Title: Exact results for metallic quantum critical points

Speakers: Hart Goldman

Series: Quantum Matter

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Abstract: I discuss how exact, non-perturbative results can be obtained for both optical transport and static susceptibilities in "Hertz-Millis" theories of Fermi surfaces coupled to critical bosons. Such models possess a large emergent symmetry and anomaly structure, which we leverage to fix these quantities. In particular, I will show that in the infrared limit, the boson self energy at zero wave vector is a constant independent of frequency, and the real part of the optical conductivity is purely a delta function Drude peak with no other corrections. I will also obtain exact relations between Fermi liquid parameters as the critical point is approached from the disordered phase.

Zoom Link: https://pitp.zoom.us/j/93340611986?pwd=cisrZmFxcEVWZVdrT2tMRVZiVTdRQT09
Exact results for metallic quantum critical points

Hart Goldman

Based on:

*arXiv:2204.07585*

*arXiv:2208.04328*
Based on work with

Zhengyan Darius Shi  
/MIT\

Dominic Else  
/Harvard → PI\

Senthil  
/MIT\
Fermi liquid theory and its discontents

• Fermi liquid theory (FLT) has been tremendously successful in describing conventional metals in terms of a Fermi surface + stable quasiparticles.

• Still many systems in nature **NOT** captured by FLT
Fermi liquid theory and its discontents

- Fermi liquid theory (FLT) has been tremendously successful in describing conventional metals in terms of a Fermi surface + stable quasiparticles.

- Still many systems in nature **NOT** captured by FLT

- **Creative name:** non-Fermi liquids (NFLs)
Fermi liquid theory and its discontents

Strange metals

Nd-LSCO

$\rho (\mu \Omega \text{cm})$

\begin{align*}
\text{T (K)} & \\
0 & 20 & 40 & 60 & 80 & 100
\end{align*}

$H = 16 \text{T}$

Nd-LSCO

$p = 0.24$

LSCO

$B (\text{T})$

\begin{align*}
0 & 20 & 40 & 60 & 80 & 50 & 50 & 25 & 25
\end{align*}

$\beta$

[Proust and Taillefer, Ann. Rev. (2019)]

[Giraldo-Gallo et al., Science (2018)]
Fermi liquid theory and its discontents

Anomalous metals

InOx

[Breznay and Kapitulnik, Science Advances (2017)]

Composite Fermi Liquids

LaAlO3/SrTiO3

[Chen et al., npj Quantum Matter (2021)]

From [Stern, Ann. Phys. (2008)], data from W. Pan (Sandia)
Suggestions of quantum criticality

- Apparent unifying feature of NFLs is "quantum critical behavior" or proximity to a (real or conjectured) quantum critical point (QCP).
- **Strange metals**: experiments suggest a scaling form,

\[
\sigma(\omega, T) = \frac{1}{T^\alpha} \sum \left( \frac{\omega}{T} \right)
\]

[van der Marel et al., Nature (2003)]

[Prochaska et al., Science (2020)]
Suggestions of quantum criticality

- Apparent unifying feature of NFLs is "quantum critical behavior" or proximity to a (real or conjectured) quantum critical point (QCP).
- **Strange metals:** experiments suggest a scaling form,

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\sigma(\omega, T) = \frac{1}{T^\alpha} \sum \left( \frac{\omega}{T} \right)
\]

- So far, no physical model of a clean metallic state has been shown to display this behavior.

[van der Marel et al., Nature (2003)]

[Prochaska et al., Science (2020)]
Quantum criticality
beyond the Landau (Fermi liquid) paradigm

- Quantum criticality presents a natural mechanism for evading the predictions of Fermi liquid theory: killing the quasiparticles.

- Quasiparticles can scatter off of gapless fluctuations at all energy scales

\[
\lim_{\omega \to 0} \frac{\Gamma(\omega)}{\omega} \neq 0
\]

In a FL:
\[
\Gamma(\omega) \sim \omega^2
\]
\[
\Rightarrow \frac{\Gamma(\omega)}{\omega} \to 0
\]
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- **Big questions:** Can we understand the ground state physics of metallic QCPs? Study tractable models? Find a clean model with $\omega/T$ conductivity scaling?
  - *Problems have vexed us for 30+ years!*
The Hertz-Millis paradigm

- Simplest class of theories of metallic QCPs was first introduced and studied by [Hertz, PRB (1976)] and [Millis, PRB (1993)].

