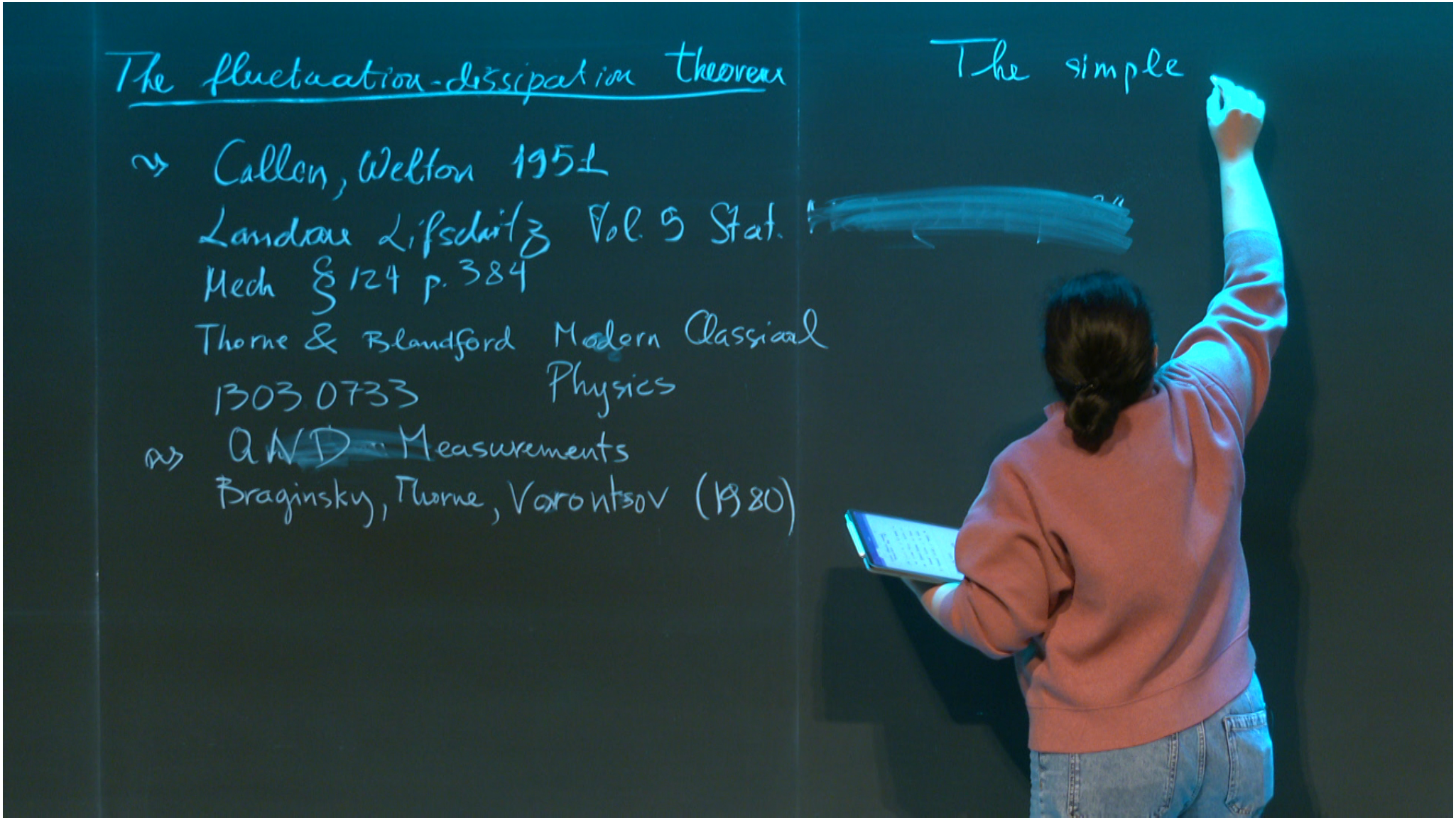
Title: Fluctuation-Dissipation theorem

Speakers: Asimina Arvanitaki

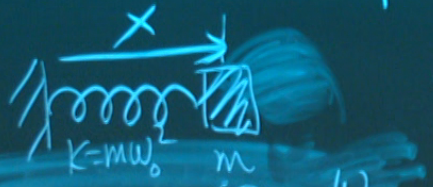
Collection: School on Table-Top Experiments for Fundamental Physics

Date: September 23, 2022 - 9:15 AM

URL: https://pirsa.org/22090026

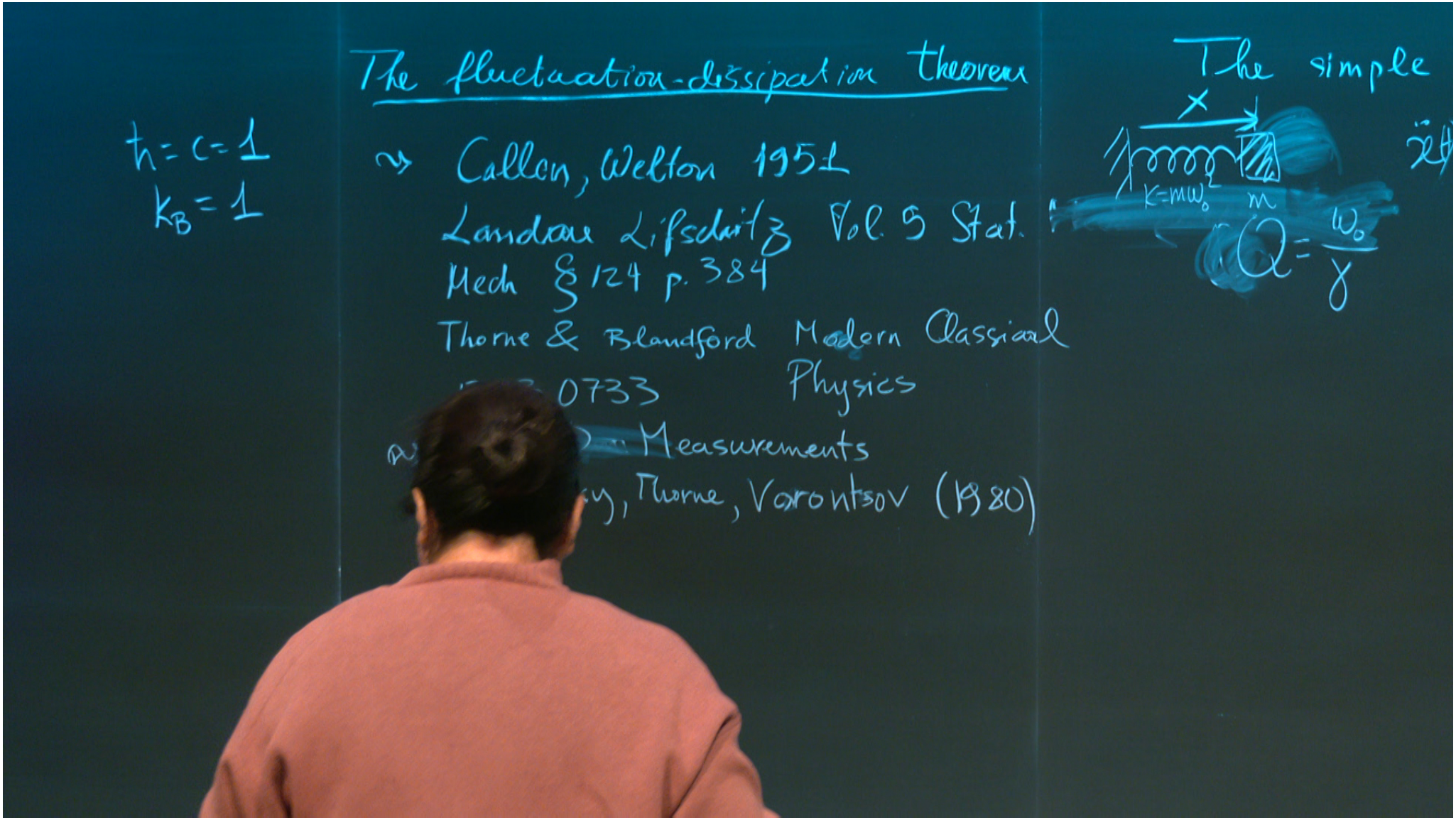

The fluctuation-dissipation theorem
The simple
Callen, Welton 1951
Landau Lifschitz Vol 5 Stat.
Mech § 124 p. 384
Thorne & Blandford Modern Classical
Physics
1303.0733
QND - Measurements
Braginsky, Thorne, Vorontsov (1980)

The simple harmonic oscillator

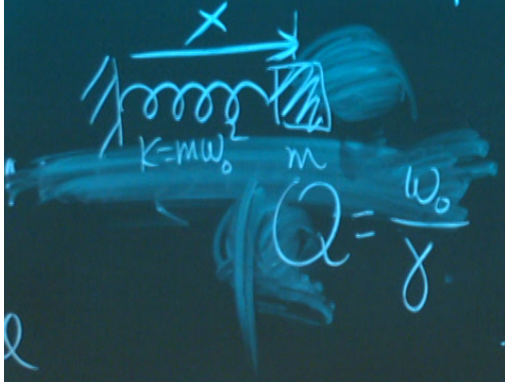


$$\ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m}$$

$$Q = \frac{\omega_0}{\gamma}$$


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The simple harmonic oscillator



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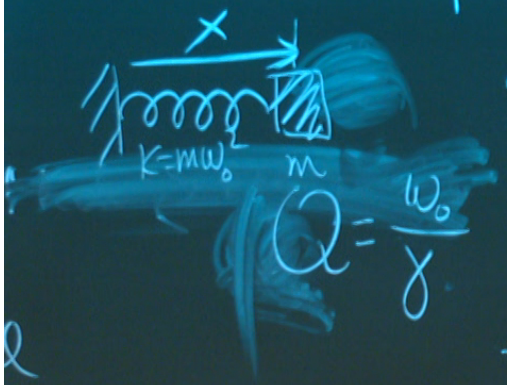
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The simple harmonic oscillator



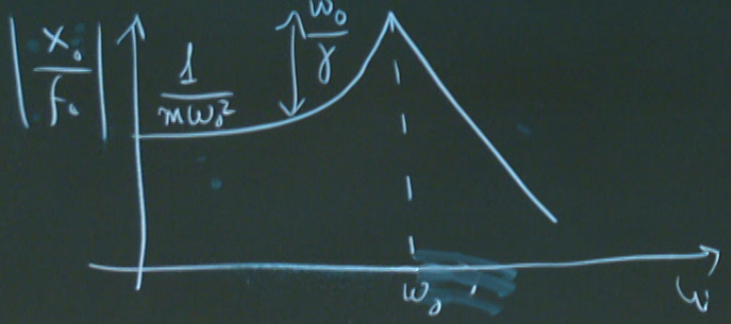
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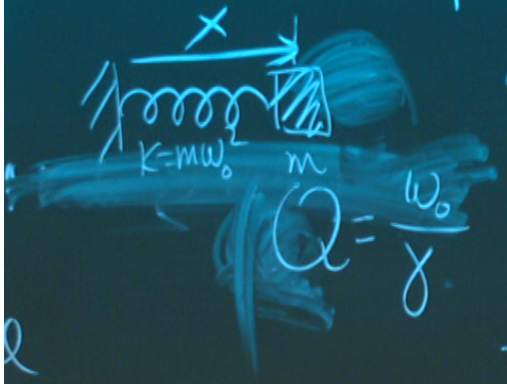
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The simple harmonic oscillator



