(t_0, t) = \underline{I}$$

$$U^{-1}(t_0, t) =$$

$$U^+(t_0, t)$$

Relation b/w  $H$  &  $U$

$$|\alpha(t)\rangle = U(t, t_0) |\alpha(t_0)\rangle$$

$$|\dot{\alpha}(t)\rangle = \dot{U}(t, t_0) |\alpha(t_0)\rangle$$

$$i\hbar \underline{\dot{|\alpha\rangle}} = U^\dagger(t_0, t) H |\alpha\rangle$$



Relation b/w  $H$  &  $U$

$$|\alpha(t)\rangle = U(t, t_0) |\alpha(t_0)\rangle$$

$$|\dot{\alpha}(t)\rangle = \dot{U}(t, t_0) |$$

$$i\hbar \underline{\underline{|\dot{\alpha}\rangle}} = U^\dagger(t_0, t) H |\alpha\rangle$$

Relation b/w  $H$  &  $U$

$$i\hbar \dot{|\alpha\rangle} = H |\alpha\rangle$$

$$|\alpha(t)\rangle = U(t, t_0) |\alpha(t_0)\rangle$$

$$|\dot{\alpha}(t)\rangle = \dot{U}(t, t_0) |\alpha(t_0)\rangle$$

$$= \dot{U}(t, t_0) U(t_0, t) |\alpha(t)\rangle$$

$$= \dot{U}(t, t_0) U^\dagger(t, t_0) |\alpha(t)\rangle$$

$$i\hbar |\dot{\alpha}(t)\rangle = \underbrace{i\hbar \dot{U}(t, t_0) U^\dagger(t, t_0)}_H |\alpha(t)\rangle$$

$$H^\dagger = H$$

$$H = \underbrace{i\hbar \dot{U}(t, t_0) U^\dagger(t, t_0)}_{\text{Hermitian}}$$

$$\Rightarrow [U(t, t_0) U^\dagger(t, t_0) = I]$$

Take derivative.

$$H = i\hbar \underbrace{\dot{U}(t, t_0) U^\dagger(t, t_0)}_{\text{Hermitian}}$$

$$H^\dagger = H$$

$$\Rightarrow [U(t, t_0) U^\dagger(t, t_0) = I]$$

Take derivative.

$\dot{U} U^\dagger \rightarrow$  antihermitian

$i$

$$\begin{aligned} \dot{U} U^\dagger + U \dot{U}^\dagger &= 0 \\ \dot{U} U^\dagger &= -U \dot{U}^\dagger \\ \checkmark \dot{U} U^\dagger &= -(\dot{U} U^\dagger)^\dagger \end{aligned}$$

$$|\alpha(t)\rangle = U(t, t_0) |\alpha(t_0)\rangle$$

$U$  follow  $\Rightarrow$   $i\hbar \dot{U} = H U$

Solution of time evolution operator

$$\begin{array}{c}
 i \hbar \dot{U} \\
 | \alpha(t) \rangle = U(t, t_0) | \alpha(t_0) \rangle \\
 U \text{ follow } \Rightarrow \boxed{i \hbar \dot{U} = H U} \\
 \begin{array}{c}
 [H(t_0), H(t)] = 0 \\
 H \quad H(t) \quad H(t) \rightarrow [H(t_0), H(t)] \neq 0
 \end{array}
 \end{array}$$

Solution of time evolution operator

$$|\alpha(t)\rangle = U(t, t_0) |\alpha(t_0)\rangle$$

U follow  $\Rightarrow$

$$i\hbar \dot{U} = H U$$

Solution of time evolution operator

$[H(t_0), H(t)] = 0$

I  $(H)$   $H(t) H(t) \rightarrow [H(t_0), H(t)] \neq 0$

II  $H(t)$

$$[H(t_0), H(t)] = 0$$

III  $H(t)$

$$\left\{ \begin{array}{l} [H(t_0), H(t)] \neq 0 \\ B_x(t_0) \quad B_y(t) \\ [S_x, S_y] \neq 0 \end{array} \right.$$

$$u = e^{iHt/\hbar}$$

operators / matrices

$$u = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{i t H}{\hbar} \right)^n$$