- **Basic idea:** Fermi surface coupled to an order parameter near a QCP.

- At the QCP, scattering with the gapless boson kills the quasiparticles at all scales.
The Hertz-Millis paradigm

- Simplest class of theories of metallic QCPs was first introduced and studied by [Hertz, PRB (1976)] and [Millis, PRB (1993)].

- **Basic idea:** Fermi surface coupled to an order parameter near a QCP.

- At the QCP, scattering with the gapless boson kills the quasiparticles at all scales.

- **We focus on simple models as a guide to understanding universal physics**, but they may not always capture observed phenomena (e.g. in cuprates).
The Hertz-Millis paradigm

\[ S = S_\psi + S_{\phi \psi} + S_\phi \]

\[ S_\psi = \int_{\omega, k} \psi^\dagger [i \omega - \epsilon(k)] \psi \]

\[ S_{\phi \psi} = \int_{k, q} g(k, q) \phi(q) \psi^\dagger(k + q) \psi(k) \]
The Hertz-Millis paradigm

\[ S = S_\psi + S_{\phi\psi} + S_\phi \]

\[ S_\psi = \int_{\omega, \mathbf{k}} \psi^\dagger [i\omega - \epsilon(\mathbf{k})] \psi \]

\[ S_{\phi\psi} = \int_{\mathbf{k}, \mathbf{q}} g(\mathbf{k}, \mathbf{q}) \phi(\mathbf{q}) \psi^\dagger(\mathbf{k} + \mathbf{q}) \psi(\mathbf{k}) \]

\[ S_\phi = \int_{\tau, \mathbf{x}} \frac{1}{2} [\lambda(\partial_\tau \phi)^2 + J(\nabla \phi)^2 + r_c \phi^2 + \ldots] \]
The treacherous path to the IR

- Fix to $d = 2$ dimensions, **no small parameter**.

- **Integrate out fermions**: $\Pi(\omega, q) = \gamma \frac{|\omega|}{|q|}$, boson becomes overdamped by decay into particle-hole pairs.
  - dynamical exponent $z = 3$.

- **Integrate out bosons**: fermion self energy, $\Sigma(\omega) = i C |\omega|^{2/3}$
  - $\Gamma(\omega)/\omega \sim \omega^{-1/3} \Rightarrow$ quasiparticles die.

- **Theory is IR sick**, usual large-$N$ (RPA) technique breaks down [Lee, PRB (2009)].

$$G_{\bar{\psi}\psi}(\omega) = \frac{1}{i \omega + i C |\omega|^{2/3}/N} \rightarrow \frac{N}{i C |\omega|^{2/3}}$$
A litany of perturbative deformations

- Much effort toward **perturbative progress** over the past decade:
  - Large-$N$, small $z - 2$, $N(z - 2)$ fixed
    [Mross et al., PRB (2010)], also: [Ye, Lee, and Zou, PRL (2022)]
  - Matrix boson large-$N$ [Mahajan et al., PRB (2013)],
    …, [Aguilera Damia et al., PRL (2019)]
  - Co-dimensional regularization
    [Dalidovich and Lee, PRB (2013)]
  - Random flavor/SYK-esque large-$N$
    [Esterlis and Schmalian, PRB (2019)],
    [Aldape et al., arXiv: 2012.00763], [Esterlis et al., PRB (2021)]
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    [Aldape et al., arXiv: 2012.00763], [Esterlis et al., PRB (2021)]

- Lead to formally controlled IR fixed points, **but many observable properties**, e.g. transport, still hard to capture.
Beyond perturbation theory

- Would like to have **general, non-perturbative constraints** on theories in the Hertz-Millis paradigm.