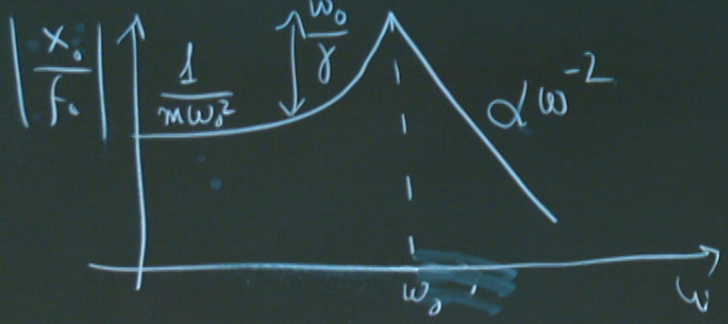
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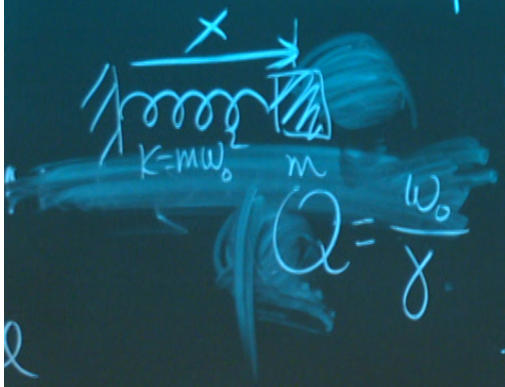
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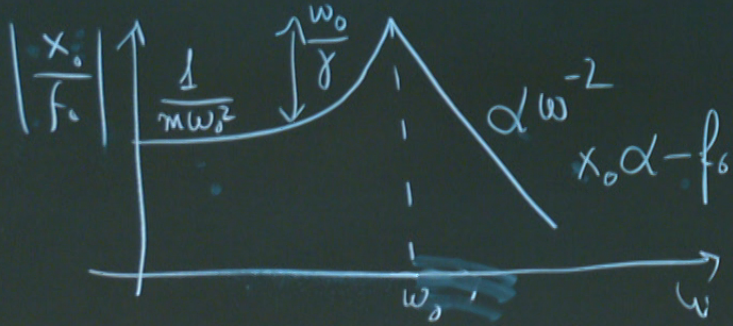
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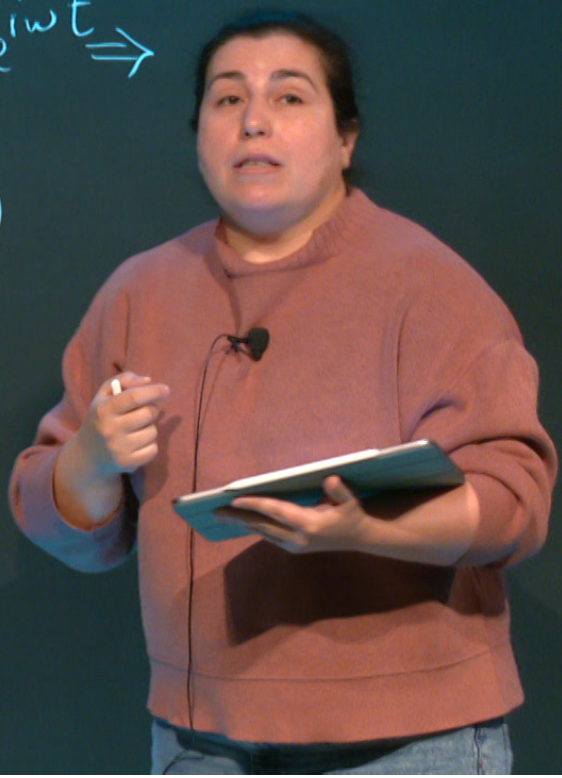
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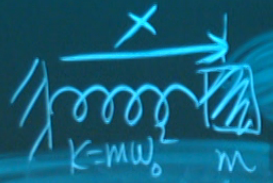
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The simple harmonic oscillator



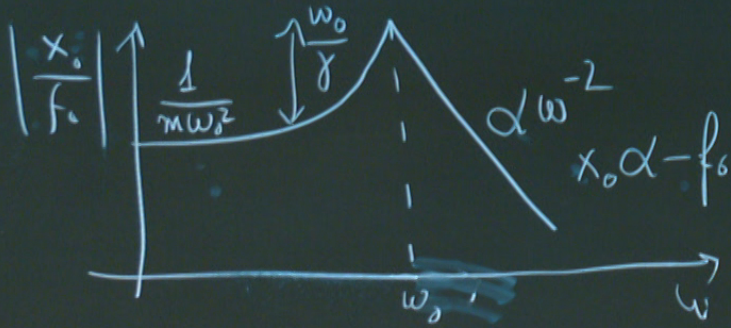
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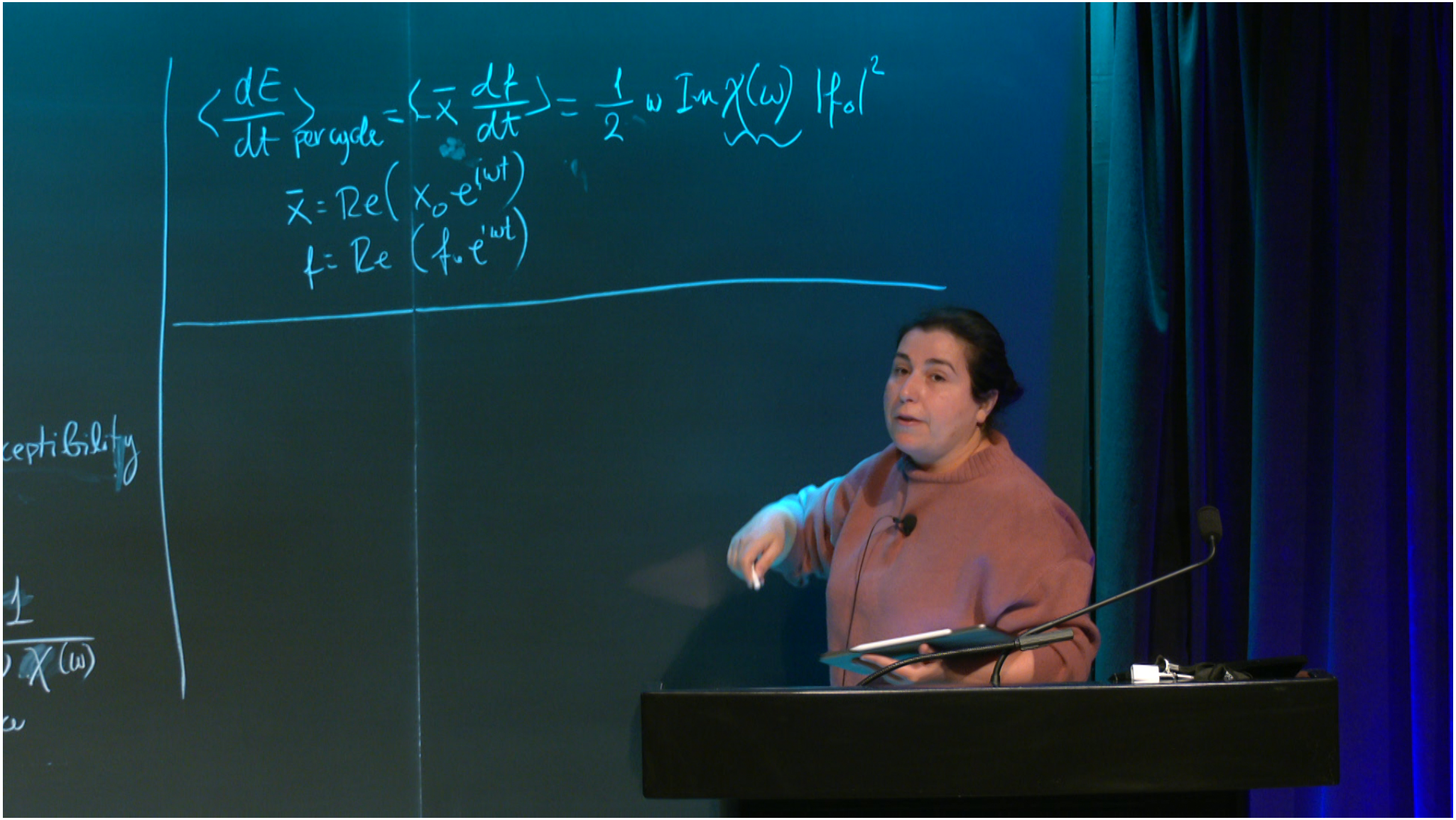