I H time independent

$H, U$  are numbers

$$i\hbar \dot{U} = HU$$

$$U = ?$$

$$\int i\hbar \frac{dU}{U} = \int H dt$$

operators / matrices

H time independent [ ]

$$U = e^{-iHt/\hbar}$$

$$U = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-itH}{\hbar} \right)^n$$

$$U = e^{-iHt/\hbar}$$

$$\left( \frac{-itH}{\hbar} \right)^{n-1}$$

rename

II  $H$  is time dependent

$$\dot{u} = -\frac{i}{\hbar} H(t) u \quad \checkmark$$

$$u = \exp \left\{ -\frac{i}{\hbar} \int_0^t H(t') dt' \right\} \rightarrow$$

$$\dot{u} = \sum_{n=1}^{\infty} \frac{1}{n!} ($$

II  $H$  is time dependent

$$\dot{u} = -\frac{i}{\hbar} H(t) u \quad \checkmark$$

$$u = \exp \left\{ -\frac{i}{\hbar} \int_0^t H(t') dt' \right\} \rightarrow$$

$$u = \sum_{n=0}^{\infty} \frac{1}{n!} \left( -\frac{i}{\hbar} \right)^n (n-1) \left[ \int_0^t H(t') dt' \right]^{n-1} \frac{d}{dt} \int_0^t H(t') dt'$$

$H(t)$

$$[H(t'), H(t)] = 0$$

$$\dot{u} = -\frac{i}{\hbar} H(t) u$$

$$\psi u = \exp \left\{ \frac{-i}{\hbar} \int_0^t H(t') dt' \right\} \quad \checkmark$$

$$\dot{u} = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \frac{-i}{\hbar} \right)^n (n-1) \left[ \int_0^t H(t') dt' \right]^{n-1} \frac{d}{dt} \underbrace{\int_0^t H(t') dt'}_{H(t)}$$

$$\boxed{\dot{u} = -\frac{i}{\hbar} H u} \quad \dot{u} = -\frac{i}{\hbar} H u$$

$$[H(t'), H(t)] = 0$$

$$H(t) H(t) = H(t) H(t)$$

Case III  $H(t)$   $[H(t), H(t)] \neq 0$   $\frac{d}{dt} [A(t) B(t)]$

$$U = \exp \left[ \frac{-i}{\hbar} \int_0^t dt' H(t') \right]$$

$$= 1 + \int_0^t dt' H(t') + \frac{1}{2} \int_0^t dt' H(t') \int_0^{t'} dt'' H(t'') + \dots$$

$$\dot{U} = -i H(t) + \frac{1}{2} H(t) \int_0^t dt'' H(t'')$$

$$\boxed{\dot{u} = -\frac{i}{\hbar} H u} \quad \boxed{\ddot{u} = -\frac{i}{\hbar} \dot{H} u}$$

$$[H(t'), H(t)] = 0$$

$$H(t) H(t) = H(t) H(t)$$

Case III       $H(t)$        $[H(t'), H(t)] \neq 0$        $\frac{d}{dt} [A(t) B(t)]$

$$u = \exp \left[ -\frac{i}{\hbar} \int_0^t dt' H(t') \right] \cdot x$$

$$= 1 + \int_0^t dt' H(t') + \frac{1}{2} \int_0^t dt' H(t') \int_0^{t'} dt'' H(t'') + \dots$$

$$\dot{u} = -\frac{i}{\hbar} H(t) u$$

$$u(t_0, t_0) = 1$$

integrate

$$u(t, t_0) = \mathbb{1} - \frac{i}{\hbar} \int_{t_0}^t H(t') u(t', t_0) dt'$$

$$t = t_0$$

integrate

$$\checkmark u(t, t_0) = \textcircled{1} - \frac{i}{\hbar} \int_{t_0}^t H(t') u(t', t_0) dt' \quad t = t_0$$

$$u(t, t_0) = 1 - \frac{i}{\hbar} \int_0^t dt H(t) + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' H(t') H(t'') + \dots$$

