Title: Quantum Theory - Lecture 220909

Speakers: Dan Wohns

Collection: Quantum Theory (2022-2023)

Date: September 09, 2022 - 10:45 AM

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Lecture 3

Friday, 9 September 2022 3:34 PM

Our learning goals:

- Announcements (visit portal: check points/feedback)
- Revisiting some questions from previous lecture
- Find the different number of ways a given occupation number configuration can be achieved
- Maximizing it
- Finding the unknown parameters
- Analyzing the results (may be upload video???)

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$\Rightarrow$ anticrossings

$$E_n = E_n^\circ + \langle n^\circ | V | n^\circ \rangle + \sum_{k \neq n} \frac{\langle n^\circ | V | k^\circ \rangle \langle k^\circ | V | n^\circ \rangle}{E_n^\circ - E_k^\circ} + \sum \frac{\quad}{(\quad)^2}$$

atom $\quad V \Rightarrow \underbrace{d}_{\text{dipole}} \cdot \mathcal{E} \rightarrow$ Strength of E.f

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$\Rightarrow$ anticrossings

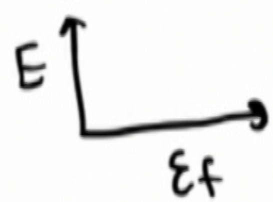
$$E_n = E_n^0 + \underbrace{\langle n^0|V|n^0\rangle} + \sum_{k\neq n} \frac{\langle n^0|\overset{d}{\hat{V}}|k^0\rangle\langle k^0|\overset{d}{\hat{V}}|n^0\rangle}{E_n^0 - E_k^0}\varepsilon + \sum \frac{\quad}{(\quad)^2}$$

atom $V \rightarrow d\cdot\varepsilon$

neutral $\rightarrow$ first order $\rightarrow$ permanent

$— j \rightarrow \dfrac{\quad}{E_j - E_i}$

$— i \rightarrow \dfrac{\quad}{E_i - E_j}$



BA

$$E_n = E_n^0 + \underbrace{\langle n^0|V|n^0\rangle} + \sum_{k \neq n} \frac{\langle n^0|\hat{V}|k^0\rangle\langle k^0|V|n^0\rangle}{E_n^0 - E_k^0}\mathcal{E}^2 + \sum \frac{\quad}{(\quad)^2}$$

atom $\quad V \to d\cdot\mathcal{E}$

neutral $\to$ first order $\to$ permanent

$— \; j \to \dfrac{\quad}{E_j - E_i}$

$— \; i \to \dfrac{\quad}{E_i - E_j}$

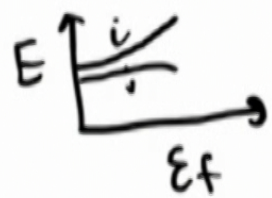
$\mathcal{E}_f$

$— \; i \Rightarrow \div$

$$E_n = E_n^0 + \underbrace{\langle n^0|V|n^0\rangle} + \sum_{k\neq n} \frac{\langle n^0|V|k^0\rangle\langle k^0|V|n^0\rangle}{E_n^0 - E_k^0}\,\varepsilon^2 + \sum \frac{\quad}{(\quad)^2}$$