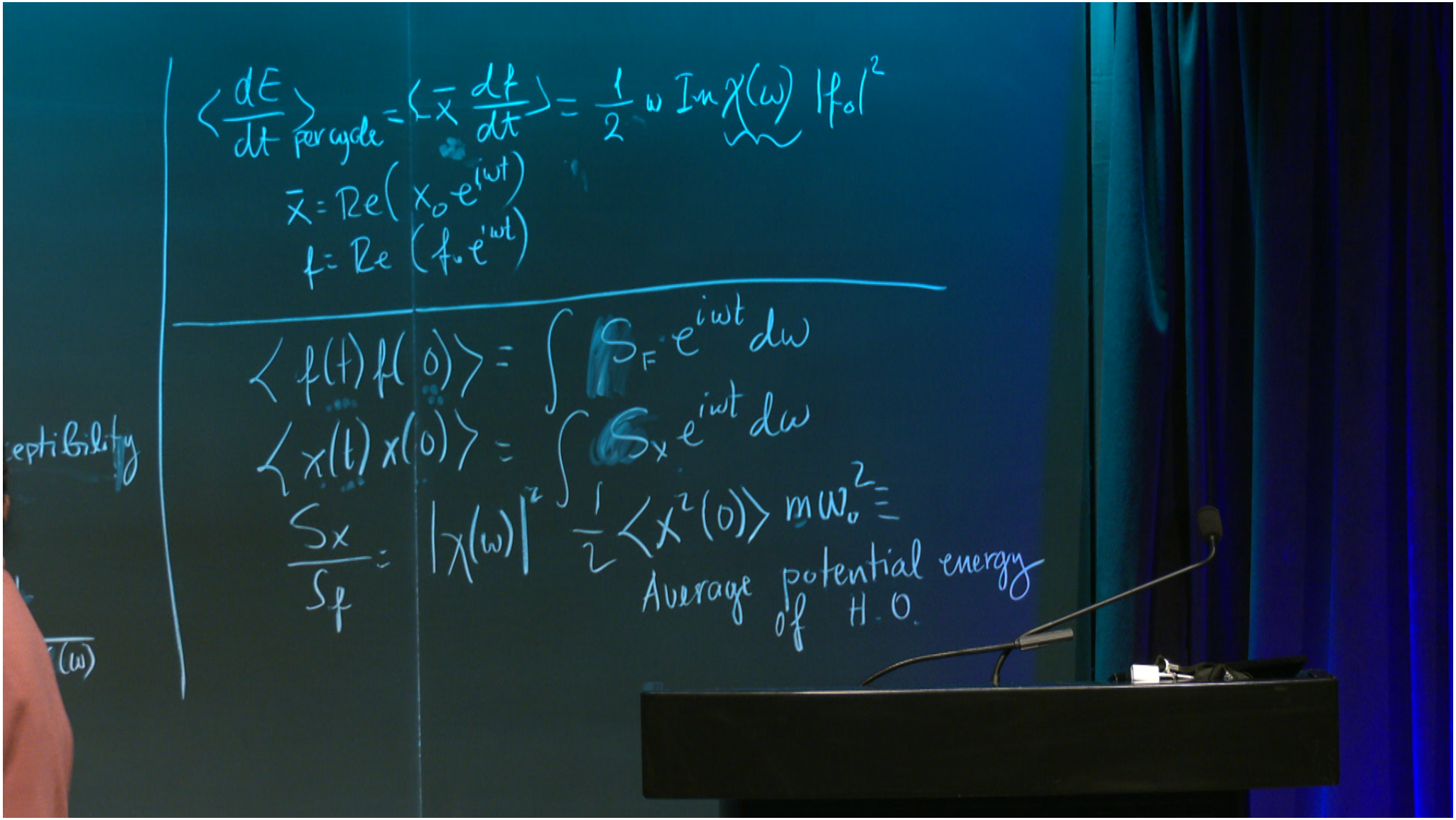
$$f_0 = Z(\omega)\,\dot{x}_\omega$$

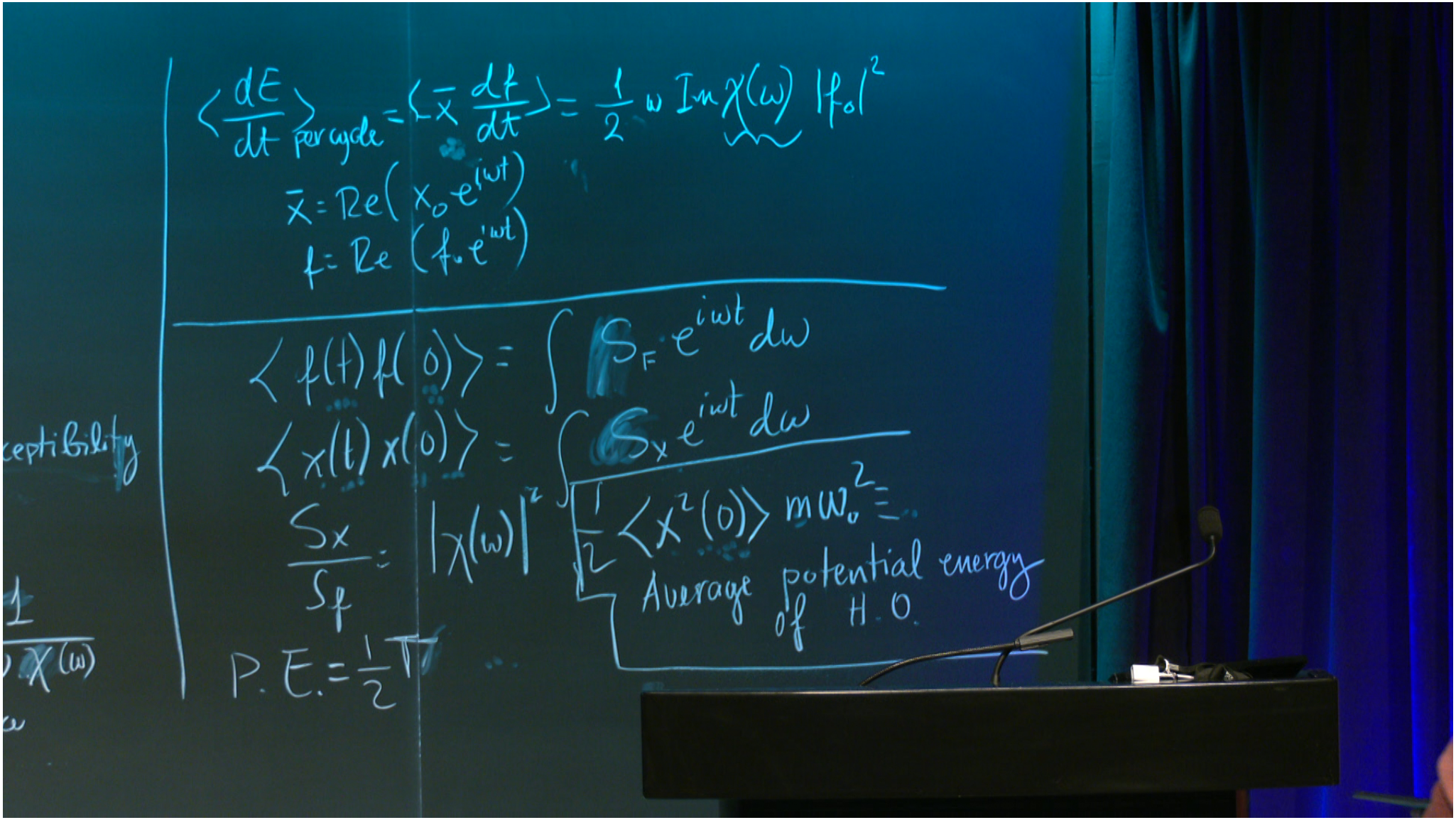
$$Z(\omega) = \frac{1}{i\omega\chi(\omega)}$$

$Z(\omega)$ ← impedance

$$\left\langle \frac{dE}{dt} \right\rangle_{\text{per cycle}} =$$


$\left\langle \frac{dE}{dt} \right\rangle_{\text{per cycle}} = \left\langle -\bar{x}\frac{df}{dt} \right\rangle = \frac{1}{2}\omega\, \mathrm{Im}\,\chi(\omega)\,|f_0|^2$
$\bar{x} = \mathrm{Re}\left(x_0 e^{i\omega t}\right)$
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ceptibility


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$\bar{x} = \mathrm{Re}(x_0 e^{i\omega t})$
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$\langle f(t) f(0) \rangle = \int S_F e^{i\omega t} d\omega$
$\langle x(t) x(0) \rangle = \int S_x e^{i\omega t} d\omega$
$\frac{S_x}{S_f} = |\chi(\omega)|^2$
$\frac{1}{2} \langle x^2(0) \rangle m\omega_0^2 =$
Average potential energy of H.O.


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Average potential energy of H.O.
P.E. $= \frac{1}{2}$
ceptibility

The fluctuation-dissipation theorem

$$\langle x^2(0)\rangle = \frac{T}{m\omega_0^2} \Rightarrow$$

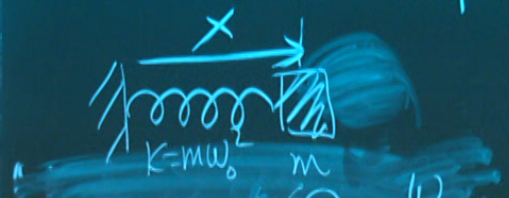
$$\int S_x \, d\omega = \frac{T}{m\omega_0^2} \Rightarrow$$

$$\int S_F \, |\chi(\omega)|^2 \, d\omega = \frac{T}{m\omega_0^2} \qquad \gamma \ll \omega_0$$

$$\frac{1}{m^2 \omega_0^2 \gamma^2} S_F(\omega=\omega_0)\,\gamma = \frac{T}{m\omega_0^2}$$

$$S_F(\omega=\omega_0) = T m \gamma$$

The simple harmonic oscill



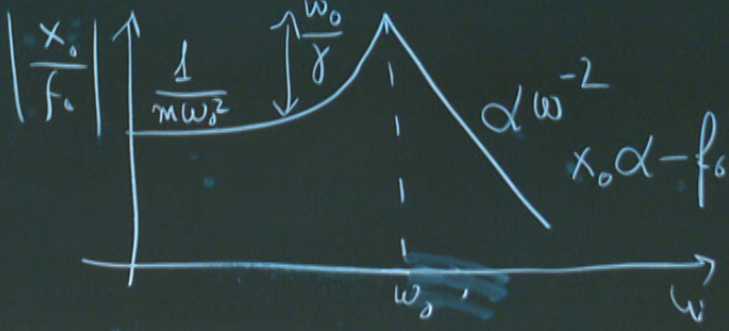
$$Q = \frac{\omega_0}{\gamma}$$

$$\ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m}$$

$$f(t) = f_0 e^{i\omega t},$$

$$-\omega^2 x_0 e^{i\omega t} + i\omega\gamma x_0 e^{i\omega t} + \omega$$

$$\frac{x_0}{f_0} = \frac{1}{m} \cdot \frac{1}{-\omega^2 + i\omega_0^2}$$



The fluctuation-dissipation theorem

$$\langle x^2(0)\rangle = \frac{I}{m\omega_0^2} \Rightarrow$$

$$\int S_x\, d\omega = \frac{I}{m\omega_0^2} \Rightarrow$$

$$\int S_F |\chi(\omega)|^2 d\omega = \frac{I}{m\omega_0^2} \qquad \gamma \ll \omega_0$$

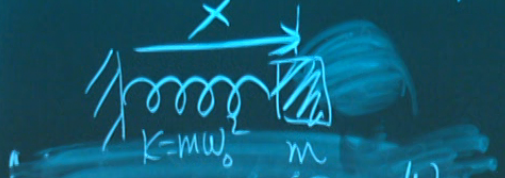
$$\frac{1}{m^2}\, S_F(\omega=\omega_0)\, \gamma = \frac{I}{m\omega_0^2}$$

$$S_F(\omega=\omega_0) = I\, m\gamma$$

$$S_F(\omega) = \frac{2I}{\omega}\, \frac{\mathrm{Im}\,\chi(\omega)}{|\chi(\omega)|^2}$$

$$S_x = \frac{2I}{\omega}\, \mathrm{Im}(\chi(\omega))$$

The simple harmonic oscille



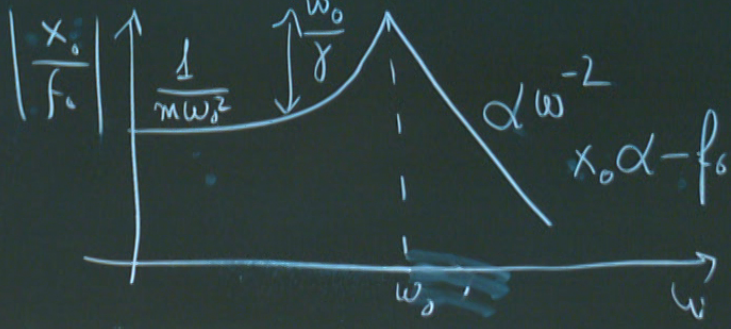
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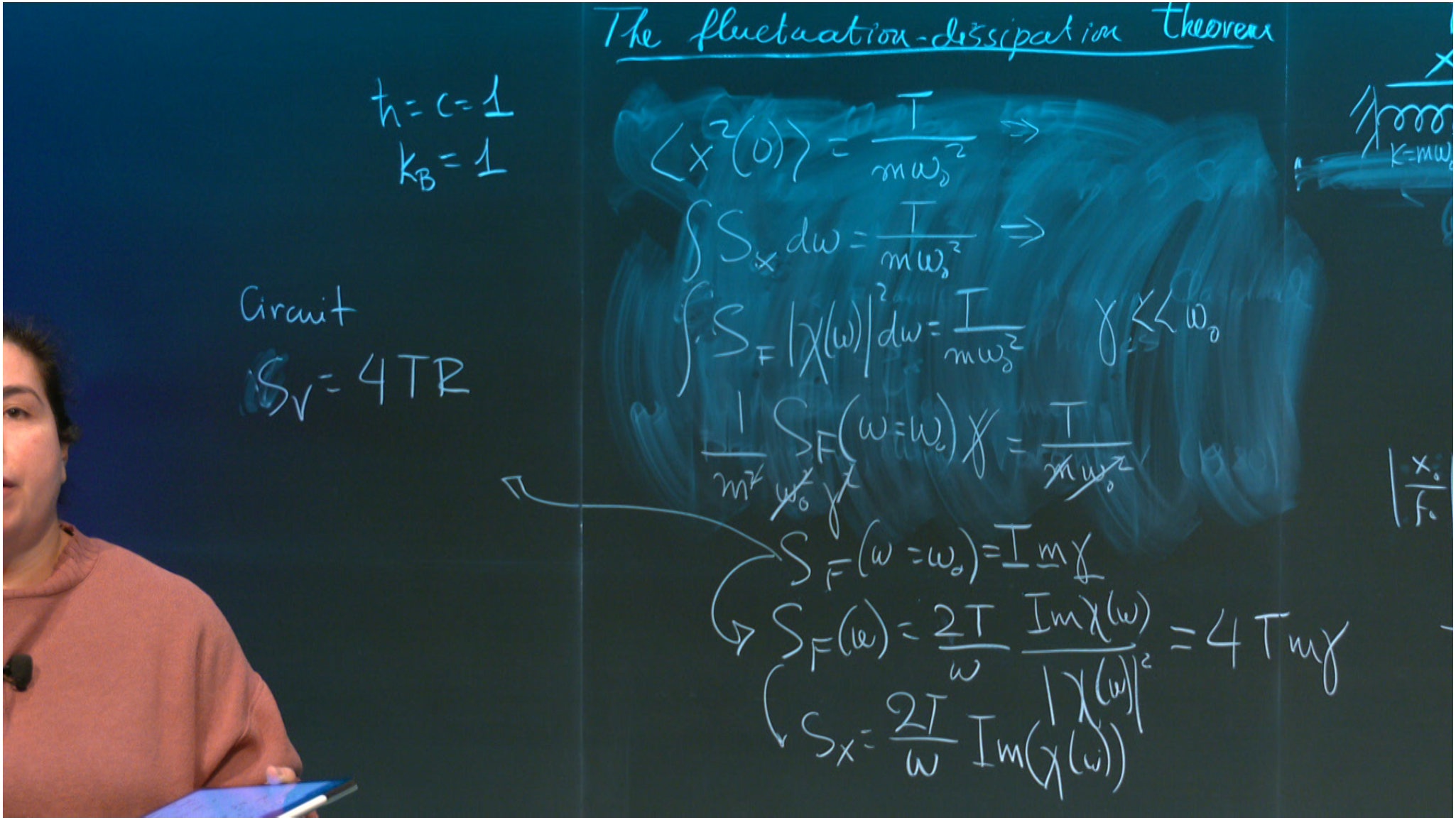
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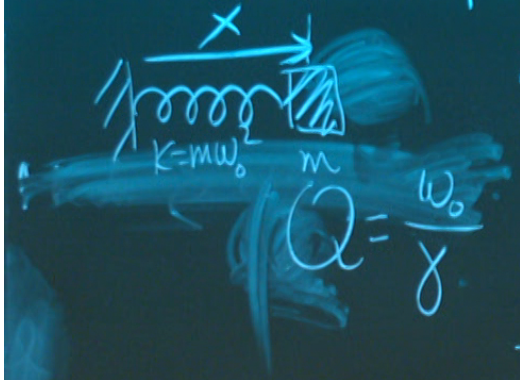
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The fluctuation-dissipation theorem
$\hbar = c = 1$
$k_B = 1$
Circuit
$S_V = 4TR$
$\langle x^2(0) \rangle = \frac{T}{m\omega_0^2} \Rightarrow$
$\int S_x \, d\omega = \frac{T}{m\omega_0^2} \Rightarrow$
$\int S_F |\chi(\omega)|^2 d\omega = \frac{T}{m\omega_0^2}$ $\gamma \ll \omega_0$
$\frac{1}{m^2 \omega_0^2 \gamma^2} S_F(\omega = \omega_0) \gamma = \frac{T}{m\omega_0^2}$
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$S_x = \frac{2T}{\omega} \mathrm{Im}(\chi(\omega))$
$K = m\omega$
$\left| \frac{x_0}{F_0} \right|$