- [Else, Thorngren, and Senthil, PRX (2021)]: emergent symmetry and anomaly structure shared by all compressible metals. **Can we leverage this philosophy to make progress?**

- **Philosophy of this talk:** start with a theory at some energy scale, $\Lambda_{\text{mid-IR}}$, derive properties that are valid either

  1. At all scales below $\Lambda_{\text{mid-IR}}$, down to the deep IR.

  2. Sitting exactly at the IR fixed point (assumes a second order transition).

- **Broader program of progress using non-perturbative principles:** can extend beyond Hertz-Millis.
Preview of exact results

- At the IR fixed point (in the clean limit), we find:

1. Exact expressions for the boson self-energy, $\Pi(q = 0, \omega)$, and optical conductivity:

$$\text{Re } \sigma(\omega, T) = D \delta(\omega)$$

Ising-nematic: $D = D_{\text{free}}$
Varma loop current: $0 < D < D_{\text{free}}$
Gauge field (e.g. HLR): $D = 0$

$\sigma(\omega \neq 0) = 0 \Rightarrow \text{Critical fluctuations do not generate conductivity}$
Preview of exact results

- At the IR fixed point (in the clean limit), we find:

2. Exact expressions for static density susceptibilities ($\omega \to 0$, then $q \to 0$) as the critical point is approached. If $\tilde{n}_\theta$ is the density at FS angle $\theta$, find

$$\chi \tilde{n}_\theta \tilde{n}_\theta, (r \to r_c) \sim \frac{\delta \chi \theta \theta'}{r - r_c} \to \infty$$

in the order parameter channel as $r \to r_c$. Divergence was expected from earlier arguments. [Maslov and Chubukov, PRB (2010)], [Mross et al., PRB (2010)].
How is this possible? Gifts from anomalies

- **Physics by analogy:** low energy degrees of freedom near the FS look like 1D chiral fermions

  \[ \epsilon(k) = v_F(\theta) k_\perp + \kappa(\theta) k_\parallel^2 + \ldots \]

- **“Chiral” anomaly:** charge is not conserved at individual \( \theta \), needs to be compensated by the rest of the FS.

- Anomalies are non-perturbative, quantum deformations of conservation laws.

- In 1D, chiral anomaly leads to tight constraints and underlies bosonization. **Same will be true here!**

Roadmap

I) The chiral anomaly and transport in 1D

II) Symmetries and anomalies of Fermi surfaces + critical bosons

III) Exact results: $\Pi(\omega)$, $\sigma(\omega)$, and $\chi_{\tilde{n}_{\theta}}^2$.

IV) A model with fixed point conductivity (time permitting)
A trivial example: a free Dirac in 1D

\[ S = \int_{t,x} \bar{\psi} \gamma^\mu (\partial_\mu + iA_\mu) \psi \]

- \( \psi = (\psi_L, \psi_R) \). Theory classically conserves charge at each Fermi point:

\[ U(1)_L : \psi_L \rightarrow e^{i\alpha_L} \psi_L \]
\[ U(1)_R : \psi_R \rightarrow e^{i\alpha_R} \psi_R \]
A trivial example: a free Dirac in 1D

\[
S = \int_{t, \mathbf{x}} \bar{\psi} \gamma^\mu (\partial_\mu + i A_\mu) \psi
\]

- Turn on background \( A_\mu \): get a current — L, R charge no longer conserved (‘t Hooft anomaly)

\[
\partial_\mu j_\mu^L = -\frac{1}{2\pi} E , \quad \partial_\mu j_\mu^R = +\frac{1}{2\pi} E
\]

\[
\begin{align*}
    j_0^L &= \rho_L = \psi_L^\dagger \psi_L \\
    j_1^L &= \psi_L^\dagger \psi_L \\
    j_0^R &= \rho_R = \psi_R^\dagger \psi_R \\
    j_1^R &= -\psi_R^\dagger \psi_R
\end{align*}
\]
A trivial example: a free Dirac in 1D

\[ S = \int_{t,x} \bar{\psi} \gamma^\mu (\partial_\mu + i A_\mu) \psi \]

- Conservation of EM current \( j_V^\mu = j_L^\mu + j_R^\mu \), means axial current, \( j_A^\mu = j_L^\mu - j_R^\mu \), is **anomalous**. Notice axial density = EM current and vice versa

\[
\begin{align*}
  j_V^0 &= \rho \\
  j_V^1 &= J \\
  \partial_\mu j_V^\mu &= 0, \quad \partial_\mu j_A^\mu = -\frac{1}{\pi} E &\iff j_A^0 &= J \\
  j_A^1 &= \rho
\end{align*}
\]
A trivial example: a free Dirac in 1D

