Title: Exact results for metallic quantum critical points

Speakers: Hart Goldman

Series: Quantum Matter

Date: August 11, 2022 - 3:00 PM

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Abstract: I discuss how exact, non-perturbative results can be obtained for both optical transport and static susceptibilities in "Hertz-Millis" theories of Fermi surfaces coupled to critical bosons. Such models possess a large emergent symmetry and anomaly structure, which we leverage to fix these quantities. In particular, I will show that in the infrared limit, the boson self energy at zero wave vector is a constant independent of frequency, and the real part of the optical conductivity is purely a delta function Drude peak with no other corrections. I will also obtain exact relations between Fermi liquid parameters as the critical point is approached from the disordered phase.

Zoom Link: https://pitp.zoom.us/j/93340611986?pwd=cisrZmFxcEVWZVdrT2tMRVZiVTdRQT09

Exact results for metallic quantum critical points

Hart Goldman

Based on:
arXiv:2204.07585
arXiv:2208.04328

GORDON AND BETTY MOORE FOUNDATION

Based on work with

Zhengyan Darius Shi

(MIT)

Dominic Else

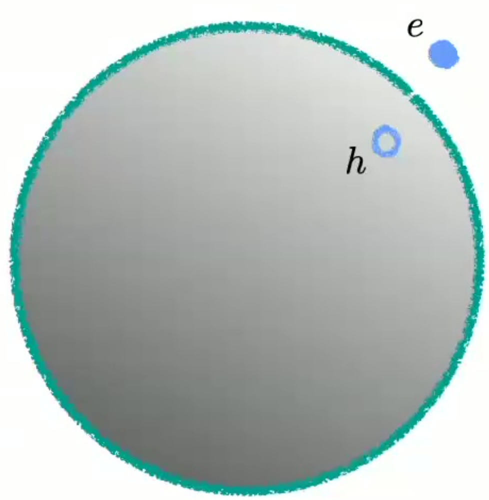
(Harvard → PI)

Senthil

(MIT)

Fermi liquid theory and its discontents

- Fermi liquid theory (FLT) has been tremendously successful in describing conventional metals in terms of a Fermi surface + stable quasiparticles.
- Still many systems in nature **NOT** captured by FLT



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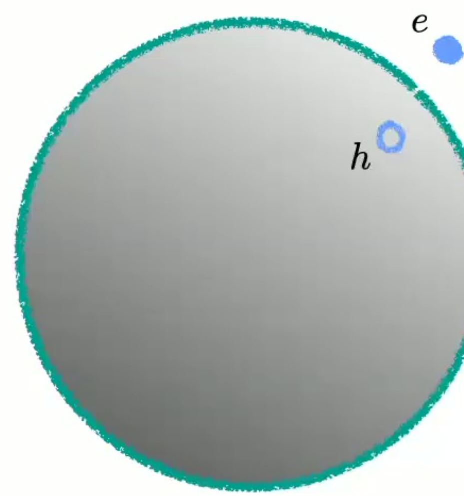
- Still many systems in nature **NOT** captured by FLT

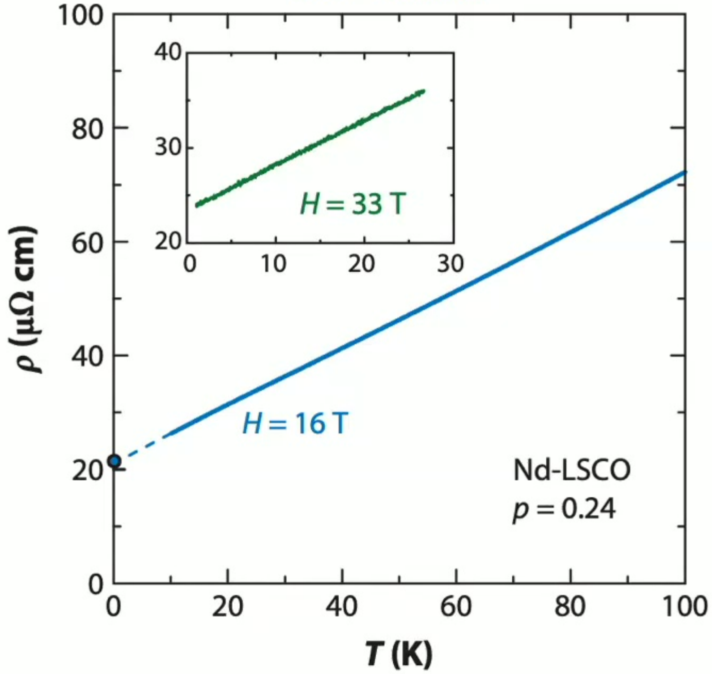
  - **Creative name:** non-Fermi liquids (NFLs)



Fermi liquid theory and its discontents

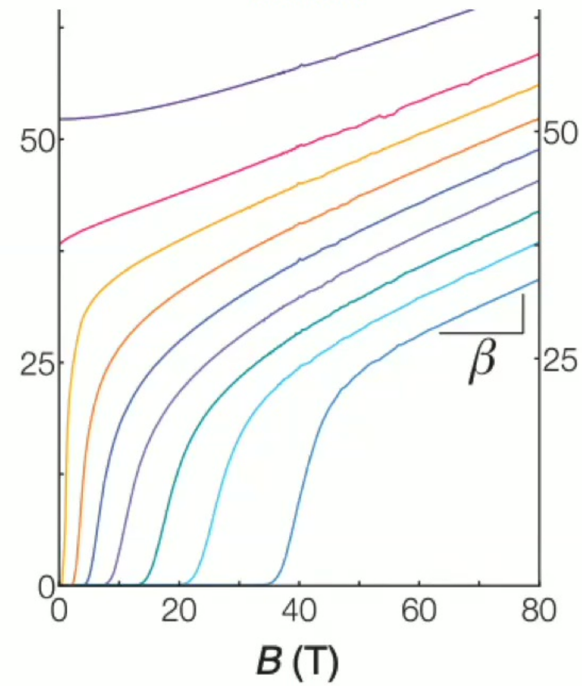
Strange metals

Nd-LSCO



[Proust and Taillefer, Ann. Rev. (2019)]

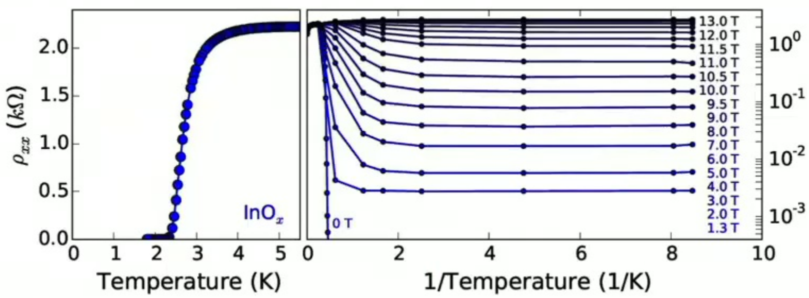
LSCO



[Giraldo-Gallo *et al.*, Science (2018)]

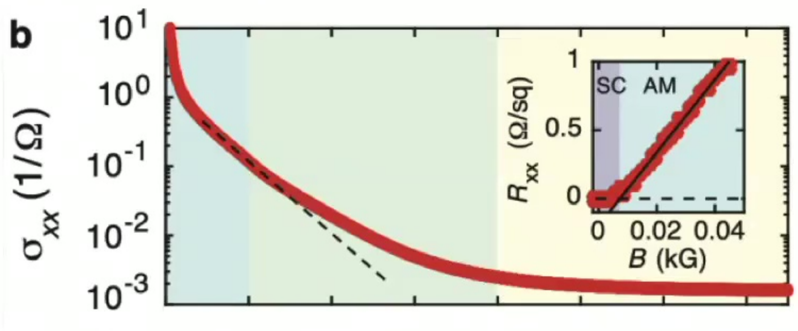
Fermi liquid theory and its discontents

Anomalous metals



InOx

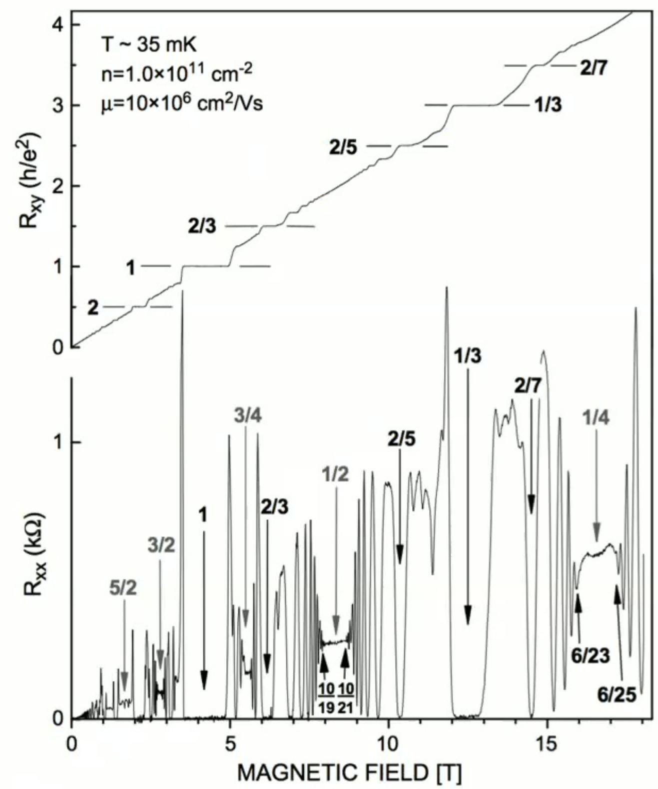
[Breznay and Kapitulnik, Science Advances (2017)]