The simple harmonic oscillator



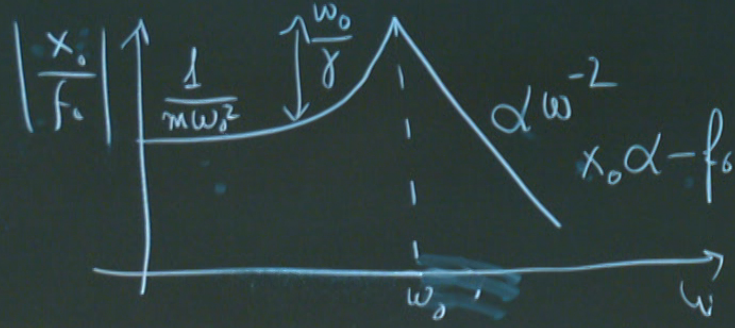
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$m\gamma$

$T \to 0 \quad \frac{2T}{\omega} \to \coth\left(\frac{\omega}{2T}\right)$

dt per cycle

$\bar{x} = \mathrm{Re}($

$f = \mathrm{Re}$

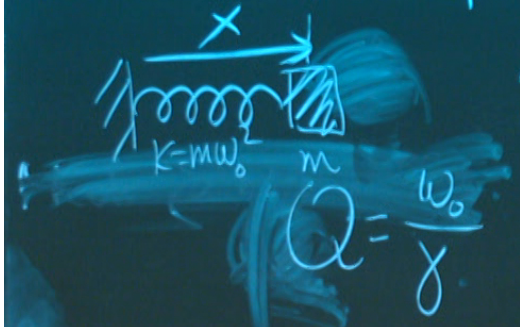
$\langle f(t) f($

$\langle x(t) x($

$\frac{S_x}{S_f} =$

P. E. $= \frac{1}{2}T$

The simple harmonic oscillator



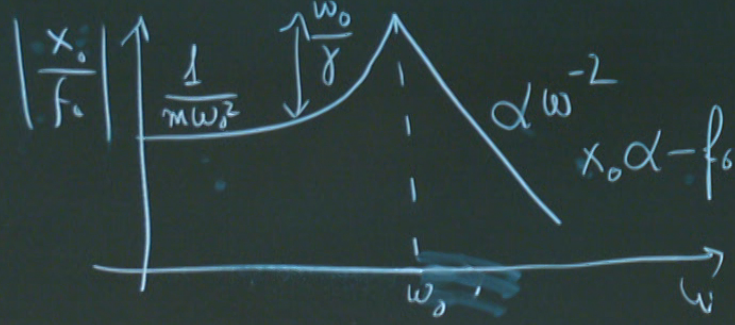
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$$\frac{x_0}{f_0}=\frac{1}{m}\cdot\frac{1}{-\omega^2+\omega_0^2+i\gamma\omega}\equiv\chi(\omega)$$ ← susceptibility



$m\gamma$

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$$\langle x^2\rangle_{T=0}=\frac{1}{m\omega_0}$$

dt per cycle

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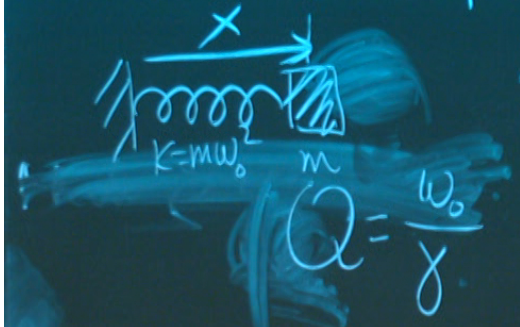
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The simple harmonic oscillator



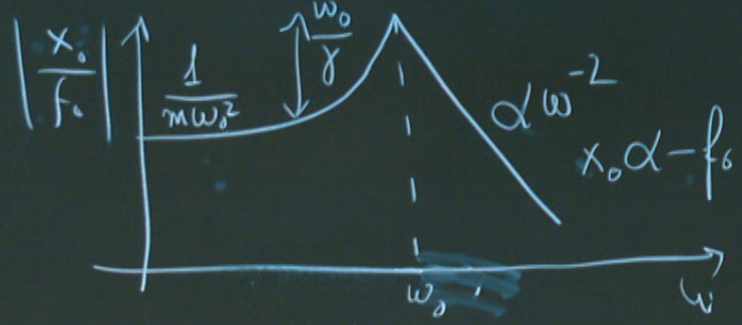
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$m\gamma$

$$\frac{2\Gamma Q}{\omega} \rightarrow \coth\left(\frac{\omega}{2T}\right)$$

$$\langle x^2 \rangle_{T=0} = \frac{1}{m\omega_0}$$

dt per cycle

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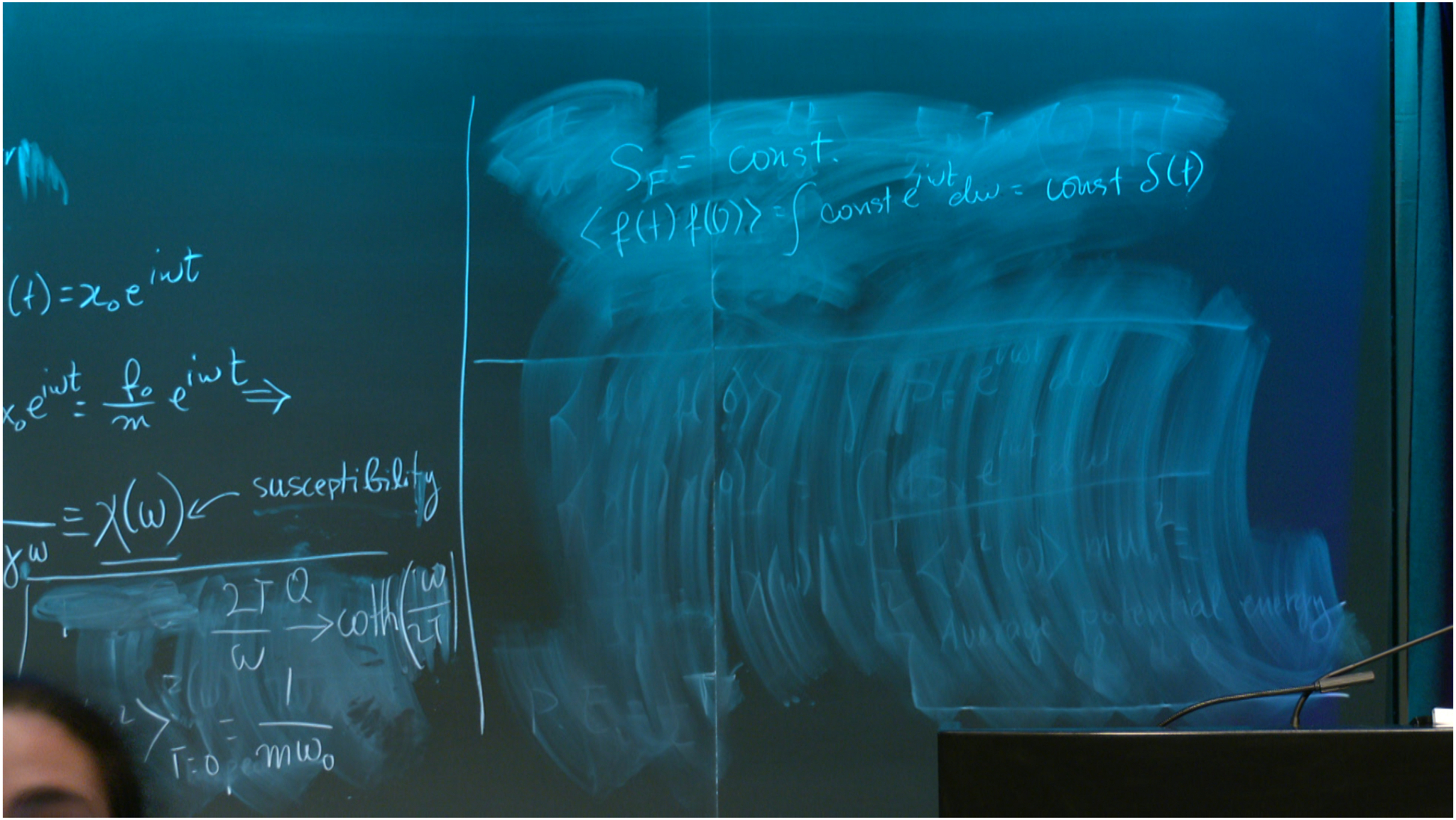
$f = \mathrm{Re}$