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    \partial_\mu j_V^\mu &= 0, \quad \partial_\mu j_A^\mu &= -\frac{1}{\pi} E \\
    j_A^0 &= J \\
    j_A^1 &= \rho
\end{align*}
\]

- Axial anomaly determines response to electric fields. Set \( q = 0 \):

\[
    i\omega J(\omega, q = 0) = -\frac{1}{\pi} E(\omega, q = 0) \Rightarrow \sigma(\omega) = \frac{1}{\pi} \frac{i}{\omega}
\]
A less trivial example: $\text{QED}_2$

- Let $A_\mu \to a_\mu$ fluctuate, get the Schwinger model ($\text{QED}_2$)

  \[ S = \int_{t,x} \left[ \bar{\psi} \gamma^\mu (\partial_\mu + ia_\mu) \psi - \frac{1}{4g^2} f^2 \right] \]

- Exactly solvable toy model of confinement ($a_\mu$ mediates a $V(r) \sim r$ potential)
A less trivial example: QED$_2$

- Let $A_\mu \rightarrow a_\mu$ fluctuate, get the **Schwinger model** (QED$_2$)

\[
S = \int_{t,x} \left[ \bar{\psi} \gamma^\mu (\partial_\mu + ia_\mu) \psi - \frac{1}{4g^2} f^2 \right]
\]

- Exactly solvable toy model of confinement ($a_\mu$ mediates a $V(r) \sim r$ potential)

- **Axial anomaly:** encodes response to emergent electric field of $a_\mu$

\[
\partial_\mu j^\mu_L = \frac{1}{2\pi} \varepsilon_{\mu\nu} \partial_\mu a_\nu, \quad \partial_\mu j^\mu_R = -\frac{1}{2\pi} \varepsilon_{\mu\nu} \partial_\mu a_\nu
\]

- Allows us to fix the propagator of $a_\mu$ exactly.
A less trivial example: QED$_2$

- Let $A_\mu \to a_\mu$ fluctuate, get the Schwinger model (QED$_2$)

\[
S = \int_{t,x} \left[ \bar{\psi} \gamma^\mu (\partial_\mu + ia_\mu) \psi - \frac{1}{4g^2} f^2 \right]
\]

- Using arguments so similar to what we will use later that we will hold off on presenting them, one can obtain:

\[
D_{xx}(\omega, q = 0) = \frac{1}{\omega^2 - g^2/\pi} \quad \text{"Plasmon\" with mass $g^2/\pi$}
\]
A less trivial example: QED$_2$

- Let $A_\mu \to a_\mu$ fluctuate, get the Schwinger model (QED$_2$)

$$S = \int_{t,x} \left[ \overline{\psi} \gamma^\mu (\partial_\mu + ia_\mu) \psi - \frac{1}{4g^2} f^2 \right]$$

- Using arguments so similar to what we will use later that we will hold off on presenting them, one can obtain:

$$D_{xx}(\omega, q = 0) = \frac{1}{\omega^2 - g^2/\pi} \quad \text{"Plasmon" with mass } g^2/\pi$$

$$\Pi_{xx}(\omega, q = 0) = \frac{1}{\pi} \quad \text{frequency independent}$$
The “mid-IR” theory

- Work with a mid-IR effective theory of a Fermi surface sliced into small patches $+ N_b$ boson species, $\phi_a$, coupling to each patch. Lives at $\Lambda_{\text{mid-IR}}$

\[
S_{\text{mid-IR}} = \sum_\theta S_{\text{patch}}(\theta) + S_\phi
\]

\[
S_{\text{patch}}(\theta) = \int_{t,x} \psi_\theta^\dagger \left\{ i\partial_t + iv_F(\theta)w(\theta) \cdot \nabla + \kappa_{ij}(\theta)\partial_i \partial_j \right\} \psi_\theta
\]

\[
+ \int_{t,x} g^a(\theta) \phi_a\psi_\theta^\dagger \psi_\theta
\]

Allow only curvature of FS:

