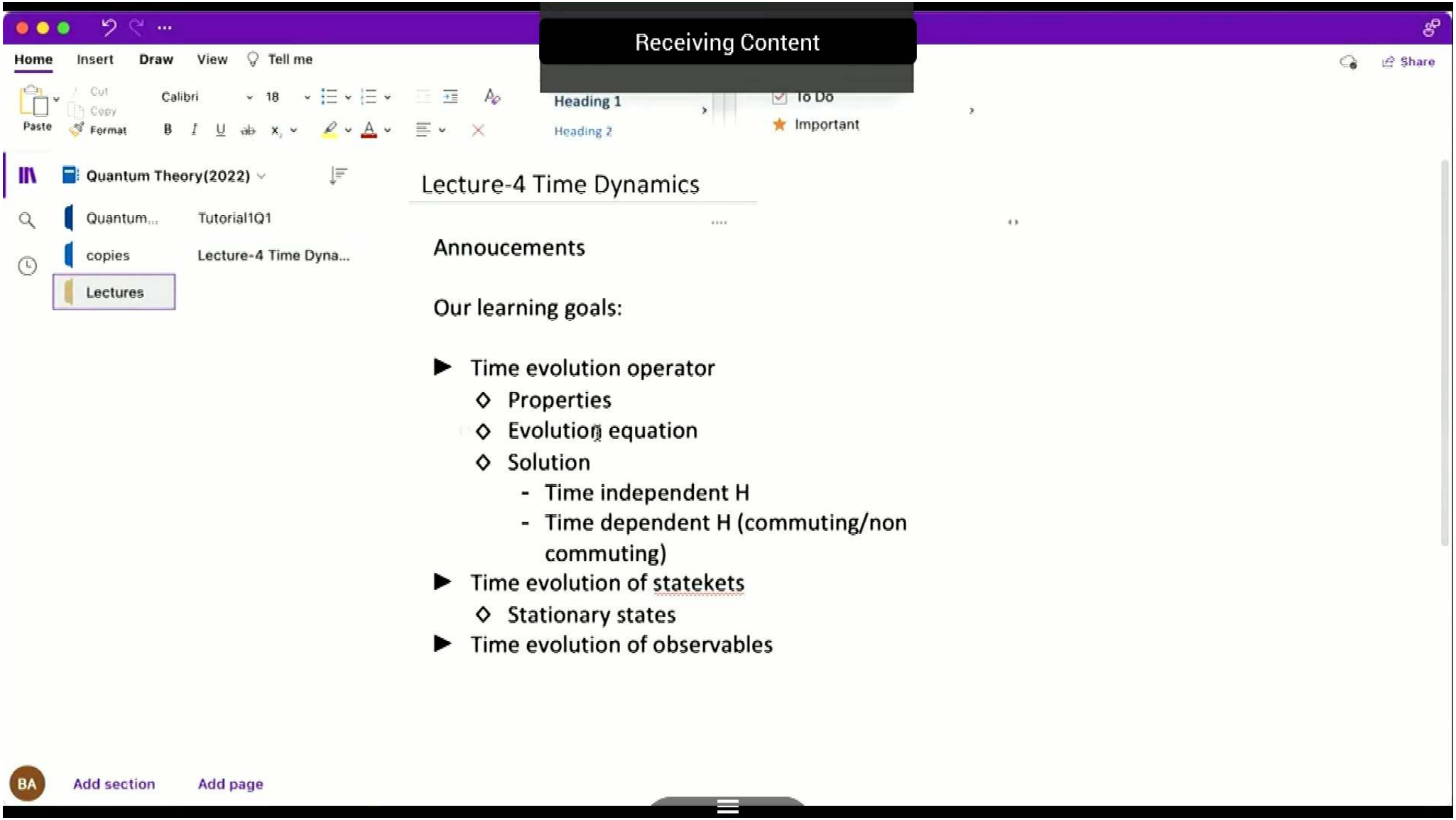
Title: Quantum Theory - Lecture 220912

