Title: Quantum Theory - Lecture 220912

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Collection: Quantum Theory (2022-2023)

Date: September 12, 2022 - 10:45 AM

URL: https://pirsa.org/22090035

Lecture-4 Time Dynamics

Annoucements

Our learning goals:

- Time evolution operator
  - Properties
  - Evolution equation
  - Solution
    - Time independent H
    - Time dependent H (commuting/non commuting)
- Time evolution of statekets
  - Stationary states
- Time evolution of observables

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Time Evolution operator.

Monday, 12 September 2022 8:27 PM

$$i\hbar \frac{\partial}{\partial t} |\alpha\rangle = H|\alpha\rangle$$

$$|\alpha(\quad = |\alpha(t_0)\rangle$$

$|\alpha\rangle \rightarrow$ state ket

$|a'\rangle \rightarrow$ basis ket

$$|\alpha\rangle = \sum_{a'} c_{a'} |a'\rangle$$

BA

1) Unitary

$$\langle \alpha(t_0) | \alpha(t_0) \rangle = 1 = \langle \alpha(t) | \alpha(t) \rangle$$

$$= \langle \alpha(t) \underbrace{U^\dagger \, U}_{I} \alpha(t) \rangle$$

2) Composite

$$| \alpha(t_2) \rangle = U(t_2, t_1 \, | \alpha(t_1) \rangle$$

2) Composite

$$|\alpha(t_2)\rangle = U(t_2,t_1)\,|\alpha(t_1)\rangle$$

$$= \underbrace{U(t_2,t_1)\,U(t_1,t_0)}\,|\alpha(t_0)\rangle$$

$$|\alpha(t_2)\rangle = \underbrace{U(t_2,t_0)}\,|\alpha(t_0)\rangle$$

Time-reversal

$$U(t_2,t_1)\,U(t$$

Time reversal

$$U(t,t_0) = U^{-1}(t_0,t)$$

$$U^{+}U = I$$

$$U^{+}(t_0,t)\,U(t_0,t) = I$$

$$U(t,t_0) = U^{+}(t_0,t)$$

$$U^{-1}(t_0,t) = U^{+}(t_0,t)$$

Relation b/w H & U

$U^{\dagger}(t_0, t)$

$i\hbar|\dot{\alpha}\rangle = H|\alpha\rangle$

$|\alpha(t)\rangle = U(t, t_0)|\alpha(t_0)\rangle$

$|\dot{\alpha}(t)\rangle = \dot{U}(t, t$

$U^{\dagger}(t_0, t)$

Relation b/w H & U

$$i\hbar|\dot{\alpha}\rangle = H|\alpha\rangle$$

$$|\alpha(t)\rangle = U(t, t_0)|\alpha(t_0)\rangle$$

$$|\dot{\alpha}(t)\rangle = \dot{U}(t, t_0)\,|$$

Relation b/w H & U $\qquad i\hbar|\dot{\alpha}\rangle = H|\alpha\rangle$

$$|\alpha(t)\rangle = U(t,t_0)\,|\alpha(t_0)\rangle$$

$$|\dot{\alpha}(t)\rangle = \dot{U}(t,t_0)\,|\alpha(t_0)\rangle$$

$$= \dot{U}(t,t_0)\;U(t_0,t)\,|\alpha(t)\rangle$$

$$= \dot{U}(t,t_0)\;U^{+}(t,t_0)\,|\alpha(t)$$

$$i\hbar|\dot{\alpha}(t)\rangle = \underbrace{i\hbar\dot{U}(t,t_0)\,U^\dagger(t,t_0)}_{H}\,|\alpha(t)\rangle$$

$$H^\dagger = H$$

$$H = \underbrace{i\hbar\ \dot{U}(t,t_0)\,U^\dagger(t,t_0)}_{\text{Hermitian}}$$

$$\Rightarrow \left[\,U(t,t_0)\,U^\dagger(t,t_0) = I\,\right]$$

Take derivative.

$$\underbrace{\qquad\qquad\qquad\qquad}_{H}$$

$$H^\dagger = H$$

$$H = \underbrace{i\hbar \; \dot{U}(t,t_0)\, U^\dagger(t,t_0)}_{\text{Hermitian}}$$

$$\Rightarrow \left[\, U(t,t_0)\, U^\dagger(t,t_0) = I \,\right]$$

Take derivative.

$$\boxed{\begin{aligned} \dot{U}U^\dagger + U\dot{U}^\dagger &= 0 \\ \dot{U}U^\dagger &= -U\dot{U}^\dagger \\ \dot{U}U^\dagger &= -(\dot{U}U^\dagger)^\dagger \end{aligned}}$$