$w_i \kappa_{ij}(\theta) = 0$
The “mid-IR” theory

- Work with a **mid-IR effective theory** of a Fermi surface sliced into small patches + \( N_b \) boson species, \( \phi_a \), coupling to each patch. Lives at \( \Lambda_{\text{mid-IR}} \)

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\]

\[
S_\phi = \frac{1}{2} \int_{t,x} \left[ \lambda (\partial_t \phi_a)^2 - r_c \phi_a \phi^a - J(\nabla \phi_a)^2 \right]
\]

V[\phi_a] = 0

Allow only curvature of FS:
\( w_i \kappa_{ij}(\theta) = 0 \)
Some key assumptions

\[ S_{\text{patch}}(\theta) = \int_{t,x} \psi_\theta^\dagger \left\{ i\partial_t + iv_F(\theta)w(\theta) \cdot \nabla + \kappa_{ij}(\theta)\partial_i\partial_j \right\} \psi_\theta + \int_{t,x} g^a(\theta) \phi_a \psi_\theta^\dagger \psi_\theta \]

• Not allowed: scattering of a fermion from patch \( \theta \) to another patch \( \theta' \) by absorbing/emitting a boson (large-angle scattering).

• Allowed: exchange of bosons between different patches (see figure).
Some key assumptions

$$S_{\text{patch}}(\theta) = \int_{t,x} \psi_\theta^\dagger \{ i \partial_t + i v_F(\theta) w(\theta) \cdot \nabla + \kappa_{ij}(\theta) \partial_i \partial_j \} \psi_\theta + \int_{t,x} g^\alpha(\theta) \phi_\alpha \psi_\theta^\dagger \psi_\theta$$

- **Not allowed:** scattering of a fermion from patch $\theta$ to another patch $\theta'$ by absorbing/emitting a boson (*large-angle scattering*).

- **Allowed:** exchange of bosons between different patches (see figure).

  - Will need to refine later for one result.
The “mid-IR” theory

\[ S_{\text{patch}}(\theta) = \int_{t,x} \psi_\theta^\dagger \left\{ i \partial_t + iv_F(\theta) \mathbf{w}(\theta) \cdot \nabla + \kappa_{ij}(\theta) \partial_i \partial_j \right\} \psi_\theta + \int_{t,x} g^a(\theta) \phi_a \psi_\theta^\dagger \psi_\theta \]

- Properties of \( g^a(\theta) \) determine the particular theory
  - Ising-nematic: \( g^a(\theta) = g^a(\theta + \pi) \)
  - “Loop current”: \( g^a(\theta) = -g^a(\theta + \pi) \)
  - Gauge field (e.g. HLR): \( g^i(\theta) = v_F(\theta) w^i(\theta) \)
Symmetries and anomalies

\[ S_{\text{patch}}(\theta) = \int_{t, \mathbf{x}} \psi_\theta^\dagger \{ i\partial_t + iv_F(\theta)w(\theta) \cdot \nabla + \kappa_{ij}(\theta)\partial_i\partial_j \} \psi_\theta + \int_{t, \mathbf{x}} g^a(\theta) \phi_a \psi_\theta^\dagger \psi_\theta \]

- Each patch looks like a chiral fermion, has $U(1)_\theta : \psi_\theta \to e^{i\alpha_\theta} \psi_\theta$ charge conservation.
- Total classical symmetry: $U(1)_{\text{patch}} = \Pi_\theta U(1)_\theta$.
- $g = 0$ : background electric fields lead to current, break charge conservation in each patch

\[ \partial_t n_\theta + \nabla \cdot j_\theta = -\frac{\Lambda_{\text{patch}}}{(2\pi)^2} w(\theta) \cdot \partial_t A \]

Patch density  Patch current  E - field \(\perp\) the FS
Symmetries and anomalies

\[ S_{\text{patch}}(\theta) = \int_{t,x} \psi_\theta^\dagger \left\{ i \partial_t + i v_F(\theta) w(\theta) \cdot \nabla + \kappa_{ij}(\theta) \partial_i \partial_j \right\} \psi_\theta + \int_{t,x} g^a(\theta) \phi_a \psi_\theta^\dagger \psi_\theta \]