atom $\quad V \rightarrow d\cdot\varepsilon$

neutral $\rightarrow$ first order $\rightarrow$ permanent

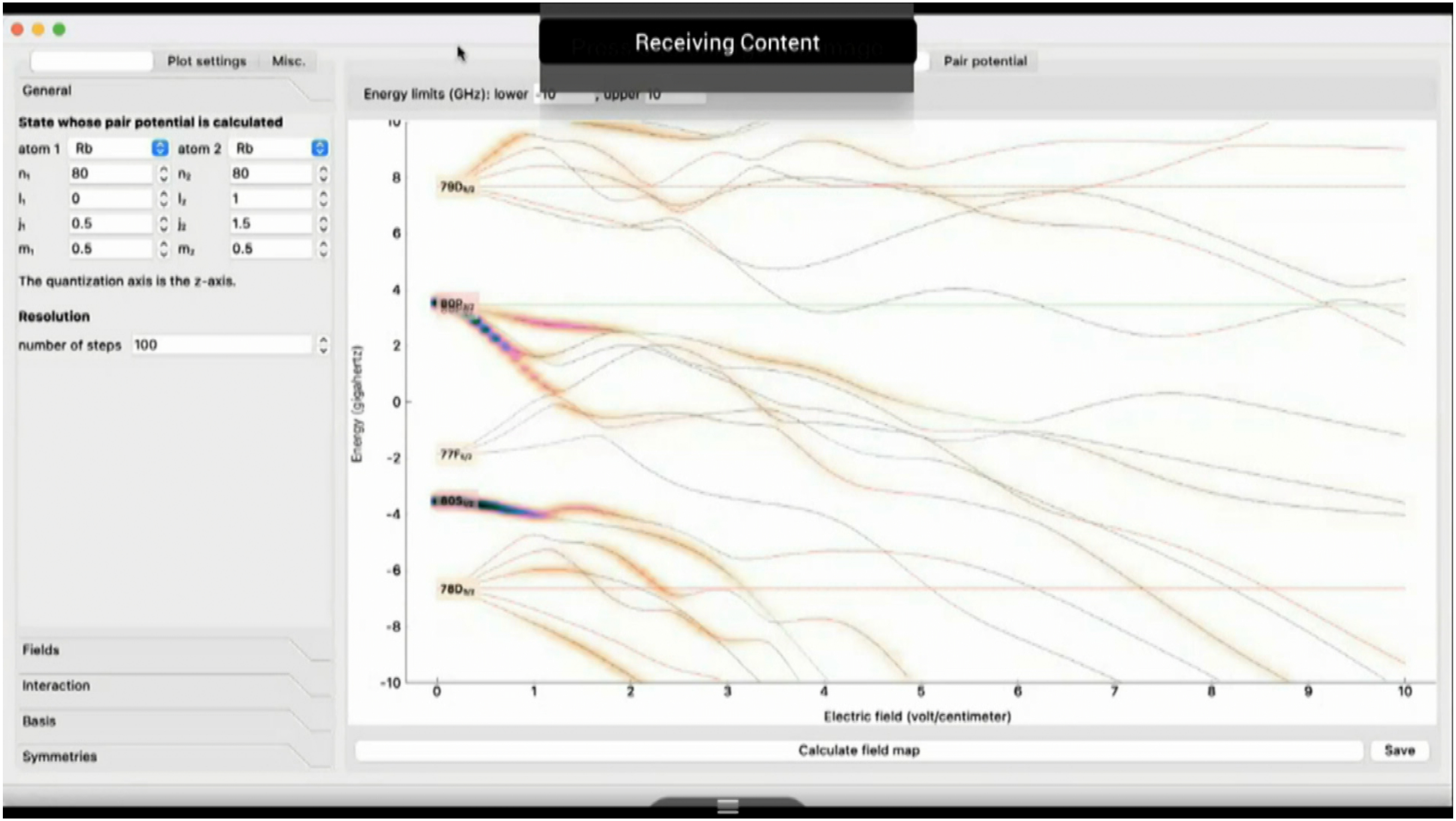


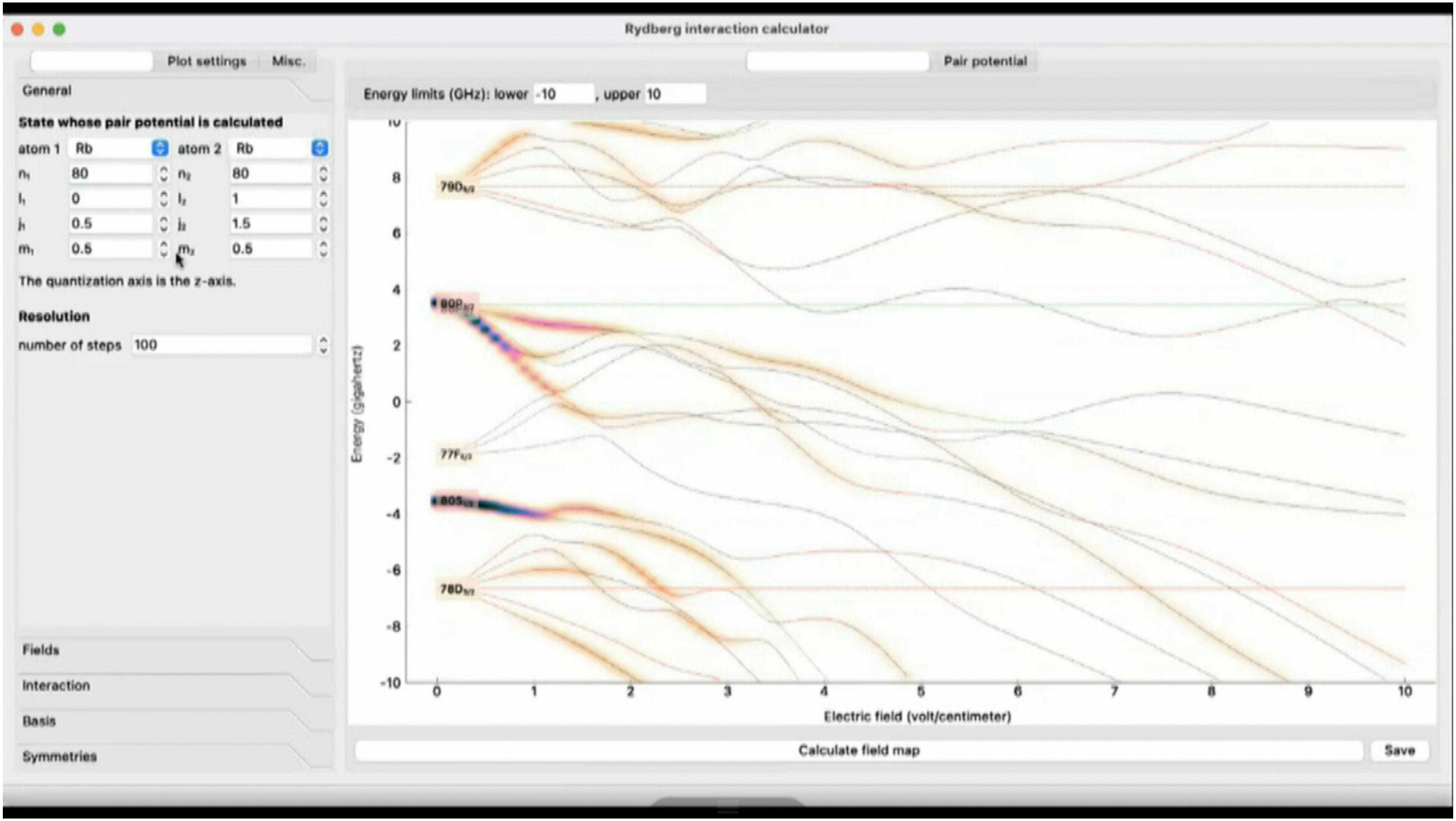
$— j \rightarrow \dfrac{}{E_j - E_i}$

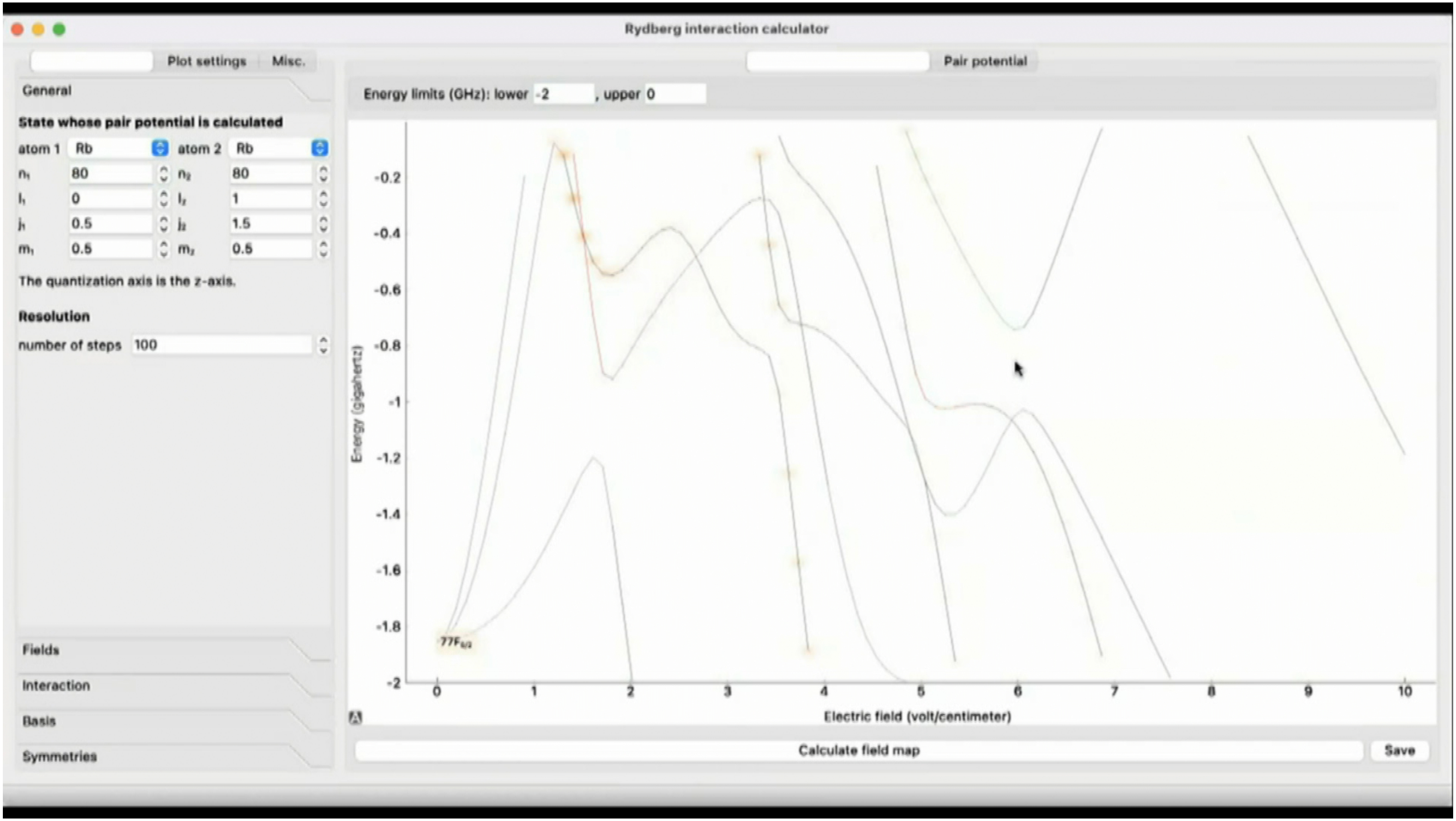
$— i \rightarrow \dfrac{}{E_i - E_j}$

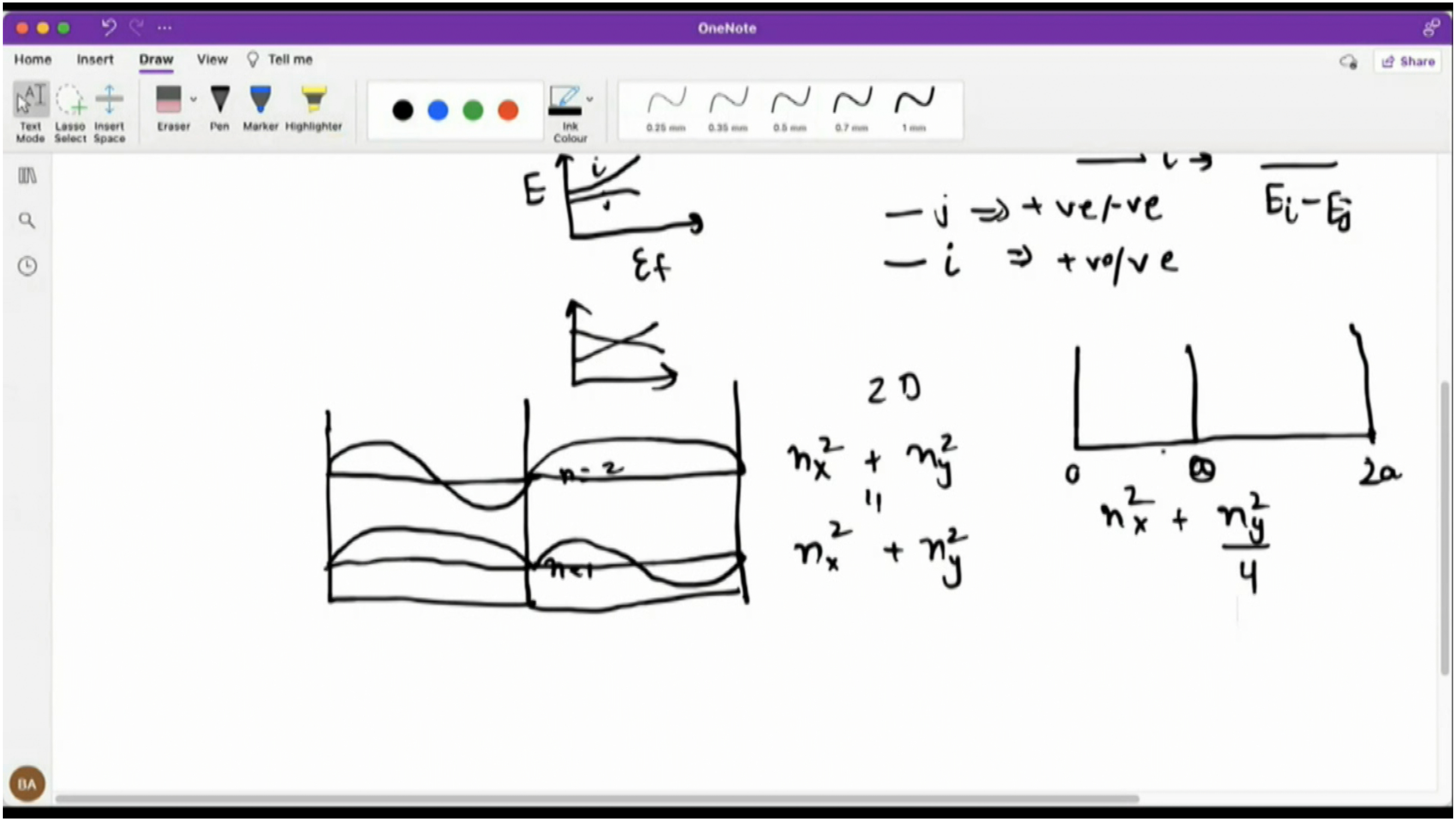
$— j \Rightarrow$ +ve/-ve

$— i \Rightarrow$ +ve/-ve


Receiving Content
Plot settings
Misc.
Pair potential
General
State whose pair potential is calculated
atom 1 Rb atom 2 Rb
n₁ 80 n₂ 80
l₁ 0 l₂ 1
j₁ 0.5 j₂ 1.5
m₁ 0.5 m₂ 0.5
The quantization axis is the z-axis.
Resolution
number of steps 100
Fields
Interaction
Basis
Symmetries
Energy limits (GHz): lower -10 , upper 10
Energy (gigahertz)
Electric field (volt/centimeter)
Calculate field map
Save


Rydberg interaction calculator
Plot settings
Misc.
Pair potential
General
State whose pair potential is calculated
atom 1 Rb
atom 2 Rb
80
80
0
1
0.5
1.5
0.5
0.5
The quantization axis is the z-axis.
Resolution
number of steps 100
Fields
Interaction
Basis
Symmetries
Energy limits (GHz): lower -10 , upper 10
Energy (gigahertz)
Electric field (volt/centimeter)
Calculate field map
Save


Rydberg interaction calculator
Plot settings
Misc.
Pair potential
General
Energy limits (GHz): lower -2 , upper 0
State whose pair potential is calculated
atom 1 Rb atom 2 Rb
n1 80 n2 80
l1 0 l2 1
j1 0.5 j2 1.5
m1 0.5 m2 0.5
The quantization axis is the z-axis.
Resolution
number of steps 100
Fields
Interaction
Basis
Symmetries
Energy (gigahertz)
Electric field (volt/centimeter)
Calculate field map
Save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i
Ef
$-j \Rightarrow +ve/-ve$
$-i \Rightarrow +ve/ve$
$E_i - E_j$
2D
n=2
n=1
$n_x^2 + n_y^2$
$n_x^2 + n_y^2$
0
2a
$n_x^2 + \frac{n_y^2}{4}$
BA

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$n_x^2 + n_y^2$ $\qquad$ $n_x + \frac{n_y}{4}$

$$\Rightarrow \quad V = \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 0 & b \\ 0 & 0 & a \\ a^* & b^* & 0 \end{pmatrix} \end{array}$$

$$\underbrace{E_1 \quad E_1}_{\text{degenerate}} \quad E_2$$

$$\underline{\langle 1 | V | 2 \rangle}$$

$<$

BA

$$V = \begin{pmatrix} 0 & 0 & a \\ a^* & b^* & 0 \end{pmatrix}$$

degenerate

$\underline{\langle 1 | V | 2 \rangle}$

$\langle 1 | V | 3 \rangle \langle 3 | V | 1 \rangle$

$$V = V_0 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow$$

formal development of Perturbation Exp.

$$\underbrace{(E_n^{(0)} - H_0)|n\rangle} = \underbrace{(\lambda V - \Delta_n)|n\rangle}$$

$$|n'\rangle \rightarrow |n^0\rangle$$

$$\phi_n \rightarrow \sum_{k \neq n} |k^0\rangle\langle k^0|$$

$$|n'\rangle \to |n^0\rangle$$

✓ $$\phi_n \to \sum_{k \neq n} |k^0\rangle\langle k^0|$$

$$|n\rangle = \frac{1}{E_n^0 - H_0} (\lambda V - \Delta_n)|n\rangle$$

✓ $$|n\rangle = \sum_k c_k |n^0\rangle$$

$$|n\rangle = \frac{1}{E_n^0 - H_0}\left(\lambda V - \Delta_n\right)|n\rangle \longrightarrow (A)$$

✓ $$|n\rangle = \sum_k c_k |n^0\rangle$$

$$\sum_k |k^0\rangle\langle k^0| = I$$

$$|n\rangle = \sum_k \frac{1}{E_n^0 - H_0} |k^0\rangle\langle k^0| \lambda V - \Delta_n |n\rangle$$

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$E_n - H_0$

✓ $|n\rangle = \sum_k c_k |n^0\rangle$

$$\sum_k |k^0\rangle\langle k^0| = I$$

$$|n\rangle = \sum_k \frac{1}{E_n - H^0} |k^0\rangle\langle k^0|\lambda V - \Delta_n|n\rangle$$

$$= \frac{1}{E_n - H^0} |n^0\rangle \underbrace{\langle n^0|\lambda V - \Delta_n|n\rangle}_{0} + \sum_{k\neq n} \frac{1}{\quad} |k^0\rangle\langle \ldots$$