$\dot{U}U^\dagger \rightarrow$ anti-hermitian

i

$$|\alpha(t)\rangle = U(t, t_0)\, |\alpha(t_0)\rangle$$

U follow $\Rightarrow$ $\boxed{i\hbar \dot{U} = H U}$

Solution of time evolution operator

$i \quad \dot{U} \quad U^\dagger$

$[H(t_0'), H(t)] = 0$

$|\alpha(t)\rangle = U(t, t_0) \, |\alpha(t_0)\rangle \qquad H \qquad H(t) \quad H(t) \longrightarrow [H(t_0'), H(t)] \neq 0$

U follow $\Rightarrow$ $\boxed{i\hbar \dot{U} = HU}$

Solution of time evolution operator

$$|\alpha(t)\rangle = U(t, t_0)\,|\alpha(t_0)\rangle$$

I $\hat{H}$ $\quad H(t) \quad H(t) \longrightarrow [H(t'), H(t)] \neq 0$

U follow $\Rightarrow$ $\boxed{i\hbar\dot{U} = HU}$

Solution of time evolution operator

II $H(t)$

$[H(t_0), H(t)] = 0$

III $H(t)$

$[H(t_0), H(t)] \neq 0$

$B_x(t_0) \quad B_y(t)$

$[S_x, S_y] \neq 0$

$$U = e^{-iHt/\hbar}$$

operators / matrices

$$U = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{it}{\hbar} H \right)^{n}$$

I H time independent

$$i\hbar \dot{U} = HU$$

$$U = ?$$

$$U = e^{-iHt/\hbar}$$

$$\checkmark \; U = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{it}{\hbar} H\right)^n$$

$$\dot{U} = -\frac{iH}{} \sum^{\infty} \quad \left(\frac{itH}{}\right)^{n-1}$$

H, U are number

$$\int i\hbar \frac{dU}{U} = \int H \, dt$$

operators / matrices

H time independent [ ]

rename

II H is time dependent

$$\dot{U} = -\frac{i}{\hbar} H(t)\, U$$

$$U = \exp\left\{-\frac{i}{\hbar}\int_0^t H(t')\,dt'\right\} \rightarrow$$

$$\dot{U} = \sum_{n=1}^{\infty} \frac{1}{n!} \Big($$

II $\quad$ H is time dependent

$$\boxed{\dot{U} = -\frac{i}{\hbar} H(t)\, U} \checkmark$$

$$\checkmark\; U = \exp\left\{-\frac{i}{\hbar}\int_0^t H(t')\,dt'\right\} \rightarrow$$

$$\dot{U} = \sum_{n=1}^{\infty} \frac{1}{n!}\left(-\frac{i}{\hbar}\right)^n (n-1) \underbrace{\left[\int_0^t H(t')\,dt'\right]^{n-1} \underbrace{\frac{d}{dt}\int_0^t H(t')\,dt'}_{H(t)}}_{[H(t'),\, H(t)] = 0}$$

$$\dot{U} = -i\, H(t)$$

$$U = \exp\left\{ -\frac{i}{\hbar} \int_0^t H(t')\, dt' \right\}$$

$$\dot{U} = \sum_{n=1}^{\infty} \frac{1}{n!} \left(-\frac{i}{\hbar}\right)^n (n-1) \left[\int_0^t H(t')\, dt' \right]^{n-1} \underbrace{\frac{d}{dt} \int_0^t H(t')\, dt'}_{H(t)}$$

$$\boxed{\dot{U} = -\frac{i}{\hbar} H U} \qquad \dot{U} = -\frac{i}{\hbar} H U$$

$$[H(t'), H(t)] = 0$$

$$H(t') H(t) = H(t) H(t')$$

$\underline{\text{Case III}}$ $\quad H(t) \quad [H(t'), H(t)] \neq 0 \quad \frac{d}{dt}[A(t)\ B(t)]$

$$U = \exp\left[\frac{-i}{\hbar}\int_0^t dt'\, H(t')\right]$$

$$= 1 + \int_0^t dt'\, H(t') + \frac{1}{2}\int_0^t dt'\, H(t')\int_0^t dt''\, H(t'') + \cdots$$

$$\dot{U} = H(t) + \frac{1}{2} H(t) \int_0 dt''\, H(t'')$$

$$\boxed{\dot{U} = -\frac{i}{\hbar} H U} \qquad \boxed{\dot{U} = -\frac{i}{\hbar} H U}$$

$$[H(t'), H(t)] = 0$$

$$H(t')H(t) = H(t)H(t')$$

Case III $\quad H(t) \qquad [H(t'), H(t)] \neq 0 \qquad \frac{d}{dt}[A(t)\,B(t)]$