LaAlO3/SrTiO3

[Chen *et al.*, npj Quantum Matter (2021)]

Composite Fermi Liquids



From [Stern, Ann. Phys. (2008)],
data from W. Pan (Sandia)

Suggestions of quantum criticality

- Apparent unifying feature of NFLs is **“quantum critical behavior”** or proximity to a (real or conjectured) quantum critical point (QCP).
  - **Strange metals:** experiments suggest a scaling form,

$$\sigma(\omega, T) = \frac{1}{T^\alpha}\,\Sigma\left(\frac{\omega}{T}\right)$$

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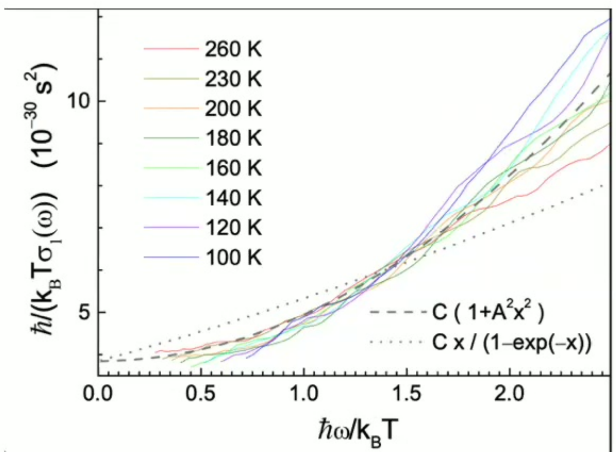
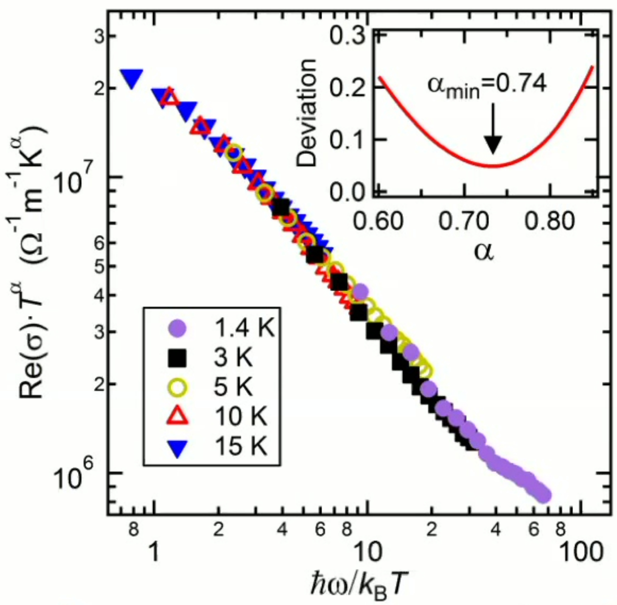


[van der Marel *et al.*, Nature (2003)]

$YbRh_2Si_2$



[Prochaska *et al.*, Science (2020)]

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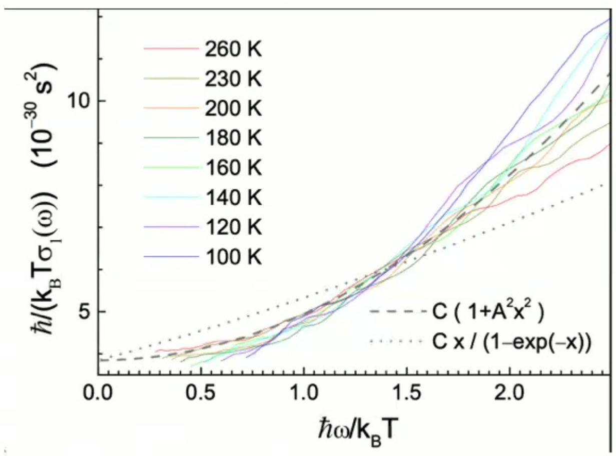
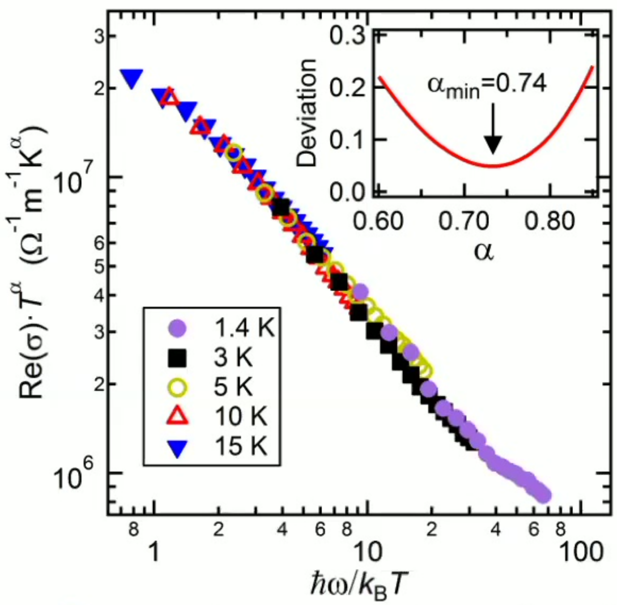


[van der Marel *et al.*, Nature (2003)]

$$\sigma(\omega, T) = \frac{1}{T^\alpha}\,\Sigma\left(\frac{\omega}{T}\right)$$

- So far, **no physical model** of a clean metallic state has been shown to display this behavior.

$YbRh_2Si_2$



[Prochaska *et al.*, Science (2020)]

Quantum criticality beyond the Landau (Fermi liquid) paradigm

- Quantum criticality presents a natural mechanism for evading the predictions of Fermi liquid theory: killing the quasiparticles.

- Quasiparticles can scatter off of gapless fluctuations at all energy scales

Quasiparticle scattering rate

$$\lim_{\omega \to 0} \frac{\Gamma(\omega)}{\omega} \neq 0$$

In a FL:
$\Gamma(\omega) \sim \omega^2$
$\Rightarrow \Gamma(\omega)/\omega \to 0$

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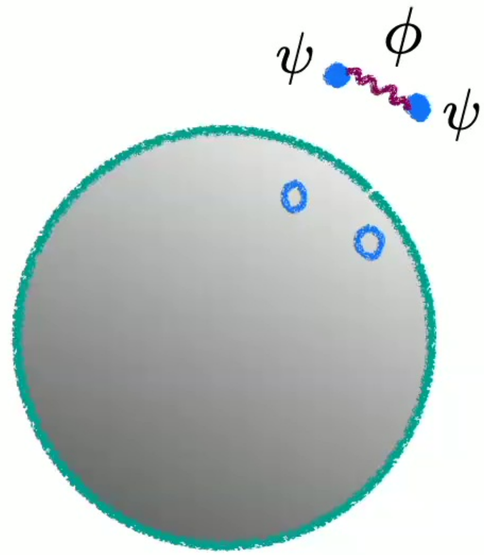
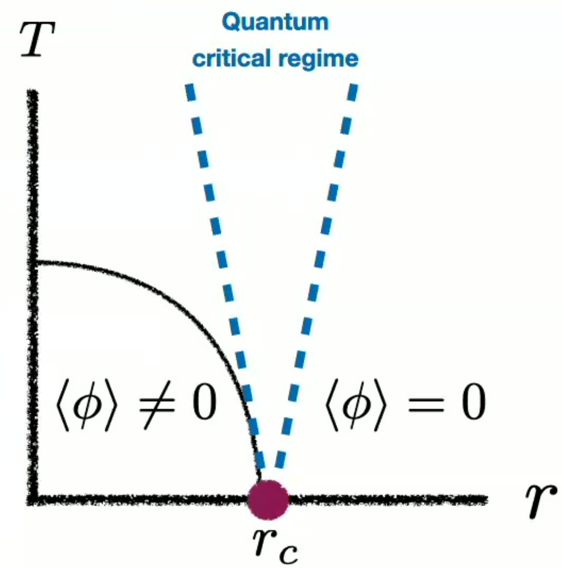
- **Big questions:** Can we understand the ground state physics of metallic QCPs? Study tractable models? Find a clean model with ω/T conductivity scaling?
  - *Problems have vexed us for 30+ years!*

The Hertz-Millis paradigm

- Simplest class of theories of metallic QCPs was first introduced and studied by [Hertz, PRB (1976)] and [Millis, PRB (1993)].
- **Basic idea:** Fermi surface coupled to an order parameter near a QCP.
- At the QCP, scattering with the gapless boson kills the quasiparticles at all scales.