$|n'\rangle$

OneNote

Tutorial1Q1

Wednesday, 7 September 2022 9:59 PM

$$\boxed{(E^0 - H_0)|l^2\rangle = V|l'\rangle - E^2|l^0\rangle \; + E'|l'\rangle} \rightarrow \text{second order.}$$

$$\langle m^0|\cancel{E^0 - H_0}|l^2\rangle$$ taking inner product with $|m^0\rangle$

$$\overset{\psi^0}{|l^0\rangle} = \alpha\,\overset{\psi_a^0}{|m^0\rangle} + \beta\,\overset{\psi_b^0}{|n^0\rangle}$$

$$\boxed{\langle m^0|V|l'\rangle = E^2\langle m^0|l^0\rangle}$$

$$|l^0\rangle = \alpha$$

$$\sum_{k^0 \neq m^0, n^0} \langle m^0|V|k^0\rangle \frac{\langle k^0|V|l^0\rangle}{E^0 - E_k^0} = E^2\alpha$$

Tutorial1Q1

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$$\boxed{(E^0 - H_0)|l^2\rangle = V|l'\rangle + E'|l'\rangle - E^2|l^0\rangle} \rightarrow \text{second order.}$$

taking inner product with $|m^0\rangle$

$\langle m^0|E^0 - H_0|l^2\rangle$

E^0

$$|l^0\rangle = \alpha|m^0\rangle + \beta|n^0\rangle$$

$\psi^0 \quad \psi_a^0 \quad \psi_b^0$

$|l^0\rangle = \alpha$

$$\boxed{\underbrace{\langle m^0|V|l'\rangle} = E^2\langle m^0|l^0\rangle}$$

$$\sum_{k^0 \neq m^0, n^0} \langle m^0|V|k^0\rangle \frac{\langle k^0|V|l^0\rangle}{E^0 - E_k^0} = E^2\alpha$$

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$$|\ell^1\rangle = \sum c_k |k^0\rangle$$

$$E^1 |\ell^1\rangle \to 0$$

$$\boxed{(E^0 - H_0)|\ell^2\rangle = V|\ell^1\rangle - E^2|\ell^0\rangle} \to \text{second order.}$$

taking inner product with $|m^0\rangle$

$$\langle m^0 | \cancel{E^0 - H_0} | \ell^2\rangle$$

$$\psi^0 \qquad \psi_a^0 \qquad \psi_b^0$$

$$|\ell^0\rangle = \alpha |m^0\rangle + \beta |n^0\rangle$$

$$|\ell^0\rangle = \alpha$$

$$\boxed{\langle m^0 | V | \ell^1\rangle = E^2 \langle m^0 | \ell^0\rangle}$$

$$\sum_{k^0 \neq m^0, n^0} \langle m^0 | V | k^0\rangle \frac{\langle k^0 | V | \ell^0\rangle}{E^0 - E_k^0} = E^2 \alpha$$

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Tutorial1Q1

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$E^1 |l^1\rangle \to 0$

$|l^1\rangle = \sum_{k \notin D} c_k |k^0\rangle$

$k \notin D \to m^0, n^0$

$$\boxed{(E^0 - H_0)|l^2\rangle = V|l^1\rangle - E^2|l^0\rangle} \to \text{second order.}$$

taking inner product with $|m^0\rangle$

$\langle m^0 | E^0 - H_0 | l^2\rangle$, $H_0 \to E^0$

ψ^0

$|l^0\rangle = \alpha |m^0\rangle + \beta |n^0\rangle$, ψ_a^0, ψ_b^0

$\langle l^0 | = \alpha$

$$\boxed{\underbrace{\langle m^0 | V | l^1\rangle} = E^2 \langle m^0 | l^0\rangle}$$

$$\sum_{k^0 \neq m^0, n^0} \langle m^0 | V | k^0\rangle \frac{\langle k^0 | V | l^0\rangle}{E^0 - E_k^0} = E^2 \alpha$$

$\cancel{\langle m^0 | E^0 - H_0 | \psi^2 \rangle}$ E^0

taking inner product with $|m^0\rangle$

ψ^0 ψ_a ψ_b

$$|\ell^0\rangle = \alpha |m^0\rangle + \beta |n^0\rangle$$

$$\langle m^0 | E' | \ell^0 \rangle = E'$$

$$\boxed{\langle m^0 | V | \ell' \rangle = E^2 \langle m^0 | \ell^0 \rangle}$$

$$|\ell^0\rangle = \alpha$$

$$\sum_{k^0 \neq m^0, n^0} \langle m^0 | V | k^0 \rangle \frac{\langle k^0 | V | \ell^0 \rangle}{E^0 - E_k^0} = E^2 \alpha$$

$$\sum_{k^0 \neq D} \frac{\langle m^0 | V | k^0 \rangle \langle k^0 | V | [\alpha |m^0\rangle + \beta |n^0\rangle]}{E^0 - E_k^0} = \alpha E^2$$

$$\alpha W_{aa} + \beta W_{ab} = \alpha E'$$

$$\alpha W_{ab} + \beta W$$

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$$\boxed{\langle m^\circ | V | l' \rangle = E^2 \langle m^\circ | l^\circ \rangle}$$

$|l^\circ\rangle = \alpha$

$= E'$

$|\psi^\circ\rangle = \alpha|\psi_a^\circ\rangle + \beta|\psi_b^\circ\rangle$

$$\sum_{k^\circ \neq m^\circ, n^\circ} \langle m^\circ | V | \frac{k^\circ\rangle \langle k^\circ | V | l^\circ \rangle}{E^\circ - E_k^\circ} = E^2 \alpha$$

$$\sum_{k^\circ \neq D} \frac{\langle m^\circ | V | k^\circ \rangle \langle k^\circ | V | [\alpha | m^\circ \rangle + \beta | n^\circ \rangle]}{E^\circ - E_k^\circ} = \alpha E^2$$

$\alpha W_{aa} + \beta W_{ab} = \alpha E'$

$\alpha W_{ab} + \beta W$

$$\sum_{k^\circ \neq D} \alpha \frac{\langle m^\circ | V | k^\circ \rangle \langle k^\circ | V | m^\circ \rangle}{E^\circ - E_k^\circ} + \sum_{k \neq D} \frac{\beta \langle m^\circ | V | k^\circ \rangle \langle k^\circ | V | n^\circ \rangle}{E_\circ - E_k^\circ} = \alpha E^2$$

BA

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$$\sum_{k^0\neq D} \frac{\langle m^0|V|k^0\rangle\langle k^0|V|\left[\alpha|m^0\rangle+\beta|n^0\rangle\right]}{E^0-E_k^0} = \alpha E$$

$$\alpha W_{aa} + \beta W_{ab} = \alpha E'$$

$$\alpha W_{ab} + \beta W$$

$$\sum_{k^0\neq D} \alpha\frac{\langle m^0|V|k^0\rangle\langle k^0|V|m^0\rangle}{E^0-E_k^0} + \sum_{k\neq D} \beta\frac{\langle m^0|V|k^0\rangle\langle k^0|V|n^0\rangle}{E_0-E_k^0} = \alpha E^2$$

$$W_{ij} = \sum_{k^0\neq D} \frac{\langle i^0|V|k^0\rangle\langle k^0|V|j^0\rangle}{E^0-E_k^0}$$

$$\alpha W_{mm} + \beta W_{mn} = \alpha E^2$$

$$\beta W_{nn} + \alpha W_{nm} = \beta E^{2}$$

$$\begin{bmatrix} W_{mm} & W_{mn} \\ W_{nm} & W_{nn} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = E^{2} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}$$

where $$W_{ij} = \sum_{k^0 \neq m^0, n^0} \frac{\langle i^0 | V | k^0 \rangle \langle k^0 | V | j^0 \rangle}{(E^0 - E^0_k)}$$

m^0, n^0 are two degenerate states

ϕ_m

one deg

$$\langle m^0 | (E^1) | l' \rangle \qquad |l'\rangle = \frac{\sum c_k |k^0\rangle}{k \notin D}$$

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$$\beta W_{nn} + \alpha W_{nm} = \beta E$$

$$\begin{bmatrix} W_{mm} & W_{mn} \\ W_{nm} & W_{nn} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = E^2 \begin{bmatrix} \alpha \\ \beta \end{bmatrix} \qquad \begin{bmatrix} \alpha' \\ \beta' \end{bmatrix}$$

$|\psi'\rangle \times$

$|\psi'\rangle$ $|\psi^0\rangle$

where $$W_{ij} = \sum_{k^0 \neq m^0 n^0} \frac{\langle i^0 | V | k^0 \rangle \langle k^0 | V | j^0 \rangle}{E^0 - E_k^0}$$

are two degenerate states

ϕ_m

one deg

$\langle m^0 | E' | l' \rangle$

$$|l'\rangle = \sum_{k \neq D} c_k |k^0\rangle$$

QSM

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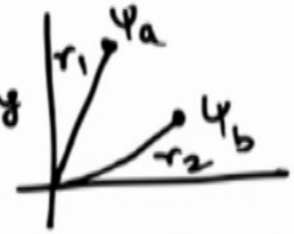
$$\Psi(r_1, r_2) = \psi_a(r_1)\,\psi_b(r_2) \rightarrow \text{disting}$$



{not interacting)

$$\Psi(r_1, r_2)_{B.} = \psi_a(r_1)\,\psi_b(r_2) \pm \psi_a(r_2)\,\psi_b(r_1)$$

{not interacting)