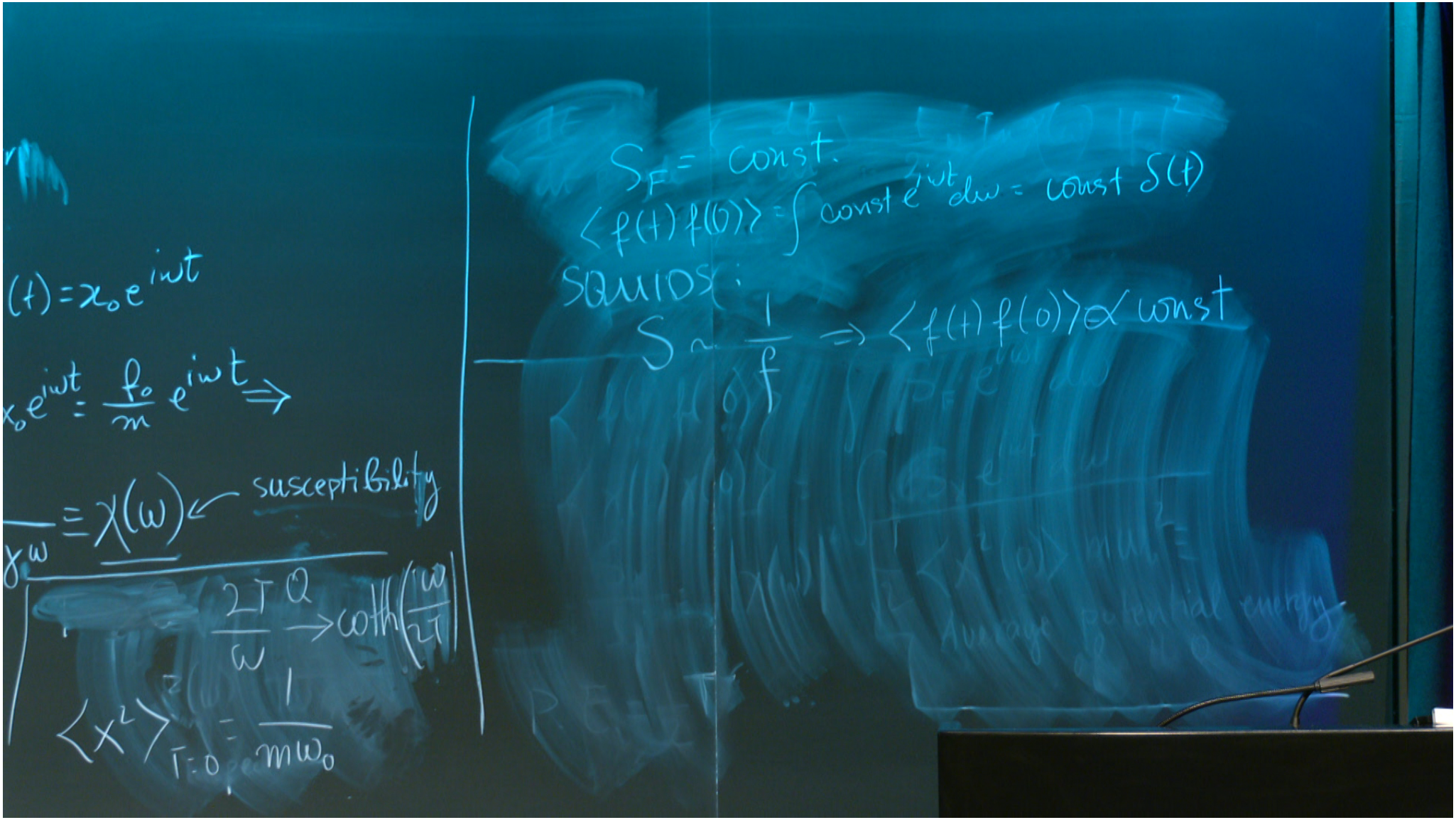
$\langle f(t) f($

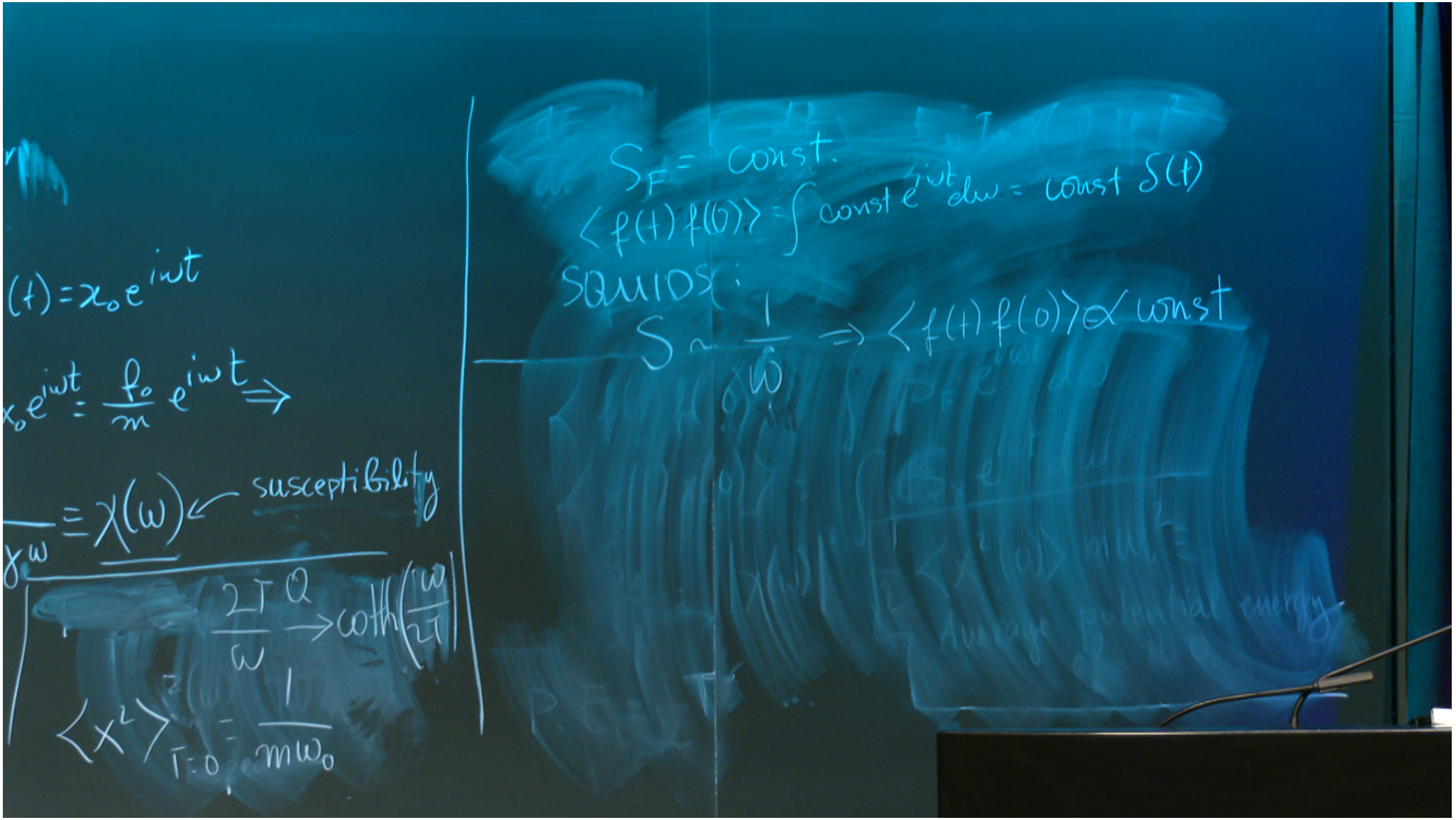
$\langle x(t) x($

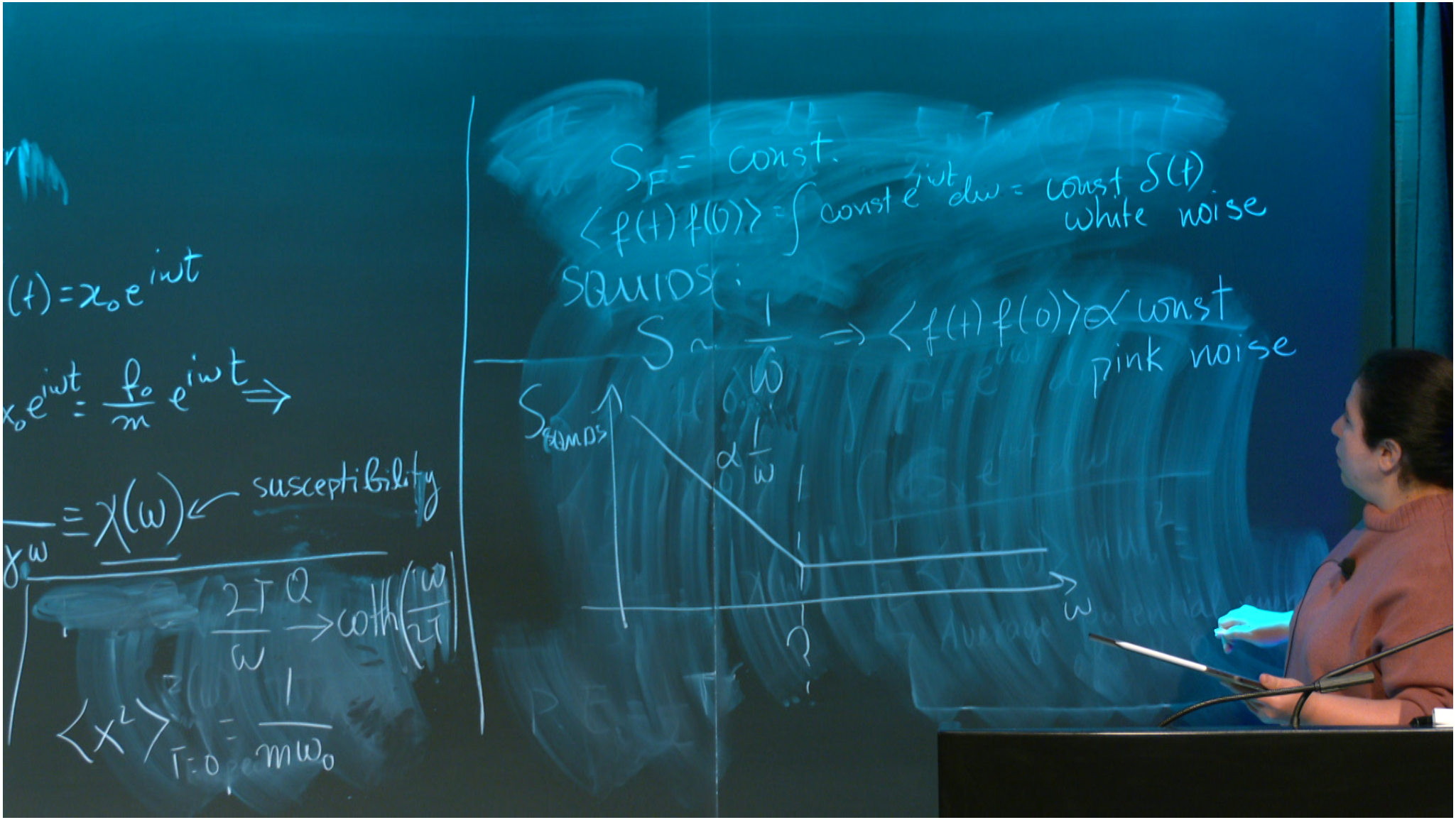
$\frac{S_x}{S_f} =$

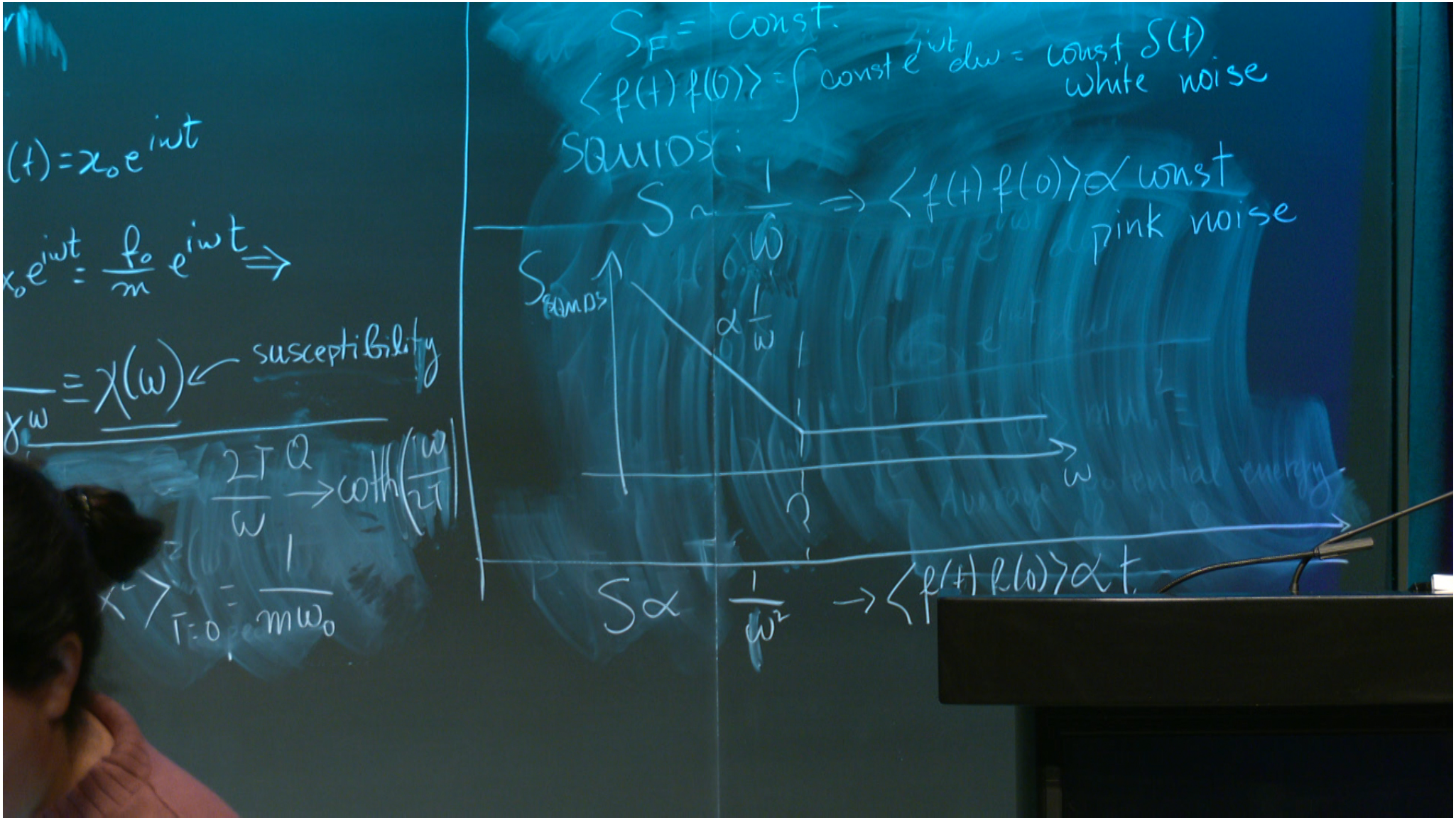
P.E. $= \frac{1}{2}T$

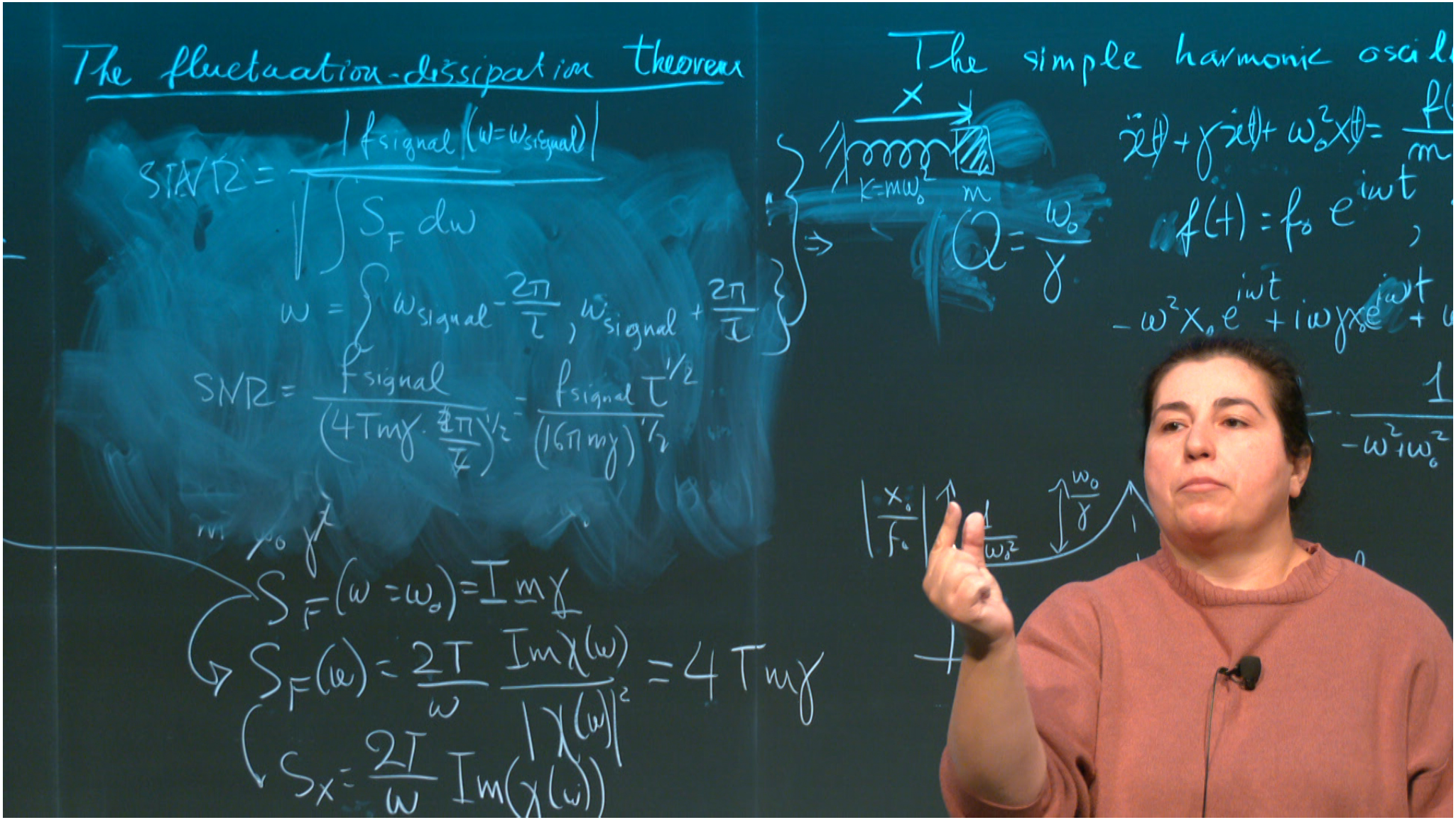

$S_F = \text{const.}$
$\langle f(t) f(0) \rangle = \int \text{const}\, e^{i\omega t}\, d\omega = \text{const}\, \delta(t)$
$(t) = x_0 e^{i\omega t}$
$x_0 e^{i\omega t} = \frac{f_0}{m} e^{i\omega t} \Rightarrow$
$= \chi(\omega)$ ← susceptibility
$\frac{2\Gamma Q}{\omega} \to \coth\left(\frac{\omega}{2T}\right)$
$\langle \quad \rangle_{T=0} = \frac{1}{m\omega_0}$
Average potential energy


$S_F = \text{const.}$
$\langle f(t) f(0) \rangle = \int \text{const}\, e^{i\omega t} d\omega = \text{const}\, \delta(t)$
SQUIDS:
$S \sim \frac{1}{f} \Rightarrow \langle f(t) f(0) \rangle \propto \text{const}$
$(t) = x_0 e^{i\omega t}$
$x_0 e^{i\omega t} = \frac{f_0}{m} e^{i\omega t} \Rightarrow$
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Average potential energy


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$\gamma\omega$
$\frac{2\Gamma Q}{\omega} \rightarrow \coth\left(\frac{\omega}{2T}\right)$
$\langle x^2 \rangle_{T=0} = \frac{1}{m\omega_0}$
Average potential energy