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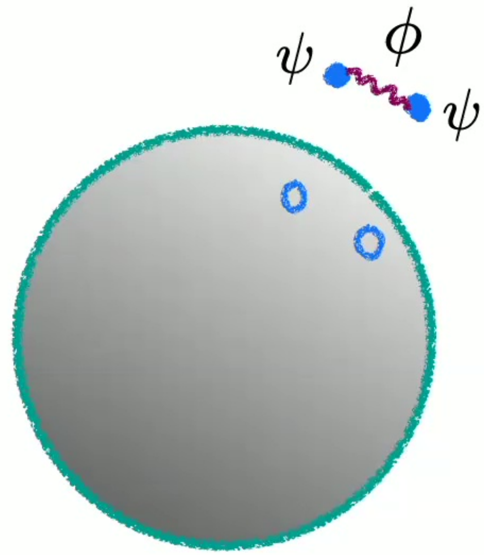
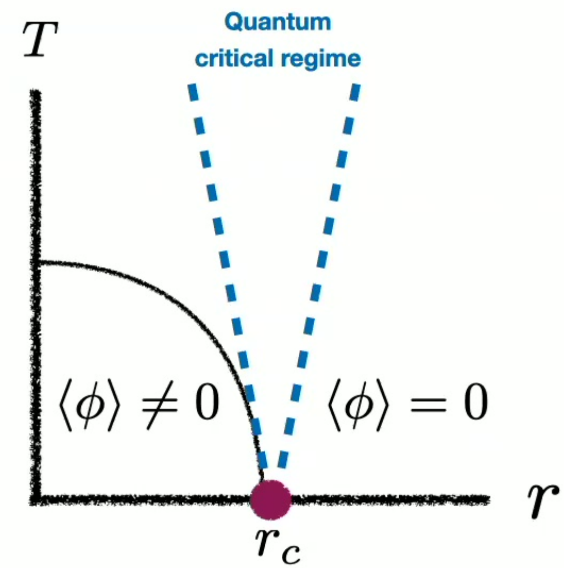
- **Basic idea:** Fermi surface coupled to an order parameter near a QCP.

- At the QCP, scattering with the gapless boson kills the quasiparticles at all scales.

- **We focus on simple models as a guide to understanding universal physics,** but they may not always capture observed phenomena (e.g. in cuprates).





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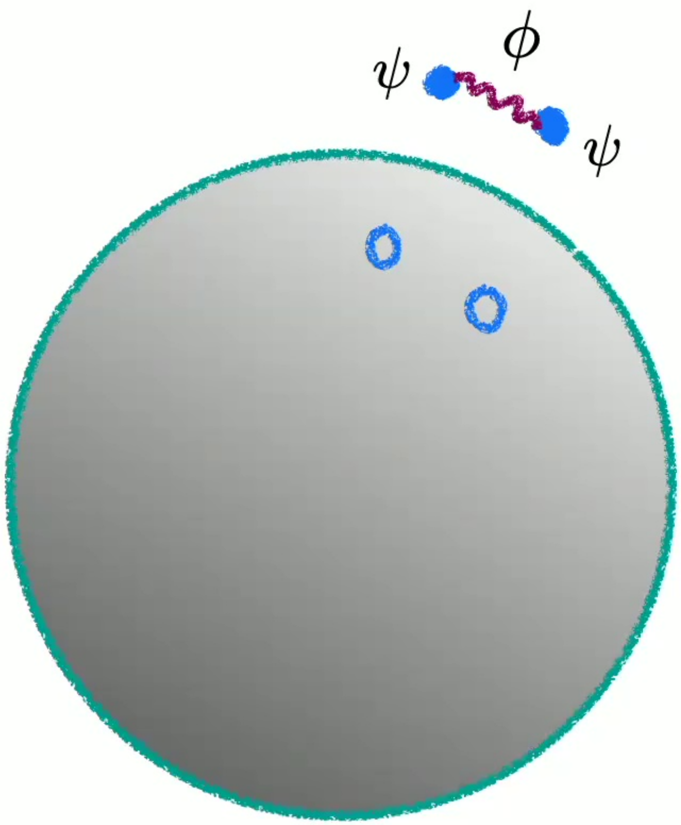
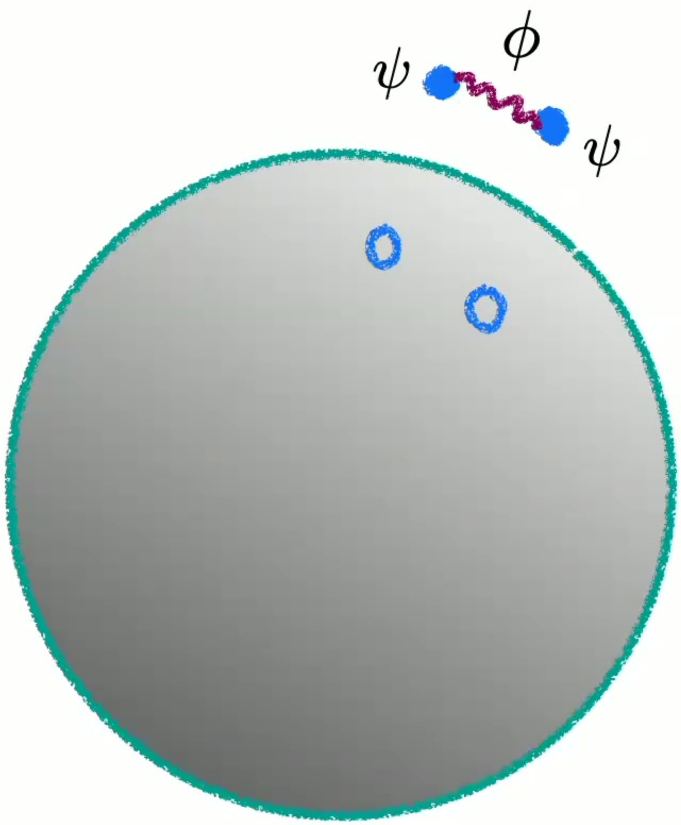


$$S = S_\psi + S_{\phi\psi} + S_\phi$$

$$S_\psi = \int_{\omega,\boldsymbol{k}} \psi^\dagger \left[i\omega - \epsilon(\boldsymbol{k})\right] \psi$$

$$S_{\phi\psi} = \int_{k,q} g(\boldsymbol{k},\boldsymbol{q})\, \phi(q)\, \psi^\dagger(k+q)\, \psi(k)$$

The Hertz-Millis paradigm



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$$S_\phi = \int_{\tau,\boldsymbol{x}} \frac{1}{2} \left[\lambda(\partial_\tau \phi)^2 + J(\boldsymbol{\nabla}\phi)^2 + r_c\, \phi^2 + \dots\right]$$

The treacherous path to the IR

- Fix to $d = 2$ dimensions, **no small parameter**.
- **Integrate out fermions:** $\Pi(\omega, \boldsymbol{q}) = \gamma \dfrac{|\omega|}{|\boldsymbol{q}|}$, boson becomes overdamped by decay into particle-hole pairs.
  - dynamical exponent $z = 3$.
- **Integrate out bosons:** fermion self energy, $\Sigma(\omega) = i\,C\,|\omega|^{2/3}$
  - $\Gamma(\omega)/\omega \sim \omega^{-1/3} \Rightarrow$ quasiparticles die.
- **Theory is IR sick,** usual large-N (RPA) technique breaks down [Lee, PRB (2009)].

$$G_{\bar{\psi}\psi}(\omega) = \frac{1}{i\omega + iC|\omega|^{2/3}/N} \to \frac{N}{iC|\omega|^{2/3}}$$