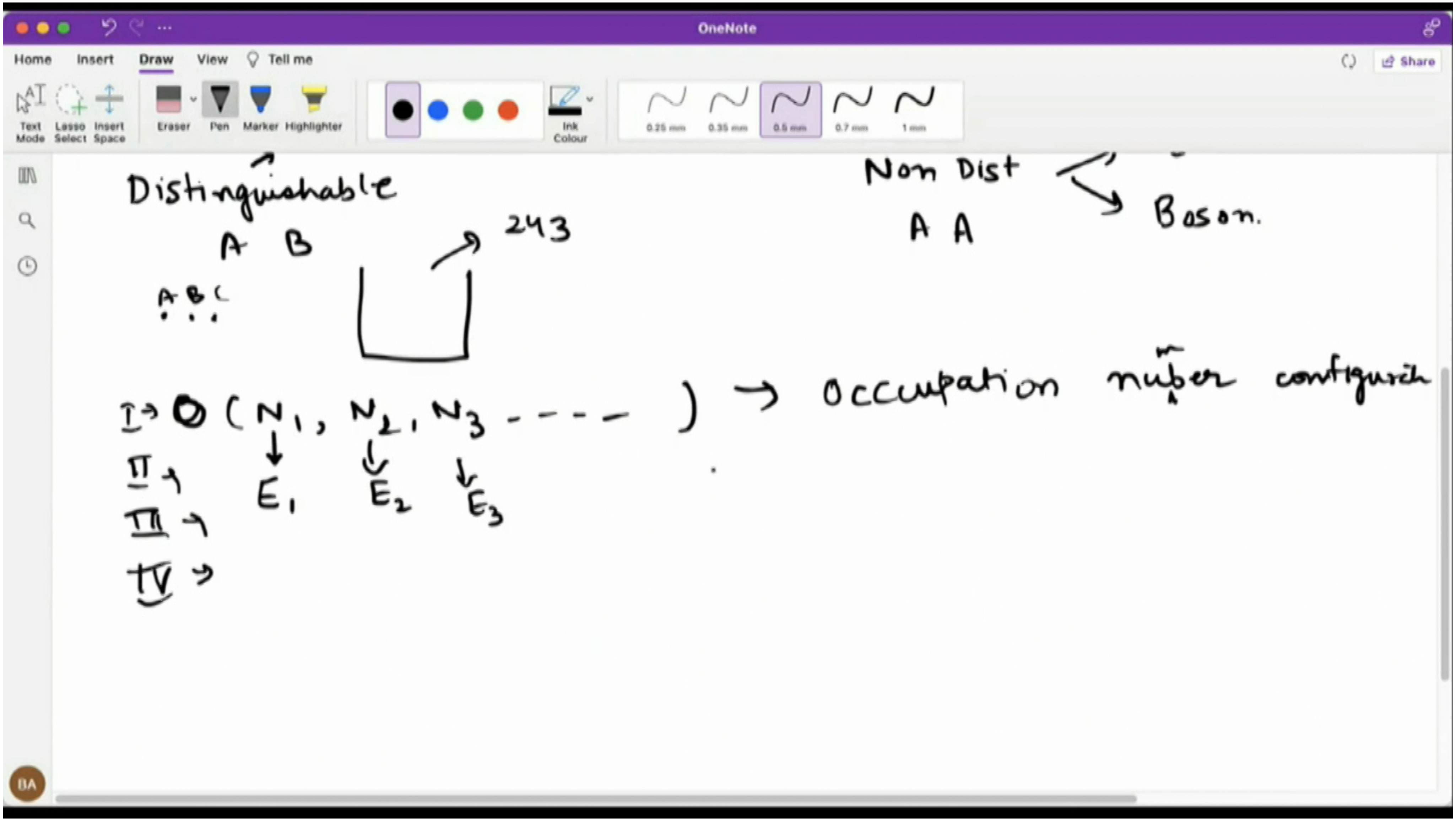
$$\Psi(r_1, r_2)_{\substack{B\\F}} = \psi_a(r_1)\psi_b(r_2) \pm \psi_a(r_2)\psi_b(r_1)$$

Distinguishable

A B

Non Dist → Fer, Boson.

A A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Distinguishable
A B
243
A B C
Non Dist
A A
Boson.
I → (N1, N2, N3 ----) → Occupation number configurati
II →
III →
IV →
E1 E2 E3
BA

A B C

I → $\bigcirc \, (N_1, N_2, N_3 \; - - - -)$ → Occupation number configuration

$$\begin{array}{ccc} n=3 & n=5 & n=7 \\ (3, & 5, & 7) \end{array}$$

II → $E_1 \quad E_2 \quad E_3$

III → d_1

IV →

$$\begin{array}{ccccccccc} (0, & 0, & 1, & 0, & 1, & 0, & 1, & 0 & - - - -) \\ E_1 & E_2 & E_3 & - & - & - & & & \end{array}$$

II → E_1 E_2 E_3 $(3, 5, 7)$

III → $\underline{E_1}$
d_1

IV →

$$(0, 0, 1, 0, 1, 0, 1, 0 - - - - -)$$
E_1 E_2 E_3 - - -

N no. of particles

$$(N_1, N_2, N_3, - - - - - -)$$

I → $(N_1, N_2, N_3 \text{ ----- })$ → Occupation number configuration

$$\begin{array}{l} N_1 \downarrow \quad N_2 \downarrow \quad N_3 \downarrow \\ \dfrac{E_1}{d_1} \quad E_2 \quad E_3 \end{array}$$

II →

III →

IV →

$$\begin{array}{ccc} n=3 & n=5 & n=7 \\ (3, & 5, & 7) \end{array}$$

$$(0,0;\ 0,1,0,1,0,1,\ 0 \text{ ----- })$$
$E_1 \ E_2 \ E_3$ - - - -

N no. of particles

$$(N_1, N_2, N_3, \text{ ------ })$$

$$\begin{array}{ccc} E_1, & E_2, & E_3 \\ d_1 & d_2 & d_3 \end{array}$$

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Distinguishable

A B → 243

A B C

A A → Boson.

Occupation number configuration

I → (0, 0, 0, 0, 0, 0, 0, 0, 3, 0, - - -)

II → (

III → (

IV → (

(1,1,1,1)

$n=3$ $n=5$ $n=7$
(3, 5, 7)

(3, 3, 15)

(5, 7, 13) → max combination ✓

(0, 0, 0, 1, 0, 1, 0, 1, 0 - - - - -)

E_0 E_1 E_2 E_3 . . . .

N no. of particles

OneNote

I → $(0,0,0,0,0,0,0,0,3,0,\ldots)$ → Occupation number configuration

II → (→ $(1,1,1)$ $\begin{array}{ccc} n=3 & n=5 & n=7 \\ (3, & 5, & 7) \end{array}$

III → (→ $(3,3,15)$

IV → (→ $(5,7,13)$ → max combination ✓

$$\begin{array}{ccccccccccc} (0, & 0, & 0, & 1, & 0, & 1, & 0, & 1, & 0 & \ldots &) \\ E_0 & E_1 & E_2 & E_3 & \ldots \end{array}$$

N no. of particles

$$\begin{array}{ccccc} (N_1, & N_2, & N_3, & \ldots &) \\ E_1, & E_2, & E_3 \end{array}$$

I → $Q(N_1, N_2, N_3, \ldots)$ → No. of ways I can make $(N_1, N_2, N_3, \ldots)$

II → Maximize Q

I. Distinguishable Particles.

Build $(N_1, N_2, N_3 \dots)$ for N particles.
2

A B C D E

$N = 5 \quad , \quad N_1 = 2$

$5 \times (5-1)$

$N(N-1)(N-2) \dots ($

$N = 5 \quad , \quad N_1 = 2$

$5 \times (5-1)$

$$\frac{N(N-1)(N-2) \cdots (N-(N_1-1))}{N_1!}$$

1 2 3 4 5

12 21 31 41

13

14

15

$2!$

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II → Maximize Q

I. Distinguishable Particles.

Build $(N_1, N_2, N_3 \ldots)$ for N particles.

$N_1 = 2$

A B C D E

$N = 5$, $N_1 = 2$

$5 \times (5-1)$

$$\frac{N(N-1)(N-2)\ldots(N-(N_1-1))(N-N_1)(N-N_1-1)\ldots 1}{N_1!\,(N-N_1)(N-N_1-1)\ldots 1}$$

$N_1!\;(N-N_1)!$

$E_1 \quad d_1 = 2$

$(N_1, N_2, \; - - - \;)$

$\downarrow$

d_1

$N_1 = 3$

$1, 2, 3$

$(d_1)^{N_1}$

$Q(N_1, \cancel{N_2}$

$\downarrow$

$(N-N_1)!\; d_2$

✓

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$N_1!\,(N-N_1)!$

$E_1 \quad d_1 = 2$

$(N_1, N_2, \ldots)$

$N_1 \to d_1$, $N_2 \to d_2$

$N_1 = 3$

$1, 2, 3$

$\frac{\cdot}{} \times \frac{\cdot}{}$

$(d_1)^{N_1}$

$N_1 \quad d_1$

$Q(N_1, N_2, \ldots$

$$\frac{N!\,(d_1)^{N_1}}{N_1!\,(N-N_1)!} \cdot \frac{(N-N_1)!\,(d_2)^{N_2}}{N_2!\,(N-N_1-N_2)!} \cdots$$

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$N_1!\,(N-N_1)!$

$E_1 \quad d_1 = 2$

$(N_1, N_2, \ldots)$

$N_1 \to d_1 \qquad N_2 \to d_2$

$N_1 = 3$

$1, 2, 3$

$\frac{\cdot}{} \times \frac{\cdot}{}$

$(d_1)^{N_1}$

$N_1 \quad d_1$

$$Q(N_1, N_2, N_3 \ldots) = \frac{N!\,(d_1)^{N_1}}{N_1!\,(N-N_1)!} \cdot \frac{(N-N_1)!\,(d_2)^{N_2}}{N_2!\,(N-N_1-N_2)!} \cdots$$

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$N_1! \, (N-N_1)!$

$E_1 \quad d_1 = 2$

$, N_2 , \; - \; - -)$

$N_1 = 3$

d_2

$\frac{\cdot}{} \times \frac{\cdot}{}$

$1, 2, 3$

N_3

$(d_1)^{N_1}$

d_1

$, N_3) - - -) \; \frac{N! \, (d_1)^{N_1}}{N_1! \, (\cancel{N-N_1})!} \cdot \frac{(\cancel{N-N_1})! \, (d_2)^{N_2}}{N_2! \, (N - \cancel{N_1} - N_2)!} \cdot \frac{(N - \cancel{N_1 - N_2})! \, (d_3)^{N}}{N_3! \, (N - \cancel{N_1} - N_1 -}$

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$$Q(N_1, N_2, N_3 \ldots) = \frac{N!\,(d_1)^{N_1}(d_2)^{N_2}\ldots}{N_1!\,N_2!\ldots\ldots}$$

$= N$

$$Q(N_1, N_2, N_3 \ldots) = \frac{N!\,(d_1)^{N_1}(d_2)^{N_2}\ldots}{N_1!\,N_2!\ldots}$$

Distinguishable particles

$$\boxed{Q(N_1, N_2, N_3 \ldots) = N! \prod_n \frac{(d_n)^{N_n}}{N_n!}}$$

13	23	32	42	52
14	24	34	43	53
15	25	35	45	54
...	...	...	...	...

$N(N-1)$ choices

↓ if 1 picked first

↓ if 2 picked first

However, we do not care in which order the particles are picked

12	21	31	41	51
13	23	32	42	52
14	24	34	43	53
15	25	35	45	54

the pairs (12, 21); (13, 31); (23, 32); ... each represent only one state.

So we need to divide by $N_1!$ (by 2 in this case)

To convince that it is true in general case, consider $N_1 = 3$

The same configuration can be build in 3! ways

$N_1! \; N_2! \; \_ \; \_\_\_$

Distinguishable particles

$$\boxed{Q(N_1, N_2, N_3 \ldots) = N! \prod_n \frac{(d_n)^{N_n}}{N_n!}}$$

Fermions

$N_1!\; N_2! \text{ --- ---}$

Distinguishable particles

$$\boxed{Q(N_1, N_2, N_3 \ldots) = N! \prod_n \frac{(d_n)^{N_n}}{N_n!}}$$

Fermions

→ indistinguishable (spinless)

→ only one particle can occupy one Energy State

$N_1!\; N_2! \;\_\_\; \cdots$

Distinguishable particles

$$\boxed{Q(N_1, N_2, N_3 \ldots) = N! \prod_n \frac{(d_n)^{N_n}}{N_n!}}$$

Fermions $\qquad \underline{\underline{N}} \qquad \underbrace{A\ \ A\ \ A}\ \ A\ \ A \quad - \ - \ -$

$N_1 \qquad\qquad\qquad\qquad\qquad\qquad N.$

$1 \qquad\qquad\qquad \_\ \times\ \_\ \times\ \_$

$1.$

Fermions

$\underline{N}$

$\underbrace{A \quad A \quad A}$ $A \quad A \quad - \quad - \; -$

N_1

$N_1 = 3$

1

$\_ \times \_ \times \_$

1.