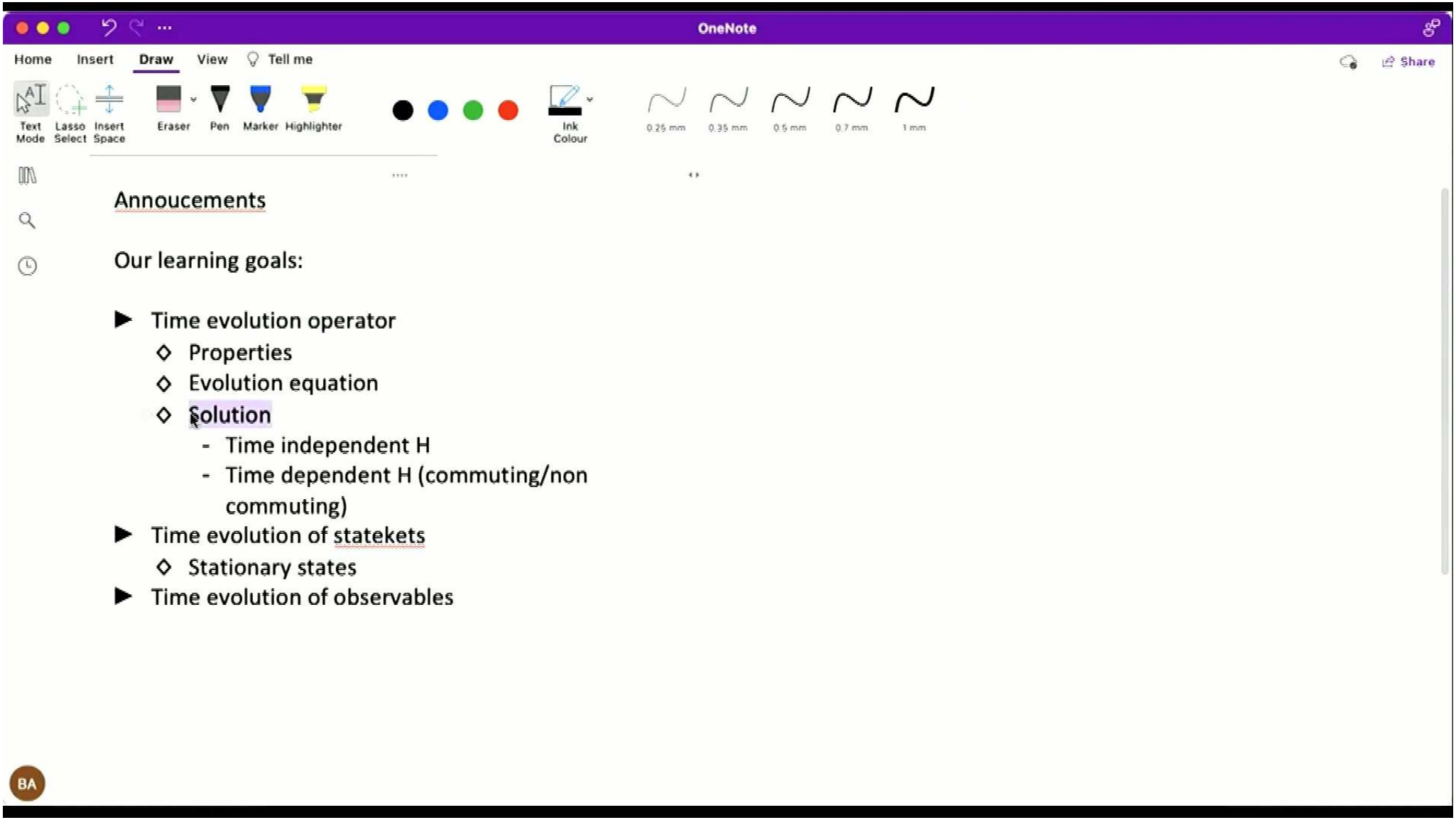
Speakers: Bindiya Arora

Collection: Quantum Theory (2022-2023)