Hertz-Millis lives here

$\Lambda_{\mathrm{mid-IR}}$

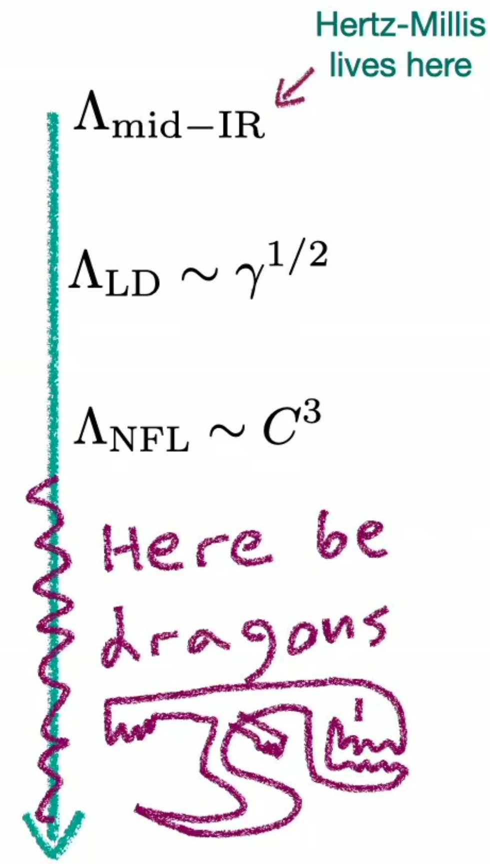
$\Lambda_{\mathrm{LD}} \sim \gamma^{1/2}$

$\Lambda_{\mathrm{NFL}} \sim C^3$

Here be dragons

A litany of perturbative deformations

- Much effort toward **perturbative progress** over the past decade:
  - Large-N, small $z-2$, $N(z-2)$ fixed [Mross *et al.,* PRB (2010)], also: [Ye, Lee, and Zou, PRL (2022)]
  - Matrix boson large-N [Mahajan *et al.,* PRB (2013)], … , [Aguilera Damia *et al.,* PRL (2019)]
  - Co-dimensional regularization [Dalidovich and Lee, PRB (2013)]
  - Random flavor/SYK-esque large-N [Esterlis and Schmalian, PRB (2019)], [Aldape *et al.*, arXiv: 2012.00763], [Esterlis *et al.*, PRB (2021)]



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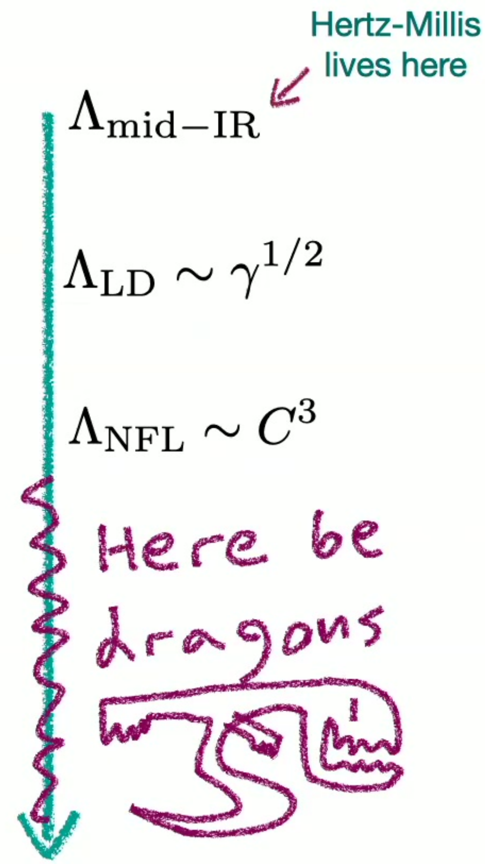
  - Random flavor/SYK-esque large-N [Esterlis and Schmalian, PRB (2019)], [Aldape *et al.*, arXiv: 2012.00763], [Esterlis *et al.*, PRB (2021)]
- Lead to formally controlled IR fixed points, **but many observable properties, e.g. transport, still hard to capture.**

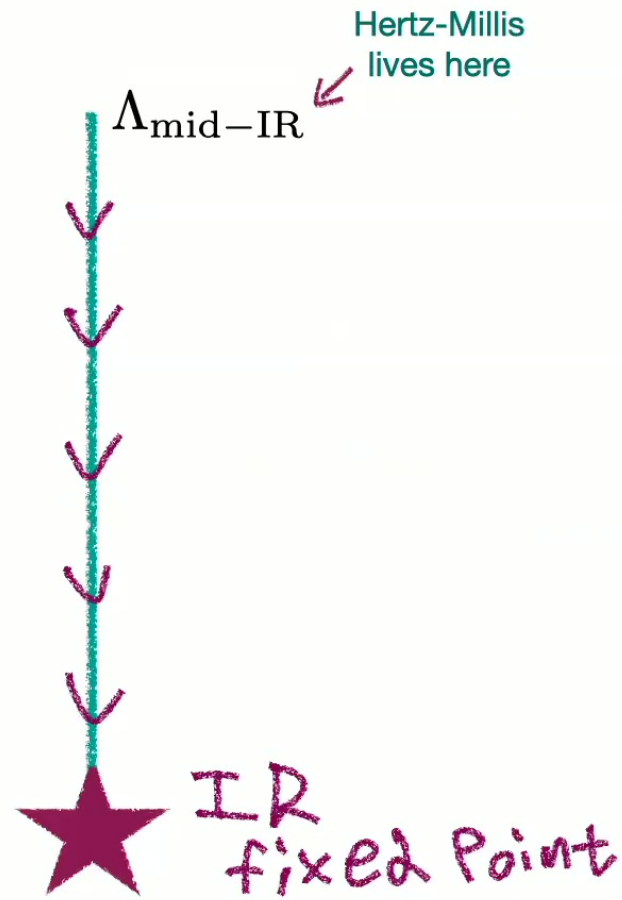


Beyond perturbation theory

- Would like to have **general, non-perturbative constraints** on theories in the Hertz-Millis paradigm.
- [Else, Thorngren, and Senthil, PRX (2021)]: emergent symmetry and anomaly structure shared by all compressible metals. **Can we leverage this philosophy to make progress?**
- **Philosophy of this talk:** start with a theory at some energy scale, $\Lambda_{\text{mid-IR}}$, derive properties that are valid either

  1. At all scales below $\Lambda_{\text{mid-IR}}$, down to the deep IR.
  2. Sitting exactly at the IR fixed point (assumes a second order transition).

- **Broader program of progress using non-perturbative principles: can extend beyond Hertz-Millis.**



Preview of exact results

- At the IR fixed point (in the clean limit), we find:

1. Exact expressions for the boson self-energy, $\Pi(\boldsymbol{q} = 0,\omega)$, and optical conductivity:

$$\mathrm{Re}\,\sigma(\omega, T) = \mathcal{D}\,\delta(\omega) \longleftarrow \text{Drude alone}$$

Ising-nematic:	Varma loop current:	Gauge field (e.g. HLR):
$\mathcal{D} = \mathcal{D}_{\mathrm{free}}$	$0 < \mathcal{D} < \mathcal{D}_{\mathrm{free}}$	$\mathcal{D} = 0$

$\sigma(\omega \neq 0) = 0 \Rightarrow$ **Critical fluctuations do not generate conductivity**

Preview of exact results

- At the IR fixed point (in the clean limit), we find:

2. Exact expressions for static density susceptibilities ($\omega \to 0$, then $\boldsymbol{q} \to 0$) as the critical point is approached. If $\tilde{n}_\theta$ is the density at FS angle θ, find

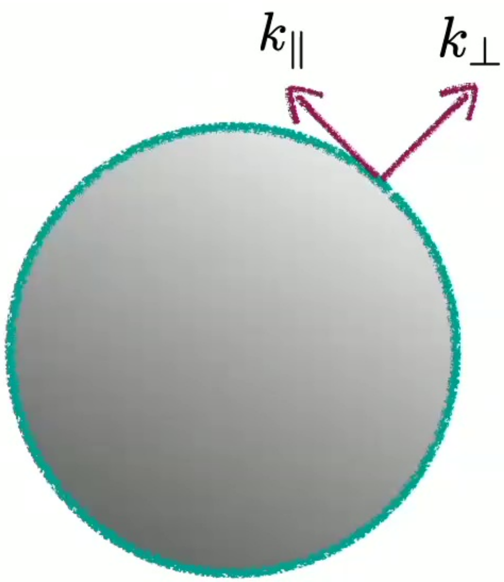
$$\chi_{\tilde{n}_\theta \tilde{n}_{\theta'}}(r \to r_c) \sim \frac{\delta\chi_{\theta\theta'}}{r - r_c} \to \infty$$

in the order parameter channel as $r \to r_c$. Divergence was expected from earlier arguments.
[Maslov and Chubukov, PRB (2010)], [Mross *et al.*, PRB (2010)].