$3 \times 2 \times 1$

N ..

$$N_1 \rightarrow d_1$$

$$\frac{d_1 \cdot (d_1-1)(d_1-2) \;----\; (d_1-(N_1-1)) \times (\qquad)}{N_1! \qquad (\qquad\qquad)}$$

$$N_1 \rightarrow d_1$$

$$\Rightarrow \frac{d_1!}{N \cdot}$$

$N_1 \to d_1$

$$\frac{d_1 \cdot (d_1-1)(d_1-2) \cdots (d_1-(N_1-1)) \times (\quad)}{N_1! \quad (\quad)}$$

N

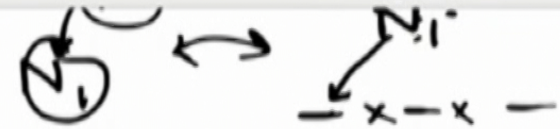
N_1

$N_1!$

— x — x —

$N_1 \to d_1$

$$\Rightarrow \frac{d_1!}{N_1!\,(d_1-N_1)!}$$



$N_1 \to d_1$

$$\Rightarrow \frac{d_1!}{N_1!\,(d_1-N_1)!} \cdot \frac{d_2!}{N_2!\,(d_2-N_1)!} \quad - - - -$$

$$Q\,(N_1, N_2, N_3 - - -) =$$

$$\frac{\quad}{N_1!\,(d_1-N_1)!}\cdot\frac{\quad}{N_2!\,(d_2-N_1)!}$$

$$Q(N_1, N_2, N_3 \ldots) = \prod_{n=1}^{\infty} \frac{d_n!}{N_n!\,(d_n-N_n)!}$$

$$d_1 \times d_1 \times d_1 - \cdots = (d_1)^{N_1}$$

Thus the number of ways to put N_1 particles, selected from a total population of N, into a bin containing d_1 distinct options is

$$\frac{N!\,(d_1)^{N_1}}{N_1!\,(N-N_1)!}$$

The same goes for bin 2 except now only $(N-N_1)$ particles are left to choose N_2 particles and fill in bin 2 containing d_2 distinct options:

$$\frac{(N-N_1)!\,(d_2)^{N_2}}{N_2!\,(N-N_1-N_2)!}$$

So

$$Q(N_1, N_2, \ldots) = \frac{N!\,(d_1)^{N_1}}{\cdots} \cdot \frac{(N-N_1)!\,(d_2)^{N_2}}{N_2!\,(N-N_1-N_2)!} \cdot \frac{(N-N_1-N_2)!\,(d_3)^{N_3}}{N_2!\,(N-N_1-N_2-N_3)!} \cdots$$

Identical Bosons:

1) There is no restriction on the number of particles that can share the same one particle state
2) Our bosons are indistinguishable so it does not matter which particle is in which state. i.e. there is only one way we can choose N_1 particles from N particles

So again our question reduces to arranging N_1 particles in d_1 bins. But now there is no restriction since more than one particle can go in one degeneracy state.

The question is for the first bin how many different ways can N_1 particles be arranged in d_1 different degenerate states. Let us understand this by an example. Let us say we have

$N_1 = 3$, $d_1 = 3$

$\psi_1 \quad \psi_1 \quad \psi_1$ {all 3 going in same degenerate state ψ}

$\underbrace{\bullet \quad \bullet \quad \bullet}$ $\times \quad \times$

... in 2 or 3

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$\underbrace{\bullet \quad \bullet \quad \bullet}_{\text{three in one}}$ × × none in 2 or 3

$\psi_1 \; \psi_1 \quad \psi_2$ { 2 going in same state and one in different}

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1) There is no restriction on the number of particles that can share the same one particle state
2) Our bosons are indistinguishable so it does not matter which particle is in which state. i.e. there is only one way we can choose N_1 particles from N particles

$N_1 \to d_1 \quad N_2 \to d_2 \quad \underbrace{(d_n)^{N_n}}$

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$\psi_1 \; \psi_1 \quad \psi_2$ { 2 going in same state and one in different} three in one

none in 2 or 3

$\psi_1 \quad \psi_1 \quad \psi_2$ {2 going in same state and one in different}

$\underbrace{\bullet \quad \bullet}_{\text{two in one}} \quad \times \quad \bullet \quad \times$

one in two

none in three

$\psi_1 \quad \psi_2 \quad \psi_3$

$\bullet \quad \times \quad \bullet \quad \times \quad \bullet$

(one in each)

Note: It is like placing N_1 balls in d_1 baskets.

A bit more complicated example:

$n_1 = 7$ and $d_1 = 5$

x x

three in one — none in 2 or 3

$\psi_1 \; \psi_1 \quad \psi_2$ { 2 going in same state and one in different }

two in one — one in two — none in three

$\psi_1 \quad \psi_2 \quad \psi_3$

Note: It is like placing N_1 balls in d_1 baskets.

(one in each)

A bit more complicated example:

$$(\psi_1\ \psi_1\ \psi_2\ \psi_3\ \psi_3\ \psi_3\ \psi_4)$$

$$\underbrace{\bullet \times \bullet \times \bullet \times \bullet}_{N_1 + d - 1}$$

Now we can write a solution:
(1) There are N_1 dots and $(d_1 - 1)$ crosses. What we need to know is how many different arrangements of these we can do.

If they are all "differently labeled this", like

$$\underbrace{a \quad b \quad c \quad d \quad . \ . \ . \ .}_{N_1 + d - 1 \text{ total}}$$

We can arrange N things in N! ways, so $N_1 + d - 1$ objects can be arranged in $(N_1 + d - 1)!$ Ways
However all dots are the same, so there are $N_1!$ Equivalent ways to permute them that do not change the state. Also, all crosses are equivalent, and permuting them $(d_1 - 1)!$ wWaysdoes not change the state either. Therefore the answer is

$$\underline{(N_1 + d_1 - 1)!}$$

$N_1 + d - 1$ total

We can arrange N things in N! ways, so N_1+d-1 objects can be arranged in $(N_1+d-1)!$ Ways
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$$\checkmark \frac{(N_1+d_1-1)!}{N_1!\,(d_1-1)!} \checkmark$$

And the total answer for identical bosons is:

$$Q(N_1, N_2, N_3, \ldots) = \prod_{n\geq 1}^{\infty} \frac{(N_n+d_n-1)!}{N_n!\,(d_n-1)!}$$

The most probable configuration:

$$Q(N_1, N_2, N_3, \ldots) = \prod_{n=1}^{\infty} \frac{(N_n + d_n - 1)!}{N_n!\,(d_n - 1)!}$$

N_n

The most probable configuration:

In thermal equilibrium, every configuration with energy E and number of particles N is equally likely. Therefore, the most probable configuration $(N_1, N_2, N_3, \ldots)$ is the one for which $Q(N_1, N_2, N_3, \ldots)$ is the largest, since it can be produced in the largest number of different ways.

Therefore, to find the most probable configuration, we need to fund when Q is a maximum assuming the following constraints:

$$\sum_{n=1}^{\infty} N_n = N \quad \Rightarrow \quad \sum_{n=1}^{\infty} N_n - N = 0$$

$$\sum_{n=1}^{\infty} N_n E_n = E \quad \Rightarrow \quad \sum_{n=1}^{\infty} N_n E_n - E = 0$$

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To maximize function $F(x_1, x_2, x_3, \ldots)$ that is subjected to constraints $f_1(x_1, x_2, x_3, ..) = 0$ and $f_2(x_1, x_2, x_3, ..) = 0$, etc, it is convenient to use the method of Lagrange multipliers.
We introduce new function

$$G(x_1, x_2, x_3 \ldots) \equiv F + \lambda_1 f_1 + \lambda_2 f_2 + \ldots$$

BA

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$$G(x_1, x_2, x_3 \dots) \equiv F + \lambda_1 f_1 + \lambda_2 f_2 + \dots$$

And set all derivatives to zero:

$$\frac{\partial G}{\partial x_n} = 0 \quad ; \quad \frac{\partial G}{\partial \lambda_n} = 0$$

λ_2

We will work with ln(Q) to turn products into sums. The maxima of Q and ln(Q) occur at the same point since ln(Q) is a monotonous function.

$$G = \ln(Q) + \underset{\lambda_1}{\alpha} \underbrace{\left(N - \sum_{n=1}^{\infty} N_n \right)}_{f_1} + \beta \underbrace{\left(E - \sum_{n=1}^{\infty} N_n E_n \right)}_{f_2}$$

$\frac{\partial G}{\partial \alpha}$ $\frac{\partial G}{\partial \beta}$

$x_n = N_n$

produced in the largest number of different ways.

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max Q

$$\checkmark \sum_{n=1}^{\infty} N_n E_n = E \quad \Rightarrow \quad \checkmark \sum_{n=1}^{\infty} N_n E_n - E = 0$$

To maximize function $F(x_1, x_2, x_3, \dots)$ that is subjected to constraints $f_1(x_1, x_2, x_3, ..) = 0$ and $f_2(x_1, x_2, x_3, ..) = 0$, etc, it is convenient to use the method of Lagrange multipliers.
We introduce new function

$$G(x_1, x_2, x_3, \dots) = F + \lambda_1 f_1 + \lambda_2 f_2 + \dots$$

We will work with ln(Q) to turn products into sums. The maxima of Q and ln(Q) occur at the same point since ln(Q) is a monotonous function.

$$G = \underbrace{\ln(Q)} + \overset{\lambda_1}{\alpha}\Big(\underbrace{N - \sum_{n=1}^{\infty} N_n}_{f_1}\Big) + \overset{\lambda_2}{\beta}\Big(\underbrace{E - \sum_{n=1}^{\infty} N_n E_n}_{f_2}\Big)$$

$$x_n \equiv N_n$$

$\frac{\partial G}{\partial N_{\cdot}} = 0$ and $\frac{\partial}{\partial} = 0$ give back the constraints, so we need to get $\frac{\partial G}{\partial N_n} = 0$

Case I: Distinguishable particle:

$$Q = N! \prod_{n=0}^{\infty} \frac{(d_n)^{N_n}}{N_n!}$$

$\infty \quad \ldots \quad \beta\Big(E - \sum^{\infty} N_n E_n\Big)$

$\partial\beta$ ... to get $\boxed{\frac{\partial G}{\partial N_n} = 0}$

Case I: Distinguishable particle:

$$Q = N! \prod_{n=1}^{\infty} \frac{(d_n)^{N_n}}{N_n!}$$

$$G = \ln\left(N! \prod_{n=1}^{\infty} \frac{(d_n)^{N_n}}{N_n!} \right) + \alpha\left(N - \sum_{n=1}^{\infty} N_n \right) + \beta\left(E - \sum_{n=1}^{\infty} N_n E_n \right)$$

$$= \ln(N!) + \sum_{n=1}^{\infty} \left\{ N_n \ln(d_n) - \ln(N!) \right\} + \alpha\left(N - \sum_{n=1}^{\infty} N_n \right)$$

$$+ \beta\left(E - \sum_{n=1}^{\infty} N_n E_n \right)$$

Using Stirling's approximation for large occupation number N_n

$$\ln(z!) \simeq z \ln z - z \qquad \text{for} \quad z \gg 1$$

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$$+ \beta \left(E - \sum_{n=1}^{\infty} N_n E_n \right)$$