- \( g \neq 0 \), view order parameter as a \( U(1)_{\text{patch}} \) "gauge field": gives rise to an "electric field,"

\[ \partial_t n_\theta + \nabla \cdot j_\theta = -\frac{\Lambda_{\text{patch}}}{(2\pi)^2} \frac{1}{v_F(\theta)} g^a(\theta) \partial_t \phi_a \]

- Anomaly deforms the conserved charge:

\[ \tilde{n}_\theta = n_\theta + \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \frac{g^a(\theta) \phi_a}{v_F(\theta)} \]

- View \( \tilde{n}_\theta \) as the physical patch charge density.
Symmetries and anomalies

\[ S_{\text{patch}}(\theta) = \int_{t,x} \psi_{\theta}^\dagger \{ i\partial_t + iv_F(\theta)w(\theta) \cdot \nabla + \kappa_{ij}(\theta) \partial_i \partial_j \} \psi_{\theta} + \int_{t,x} g^a(\theta) \phi_a \psi_{\theta}^\dagger \psi_{\theta} \]

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- View \( \tilde{n}_\theta \) as the physical patch charge density.

- Inversion symmetry: \( \Sigma_{\theta} n_\theta = \Sigma_{\theta} \tilde{n}_\theta = \rho_{\text{EM}}, \rho_{\text{EM}} \) conserved total charge.

- Technical note: \( n_\theta = \psi_{\theta}^\dagger \psi_{\theta} \) also a choice (regularization).
The boson propagator

- Now we are prepared to get our first “exact result,” basically follows Schwinger model. Set \( q = 0 \):

\[
\partial_t (\psi_\theta^\dagger \psi_\theta) = - \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \frac{g^a(\theta)}{v_F(\theta)} \partial_t \phi_a
\]

\[
\text{Equation of motion:}
\]

\[
(\lambda \partial_t^2 + r_c) \phi_a = \sum_\theta g^a(\theta) \psi_\theta^\dagger \psi_\theta
\]

Together:

\[
[(-\lambda \omega^2 + r_c) \delta^{ab} + \Pi^{ab}] \phi_b(\omega, q = 0) = h^a(\omega)
\]

\[
\Pi^{ab} = \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \sum_\theta \frac{g^a(\theta)g^b(\theta)}{v_F(\theta)}
\]
The boson propagator

- Means we exactly know the boson self-energy and propagator at $q = 0$

$$\Pi^{ab}(\omega, q = 0) = \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \sum_\theta \frac{g^a(\theta)g^b(\theta)}{v_F(\theta)} \quad D_{ab}(\omega, q = 0) = \left[-\lambda \omega^2 \mathbb{I} + \Pi\right]_{ab}^{-1}$$

- **Self-energy is frequency-independent!** Valid at all scales below $\Lambda_{\text{mid-IR}}$.
- There is a “plasmon” pole, but not necessarily physical.
- Non-trivial frequency scaling is from irrelevant operators, **is not intrinsic to the fixed point**. Consistent with expectation from RPA [Kim, Furusaki, Wen, and Lee, PRB (1994)]

- **What happened to $r_c$?** Can compute static boson susceptibility:

$$\chi_{\phi_a\phi_b}(\omega = 0, q \to 0) = 1/r_c \to \infty \text{ as } r_c \to 0$$
The boson propagator

- Means we **exactly** know the boson self-energy and propagator at \( q = 0 \)

\[
\Pi^{ab}(\omega, q = 0) = \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \sum_\theta \frac{g^a(\theta)g^b(\theta)}{v_F(\theta)} \quad D_{ab}(\omega, q = 0) = \left[ -\lambda \omega^2 \mathbb{I} + \Pi \right]^{-1}_{ab}
\]

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- **What happened to** \( r_c \)? Can compute static boson susceptibility:

\[ \chi_{\phi_a\phi_b}(\omega = 0, q \to 0) = \frac{1}{r_c} \to \infty \text{ as } r_c \to 0 \]

  - Related the chosen UV regularization, follows from “gauge invariance” of \( g^a(\theta)\phi_a \)
The optical conductivity

- Decompose the current into parallel and perpendicular components

\[ J^i_\perp = \sum_\theta v_F(\theta) w^i(\theta) \psi_\theta^\dagger \psi_\theta \]

\[ J^i_\parallel = \sum_\theta \frac{i}{2} \kappa^{ij}(\theta) \left[ \psi_\theta^\dagger \partial_j \psi_\theta - \partial_j \psi_\theta^\dagger \psi_\theta \right] \]

Current \(\perp\) FS, what we will focus on

Current \(\parallel\) FS

Momentum \(\parallel\) FS in patch \(\theta\)

- \(J_\parallel\) does not contribute to conductivity at the IR fixed point: critical boson fluctuations have \(q >> \omega\), couple strongest to antipodal patches tangent to \(q\) [Meltlitski, Sachdev, PRB (2010)].