$$\dots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \dots \int_0^{t^{(n-1)}} dt^{(n)} H(t') H(t'') \dots H(t^{(n)})$$



$$\begin{aligned}
 u(t, t_0) = & 1 - \frac{i}{\hbar} \int_0^t dt' H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' \underbrace{H(t') H(t'')}_{\text{Time } 0} + \dots \\
 & \dots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \dots \int_0^{t^{(n-1)}} dt^{(n)} H(t') H(t'') \dots H(t^{(n)})
 \end{aligned}$$

$$0 < t' < t$$

$$0 < t'' < t' < t$$

$$0 < t^{(n)} < t^{(n-1)} < \dots < t' < t$$

$$\begin{aligned}
 u(t, t_0) = & 1 - \frac{i}{\hbar} \int_0^t dt H(t) + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt \int_0^{t'} dt'' \underbrace{H(t') H(t'')}_{\text{Time } 0} + \dots \\
 & \dots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt \int_0^{t'} dt'' \int_0^{t''} dt''' \dots \int_0^{t^{n-1}} dt^n H(t) H(t'') \dots H(t^n)
 \end{aligned}$$

$$0 < t' < t$$

$$0 < t'' < t' < t$$

$$0 < t^n < t^{n-1} \dots < t < t$$





$\checkmark$   $U(t, t_0) = \mathbb{1} - \frac{i}{\hbar} \int_{t_0}^t H(t') U(t', t_0) dt'$

$H(t)$   
 $[H(t'), H(t'')] \neq 0$   
 $|\alpha(t)\rangle \rightarrow |\alpha(t_0)\rangle$

$\checkmark$   $U(t, t_0) = 1 - \frac{i}{\hbar} \int_0^t dt H(t) + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' \underbrace{H(t') H(t'')}_{\text{Time order}} + \dots$

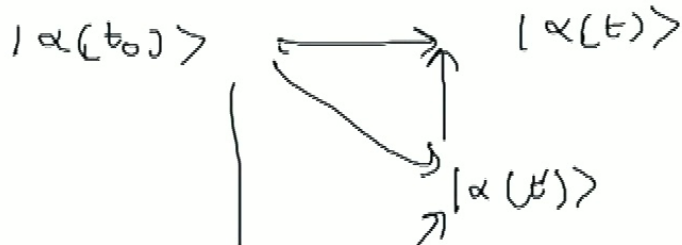
$\dot{U} = -\frac{i}{\hbar} H U$

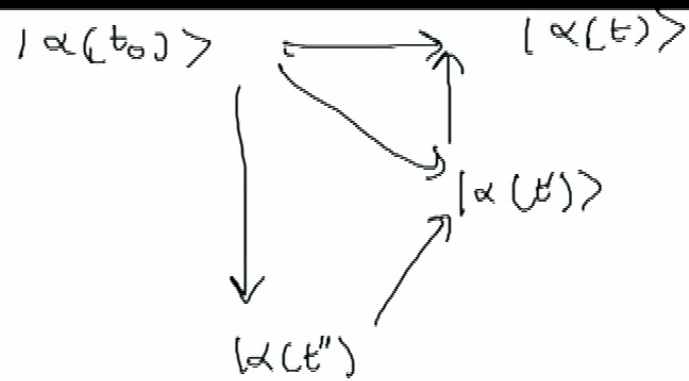
$\dots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \dots \int_0^{t^{n-1}} dt^n H(t')$

$0 < t' < t$

$0 < t'' < t' < t$

$0 < t^n < t^{n-1} \dots < t' < t$





positive time ordered exponential (Dyson series) .