How is this possible? Gifts from anomalies

- **Physics by analogy:** low energy degrees of freedom near the FS look like 1D chiral fermions

$$\epsilon(\boldsymbol{k}) = v_F(\theta)\, k_\perp + \kappa_\parallel(\theta)\, k_\parallel^2 + \ldots$$



- **"Chiral" anomaly:** charge is not conserved at individual θ, needs to be compensated by the rest of the FS.
- Anomalies are non-perturbative, quantum deformations of conservation laws.
- In 1D, chiral anomaly leads to tight constraints and underlies bosonization. **Same will be true here!**
- Similar ideas have been applied to Fermi liquids in the past [Luther, PRB (1979)], [Haldane, Journ. Phys. C (1981)], [Houghton and Marston, PRB (1993)], [Castro Neto and Fradkin, PRL (1994)], …, [Delacretaz *et al.*, arXiv:2203.05004]

Roadmap

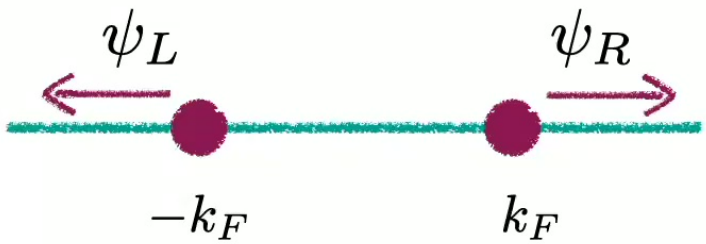
I) The chiral anomaly and transport in 1D

II) Symmetries and anomalies of Fermi surfaces + critical bosons

III) Exact results: $\Pi(\omega)$, $\sigma(\omega)$, and $\chi_{\tilde{n}_\theta \tilde{n}_{\theta'}}$.

IV) A model with fixed point conductivity (time permitting)

A trivial example: a free Dirac in 1D



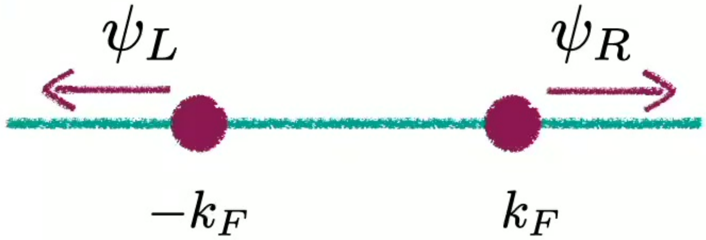
$$S = \int_{t,\boldsymbol{x}} \bar{\psi}\,\gamma^\mu(\partial_\mu + iA_\mu)\psi$$

- $\psi = (\psi_L, \psi_R)$. Theory classically conserves charge at each Fermi point:

$$U(1)_L : \psi_L \rightarrow e^{i\alpha_L}\psi_L$$

$$U(1)_R : \psi_R \rightarrow e^{i\alpha_R}\psi_R$$

A trivial example: a free Dirac in 1D



$$S = \int_{t,\boldsymbol{x}} \bar{\psi}\, \gamma^{\mu} (\partial_{\mu} + i A_{\mu}) \psi$$

- **Turn on background** A_μ **:** get a current — L, R charge no longer conserved ('t Hooft anomaly)

$$\partial_\mu j_L^\mu = -\frac{1}{2\pi} E \,, \partial_\mu j_R^\mu = +\frac{1}{2\pi} E$$

$$j_L^0 = \rho_L = \psi_L^\dagger \psi_L \qquad j_R^0 = \rho_R = \psi_R^\dagger \psi_R$$

$$j_L^1 = \psi_L^\dagger \psi_L \qquad j_R^1 = -\psi_R^\dagger \psi_R$$

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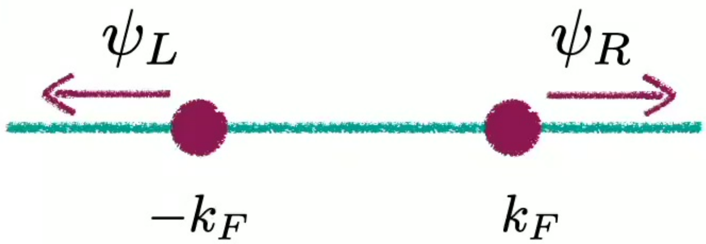

$$S = \int_{t,\boldsymbol{x}} \bar{\psi}\, \gamma^\mu (\partial_\mu + i A_\mu) \psi$$

- Conservation of EM current $j^\mu_V = j^\mu_L + j^\mu_R$, means axial current, $j^\mu_A = j^\mu_L - j^\mu_R$, is **anomalous.** Notice axial density = EM current and *vice versa*

$$\begin{matrix} j^0_V = \rho \\ j^1_V = J \end{matrix} \longrightarrow \partial_\mu j^\mu_V = 0\,,\ \partial_\mu j^\mu_A = -\frac{1}{\pi} E \longleftarrow \begin{matrix} j^0_A = J \\ j^1_A = \rho \end{matrix}$$

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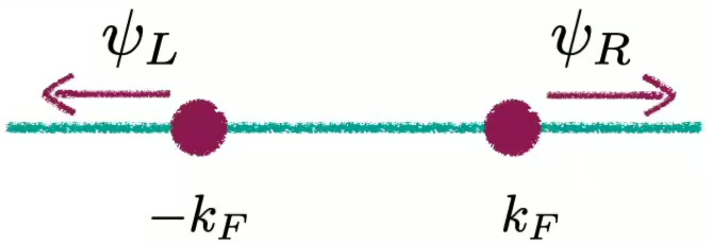

$$S = \int_{t,\boldsymbol{x}} \bar{\psi}\,\gamma^\mu(\partial_\mu + iA_\mu)\psi$$

- Conservation of EM current $j_V^\mu = j_L^\mu + j_R^\mu$, means axial current, $j_A^\mu = j_L^\mu - j_R^\mu$, is **anomalous.** Notice axial density = EM current and *vice versa*

$$\begin{array}{l} j_V^0 = \rho \\ j_V^1 = J \end{array} \longrightarrow \partial_\mu j_V^\mu = 0\,,\ \partial_\mu j_A^\mu = -\frac{1}{\pi}E \longleftarrow \begin{array}{l} j_A^0 = J \\ j_A^1 = \rho \end{array}$$

- **Axial anomaly determines response to electric fields.** Set $q = 0$ **:**

$$i\omega\, J(\omega, q=0) = -\frac{1}{\pi}E(\omega, q=0) \Rightarrow \sigma(\omega) = \frac{1}{\pi}\frac{i}{\omega}$$

A less trivial example: QED_2

- Let $A_\mu \to a_\mu$ fluctuate, get **the Schwinger model** (QED_2)

$$S = \int_{t,x} \left[\bar{\psi}\gamma^\mu(\partial_\mu + i a_\mu)\psi - \frac{1}{4g^2} f^2 \right]$$

- Exactly solvable toy model of confinement (a_μ mediates a $V(r) \sim r$ potential)

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- Exactly solvable toy model of confinement (a_μ mediates a $V(r) \sim r$ potential)
- **Axial anomaly:** encodes response to emergent electric field of a_μ

$$\partial_\mu j_L^\mu = \frac{1}{2\pi}\varepsilon_{\mu\nu}\partial_\mu a_\nu \, , \partial_\mu j_R^\mu = -\frac{1}{2\pi}\varepsilon_{\mu\nu}\partial_\mu a_\nu$$

- Allows us to fix the propagator of a_μ **exactly**.

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$$S = \int_{t,x} \left[\bar{\psi}\gamma^\mu(\partial_\mu + i a_\mu)\psi - \frac{1}{4g^2} f^2 \right]$$

- Using arguments so similar to what we will use later that we will hold off on presenting them, one can obtain:

$$D_{xx}(\omega, q = 0) = \frac{1}{\omega^2 - g^2/\pi}$$

"Plasmon" with mass g^2/π

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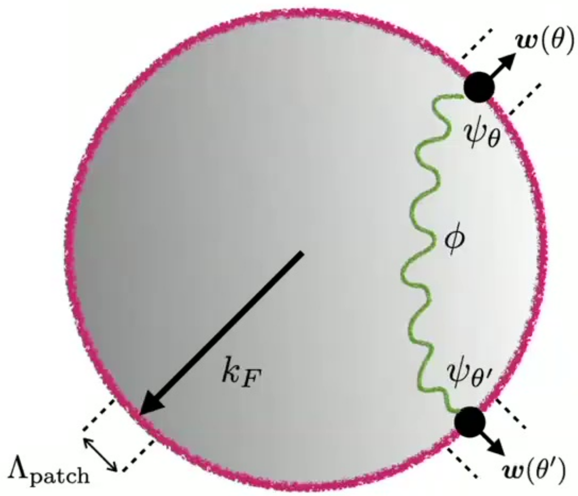
$$D_{xx}(\omega, q=0) = \frac{1}{\omega^2 - g^2/\pi} \leftarrow \text{"Plasmon" with mass } g^2/\pi$$

$$\Pi_{xx}(\omega, q=0) = \frac{1}{\pi} \leftarrow \text{frequency independent}$$