\(\Rightarrow\) Emergent intra-patch momentum conservation: antipodal pairs of patches conserve momentum separately at low energies. Means \(J_\parallel\) conserved.
The optical conductivity

- $J_\perp$ correlators are determined by the anomaly equation, which gives:

$$J^i_\perp(\omega, q = 0) = -\sum_\theta \frac{\Lambda_{\text{patch}}}{(2\pi)^2} v_F(\theta) w^i(\theta) \left[ \frac{g^a(\theta)}{v_F(\theta)} \phi_a + w^j(\theta) A_j \right]$$

$$\sigma^{ij}(\omega) = -\frac{1}{i\omega} \frac{\delta\langle J^i_\perp(\omega) \rangle_A}{\delta A_j} = \frac{i}{\omega} \left[ \frac{D_{ij}^{(0)}}{\pi} - V^{a,i} V^{b,j} i \langle \phi_a(-\omega) \phi_b(\omega) \rangle \right]$$

$$D_{ij}^{(0)} = \sum_\theta \frac{\Lambda_{\text{patch}}}{4\pi} v_F(\theta) w^i(\theta) w^j(\theta) \quad V^{a,i} = \sum_\theta \frac{\Lambda_{\text{patch}}}{(2\pi)^2} g^a(\theta) w^i(\theta)$$

- Current probe
- Drude weight of free Fermi gas
- Vertex factor
The optical conductivity

- **But we have just calculated the boson propagator!** Define the IR limit as $\lambda \to 0$ and let $v_F$ be uniform. Then

\[
\sigma^{ij}(\omega) = \frac{i}{\omega} \frac{\Lambda_{\text{patch}}}{(2\pi)^2} v_F \left( \text{Tr}_\theta [w^i w^j] - \text{Tr}_\theta [g^a w^i] (\text{Tr}_\theta [gg])_{ab}^{-1} \text{Tr}_\theta [g^b w^j] \right)
\]

$\text{Tr}_\theta [fg] \equiv \sum_\theta f(\theta)g(\theta)$

- **Drude weight of free Fermi gas**
- **Coupling to boson reduces Drude weight**

- **Ising-nematic**, $g^a(\theta) = g^a(\theta + \pi)$: second term vanishes b/c $w^i(\theta) = -w^i(\theta + \pi)$

- "**Loop current,"** $g^a(\theta) = -g^a(\theta + \pi)$: Drude weight reduced, not necessarily zero

- **Gauge field**, $g^i(\theta) = v_F(\theta) w^i(\theta)$: conductivity vanishes b/c charge is gauged away

- Again, we find no non-trivial frequency scaling at the IR fixed point. **Critical fluctuations do not generate conductivity in Hertz-Millis theories!**
Confessing our sins

• The arguments leading to the conductivity involved a few (weak) assumptions:

1. **Validity of the mid-IR theory and** $U(1)_{\text{patch}}$ **symmetry/anomaly** [sufficient for deriving $\Pi(\omega)$]

   Absence of irrelevant operators, e.g.
   
   (i) inter-patch scattering,
   
   (ii) curvature of the dispersion, $w_i(\theta) \kappa_{ij}(\theta) = 0$
   
   (iii) form factor variation inside a patch, $g(k; \theta)$
   
   (iv) Boson self-interactions, $V[\phi]$

   Presence of these terms leads to “corrections to scaling” in $\Pi(\omega), \sigma(\omega)$.

2. **Emergent intra-patch momentum conservation at the IR fixed point:** Critical boson fluctuations ($\omega < q$) only see patches tangent to their momentum. Bosons with $q = 0$ couple to all patches. [sufficient, with (1), for deriving $\sigma(\omega)$; will not need this later]
Static susceptibilities

- Earlier, I said we could compute \( \chi_{\phi_a \phi_b} = \frac{1}{r - r_c} \delta_{ab} \) exactly (with \( r_c = 0 \)).
  