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$H(t)$   
 $[H(t), H(t')] \neq 0$

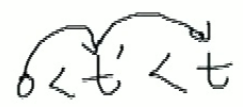
$|\alpha(t)\rangle \rightarrow |\alpha(t_0)\rangle$

$$U(t, t_0) = 1 - \frac{i}{\hbar} \int_0^t dt H(t) + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' \underbrace{H(t') H(t'')}_{\text{Time order}} + \dots$$

$$\dots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \dots \int_0^{t^{(n-1)}} dt^{(n)} H(t') H(t'') \dots H(t^{(n)})$$

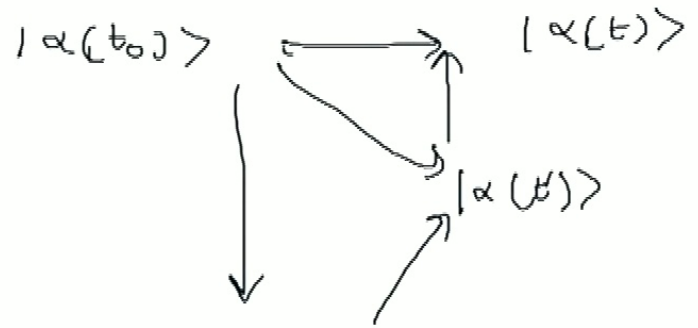
$\downarrow_{n+1}$

$$\dot{U} = -\frac{i}{\hbar} H U$$



$$0 < t'' < t' < t$$

$$0 < t^{(n)} < t^{(n-1)} \dots < t < t$$



positive time ordered exponential (Dyson series) .

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$$[H(t'), H(t)] = 0$$

$$\exp(\dots) \rightarrow [H(t'), H(t)] = 0$$

SP

H is not time dependent

Observables associated

$$\langle A(t_0) \rangle = \langle \alpha(t_0) | A | \alpha(t_0) \rangle$$

$$\langle A(t) \rangle = \langle \alpha(t) | A | \alpha(t) \rangle$$

$$= \langle \alpha(t_0) U^\dagger A U | \alpha(t_0) \rangle$$

Approach  $\rightarrow \langle \alpha(t_0) | U^\dagger A U | \alpha(t_0) \rangle \rightarrow \text{HP}$ .

Schrodinger picture

$$|\alpha(t)\rangle = \sum_{a'} c_{a'} |a'\rangle$$

$$H|a'\rangle = E_{a'}|a'\rangle$$

$$U = e^{-iHt}$$

$$U = \exp\left(-i\frac{Ht}{\hbar}\right)$$
$$|\alpha(t)\rangle = \sum_{a'} \exp\left(-i\frac{Ht}{\hbar}\right) c_{a'} |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_a t/\hbar} \sum_{a'} c_a(t_0) |a'\rangle$$

$$|\alpha(t_0)\rangle = |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_a t/\hbar} |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_{\alpha}t/\hbar} \sum_{a'} c_a(t_0) |a'\rangle$$

$$|\alpha(t_0)\rangle = |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_{\alpha}t/\hbar} |a'\rangle$$

---

Expectation value of observables

$$|\alpha(t_0)\rangle = |a\rangle$$

$$\begin{aligned}\langle B \rangle &= \langle a' | B | a' \rangle \\ &= \langle a' | e^{+iE_{a'}t/\hbar} | B | e^{-iE_{a'}t/\hbar} | a' \rangle \\ &= \langle a' | B | a' \rangle\end{aligned}$$

$$\langle B \rangle = \sum_{a' a''} c_{a'}^* c_{a''} e^{-i(E_{a''} - E_{a'}) t / \hbar} \langle a' | B | a'' \rangle$$

(td)

$$|\alpha(t)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

Harmonic Oscillator

$n=0$

$$E_0 = \frac{\hbar\omega}{2}$$

$n=1$

t?

$|\alpha(t)\rangle$  using time evolution operator (at time independent)



$$| \alpha(t) \rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

$$n=0 \quad E_0 = \frac{\hbar\omega}{2}$$

$$n=1$$

$t?$   $| \alpha(t) \rangle$  using time evolution operator (time independent)

$$| \alpha(t) \rangle$$

at  $t$  find  $\langle x \rangle_s =$

$$\textcircled{a} \quad |\alpha(t)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$$

Harmonic Oscillator

$$n=0 \quad E_0 = \frac{\hbar\omega}{2}$$

$$n=1$$

$t?$   $|\alpha(t)\rangle$  using time evolution operator (time independent)

$$|\alpha(t)\rangle$$

$$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}} (a + a^\dagger)$$

at t find  $\langle x \rangle_s =$