The "mid-IR" theory

- Work with a **mid-IR effective theory** of a Fermi surface sliced into small patches + N_b boson species, ϕ_a , coupling to each patch. Lives at $\Lambda_{\text{mid-IR}}$



$$S_{\text{mid-IR}} = \sum_\theta S_{\text{patch}}(\theta) + S_\phi$$

Allow only curvature of FS:
$w_i \kappa_{ij}(\theta) = 0$

$$S_{\text{patch}}(\theta) = \int_{t,\boldsymbol{x}} \psi_\theta^\dagger \{i\partial_t + iv_F(\theta)\boldsymbol{w}(\theta)\cdot\boldsymbol{\nabla} + \kappa_{ij}(\theta)\partial_i\partial_j\}\psi_\theta$$

$$+ \int_{t,\boldsymbol{x}} g^a(\theta)\,\phi_a\psi_\theta^\dagger\psi_\theta$$

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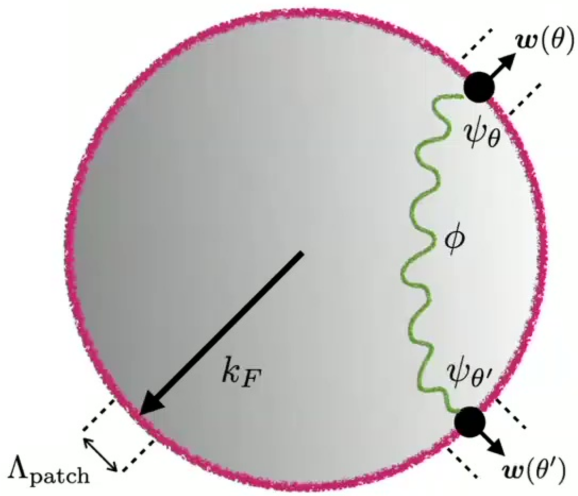

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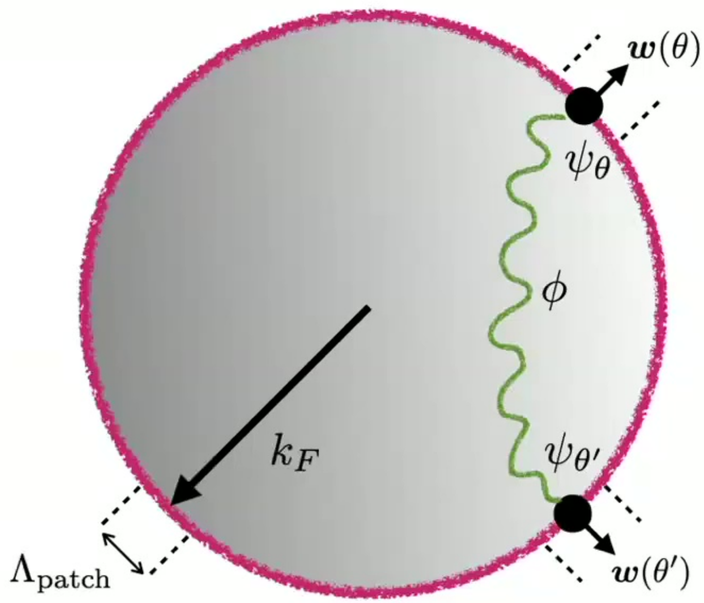
$$S_\phi = \frac{1}{2} \int_{t,\boldsymbol{x}} \left[\lambda (\partial_t \phi_a)^2 - r_c \phi_a \phi^a - J (\boldsymbol{\nabla} \phi_a)^2 \right]$$

$V[\phi_a] = 0$

Some key assumptions

$$S_{\text{patch}}(\theta) = \int_{t,\boldsymbol{x}} \psi^\dagger_\theta \left\{ i\partial_t + iv_F(\theta)\boldsymbol{w}(\theta)\cdot\boldsymbol{\nabla} + \kappa_{ij}(\theta)\partial_i\partial_j \right\} \psi_\theta + \int_{t,\boldsymbol{x}} g^a(\theta)\,\phi_a \psi^\dagger_\theta \psi_\theta$$



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- **Allowed:** exchange of bosons between different patches (see figure).

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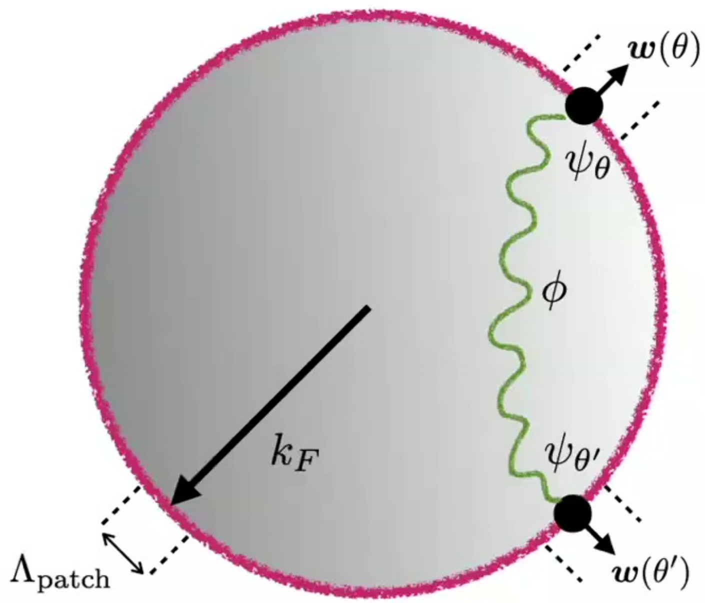
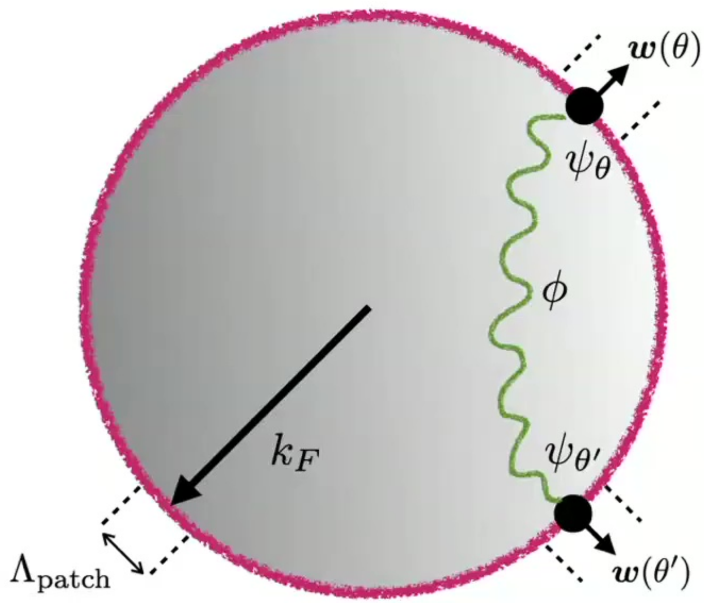


- **Not allowed**: scattering of a fermion from patch θ to another patch θ' by absorbing/emitting a boson (*large-angle scattering*).
- **Allowed:** exchange of bosons between different patches (see figure).
  - Will need to refine later for one result.

The “mid-IR” theory

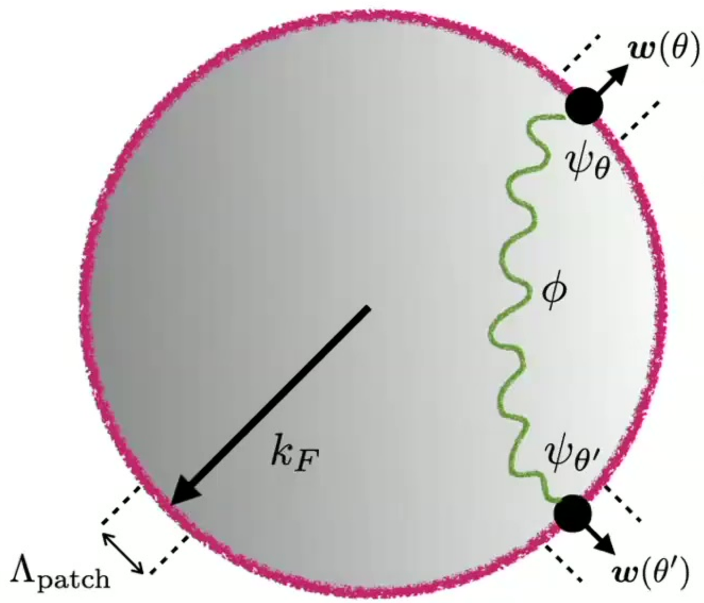
$$S_{\mathrm{patch}}(\theta) = \int_{t,\boldsymbol{x}} \psi_\theta^\dagger \left\{ i\partial_t + i v_F(\theta) \boldsymbol{w}(\theta) \cdot \boldsymbol{\nabla} + \kappa_{ij}(\theta)\partial_i\partial_j \right\} \psi_\theta + \int_{t,\boldsymbol{x}} g^a(\theta)\, \phi_a \psi_\theta^\dagger \psi_\theta$$