$N \gg 1 \quad N_n \gg 1$

✓ Using Stirling's approximation for large occupation number N_n

$$\ln(z!) \simeq z \ln z - z \quad \text{for} \quad \boxed{z \gg 1}$$

$$G = \ln(N!) + \sum_{n=1}^{\infty} \left\{ N_n \ln(d_n) - N_n \ln(N_n) + N_n - \alpha N_n - \beta E_n N_n \right\} + \alpha N - \beta E$$

$$\frac{\partial G}{\partial N_n} = \ln(d_n) - \ln(N_n) - \frac{N_n}{N_n} + 1 - \alpha - \beta E_n$$

$$= \ln(d_n) - \ln(N_n) - \alpha - \beta E_n$$

$$\frac{\partial G}{\partial N_n} = \ln(d_n) - \ln(N_n) - \frac{N_n}{N_n} + 1$$

$$0 = \ln(d_n) - \ln(N_n) - \alpha - \beta E_n$$

$$-\alpha - \beta E_n = \ln\left(\frac{N_n}{d_n}\right)$$

$$\boxed{N_n = d_n \, e^{-(\alpha + \beta E_n)}}$$

CASEII: Identical fermions $(N_n >> 1)$ and $(d_n >> N_n)$

$$Q = \prod_{i=1}^{\infty} \frac{d_n!}{N_n! \, (d_n - N_n)!}$$

$$G = \sum_{n=1}^{\infty} \left\{ \ln(d_n!) - \ln(N_n!) - \ln(d_n - N_n)! + \alpha\left[N - \sum_{n=1}^{\infty} N_n\right] + \beta\left[E - \sum_{n=1}^{\infty} N_n E_n\right]\right.$$

Using Stirling's approximation for large occupation number N_n

$$\ln(z!) \simeq z\ln z - z \quad \text{for} \quad \boxed{z \gg 1}$$

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$$-\alpha - \beta E_n = \ln\left(\frac{N_n}{d_n}\right)$$

$$\boxed{N_n = d_n\, e^{-(\alpha + \beta E_n)}}$$

$$0 = \ln\left(\frac{d_n - N_n}{N_n}\right) - \alpha - \beta E_n$$

$$\ln\left(\frac{d_n}{N_n} - 1\right) = \alpha + \beta E_n$$

$$\frac{d_n}{N_n} = e^{\alpha + \beta E_n} + 1$$

$$\boxed{N_n = \frac{d_n}{e^{\alpha+\beta E_n} + 1}}$$

CASEIII: Identical bosons $N_n >> 1$

$$Q = \prod^{\infty} \frac{(N_n + d_n - 1)!}{\quad}$$

$$1 + \frac{d_n - 1}{N_n} = e^{\alpha + \beta E_n} \Rightarrow N_n = \frac{d_n - 1}{e^{(\alpha + \beta E_n)} - 1}$$

Physical significance of α and β

α and β are the constants involved in the theory of Lagrange multipliers and they constraint the conservation of total number of particles and total energy. Mathematically they can be calculated from the equations of constraints if one knows N_n and d_n. It is also important that E_n are also known.

Let us work this out for the case of infinite square well. For which the allowed energies are:

$$E_k = \frac{\hbar^2 k^2}{2m}, \text{ where } \vec{k} = \left(\frac{\pi n_x}{l_x}, \frac{\pi n_y}{l_y}, \frac{\pi n_z}{l_z} \right)$$

Let us now calculate the degeneracy of the energy E_k i.e. how many number of states with energy E_k

Imagine a 3-D space with axis k_x, k_y, k_z with planes drawn at

$$k = \pi \quad 2\pi$$

$$0 = \ln(N_n + d_n - 1) - \ln(N_n) - \alpha - \beta E_n$$

$$\ln\left(\frac{N_n + d_n - 1}{N_n}\right) = \alpha + \beta E_n$$

$$1 + \frac{d_n - 1}{N_n} = e^{\alpha + \beta E_n} \quad \Rightarrow \quad N_n = \frac{d_n - 1}{e^{(\alpha + \beta E_n)} - 1}$$

$\rightarrow$ can be dropped

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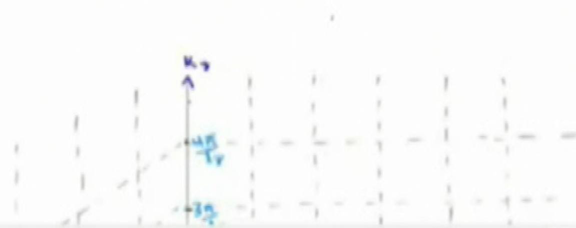
Imagine a 3-D space with axis k_x, k_y, k_z with planes drawn at

$$k_x = \frac{\pi}{\ell_x}, \frac{2\pi}{\ell_x}, \dots$$



$$k_y = \pi \quad 2\pi$$

Physical significance of α and β

α and β are the constants involved in the theory of Lagrange multipliers and they constraint the conservation of total number of particles and total energy. Mathematically they can be calculated from the equations of constraints if one knows N_n and d_n. It is also important that E_n are also known.

Let us work this out for the case of infinite square well. For which the allowed energies are:

$$E_k = \frac{\hbar^2 k^2}{2m} \quad , \text{ where } \vec{k} = \left(\frac{\pi n_x}{\ell_x}, \frac{\pi n_y}{\ell_y}, \frac{\pi n_z}{\ell_z} \right)$$

Let us now calculate the degeneracy of the energy E_k i.e. how many number of states with energy E_k

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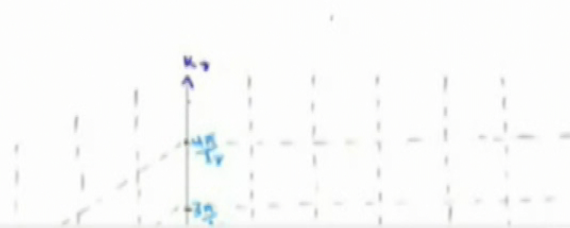

$$k_{.} = \pi \qquad 2\pi$$

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$$\sum N_n = N$$
$$\sum E_n N_n = E$$

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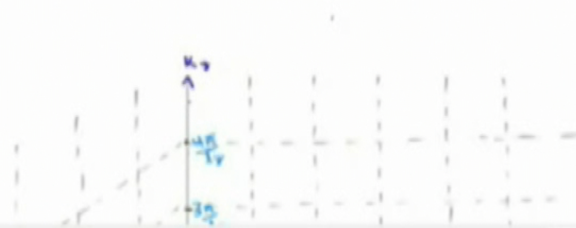

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Physical significance of α and β

α and β are the constants involved in the theory of Lagrange multipliers and they constraint the conservation of total number of particles and total energy. Mathematically they can be calculated from the equations of constraints if one knows N_n and d_n. It is also important that E_n are also known.

$$\alpha \quad \sum N_n = N \qquad d_n$$

$$\beta \quad \sum E_n N_n = E$$

$$? \quad d_n$$

Let us work this out for the case of infinite square well. For which the allowed energies are:

$$E_k = \frac{\hbar^2 k^2}{2m} \, , \quad \text{where } \vec{k} = \left(\frac{\pi n_x}{l_x}, \frac{\pi n_y}{l_y}, \frac{\pi n_z}{l_z} \right)$$

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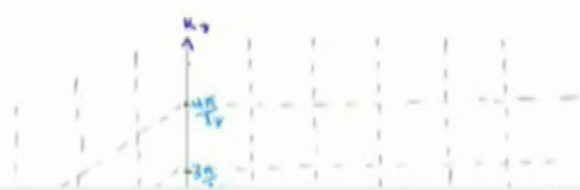

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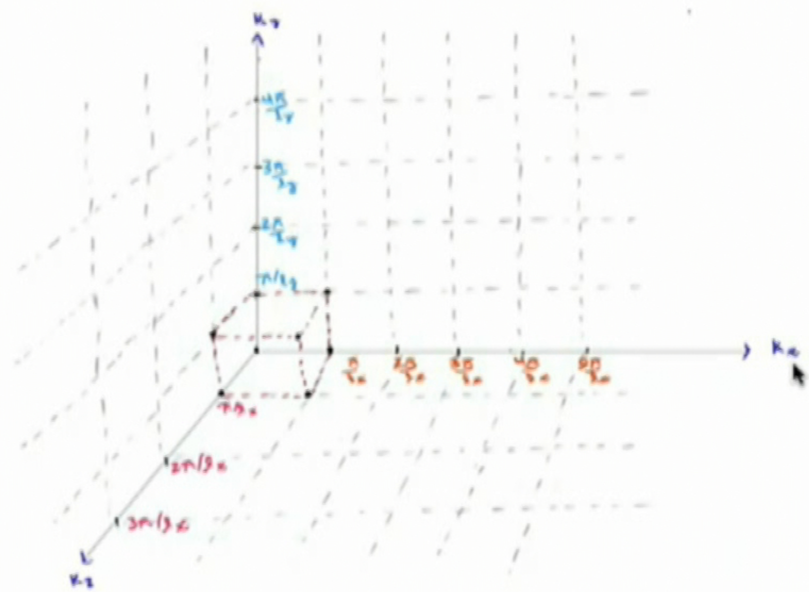

Each intersection represents a distinct one particle state.

Each state occupies $\frac{\pi^3}{V}$ volume of k space.

Each block in this grid represents each state

... π^3

$2m$ $\quad k_x \quad k_y \quad k_z$ $\quad$? (dn)

Let us now calculate the degeneracy of the energy E_k i.e. how many number of states with energy E_k

Imagine a 3-D space with axis k_x, k_y, k_z with planes drawn at

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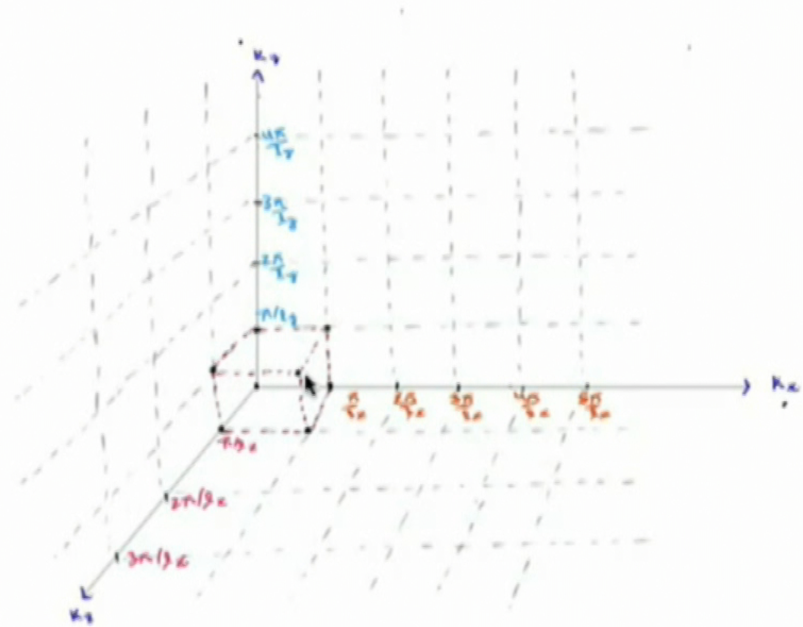
$$k_z = \frac{\pi}{l_z}, \frac{2\pi}{l_z}, \ldots$$



Each intersection represents a distinct one particle state.