Date: September 12, 2022 - 10:45 AM

URL: https://pirsa.org/22090035


Receiving Content
Home
Insert
Draw
View
Tell me
Share
Paste
Cut
Copy
Format
Calibri
18
Heading 1
Heading 2
To Do
Important
Quantum Theory(2022)
Quantum...
Tutorial1Q1
copies
Lecture-4 Time Dyna...
Lectures
Lecture-4 Time Dynamics
Annoucements
Our learning goals:
Time evolution operator
Properties
Evolution equation
Solution
Time independent H
Time dependent H (commuting/non commuting)
Time evolution of statekets
Stationary states
Time evolution of observables
BA
Add section
Add page


OneNote
Home
Insert
Draw
View
Tell me
Share
Text Mode
Lasso Select
Insert Space
Eraser
Pen
Marker
Highlighter
Ink Colour
0.25 mm
0.35 mm
0.5 mm
0.7 mm
1 mm
Annoucements
Our learning goals:
Time evolution operator
Properties
Evolution equation
Solution
Time independent H
Time dependent H (commuting/non commuting)
Time evolution of statekets
Stationary states
Time evolution of observables
BA

# Time Evolution operator.

Monday, 12 September 2022 8:27 PM

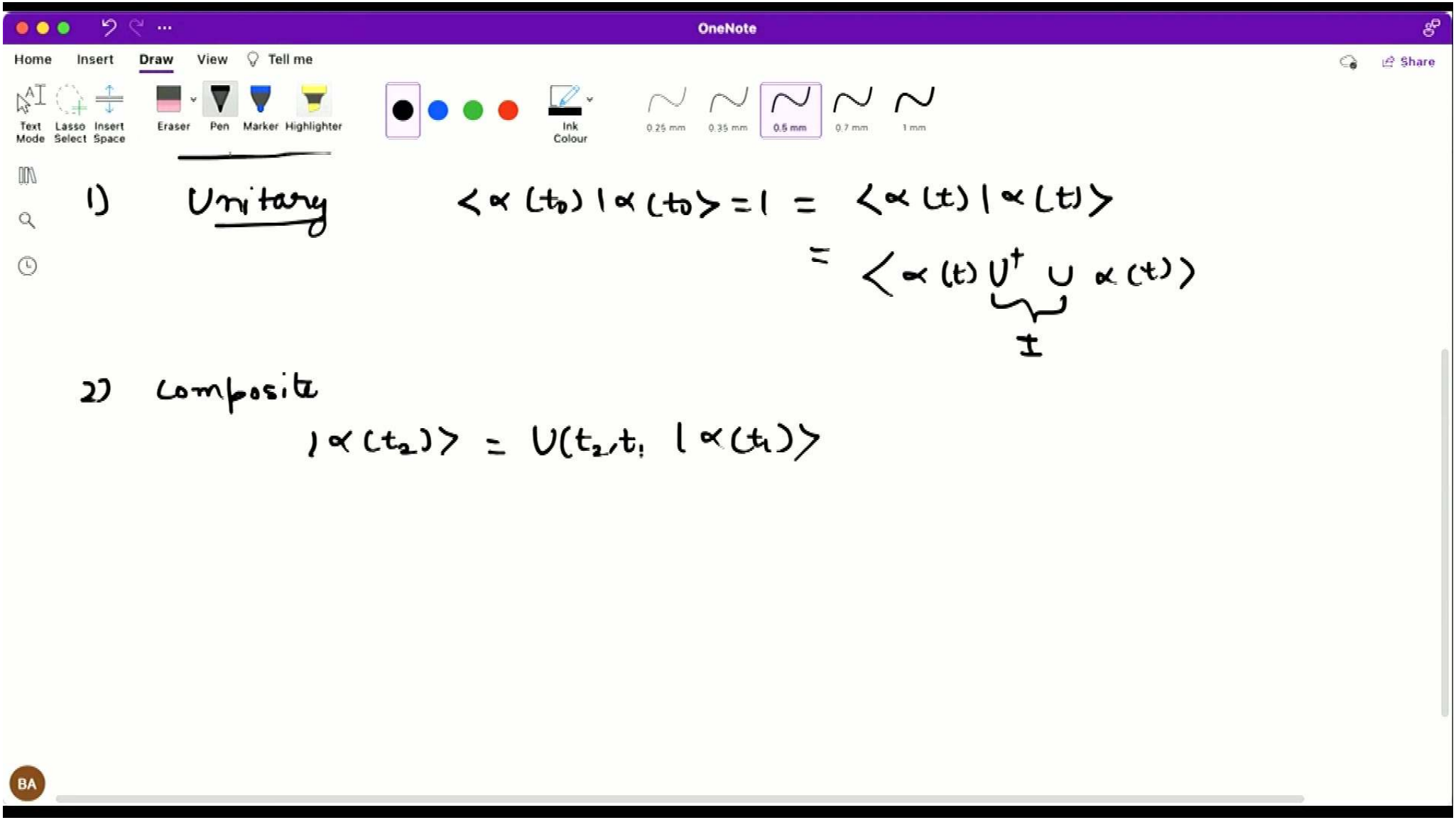
$$i\hbar \frac{\partial}{\partial t} |\alpha\rangle = H|\alpha\rangle$$