- Properties of $g^a(\theta)$ determine the particular theory
  - Ising-nematic: $g^a(\theta) = g^a(\theta + \pi)$
  - “Loop current”: $g^a(\theta) = -\, g^a(\theta + \pi)$
    - Gauge field (e.g. HLR): $g^i(\theta) = v_F(\theta)\, \boldsymbol{w}^i(\theta)$

Symmetries and anomalies

$$S_{\text{patch}}(\theta) = \int_{t,\boldsymbol{x}} \psi_\theta^\dagger \{ i\partial_t + i v_F(\theta)\boldsymbol{w}(\theta)\cdot\boldsymbol{\nabla} + \kappa_{ij}(\theta)\partial_i\partial_j \} \psi_\theta + \int_{t,\boldsymbol{x}} g^a(\theta)\, \phi_a \psi_\theta^\dagger \psi_\theta$$



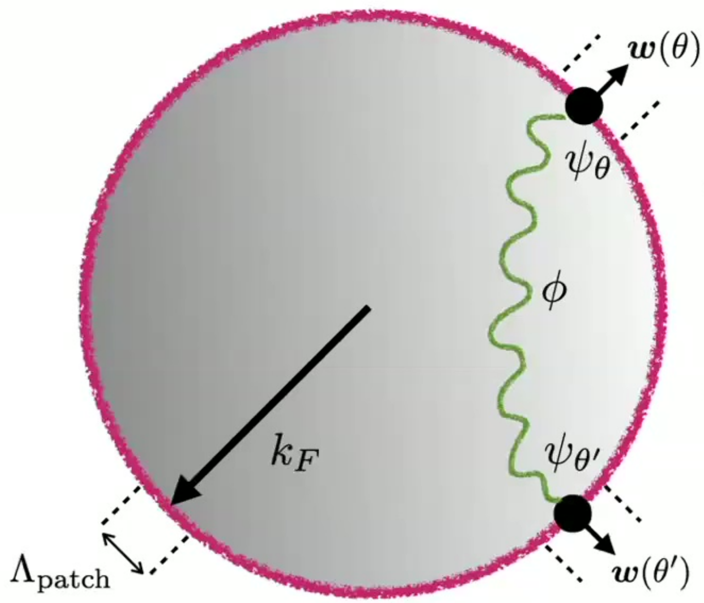
- Each patch looks like a chiral fermion, has $U(1)_\theta : \psi_\theta \to e^{i\alpha_\theta}\psi_\theta$ charge conservation.
- Total classical symmetry: $U(1)_{\text{patch}} = \Pi_\theta\, U(1)_\theta$.
- $g = 0$: background electric fields lead to current, break charge conservation in each patch

$$\partial_t n_\theta + \boldsymbol{\nabla}\cdot\boldsymbol{j}_\theta = -\frac{\Lambda_{\text{patch}}}{(2\pi)^2}\,\boldsymbol{w}(\theta)\cdot\partial_t \boldsymbol{A}$$

Patch density Patch current E - field ⊥ the FS

Symmetries and anomalies

$$S_{\rm patch}(\theta) = \int_{t,\boldsymbol{x}} \psi_\theta^\dagger \left\{ i\partial_t + i v_F(\theta)\boldsymbol{w}(\theta)\cdot\boldsymbol{\nabla} + \kappa_{ij}(\theta)\partial_i\partial_j \right\} \psi_\theta + \int_{t,\boldsymbol{x}} g^a(\theta)\,\phi_a \psi_\theta^\dagger \psi_\theta$$



- $g \neq 0$, **view order parameter as a** $U(1)_{\rm patch}$ **"gauge field":** gives rise to an "electric field,"

$$\partial_t n_\theta + \boldsymbol{\nabla}\cdot\boldsymbol{j}_\theta = -\frac{\Lambda_{\rm patch}}{(2\pi)^2}\frac{1}{v_F(\theta)} g^a(\theta)\,\partial_t\phi_a$$

- Anomaly deforms the conserved charge:

$$\tilde{n}_\theta = n_\theta + \frac{\Lambda_{\rm patch}}{(2\pi)^2}\frac{g^a(\theta)\phi_a}{v_F(\theta)}$$

$\partial_t \tilde{n}_\theta + \boldsymbol{\nabla}\cdot\boldsymbol{j}_\theta = 0$

- View $\tilde{n}_\theta$ as the physical patch charge density.

Symmetries and anomalies

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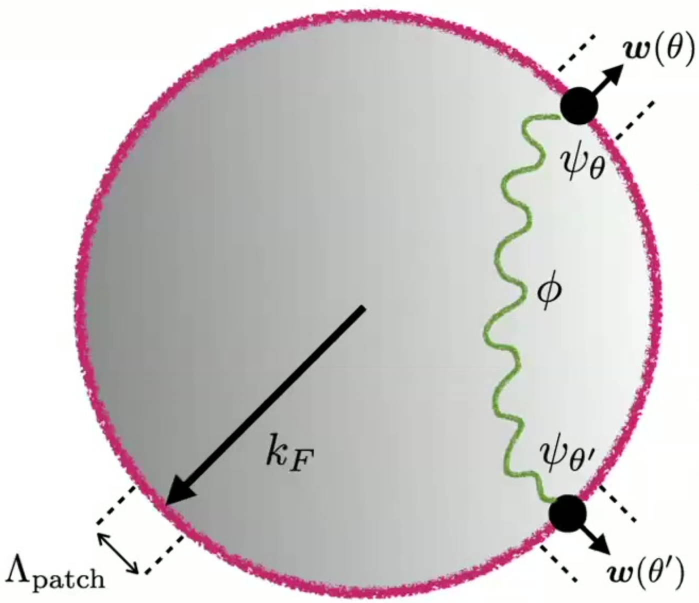

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- View $\tilde{n}_\theta$ as the physical patch charge density.
- Inversion symmetry: $\Sigma_\theta n_\theta = \Sigma_\theta \tilde{n}_\theta = \rho_{\text{EM}}$, ρ_{EM} conserved total charge.
- *Technical note:* $n_\theta = \psi_\theta^\dagger \psi_\theta$ also a choice (regularization).

The boson propagator

- Now we are prepared to get our first "exact result," basically follows Schwinger model. Set $\boldsymbol{q} = 0$:

$U(1)_{\text{patch}}$ anomaly:

$$\partial_t(\psi_\theta^\dagger \psi_\theta) = -\frac{\Lambda_{\text{patch}}}{(2\pi)^2} \frac{g^a(\theta)}{v_F(\theta)} \partial_t \phi_a$$

Equation of motion:

$$(\lambda \partial_t^2 + r_c)\phi_a = \sum_\theta g^a(\theta) \psi_\theta^\dagger \psi_\theta$$

Together:

source $h^a \phi_a$

$$\left[(-\lambda\omega^2 + r_c)\delta^{ab} + \Pi^{ab}\right] \phi_b(\omega, \boldsymbol{q} = \mathbf{0}) = h^a(\omega)$$

$$\Pi^{ab} = \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \sum_\theta \frac{g^a(\theta) g^b(\theta)}{v_F(\theta)}$$

The boson propagator

- Means we **exactly** know the boson self-energy and propagator at $\boldsymbol{q} = 0$

$$\Pi^{ab}(\omega, \boldsymbol{q} = 0) = \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \sum_{\theta} \frac{g^a(\theta) g^b(\theta)}{v_F(\theta)} \qquad D_{ab}(\omega, \boldsymbol{q} = 0) = \left[-\lambda\, \omega^2\, \mathbb{I} + \Pi\right]^{-1}_{ab}$$

- **Self-energy is frequency-independent!** Valid at all scales below $\Lambda_{\text{mid-IR}}$.
- There is a "plasmon" pole, but not necessarily physical.
- Non-trivial frequency scaling is from irrelevant operators, **is not intrinsic to the fixed point.** Consistent with expectation from RPA [Kim, Furusaki, Wen, and Lee, PRB (1994)]
- **What happened to r_c ?** Can compute static boson susceptibility: $\chi_{\phi_a \phi_b}(\omega = 0, \boldsymbol{q} \to 0) = 1/r_c \to \infty$ as $r_c \to 0$

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  - Related the chosen UV regularization, follows from "gauge invariance" of $g^a(\theta)\phi_a$

The optical conductivity

- Decompose the current into parallel and perpendicular components

$$J^i_\perp = \sum_\theta v_F(\theta) w^i(\theta)\, \psi^\dagger_\theta \psi_\theta \qquad J^i_{||} = \sum_\theta \frac{i}{2}\, \kappa^{ij}(\theta) \left[\psi^\dagger_\theta \partial_j \psi_\theta - \partial_j \psi^\dagger_\theta\, \psi_\theta\right]$$

Current $\perp$ FS, what we will focus on

Current || FS

Momentum || FS in patch θ

- $J_{||}$ **does not contribute to conductivity at the IR fixed point:** critical boson fluctuations have $q >> \omega$, couple strongest to antipodal patches tangent to q [Metlitski, Sachdev, PRB (2010)].