  Follows from “gauge invariance” of \( g^a(\theta) \phi_a \).

- Also get \( \chi_{n_\theta \phi_a} = 0 \) from path integral manipulations.

- These results plus similar anomaly arguments (this time with \( \omega = 0 \), finite \( q_\perp \)) allows us to compute the static density susceptibility in a patch,

\[
\chi_{n_\theta n_{\theta'}} = \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \frac{1}{v_F(\theta)} \delta_{\theta \theta'}
\]
Static susceptibilities

- Susceptibility of $\tilde{n}_\theta = n_\theta - \Lambda_{\text{patch}} g^a \phi_a / (2\pi)^2 v_F$, which we said is always conserved in the absence of external fields, is more interesting.

- Tune into the disordered phase, $r - r_c > 0$. Get a Fermi liquid with conserved density $\tilde{n}_\theta$. Landau parameters determined by the coupling to the boson. Find

$$\chi \tilde{n}_\theta \tilde{n}_{\theta'} = \chi n_\theta n_{\theta'} + \frac{(\Lambda_{\text{patch}})^2}{(2\pi)^4} \frac{g^a(\theta)g^b(\theta')}{v_F(\theta)v_F(\theta')} \chi \phi_a \phi_b$$

$$= \frac{1}{v_F(\theta)} \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \delta_{\theta \theta'} + \frac{\Lambda_{\text{patch}}(\theta)\Lambda_{\text{patch}}(\theta')}{(2\pi)^4} \frac{g^a(\theta)g_a(\theta')}{v_F(\theta)v_F(\theta')} \frac{1}{r - r_c}$$

- $\chi \tilde{n}_\theta \tilde{n}_{\theta'} \to \infty$ as criticality is approached, $r - r_c \to 0$. Landau parameters diverge. Consistent with perturbative expectations [Maslov and Chubukov, PRB (2010)], [Mross et al., PRB (2010)]
In search of quantum critical conductivity

- **Tuesday night’s arXiv:** [Shi, Else, HG, and Senthil, arXiv:2208.04328]

- Have $N$ bosons, $N$ fermions with Gaussian random couplings $g_{IJK}$, $N \to \infty$ gives a controlled, “Eliashberg” NFL fixed point

  $$g_{IJK}(\theta) \phi_I \psi_J^\dagger \psi_K$$

- **Anomaly loosened:** boson does not quite couple to a conserved charge.

- Means $\Pi(\omega)$ not fixed to a constant, incompatible with $N = 1$ theory. One finds the Eliashberg result:

  $$\Pi(\omega) \sim c_\omega \omega^{(z-2)/z} + \mathcal{O}(1/N)$$
In search of quantum critical conductivity

- Implications:
  1. Non-vanishing fixed point conductivity (loop current only):
     \[ \sigma(\omega > 0) \sim \omega^{-2/z} = \omega^{-2/3} \text{ for } z = 3. \]
  2. \( \omega \to 0 \) and \( N \to \infty \) limits do not commute.

- Calculate conductivity using our "anomaly-assisted large \( N \)" technique + memory matrix. Final answer is

\[
\sigma(\omega) = N \frac{i}{\omega} \left[ \frac{D_0}{\pi} - \frac{\delta D}{1 + i NC_z \omega^{(z-2)/z} / \delta D} + \ldots \right]
\]

\[
= N \frac{i}{\omega} \frac{D - \delta D}{\pi} + N^2 C_z \omega^{-2/z} \quad \text{if } \omega^{(2-z)/z} << 1/N
\]

The future

- Big question: Are there (clean) theories of non-Fermi liquids with IR fixed point conductivity?

  ⇒ Need better models!

- Magnetotransport

- Finite-\(Q\) order parameters

- Higher dimensional bosonization of Hertz-Millis theories

\[ \Lambda_{\text{mid-IR}} \]
\[ \Lambda_{\text{LD}} \sim \gamma^{1/2} \]
\[ \Lambda_{\text{NFL}} \sim C^3 \]

Here be dragons