$$U = \exp\left[-\frac{i}{\hbar}\int_0^t dt'\, H(t')\right] \times$$

$$= 1 + \int_0^t dt'\, H(t') + \frac{1}{2}\int_0^t dt'\, H(t')\int_0^t dt''\, H(t'') + \cdots$$

$$\dot{U} = -\frac{i}{\hbar} H(t) U \qquad U(t_0, t_0) = 1$$

integrate

$$t = t_0$$

$$U(t, t_0) = 1 - \frac{i}{\hbar} \int_{t_0}^{t} H(t') U(t', t_0) \, dt'$$

integrate

$t = t_0$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0}^{t} H(t')\, U(t',t_0)\, dt'$$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\, H(t') + \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, H(t')\, H(t'') + \cdots$$

$$\cdots \left(-\frac{i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots\cdots \int_0^{t^{n-1}} dt^n\, H(t')\, H(t'') \cdots H($$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt' \, H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt'' \, \underbrace{H(t')\,H(t'')}_{\text{Time } o} + \cdots$$

$$\cdots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots\cdots \int_0^{t^{n-1}} dt^n \, H(t') H(t'') \cdots H(t^n)$$

$$0 < t' < t \qquad 0 < t'' < t' < t$$

$$0 < t^n < t^{n-1} \cdots < t' < t$$

h t_0

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\, H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, \underbrace{H(t')\, H(t'')}_{\text{Time o}} + \cdots$$

$$\cdots \left(\frac{-i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} \underbrace{dt'''}_{n'} \cdots\cdots \int_0^{t^{n-1}} dt^n\, H(t')\, H(t'') \cdots H(t^n)$$

$U(t$

$$0 < t' < t \qquad 0 < t'' < t' < t$$

$$0 < t^n < t^{n-1} \cdots < t' < t$$

$$\mathcal{U}(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\, H(t') + \left(-\frac{i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, \underbrace{H(t')\,H(t'')}_{\text{Time o}} + \cdots$$

$$\cdots \left(-\frac{i}{\hbar}\right)^{n-1} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots\cdots \int_0^{t^{n-1}} dt^n\, H(t')\,H(t'') \cdots H(t^n)$$

$T_{n'}$ $\quad$ $U(t$

$0 < t' < t \qquad 0 < t'' < t' < t$

$0 < t^n < t^{n-1} \cdots < t' < t$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\, H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, \underbrace{H(t')\, H(t'')}_{\text{Time order}} + \cdots$$

$$\cdots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots \int_0^{t^{n-1}} dt^n\, H(t')\, H(t'') \cdots H(t^n)$$

t_{n+1} $U(t$

$0 < t' < t$ $\qquad 0 < t'' < t' < t$

$$0 < t^n < t^{n-1} \cdots < t' < t$$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_{t_0} H(t')\, U(t',t_0)\, dt'$$

$H(t)$ $\quad |\alpha(t)\rangle \to |\alpha(t_0)\rangle$

$[H(t), H(t')] \neq 0$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\, H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, \underbrace{H(t')\, H(t'')}_{\text{Time order}} + \; ---$$

$$\dot{U} = \frac{-i}{\hbar} H U$$

$$--- \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \; ---- \int_0^{t^{n-1}} dt^n\, H(t') \; ---$$

t_{n+1}

$0 < t' < t \qquad 0 < t'' < t' < t$

$0 < t^n < t^{n-1} \; --- \; < t' < t$

$$\begin{array}{ccc} |\alpha(t_0)\rangle & \longrightarrow & |\alpha(t)\rangle \\ & \searrow & \uparrow \\ & & |\alpha(t')\rangle \end{array}$$

$$
\begin{tikzcd}
|\alpha(t_0)\rangle \arrow[r] \arrow[dr] \arrow[dd] & |\alpha(t)\rangle \\
 & |\alpha(t')\rangle \arrow[u] \\
|\alpha(t'') \arrow[ur] &
\end{tikzcd}
$$

positive time ordered exponential (Dyson series).

$H(t)$

$[H(t), H(t')] \neq 0$ ✓

$|\alpha(t)\rangle \to |\alpha(t_0)\rangle$

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0^t dt'\, H(t') + \left(\frac{-i}{\hbar}\right)^2 \int_0^t dt' \int_0^{t'} dt''\, \underbrace{H(t')\, H(t'')}_{\text{Time order}} + \cdots$$

$$\dot{U} = \frac{-i}{\hbar} H U$$

$$\cdots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' \cdots \int_0^{t^{n-1}} dt^n\, H(t')H \cdots H$$