Each state occupies π^3 ... volume of k space

So I look at this grid

Let us now calculate the degeneracy of the energy E_k i.e. how many number of states with energy E_k

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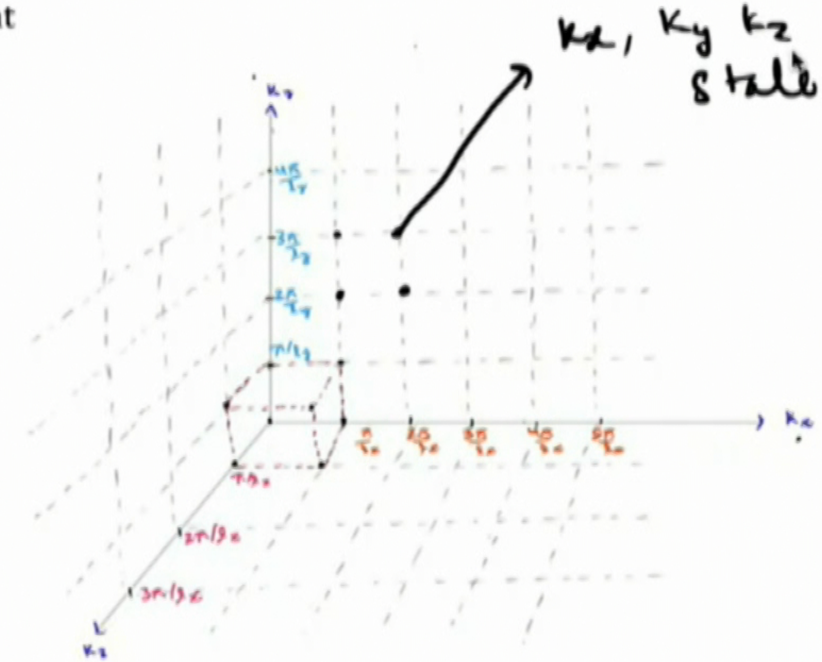

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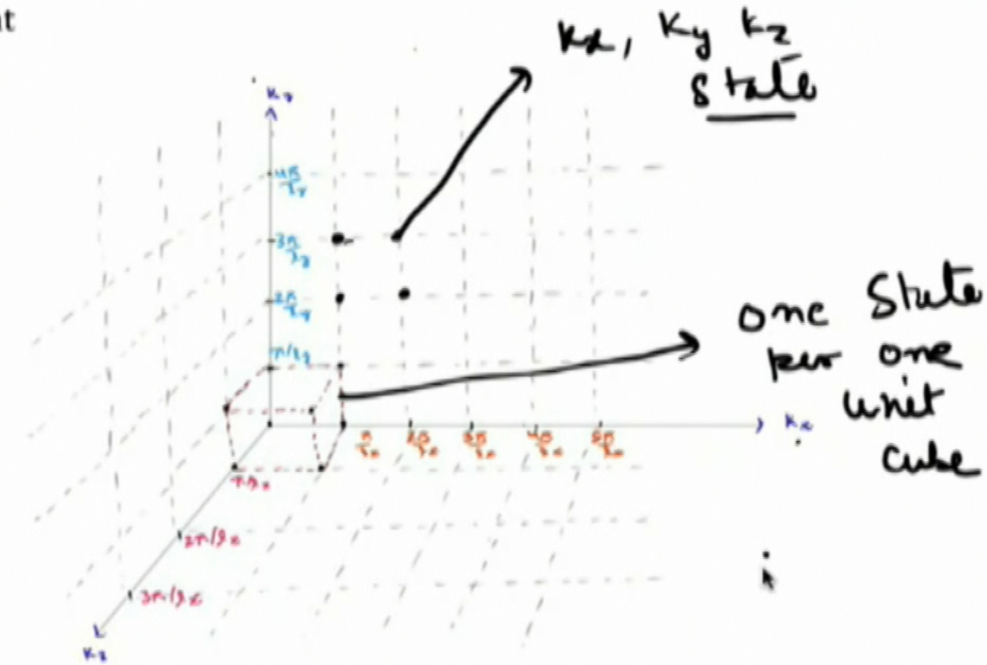
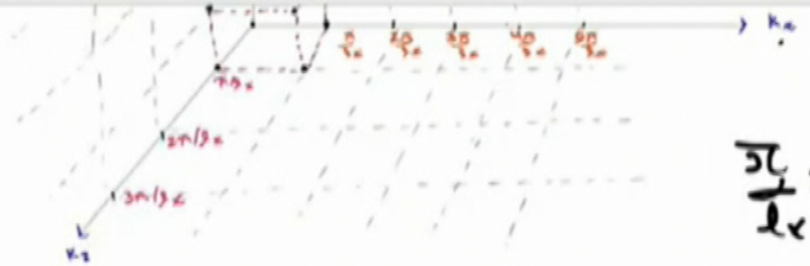


Each intersection represents a distinct one particle state.

Each state occupies $\frac{\pi^3}{V}$ volume of k space.

Each block in this grid represents each state

$\frac{\;}{l_z}$, $\frac{\;}{l_z}$, ...



Cube

$\frac{\pi}{l_x} \times \frac{\pi}{l_y} \times \frac{\pi}{l_z}$

Each intersection represents a distinct one particle state.

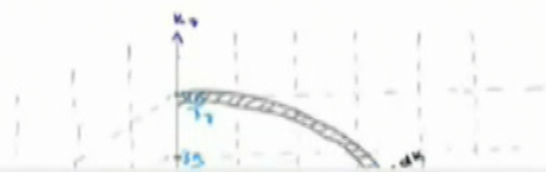
Each state occupies $\frac{\pi^3}{V}$ volume of k space.

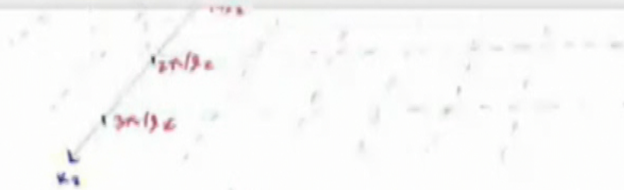
Each block in this grid represents each state

Hence each state occupies $\frac{\pi^3}{l_x l_y l_z}$

$\frac{\pi^3}{V}$ volume

Let us calculate the volume of our bin (spherical shell in the first octant)





$$\frac{\pi}{l_x} \times \frac{\pi}{l_y} \times \frac{\pi}{l_z}$$

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✓ $\frac{\pi^3}{V}$ volume

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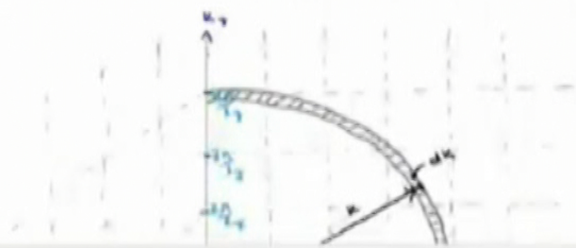

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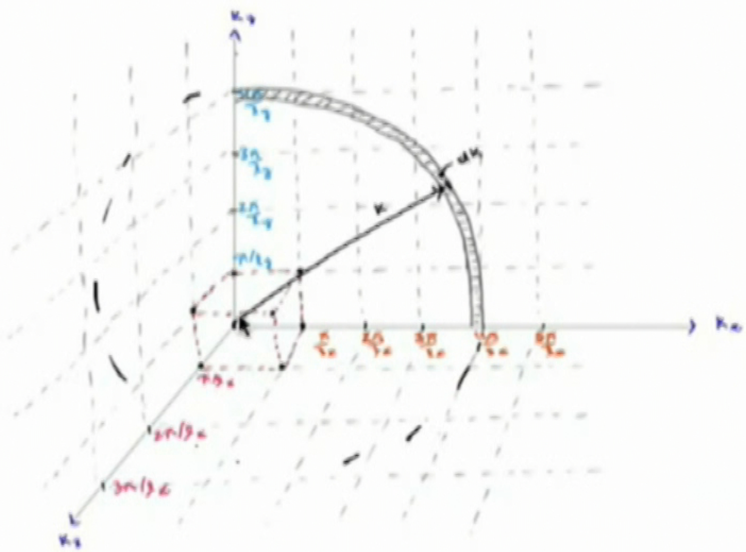

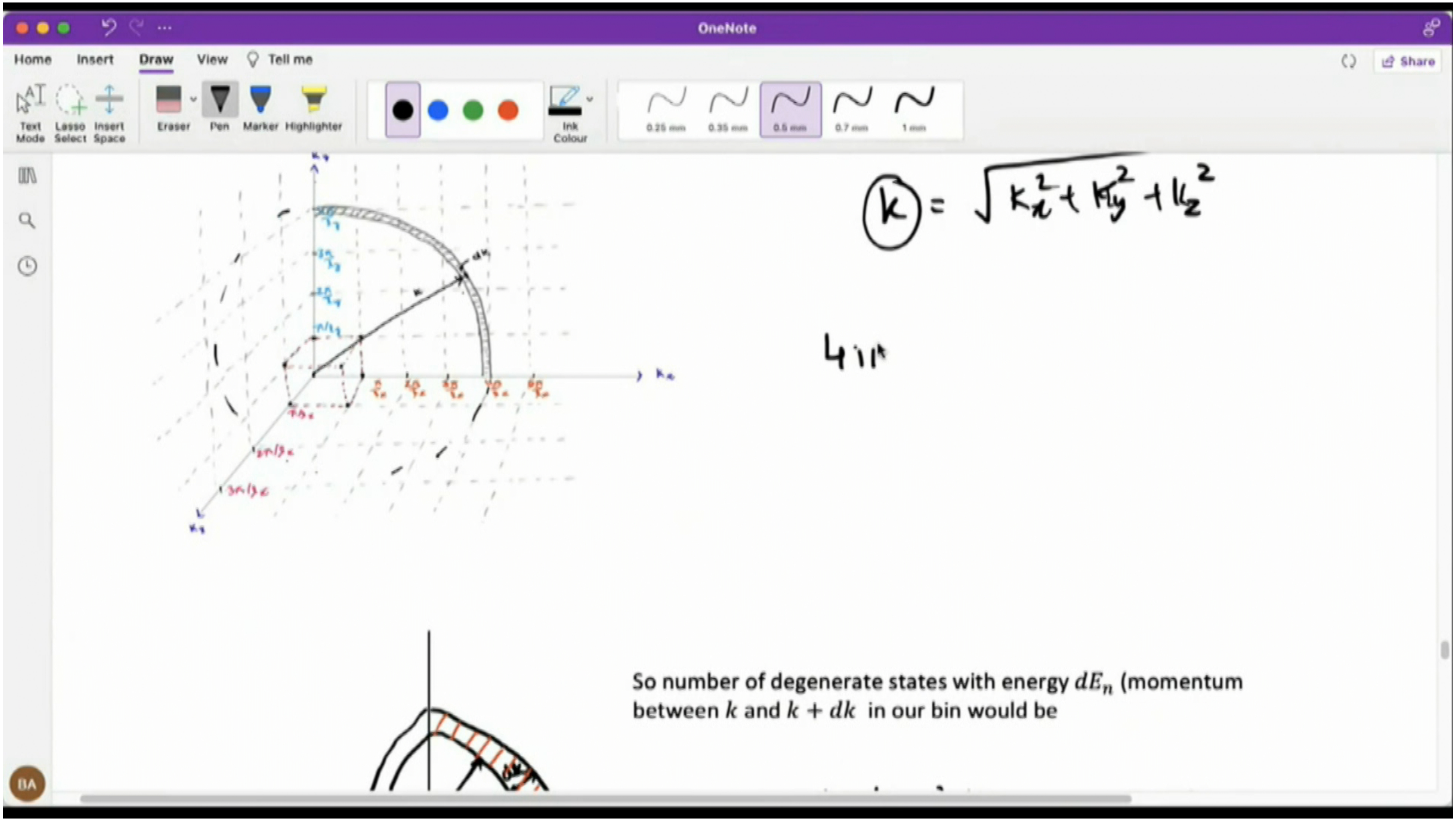
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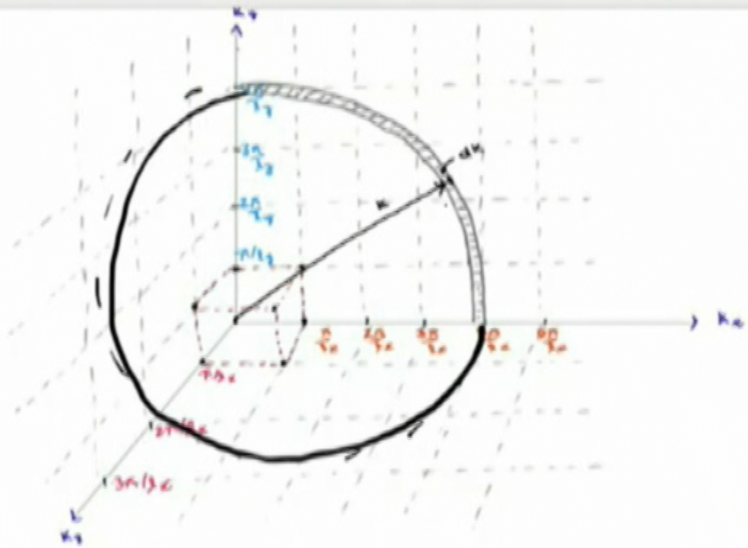
Ink Colour
$k = \sqrt{k_x^2 + k_y^2 + k_z^2}$
4π
So number of degenerate states with energy dE_n (momentum between k and $k + dk$ in our bin would be