$$|\dot{\alpha}(t_0)\rangle = |\alpha(t_0)\rangle$$

$|\alpha\rangle \rightarrow$ state ket

$|a'\rangle \rightarrow$ basis ket

$$|\alpha\rangle = \sum_{a'} C_{a'} |a'\rangle$$



1) Unitary $\langle \alpha(t_0) | \alpha(t_0) \rangle = 1 = \langle \alpha(t) | \alpha(t) \rangle$

$$= \langle \alpha(t) \underbrace{U^\dagger U}_{I} \alpha(t) \rangle$$

2) Composite

$$|\alpha(t_2)\rangle = U(t_2, t_1 \, |\alpha(t_1)\rangle$$

2) Composite

$$|\alpha(t_2)\rangle = U(t_2,t_1)\,|\alpha(t_1)\rangle$$

$$= U(t_2,t_1)\,U(t_1,t_0)\,|\alpha(t_0)\rangle$$

$$|\alpha(t_2)\rangle = U(t_2,t_0)\,|\alpha(t_0)\rangle$$

Time-reversal

$$U(t_2,t_1)\;U(t_1$$

Time reversal

$$U(t, t_0) = U^{-1}(t_0, t)$$

$$U^{\dagger} U = \mathbb{I}$$

$$U^{\dagger}(t_0, t)\, U(t_0 t) = \mathbb{I}$$

$$U^{-1}(t_0 t) = U^{\dagger}(t_0 t)$$

$$U(t, t_0) = U^{\dagger}(t_0, t)$$

Relation b/w H & U

$$|\alpha(t)\rangle = U(t, t_0)\,|\alpha(t_0)\rangle$$

$$|\dot{\alpha}(t)\rangle = \dot{U}(t, t$$

$$U^{+}(t_0, t)$$

$$i\hbar|\dot{\alpha}\rangle = H|\alpha\rangle$$

$U^{\dagger}(t_0, t)$

Relation b/w H & U

$i\hbar|\dot{\alpha}\rangle = H|\alpha\rangle$

$|\alpha(t)\rangle = U(t, t_0)\,|\alpha(t_0)\rangle$

$|\dot{\alpha}(t)\rangle = \dot{U}(t, t_0)\,|$

Relation b/w H & U

$$i\hbar|\dot{\alpha}\rangle = H|\alpha\rangle$$

$$|\alpha(t)\rangle = U(t,t_0)\,|\alpha(t_0)\rangle$$

$$|\dot{\alpha}(t)\rangle = \dot{U}(t,t_0)\,|\alpha(t_0)\rangle$$

$$= \dot{U}(t,t_0)\;U(t_0,t)\,|\alpha(t)\rangle$$

$$= \dot{U}(t,t_0)\;U^{+}(t,t_0)\,|\alpha(t)\rangle$$

$$i\hbar\,|\alpha'(t)\rangle = \underbrace{i\hbar\dot{U}(t,t_0)\,U^\dagger(t,t_0)}_{H}\,|\alpha(t)\rangle$$