$0 < t' < t \qquad 0 < t'' < t' < t$

$0 < t^n < t^{n-1} \cdots < t' < t$

$$\begin{tikzcd} |\alpha(t_0)\rangle \arrow[r] \arrow[dr] \arrow[d] & |\alpha(t)\rangle \\ {} & |\alpha(t')\rangle \arrow[u] \end{tikzcd}$$

positive time ordered exponential (Dyson series).

$$U(t,t_0) = 1 - \frac{i}{\hbar}\int_0 dt'\, H(t') + \left(\frac{-i}{\hbar}\right)\int_0 dt' \int_0 \cdots$$

Time order

$$\dot{U} = -\frac{i}{\hbar} H U$$

$$\cdots \left(\frac{-i}{\hbar}\right)^n \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} \underbrace{dt'''}_{n+1} \cdots \int_0^{t^{n-1}} dt^n\, H(t')H(t'') \cdots H(t^n)\, U(t$$

$0 < t' < t \qquad 0 < t'' < t' < t$

$0 < t^n < t^{n-1} \cdots < t' < t$

$$\begin{tikzcd}
{|\alpha(t_0)\rangle} \arrow[rr] \arrow[drr] \arrow[dd] & & {|\alpha(t)\rangle} \\
& & {|\alpha(t')\rangle} \arrow[u] \\
{|\alpha(t'')\rangle} \arrow[urr] & &
\end{tikzcd}$$

$[H(t'), H(t)] \neq 0$

$\exp(\quad) \to [H(t'), H(t)] = 0$

SP H is not time dependent

Observables associated

$$\langle A(t_0) \rangle = \langle \alpha(t_0) | A | \alpha(t_0) \rangle$$

$$\langle A(t) \rangle = \langle \alpha(t) | A | \alpha(t) \rangle$$

$$= \langle \alpha(t_0) U^\dagger A U \alpha(t_0) \rangle$$

Approach $\rightarrow$ $\langle \alpha(t_0) | \, U^\dagger A U \, | \alpha(t_0) \rangle$ $\rightarrow$ HP.

<u>Schrodinger picture</u>

$$|\alpha(t)\rangle = \sum_{a'} c_{a'} |a'\rangle \qquad H|a'\rangle = E_{a'} |a'\rangle$$

$$U = ex$$

a'

$$U = \exp\left(-i\frac{Ht}{\hbar}\right)$$

$$|\alpha(t)\rangle = \sum_{a'} \exp\left(-i\frac{Ht}{\hbar}\right) c_{a'}\, |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_a t/\hbar} \sum_{a'} C_a(t_0)\, |a'\rangle$$

$$|\alpha(t_0)\rangle = |a'\rangle$$

$$|\alpha(t)\rangle = \left|e^{-iE_a t/\hbar}\right|^{2} \Rightarrow 1 \quad |a'\rangle$$

$$|\alpha(t)\rangle = e^{-iE_a t/\hbar} \sum_{a'} C_a(t_0) \, |a'\rangle$$

$$|\alpha(t_0)\rangle = |a'\rangle$$

$$|\alpha(t)\rangle = \left[e^{-iE_{a'} t/\hbar} \right] |a'\rangle$$

Expectation value of observables $\qquad |\alpha(t_0)\rangle = |a'\rangle$

$$\langle B\rangle = \langle a' | B | a'\rangle$$

$$= \langle a' e^{+iE_{a'}t/\hbar} | B | e^{-iE_{a'}t/\hbar} | a'\rangle$$

$$= \langle a' | B | a'\rangle$$

$$\langle B \rangle = \sum_{a'a''} c_{a'}^{*} c_{a''} e^{-i(E_{a''} - E_{a'})t/\hbar} \langle a' | B | a'' \rangle$$

Harmonic Oscillator

t_0 $|\alpha(t_0)\rangle = \frac{1}{\sqrt{2}} \left(|0\rangle + |1\rangle \right)$

$n=0$ $E_0 = \frac{\hbar\omega}{2}$

$n=1$

t ?

$|\alpha(t)\rangle$ using time evolution operator (H time independent)

[redacted]

t_0 $|\alpha(t_0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$ $n=0$ $E_0 = \frac{\hbar\omega}{2}$

$n=1$

t ?

$|\alpha(t)\rangle$ using time evolution operator (H time independe

$|\alpha(t)\rangle$

at t find $\langle x\rangle_S$ =

Harmonic Oscillator

t_0 $|\alpha(t_0)\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + |1\rangle\right)$

$n=0 \qquad E_0 = \frac{\hbar\omega}{2}$

$n=1$

t ?

$|\alpha(t)\rangle$ using time evolution operator (H time independent)

$\hat{x} = \sqrt{\frac{\hbar}{2m\omega}}\left[a + a^\dagger\right]$

$|\alpha(t)\rangle$

at t find $\langle x\rangle_S$ =