$S_F = \text{const.}$
$\langle f(t) f(0) \rangle = \int \text{const}\, e^{i\omega t} d\omega = \text{const}\, \delta(t)$
white noise
SQUIDS:
$S \sim \frac{1}{\omega} \Rightarrow \langle f(t) f(0) \rangle \propto \text{const}$
pink noise
S_{SQUIDS}
$\propto \frac{1}{\omega}$
ω
$x(t) = x_0 e^{i\omega t}$
$x_0 e^{i\omega t} = \frac{f_0}{m} e^{i\omega t} \Rightarrow$
$= \chi(\omega)$ ← susceptibility
$\gamma\omega$
$\frac{2\Gamma Q}{\omega} \rightarrow \coth\left(\frac{\omega}{2T}\right)$
$\langle x^2 \rangle_{T=0} = \frac{1}{m\omega_0}$


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$\frac{2\Gamma Q}{\omega} \to \coth\left(\frac{\omega}{2T}\right)$
$\langle x^2 \rangle_{T=0} = \frac{1}{m\omega_0}$
$S \propto \frac{1}{\omega^2} \to \langle f(t) f(0) \rangle \propto t$


The fluctuation-dissipation theorem
The simple harmonic oscil
SNR2 =
S_F dω
ω = { ω_signal − 2π/τ , ω_signal + 2π/τ }
SNR2 =
S_F(ω = ω_0) = Im γ
S_F(ω) = 2T/ω · Im χ(ω)/|χ(ω)|² = 4Tmγ
S_x = 2T/ω Im(χ(ω))
K = mω_0²
m
Q = ω_0/γ
f(t) = f_0 e^{iωt}
ω_0
γ

The fluctuation-dissipation theorem

$$SNR^2 = \frac{|f_{signal}(\omega=\omega_{signal})|}{\sqrt{\int S_F \, d\omega}}$$

$$\omega = \left\{ \omega_{signal} - \frac{2\pi}{\tau}, \omega_{signal} + \frac{2\pi}{\tau} \right\}$$