  $\Rightarrow$ **Emergent intra-patch momentum conservation:** antipodal pairs of patches conserve momentum separately at low energies. Means $J_{||}$ conserved.

The optical conductivity

- $J_\perp$ correlators are determined by the anomaly equation, which gives:

$$J^i_\perp(\omega, \boldsymbol{q}=0) = -\sum_\theta \frac{\Lambda_{\text{patch}}}{(2\pi)^2} v_F(\theta) w^i(\theta) \left[\frac{g^a(\theta)}{v_F(\theta)}\phi_a + w^j(\theta) A_j\right]$$

Current probe

$$\sigma^{ij}(\omega) = -\frac{1}{i\omega}\frac{\delta\langle J^i_\perp(\omega)\rangle_A}{\delta A_j} = \frac{i}{\omega}\left[\frac{\mathcal{D}^{ij}_{(0)}}{\pi} - V^{a,i}V^{b,j}\, i\langle\phi_a(-\omega)\phi_b(\omega)\rangle\right]$$

$$\mathcal{D}^{ij}_{(0)} = \sum_\theta \frac{\Lambda_{\text{patch}}}{4\pi} v_F(\theta) w^i(\theta) w^j(\theta) \qquad V^{a,i} = \sum_\theta \frac{\Lambda_{\text{patch}}}{(2\pi)^2} g^a(\theta) w^i(\theta)$$

Drude weight of free Fermi gas

Vertex factor

The optical conductivity

- **But we have just calculated the boson propagator!** Define the IR limit as $\lambda \to 0$ and let v_F be uniform. Then

$$\sigma^{ij}(\omega) = \frac{i}{\omega}\frac{\Lambda_{\text{patch}}}{(2\pi)^2}\, v_F \left(\mathrm{Tr}_\theta[w^i w^j] - \mathrm{Tr}_\theta[g^a w^i](\mathrm{Tr}_\theta[gg])^{-1}_{ab}\,\mathrm{Tr}_\theta[g^b w^j]\right)$$

$\mathrm{Tr}_\theta[fg] \equiv \sum_\theta f(\theta)g(\theta)$

Drude weight of free Fermi gas

Coupling to boson reduces Drude weight

- **Ising-nematic,** $g^a(\theta) = g^a(\theta+\pi)$ **:** second term vanishes b/c $w^i(\theta) = -\,w^i(\theta+\pi)$
- **"Loop current,"** $g^a(\theta) = -\,g^a(\theta+\pi)$ **:** Drude weight reduced, not necessarily zero
- **Gauge field,** $g^i(\theta) = v_F(\theta)\,w^i(\theta)$ **:** conductivity vanishes b/c charge is gauged away
- Again, we find no non-trivial frequency scaling at the IR fixed point. **Critical fluctuations do not generate conductivity in Hertz-Millis theories!**

Confessing our sins

- The arguments leading to the conductivity involved a few (weak) assumptions:

  1. **Validity of the mid-IR theory and $U(1)_{\text{patch}}$ symmetry/anomaly** [sufficient for deriving $\Pi(\omega)$]

     Absence of irrelevant operators, e.g.

     (i) inter-patch scattering,

     (ii) curvature of the dispersion, $w_i(\theta)\,\kappa_{ij}(\theta) = 0$

     (iii) form factor variation inside a patch, $g(\mathbf{k};\theta)$

     (iv) Boson self-interactions, $V[\phi]$

     Presence of these terms leads to "corrections to scaling" in $\Pi(\omega), \sigma(\omega)$.

  2. **Emergent intra-patch momentum conservation at the IR fixed point:** Critical boson fluctuations ($\omega << q$) only see patches tangent to their momentum. Bosons with $q = 0$ couple to all patches. [sufficient, with (1), for deriving $\sigma(\omega)$; will not need this later]

Static susceptibilities

- Earlier, I said we could compute $\chi_{\phi_a\phi_b} = \frac{1}{r - r_c}\delta_{ab}$ exactly (with $r_c = 0$). Follows from "gauge invariance" of $g^a(\theta)\phi_a$.
- Also get $\chi_{n_\theta\phi_a} = 0$ from path integral manipulations.
- These results plus similar anomaly arguments (this time with $\omega = 0$, finite $q_\perp$) allows us to compute the static density susceptibility in a patch,

$$\chi_{n_\theta n_{\theta'}} = \frac{\Lambda_{\text{patch}}}{(2\pi)^2}\frac{1}{v_F(\theta)}\delta_{\theta\theta'}$$

Static susceptibilities

- Susceptibility of $\tilde{n}_\theta = n_\theta - \Lambda_{\text{patch}}\, g^a \phi_a / (2\pi)^2 v_F$, which we said is always conserved in the absence of external fields, is more interesting.

- Tune into the disordered phase, $r - r_c > 0$. Get a Fermi liquid with conserved density $\tilde{n}_\theta$, Landau parameters determined by the coupling to the boson. Find

$$\chi_{\tilde{n}_\theta \tilde{n}_{\theta'}} = \chi_{n_\theta n_{\theta'}} + \frac{(\Lambda_{\text{patch}})^2}{(2\pi)^4} \frac{g^a(\theta) g^b(\theta')}{v_F(\theta) v_F(\theta')} \chi_{\phi_a \phi_b}$$

$$= \frac{1}{v_F(\theta)} \frac{\Lambda_{\text{patch}}}{(2\pi)^2} \delta_{\theta\theta'} + \frac{\Lambda_{\text{patch}}(\theta) \Lambda_{\text{patch}}(\theta')}{(2\pi)^4} \frac{g^a(\theta) g_a(\theta')}{v_F(\theta) v_F(\theta')} \frac{1}{r - r_c}$$

- $\chi_{\tilde{n}_\theta \tilde{n}_{\theta'}} \to \infty$ **as criticality is approached,** $r - r_c \to 0$**.** Landau parameters diverge. Consistent with perturbative expectations [Maslov and Chubukov, PRB (2010)], [Mross *et al.*, PRB (2010)]

In search of quantum critical conductivity

- **Tuesday night's arXiv: [Shi, Else, HG, and Senthil, arXiv:2208.04328]** Transport in the random flavor expansion of [Esterlis and Schmalian, PRB (2019)], [Aldape *et al.*, arXiv: 2012.00763], [Esterlis *et al.*, PRB (2021)].

- Have *N* bosons, *N* fermions with Gaussian random couplings g_{IJK}, $N \to \infty$ gives a controlled, "Eliashberg" NFL fixed point

$$g_{IJK}(\theta)\,\phi_I \psi^\dagger_J \psi_K$$

- **Anomaly loosened:** boson does not quite couple to a conserved charge.

- Means $\Pi(\omega)$ not fixed to a constant, incompatible with $N = 1$ theory. One finds the Eliashberg result:

$$\Pi(\omega) \sim c_z\,\omega^{(z-2)/z} + \mathcal{O}(1/N)$$

In search of quantum critical conductivity

- Implications:
    1. **Non-vanishing fixed point conductivity (loop current only):** $\sigma(\omega > 0) \sim \omega^{-2/z} = \omega^{-2/3}$ for $z = 3$.
    2. $\omega \to 0$ and $N \to \infty$ limits do not commute.

- Calculate conductivity using our **"anomaly-assisted large N "** technique + memory matrix. **Final answer is**

$$\sigma(\omega) = N\frac{i}{\omega}\left[\frac{\mathcal{D}_0}{\pi} - \frac{\delta\mathcal{D}}{1 + iN\mathcal{C}_z\,\omega^{(z-2)/z}/\delta\mathcal{D}} + \dots\right]$$

$$= N\frac{i}{\omega}\frac{\mathcal{D} - \delta\mathcal{D}}{\pi} + N^2\,\mathcal{C}_z\,\omega^{-2/z} \quad \text{if } \omega^{(2-z)/z} << 1/N$$

[Shi, Else, HG, and Senthil, arXiv:2208.04328]

The future

- **Big question**: Are there (clean) theories of non-Fermi liquids with IR fixed point conductivity?

    ⇒ ***<u>Need better models!</u>***

- Magnetotransport

- Finite-**Q** order parameters

- Higher dimensional bosonization of Hertz-Millis theories