$$H^\dagger = H$$

$$H = \underbrace{i\hbar\,\dot{U}(t,t_0)\,U^\dagger(t,t_0)}_{\text{Hermitian}}$$

$$\Rightarrow \boxed{U(t,t_0)\,U^\dagger(t,t_0) = I}$$

Take derivative.

$H$

$H^\dagger = H$

$$H = i\hbar\, \dot{U}(t, t_0)\, U^\dagger(t, t_0)$$

Hermitian

$$\Rightarrow \left[ U(t, t_0)\, U^\dagger(t, t_0) = I \right]$$

Take derivative.

$\dot{U}U^\dagger \rightarrow$ antihermitian

$i$

$$\dot{U}U^\dagger + U\dot{U}^\dagger = 0$$

$$\dot{U}U^\dagger = -U\dot{U}^\dagger$$

$$\dot{U}U^\dagger = -(\dot{U}U^\dagger)^\dagger$$

$|\alpha(t)\rangle = U(t, t_0)\,|\alpha(t_0)\rangle$

$U$ follow $\Rightarrow$ $\boxed{i\hbar\dot{U} = HU}$.

Solution of time evolution operator

$i\dot{U}U^{\dagger}$

$|\alpha(t)\rangle = U(t, t_0)\,|\alpha(t_0)\rangle$

$[H(t_0), H(t)] = 0$

$H \quad H(t) \quad H(t) \longrightarrow [H(t'), H(t)] \neq 0$

$U$ follow $\Rightarrow$ $\boxed{i\hbar\dot{U} = HU}$

Solution of time evolution operator

$|\alpha(t)\rangle = U(t, t_0)\ |\alpha(t_0)\rangle$

U follow $\Rightarrow$ $\boxed{i\hbar\dot{U} = HU}$

Solution of time evolution operator

I $\hat{H}$ $H(t)$ $H(t)$ $\rightarrow$ $[H(t), H(t)] = 0$

II $H(t)$

$[H(t_0), H(t)] = 0$

III $H(t)$

$\rightarrow [H(t_0), H(t)] \neq 0$

$B_x(t_0)$ $B_y(t)$

$[S_x, S_y] \neq 0$

operators / matrices

$$U = e^{-iHt/\hbar}$$

$$U = \sum_{n=0}^{\infty} \frac{1}{n!} \left( \frac{-iHt}{\hbar} \right)^n$$

I H time independent

$$i\hbar \dot{U} = HU$$

$U = ?$

H, U are numbers

$$\int i\hbar \frac{dU}{U} = \int H\, dt$$

operators / matrices

H time independent [ ]

$U = e^{iHt/\hbar}$

$$U = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{it}{\hbar} H\right)^n$$

$$\dot{U} = -\frac{iH}{\hbar} \sum_{n=1}^{\infty} \frac{1}{(n-1)!} \left(\frac{itH}{\hbar}\right)^{n-1}$$

rename

II H is time dependent

$$\dot{U} = -\frac{i}{\hbar} H(t)\, U \quad \checkmark$$

$$U = \exp\left\{ -\frac{i}{\hbar} \int_0^t H(t')\, dt' \right\} \rightarrow$$

$$\dot{U} = \sum_{n=1}^{\infty} \frac{1}{n!} \left( \right.$$

II H is time dependent

$$\dot{U} = \frac{-i}{\hbar} H(t)\, U \quad \checkmark$$

$$U = \exp\left\{ \frac{-i}{\hbar} \int_0^t H(t')\, dt' \right\} \rightarrow$$

$$\dot{U} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(\frac{-i}{\hbar}\right)^n (n-1) \left[ \int_0^t H(t')\, dt' \right]^{n-1} \underbrace{\frac{d}{dt} \int_0^t H(t')\, dt'}_{\dot{H}(t)}$$