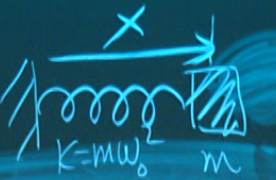
$$SNR^2 = \frac{f_{signal}}{\left(4Tm\gamma \cdot \frac{4\pi}{\tau}\right)^{1/2}} = \frac{f_{signal}\,\tau^{1/2}}{(16\pi m\gamma T)^{1/2}} = 1$$

$$S_F(\omega=\omega_0) = Im\gamma$$

$$S_F(\omega) = \frac{2T}{\omega}\frac{Im\chi(\omega)}{|\chi(\omega)|^2} = 4Tm\gamma$$

$$S_x = \frac{2T}{\omega} Im(\chi(\omega))$$

The simple harmonic oscil



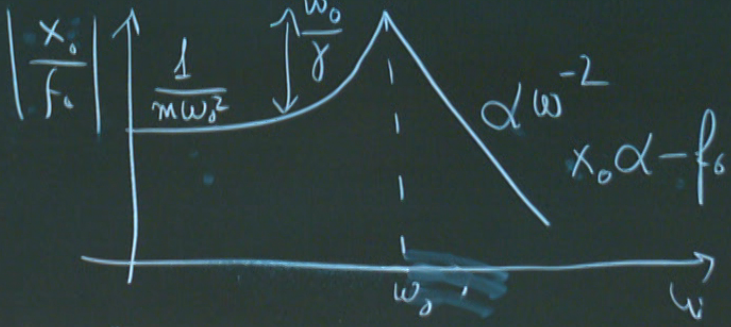
$$Q = \frac{\omega_0}{\gamma}$$

$$\ddot{x}(t) + \gamma\dot{x}(t) + \omega_0^2 x(t) = \frac{f(}{m}$$

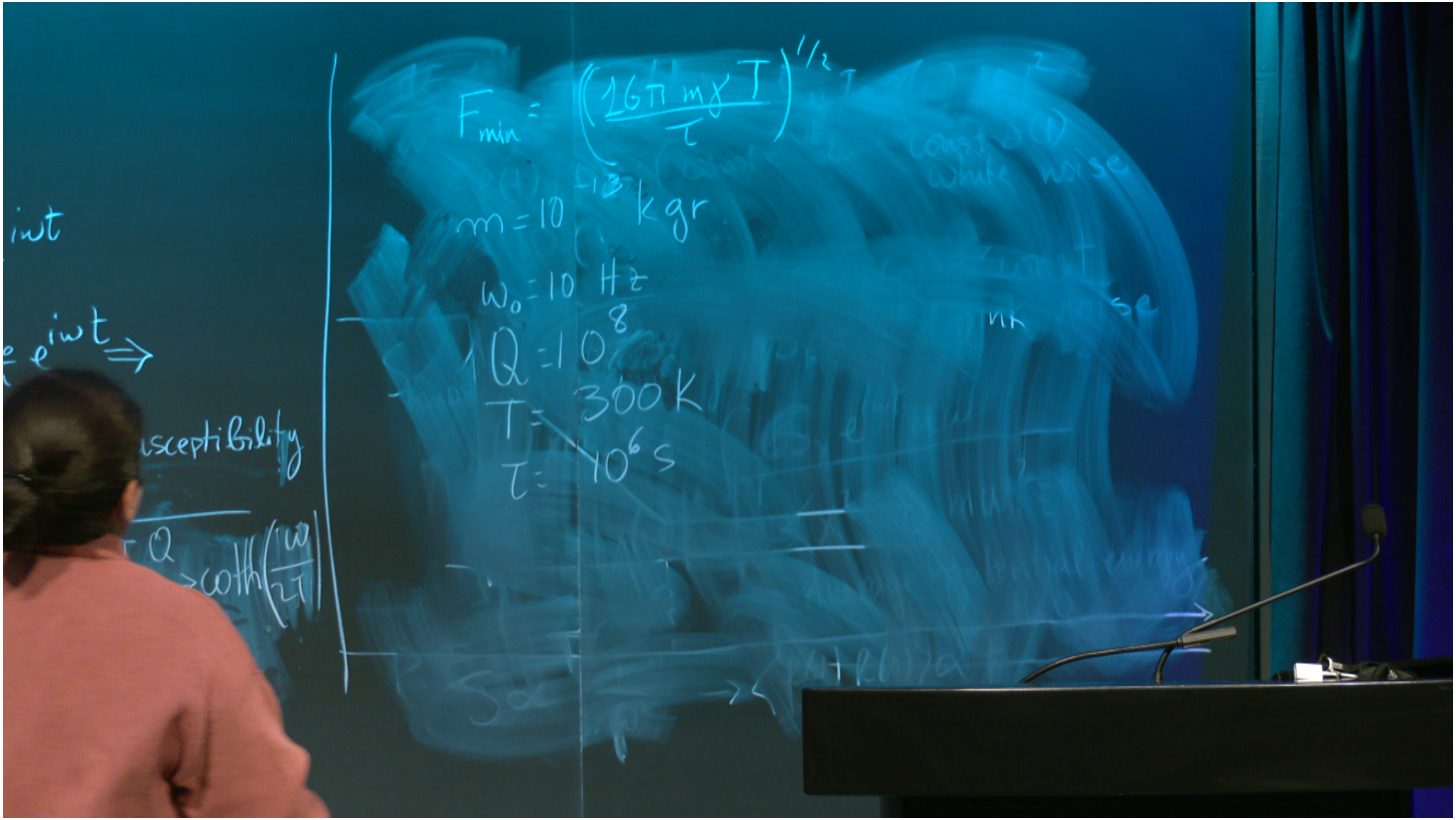
$$f(t) = f_0 e^{i\omega t},$$

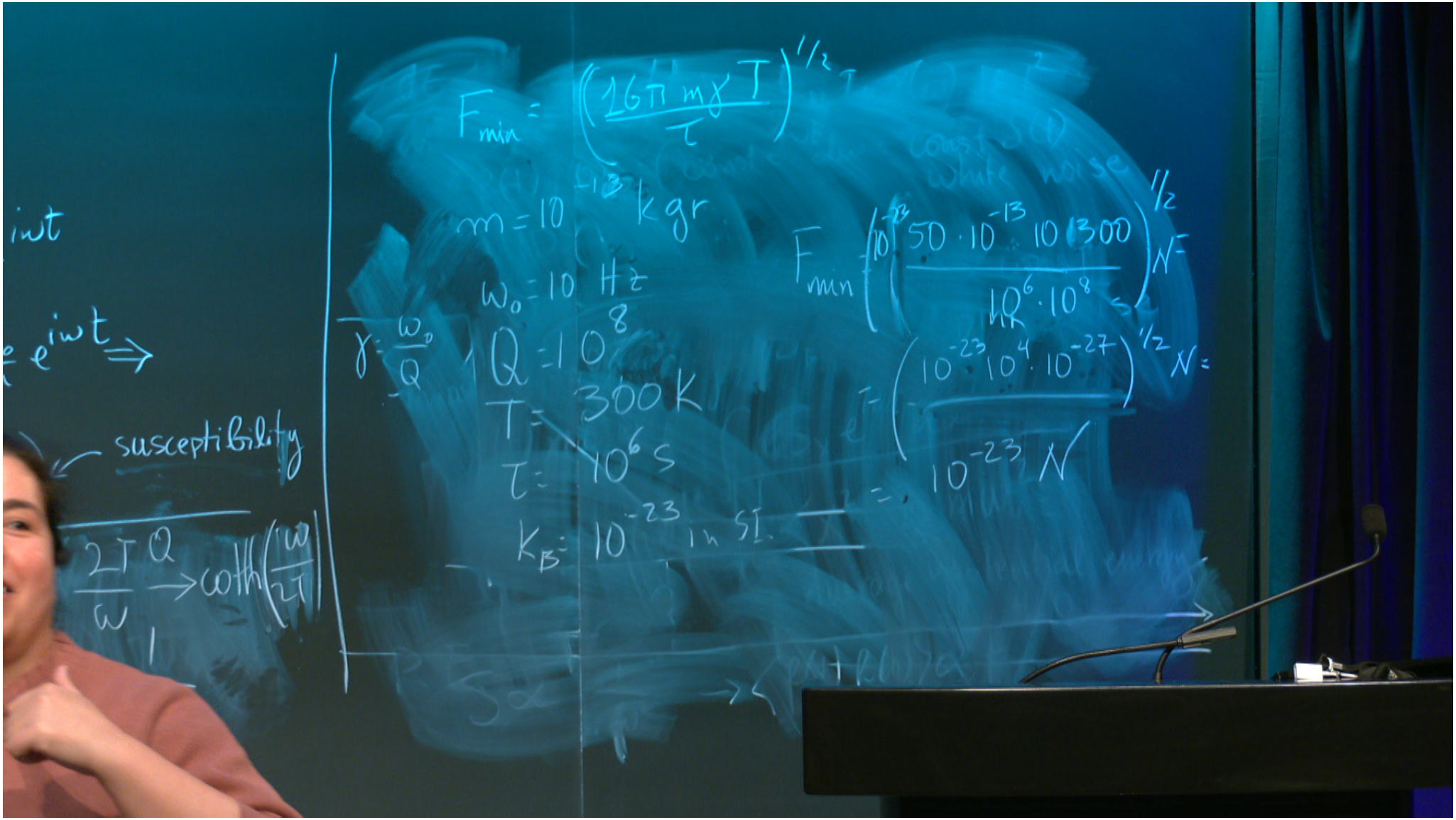
$$-\omega^2 x_0 e^{i\omega t} + i\omega\gamma x_0 e^{i\omega t} + \omega$$

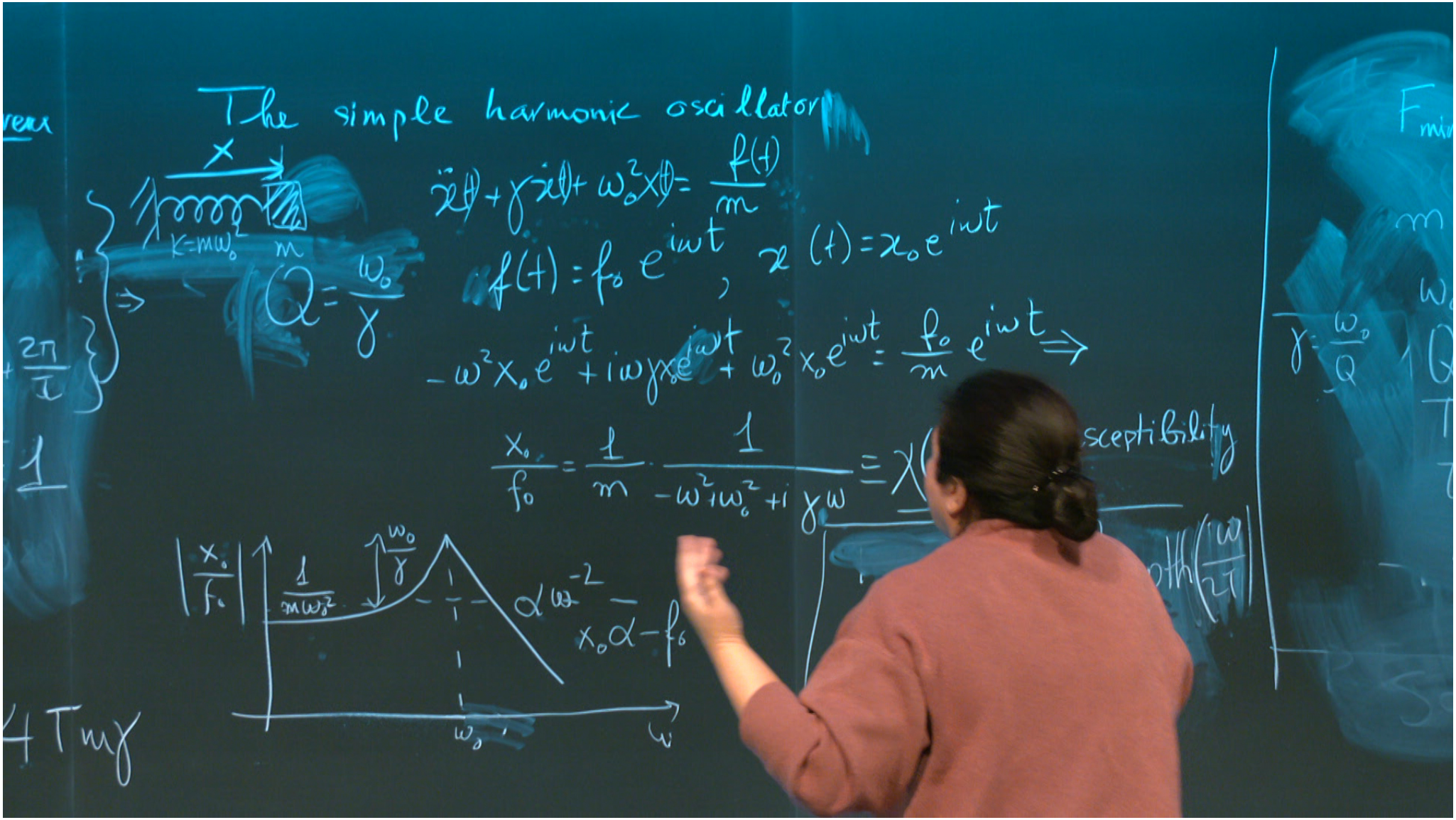
$$\frac{x_0}{f_0} = \frac{1}{m} \cdot \frac{1}{-\omega^2 + i\omega_0^2}$$