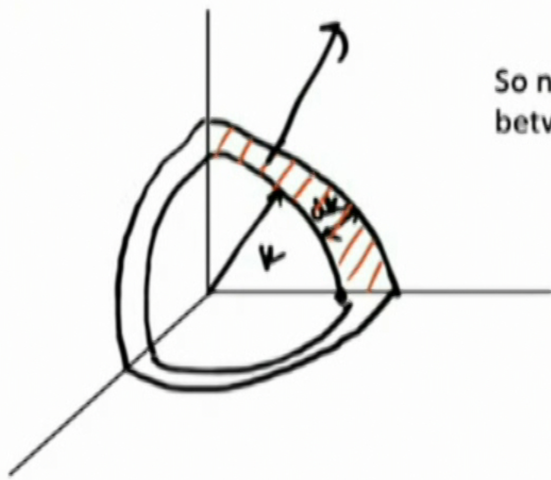


$$k = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

$$4\pi k^2 dk$$

So number of degenerate states with energy dE_n (momentum between k and $k + dk$ in our bin would be

$$\frac{1}{8}\,4\pi k^2\,dk \rightarrow \qquad \frac{\pi^3}{V} \rightarrow 1$$



So number of degenerate states with energy dE_n (momentum between k and $k + dk$ in our bin would be

$$d_n = \frac{1}{8}\,\frac{4\pi k^2\,dk}{(\pi^3/V)} = \frac{V}{2\pi^2}\,k^2\,dk$$

We have now have E_n and d_n. For **DISTINGUISHABLE PARTICLES:**

$$n(E_n) = e^{-(\alpha+\beta E_n)}$$

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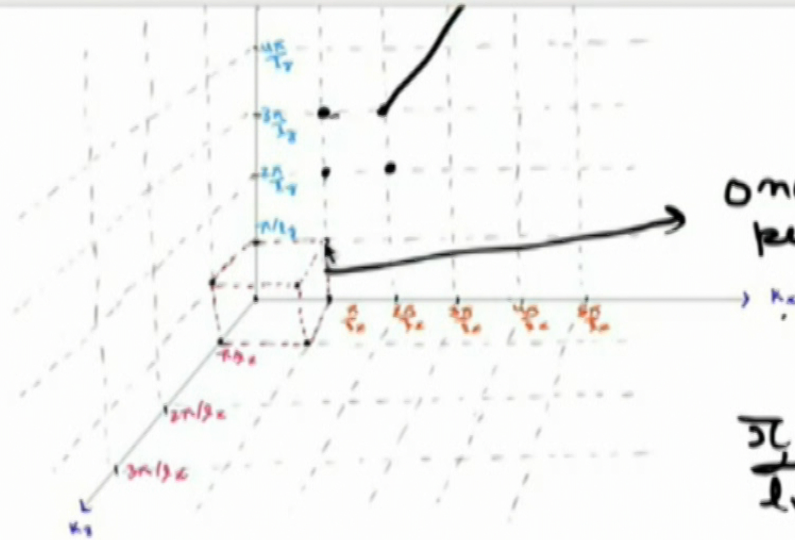
$$\ell_x , \quad \ell_x ,$$

$$k_y = \frac{\pi}{\ell_y}, \frac{2\pi}{\ell_y}, \ldots$$

$$k_z = \frac{\pi}{\ell_z}, \frac{2\pi}{\ell_z}, \ldots$$



one State per one unit cube

$$\frac{\pi}{\ell_x} \times \frac{\pi}{\ell_y} \times \frac{\pi}{\ell_z}$$

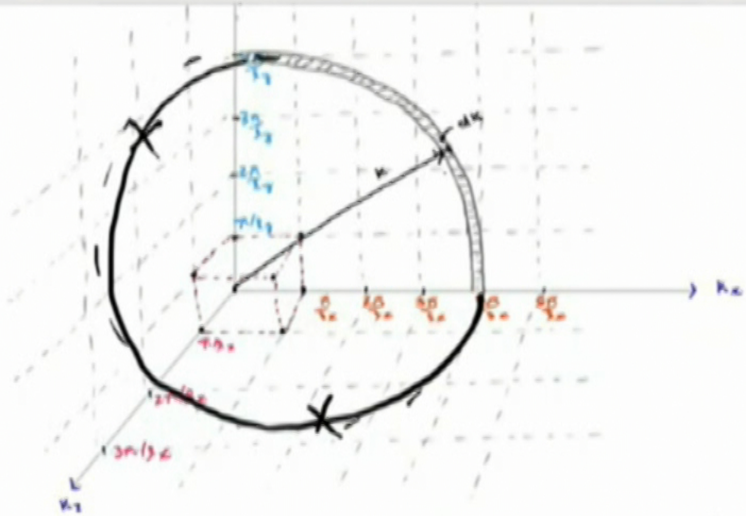
Each intersection represents a distinct one particle state.

Each state occupies $\frac{\pi^3}{V}$ volume of k space.

Each block in this grid represents each state

Hence each state occupies $\frac{\pi^3}{\ell_x \ell_y \ell_z}$

$\checkmark \frac{\pi^3}{V}$ volume



$$(k) = \sqrt{k_x^2 + k_y^2 + k_z^2}$$

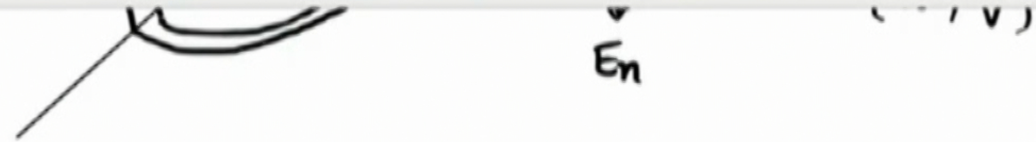
41

$$\left\{ \frac{4\pi k^2 dk}{8} \right.$$

$$\frac{1}{8} 4\pi k^2 dk \to$$

$$\frac{\pi^3}{V} \to 1$$

So number of degenerate states with energy dE_n (momentum between k and $k + dk$ in our bin would be



We have now have E_n and d_n. For **DISTINGUISHABLE PARTICLES:**

$$n(E_n) = e^{-(\alpha + \beta E_n)}$$

Evaluating the first constraint

$$N = \int n(E_n)\, d_n = \frac{V}{2\pi^2} \int_0^\infty e^{-(\alpha + \beta \hbar^2 k^2/2m)}\, k^2\, dk$$

$$N = \frac{V}{2\pi^2}\, e^{-\alpha} \left(\frac{m}{2\pi \beta \hbar^2} \right)^{3/2}$$

$$e^{-\alpha} = \frac{N}{V} \left(\frac{2\pi \beta \hbar^2}{m} \right)^{3/2}$$

The second constraint $\infty \quad -\beta\hbar^2 k^2/2m \quad k^4\, dk$

We have now have E_n and d_n. For **DISTINGUISHABLE PARTICLES:**

$$N_n = d_n e^{-(\alpha+\beta E_n)}$$

Evaluating the first constraint

$$N = \int n(E_n)\, d_n = \frac{V}{2\pi^2}\int_0^\infty e^{-(\alpha+\beta\hbar^2k^2/2m)}\, k^2\, dk \rightarrow$$

$$N = \frac{V}{2\pi^2}\, e^{-\alpha}\left(\frac{m}{2\pi\beta\hbar^2}\right)^{3/2}$$

$$e^{-\alpha} = \frac{N}{V}\left(\frac{2\pi\beta\hbar^2}{m}\right)^{3/2}$$

The second constraint

$$E = \int^\infty N\, E_n\, dk = \frac{V}{\ } \int^\infty e^{-\alpha}\, \frac{\hbar^2}{\ }\, e^{-\beta\hbar^2k^2/2m}\, k^4\, dk$$

So
$$\frac{N}{V}\left(\frac{2\pi\beta\hbar^2}{m}\right)^{3/2} = \frac{2E\beta}{3V}\left(\frac{2\pi\beta\hbar^2}{m}\right)^{3/2}$$

$$\boxed{E = \frac{3N}{2\beta}}$$

Comparing with the classical formula for the average energy of an atom at temperature T

$$E = \frac{3}{2}k_B T$$

$$\boxed{\beta = \frac{1}{k_B T}}$$

K_B is Boltzman constant and β is related to the temperature. The same conclusion can be infered for other particles as well.

It is customary to replace α by chemical potential $\mu(T)$

$$\mu(t) = -\alpha k_B T \quad \Rightarrow \quad \alpha = -\frac{\mu(t)}{k_B T}$$

$\mu(t) \rightarrow$ chemical p^l

Thus one can rewrite our equations for the most probable number of particles in a particular state with energy ϵ.

$$n(\epsilon) = \frac{N_n}{d_n}$$

$N_n \rightarrow$ total no. of particles with energy ϵ

$d_n \rightarrow$ degeneracy

$$n(\epsilon) = \begin{cases} e^{-(\epsilon-\mu)/k_B T} & \text{Maxwell Boltzman} \\ \dfrac{1}{e^{(\epsilon-\mu)/k_B T} + 1} & \text{Fermi Dirac} \\ \dfrac{1}{e^{(\epsilon-\mu)/k_B T} - 1} & \text{Bose Einstien} \end{cases}$$

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$n(\epsilon) = \frac{N_n}{d_n}$ $N_n \rightarrow$ total no. of particles with energy ϵ

$d_n \rightarrow$ degeneracy

$n(\epsilon) \downarrow$ no. density

$$n(\epsilon) = \begin{cases} e^{-(\epsilon-\mu)/k_BT} & \text{Maxwell Boltzman} \\ \dfrac{1}{e^{(\epsilon-\mu)/k_BT}+1} & \text{Fermi Dirac} \\ \dfrac{1}{e^{(\epsilon-\mu)/k_BT}-1} & \text{Bose Einstien} \end{cases}$$

Behavior of **Fermi Dirac distribution:**

$$n(\epsilon) = \frac{1}{e^{(\epsilon-\mu)/k_BT}+1}$$

if $\epsilon - \mu < 0$

BA


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$\frac{1}{\exp\left(\frac{x}{T}\right)}$
$\frac{1}{\exp\left(\frac{x}{T}\right)-1}$
$\frac{1}{\exp\left(\frac{x}{T}\right)+1}$
$T = 1.8$
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