$$\left[ H(t'), H(t) \right] = 0$$

$$\dot{U} = -i H \dot{H}$$

$$U = \exp\left\{ -\frac{i}{\hbar} \int_0^t H(t')\, dt' \right\}$$

$$\dot{U} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n (n-1) \left[ \int_0^t H(t')\, dt' \right]^{n-1} \underbrace{\frac{d}{dt} \int_0^t H(t')\, dt'}_{\dot{H}(t)}$$

$$\boxed{\dot{U} = -\frac{i}{\hbar} H U} \qquad \dot{U} = -\frac{i}{\hbar} \dot{H} U$$

$$\left[ H(t'), H(t) \right] = 0$$

$$H(t') H(t) = H(t) H(t')$$

Case III $H(t)$ $[H(t), H(t)] \neq 0$ $\frac{d}{dt}[A(t)\,B(t)]$

$$U = \exp\left[\frac{-i}{\hbar}\int_0^t dt' H(t')\right]$$

$$= 1 + \int_0^t dt' H(t') + \frac{1}{2}\int_0^t dt' H(t') \int_0^t dt'' H(t'') + \cdots$$

$$\dot{U} = H(t) + \frac{1}{2} H(t) \int_0^t dt'' H(t'')$$

$$\dot{U} = \frac{-i}{\hbar} H U$$

$$\dot{U} = \frac{-i}{\hbar} H U$$

$$[H(t'), H(t)] = 0$$

$$H(t') H(t) = H(t) H(t')$$

**Case III** $H(t)$ $\quad [H(t'), H(t)] \neq 0$ $\quad \frac{d}{dt}[A(t)\, B(t)]$

$$U = \exp\left[\frac{-i}{\hbar}\int_0^t dt' H(t')\right] \cdot X$$

$$= 1 + \int_0^t dt' H(t') + \frac{1}{2}\int_0^t dt' H(t') \int_0^t dt'' H(t'') + \cdots$$

$$\dot{U} = -\frac{i}{\hbar} H(t) U$$

$U(t_0, t_0) = 1$

integrate

$t = t_0$

$$U(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} H(t') U(t', t_0) \, dt'$$

integrate $t = t_0$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H(t')\,U(t',t_0)\,dt'$$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\,H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\,H(t')\,H(t'') + \cdots$$

$$\cdots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots \int_0^{t^{n-1}} dt^n\,H(t')\,H(t'')\cdots H(\cdots$$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\, H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, H(t')\,H(t'') + \cdots$$

(Time ordering)

$$\cdots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots \int_0^{t^{n-1}} dt^n\, H(t')\,H(t'')\cdots H(t^n)$$

$0 < t' < t$

$0 < t'' < t' < t$

$0 < t^n < t^{n-1} \cdots < t' < t$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt' H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' \underbrace{H(t')H(t'')}_{\text{Time ordered}} + \cdots$$

$$\cdots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots \int_0^{t^{n-1}} dt^n H(t')H(t'')\cdots H(t^n)$$

$$0 < t' < t \qquad 0 < t'' < t' < t$$

$$0 < t^n < t^{n-1} \cdots < t' < t$$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt' H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' H(t') H(t'') + \cdots$$

Time ordering

$$\cdots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt'' \cdots \int_0^{t^{n-1}} dt^n H(t') H(t'') \cdots H(t^n)$$

$n!$

$0 < t' < t$ $\quad 0 < t'' < t' < t$

$< t' < t$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt' H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' H(t') H(t'') + \cdots$$

Time order

$$\cdots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt'' \cdots \int_0^{t^{n-1}} dt^n H(t') H(t'') \cdots H(t^n)$$

$n+1$

$U(t)$

$0 < t' < t$

$0 < t'' < t' < t$

$0 < t^n < t^{n-1} \cdots < t' < t$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H(t')\,U(t',t_0)\,dt'$$