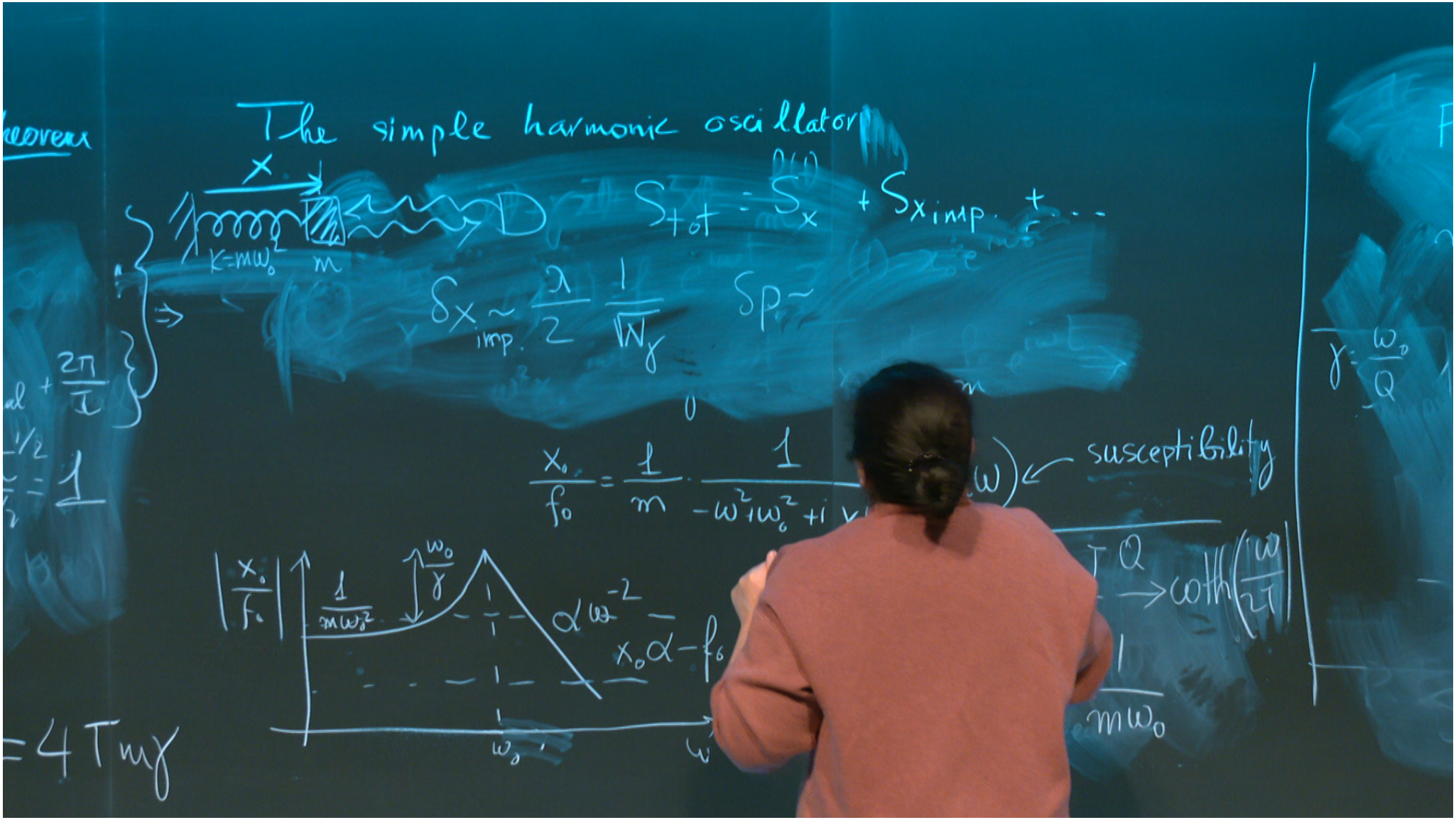


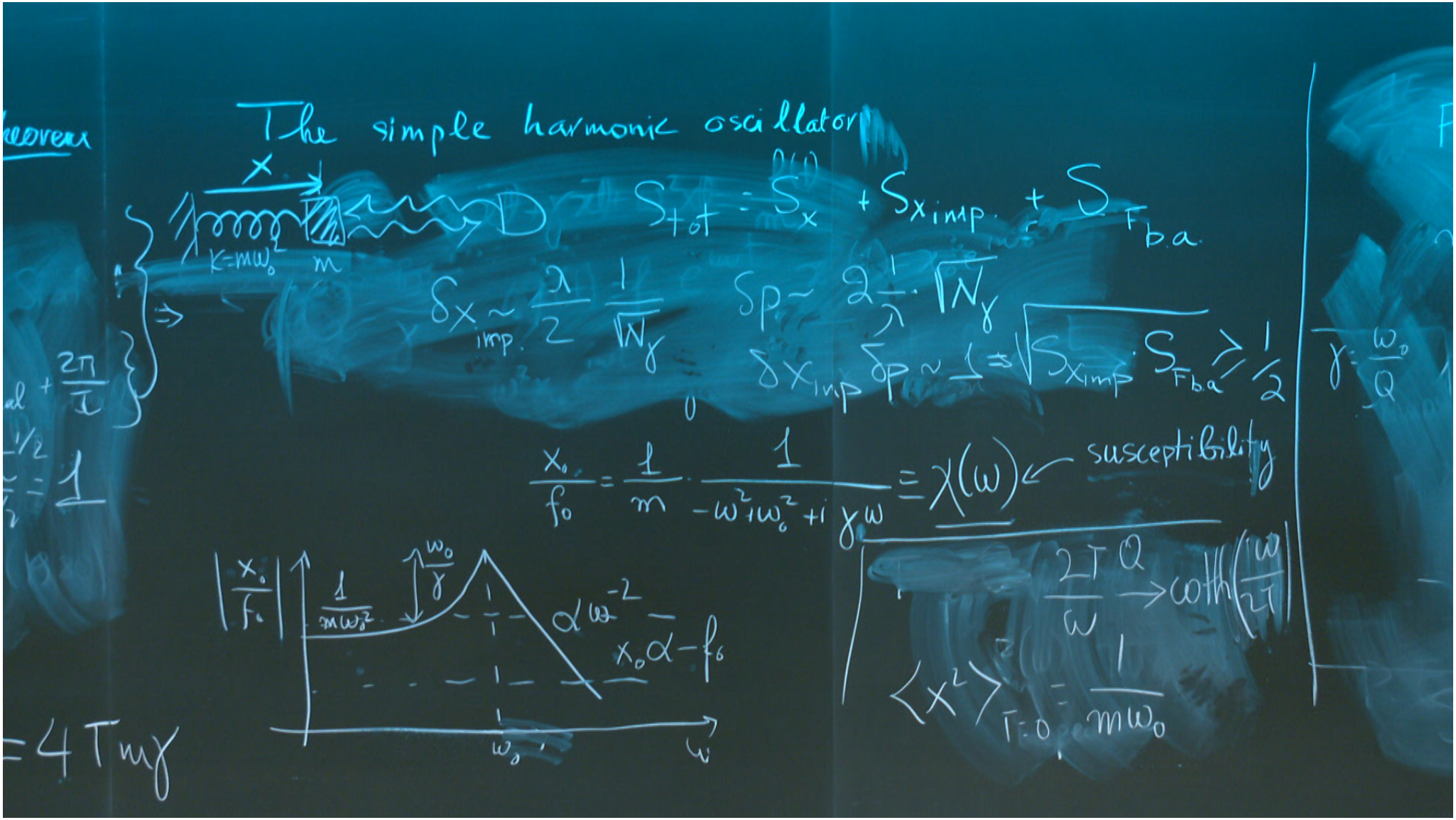

$F_{min} = \left(\frac{16\pi m \gamma T}{\tau}\right)^{1/2}$
susceptibility
$\frac{2TQ}{\omega} \to \coth\left(\frac{\omega}{2T}\right)$
$= \frac{1}{m\omega_0}$
$T=0$


$F_{min} = \left(\frac{16\pi m \gamma T}{\tau}\right)^{1/2}$
$m = 10^{-12}$ kgr
$\omega_0 = 10$ Hz
$Q = 10^8$
$T = 300$ K
$\tau = 10^6$ s
susceptibility
$e^{i\omega t} \Rightarrow$
$\coth\left(\frac{\omega}{2T}\right)$
white noise


$F_{min} = \left(\frac{16\pi m \gamma T}{\tau}\right)^{1/2}$
$m = 10^{-13}$ kgr
$\omega_0 = 10$ Hz
$\gamma = \frac{\omega_0}{Q}$
$Q = 10^8$
$T = 300$ K
$\tau = 10^6$ s
$k_B = 10^{-23}$ in SI.
$F_{min} = \left(10^{-23} \frac{50 \cdot 10^{-13} \cdot 10 \cdot 300}{10^6 \cdot 10^8}\right)^{1/2} N =$
$= \left(\frac{10^{-23} \cdot 10^4 \cdot 10^{-27}}{\quad}\right)^{1/2} N =$
$= 10^{-23}$ N
susceptibility
$\frac{2TQ}{\omega} \rightarrow \coth\left(\frac{\omega}{2T}\right)$
$e^{i\omega t} \Rightarrow$
$i\omega t$


The simple harmonic oscillator
x
$K = m\omega_0^2$
m
$Q = \frac{\omega_0}{\gamma}$
$\ddot{x}(t) + \gamma \dot{x}(t) + \omega_0^2 x(t) = \frac{f(t)}{m}$
$f(t) = f_0 e^{i\omega t}$, $x(t) = x_0 e^{i\omega t}$
$-\omega^2 x_0 e^{i\omega t} + i\omega\gamma x_0 e^{i\omega t} + \omega_0^2 x_0 e^{i\omega t} = \frac{f_0}{m} e^{i\omega t} \Rightarrow$
$\frac{x_0}{f_0} = \frac{1}{m} \cdot \frac{1}{-\omega^2 + \omega_0^2 + i\gamma\omega} = \chi($
sceptibility
$\left|\frac{x_0}{f_0}\right|$
$\frac{1}{m\omega_0^2}$
ω_0
γ
$\propto \omega^{-2}$
$x_0 \propto f_0$
ω_0
ω
$\gamma = \frac{\omega_0}{Q}$
4Tmγ


The simple harmonic oscillator
susceptibility


The simple harmonic oscillator
x
$K = m\omega_0^2$
m
$S_{tot} = S_x + S_{x\,imp.} + S_{F\,b.a.}$
$\delta x_{imp.} \sim \frac{\lambda}{2} \frac{1}{\sqrt{N_\gamma}}$
$\delta p \sim 2\frac{1}{\lambda} \cdot \sqrt{N_\gamma}$
$\delta x_{imp} \delta p \sim 1 \Rightarrow \sqrt{S_{x\,imp} \cdot S_{F\,b.a}} \geq \frac{1}{2}$
$\frac{x_0}{f_0} = \frac{1}{m} \cdot \frac{1}{-\omega^2 + \omega_0^2 + i\gamma\omega} = \chi(\omega)$ ← susceptibility
$\gamma = \frac{\omega_0}{Q}$
$\frac{2\pi Q}{\omega} \to \coth\left(\frac{\omega}{2T}\right)$
$\langle x^2 \rangle_{T=0} = \frac{1}{m\omega_0}$
$\left|\frac{x_0}{f_0}\right|$
$\frac{1}{m\omega_0^2}$
$\frac{\omega_0}{\gamma}$
$\propto \omega^{-2}$
$x_0 \propto f_0$
ω_0
ω
$= 4Tm\gamma$