$|\alpha(t)\rangle \to |\alpha(t_0)\rangle$

$H(t)$, $[H(t),H(t')] \neq 0$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\,H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, H(t')\,H(t'') + \cdots$$

Time order

$$i\dot{U} = -\frac{i}{\hbar} H U$$

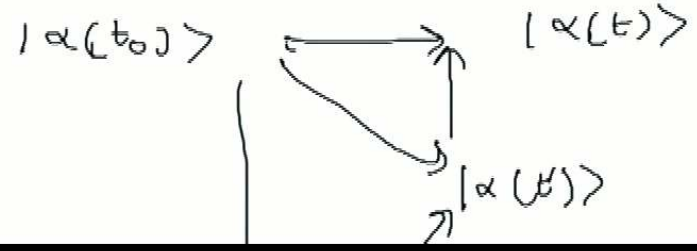
$$\cdots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots \int_0^{t^{n-1}} dt^n\, H(t') \cdots$$

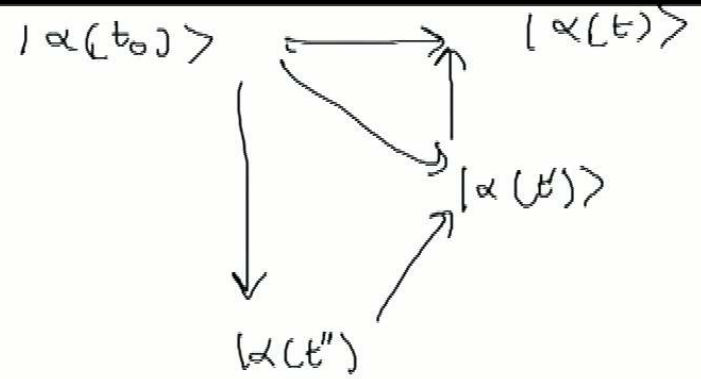
$n+1$

$0 < t' < t$

$0 < t'' < t' < t$

$0 < t^n < t^{n-1} \cdots < t' < t$






positive time ordered exponential (Dyson series).

$|\alpha(t)\rangle \to |\alpha(t_0)\rangle$

$H(t)$, $[H(t), H(t')] \neq 0$

$$U(t, t_0) = 1 - \frac{i}{\hbar}\int_0^t dt' H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' H(t') H(t'') + \cdots$$

Time order

$$i\dot{U} = -\frac{i}{\hbar} H U$$

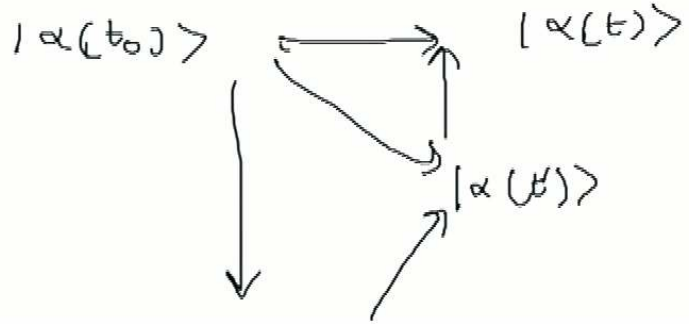
$$\cdots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots \int_0^{t^{n-1}} dt^n H(t') H \cdots H$$

$n+1$

$0 < t' < t$

$0 < t'' < t' < t$

$0 < t^n < t^{n-1} \cdots < t' < t$



positive time ordered exponential (Dyson series).

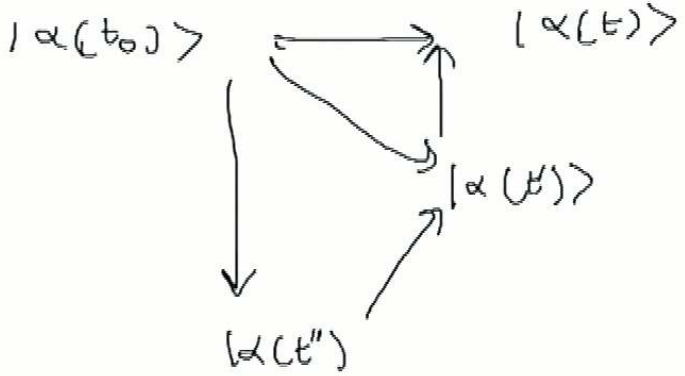
$$\dot{u} = \frac{-i}{\hbar} H u$$

$$\cdots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots \int_0^{t^{n-1}} dt^n \, H(t') H(t'') \cdots H(t^n) \, U(t$$

$$0 < t' < t \qquad 0 < t'' < t' < t$$

$$0 < t^n < t^{n-1} \cdots < t' < t$$

$$\begin{tikzcd}
|\alpha(t_0)\rangle \arrow[rr] \arrow[dr] \arrow[dd] & & |\alpha(t)\rangle \\
& |\alpha(t')\rangle \arrow[ur] & \\
|\alpha(t'')\rangle \arrow[ur] & &
\end{tikzcd}$$



$[H(t'), H(t)] \neq 0$

$\exp(\;\;) \rightarrow [H(t'), H(t)] = 0$

SP H is not time dependent

Observables associated

$\langle A(t_0) \rangle = \langle \alpha(t_0) | A | \alpha(t_0) \rangle$

$\langle A(t) \rangle = \langle \alpha(t) | A | \alpha(t) \rangle$

$= \langle \alpha(t_0) U^{\dagger} A U \alpha(t_0) \rangle$

Approach → $\langle \alpha(t_0) | U^\dagger A U | \alpha(t_0) \rangle$ → HP.

Schrodinger picture

$$|\alpha(t)\rangle = \sum_{a'} C_{a'} |a'\rangle \qquad H|a'\rangle = E_{a'}|a'\rangle$$

$U = ex$

$$U = \exp\left(-i \frac{H t}{\hbar}\right)$$

$$|\alpha(t)\rangle = \sum_{a'} \exp\left(-i \frac{H t}{\hbar}\right) C_{a'} |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_a t/\hbar} \sum_{a'} (a(t_0)\,|a'\rangle$$

$$|\alpha(t_0)\rangle = |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_a t/\hbar} |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_a t/\hbar} \sum_{a'} (a(t_0)|a'\rangle$$

$$|\alpha(t_0)\rangle = |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_{a'}t/\hbar}|a'\rangle$$

Expectation value of observables $|\alpha(t_0)\rangle = |a'\rangle$

$$\langle B \rangle = \langle a' | B | a' \rangle$$

$$= \langle a' e^{+iE_{a'}t/\hbar} | B | e^{-iE_{a'}t/\hbar} | a' \rangle$$

$$= \langle a' | B | a \rangle$$

$$\langle B \rangle = \sum_{a' a''} C_{a'}^{*} C_{a''} \, e^{-i(E_{a''} - E_{a'}) t/\hbar} \, \langle a' | B | a'' \rangle$$

---

Harmonic Oscillator

(b) $|\alpha(t_0)\rangle = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$

$n = 0 \qquad E_0 = \frac{\hbar\omega}{2}$

$n = 1$

$t$ ?

$|\alpha(t)\rangle$ using time evolution operator ($H$ time independent)

(td) $|\alpha(t_0)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$n=0 \qquad E_0 = \frac{\hbar\omega}{2}$

$n=1$

$t$ ?

$|\alpha(t)\rangle$ using time evolution operator ( time independe

$|\alpha(t)\rangle$

at $t$ find $\langle x \rangle_S$ =

Harmonic oscillator

$n=0 \qquad E_0 = \frac{\hbar\omega}{2}$

$n=1$

(t2) $|\alpha(t_0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$t$ ?

$|\alpha(t)\rangle$ using time evolution operator ($H$ time independent)

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left[a + a^\dagger\right]$

$|\alpha(t)\rangle$

at $t$ find $\langle x \rangle_S$ =