Title: Lie superalgebras and S-duality

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Abstract: We present a series of (partly proven) conjectures describing geometric realizations of categories of (finite-dimensional) representations of quantum super-groups U_q(g) corresponding to Lie super-algebras g with reductive even part and a non-degenerate invariant form.

We shall also discuss the meaning of these conjectures from the point of view of local geometric Langlands correspondence as well as a connection to the work of Ben-Zvi, Sakellaridis and Venkatesh.

Based on joint works with M.Finkelberg, V.Ginzburg and R.Travkin as well as the work of R.Travkin and R.Yang.

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PI talk 2022

Thursday, June 30, 2022 11:56 AM

1. Begin by recalling usual geometric Satake equivalence and discuss why it is bad
2. Replace it with something good (known)
3. Generalize part 3 (Lie superalgebras will appear)
4. Discuss relation to Do

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① Discuss relation to David's talk

G – reductive alg. group, connected $/\mathbb{C}$

$G^\vee$ – Langlands dual

$K = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]]$

$Gr_G = G(K)/G(\mathcal{O})$

Re

G reductive alg. group, connected / $\mathbb{C}$

$G^\vee$ – Langlands dual

$K = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]]$

$Gr_G = G(K)/G(\mathcal{O})$

<u>Theorem</u> $\mathrm{Perv}_{G(\mathcal{O})}(Gr_G) =$

$= D\text{-mod}_{G(\mathcal{O})}(Gr_G) \simeq \mathrm{Rep}\, G^\vee$

D.

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$$\mathrm{Gr}_G = G(K)/G(\mathcal{O})$$

Theorem $\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G) =$

$$= D\text{-mod}_{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{Rep}\, G^\vee$$

Drawbacks of this statement:

① Doesn't hold as stated on the derived level

$= D\text{-mod}_{G(\mathcal{O})}(\mathrm{Gr}_G) \simeq \mathrm{Rep}\, G$

Drawbacks of this statement:

(1) Doesn't hold as stated on the derived level

(2) Doesn't extend to quantum level

$\mathrm{Rep}\, G^\vee$ has a natural deformation

$\mathrm{Rep}_q G^\vee$ – repn's of the quantum

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$\mathrm{Rep}_q G^\vee$ – repn's of the quantum

(a form on $\mathfrak{g} = \mathrm{Lie}\, G$ is fixed

integral, even) $(\cdot,\cdot)$

$(\cdot,\cdot)$ defines some line bundle $\mathcal{L}$ (determinant)

on Gr_G

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(a form on $\mathfrak{g} = \mathrm{Lie}\, G$ is fixed integral, even) $\langle \cdot , \cdot \rangle$

$\langle \cdot , \cdot \rangle$ defines some line bundle $\mathcal{L}$ (determinant)

on Gr_G

Try to study $\mathrm{Perv}_{G(\mathcal{O})}(\mathring{\mathcal{L}})_q$

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathring{\mathcal{L}})_q \simeq \mathrm{Rep}_q\, G^\vee$$

on Gr_G

Try to study $\mathrm{Perv}_{G(\mathcal{O})}(\mathring{\mathcal{L}})_q$

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathring{\mathcal{L}})_q \simeq \mathrm{Rep}_q\, G^\vee$$

LHS is Vect is q is generic

<u>Remark</u> If $q^\ell = 1$ Then LHS $\simeq \mathrm{Rep}\, H$

H – some red. group $\mathrm{rk}\, H = \mathrm{rk}\, G$

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H – some red. group $\mathrm{rk}\, H = \mathrm{rk}\, G$

Better statement FLE – fundamental local equivalence

Lurie, Gaitsgory, Gaitsgory – Lysenko

$$\begin{array}{c} \mathrm{Whit}(\mathrm{Gr}_G) \\ \| \\ (N(K), \chi)\text{-equiva} \end{array}$$

$\mathrm{Whit}(\mathrm{Gr}_G)$ $\qquad$ $N \subset G$ max. unip.

$\parallel$

$(N(K), \chi)$-equivariant D-module

χ - generic character

$$\mathrm{Whit}(\mathrm{Gr}_G) \simeq \mathrm{Rep}\, G^\vee$$

True on f

$\mathrm{Whit}(\mathrm{Gr}_G)$ $\qquad N \subset G$

$\quad\shortparallel \qquad\qquad\qquad$ max. unip.

$(N(K), \chi)$-equivariant D-module

χ - generic character

$$\mathrm{Whit}(\mathrm{Gr}_G) \simeq \mathrm{Rep}\, G^\vee$$

True on derived level

True with q.

$$\mathrm{Whit}_q(\mathrm{Gr}_G) \simeq \mathrm{Rep}_q\, G^\vee$$

True on derived level

True with q.

$$\mathrm{Whit}_q(\mathrm{Gr}_G) \simeq \mathrm{Rep}_q\, G^\vee$$

Discuss some series of generalizations (we'll construct examples of dual boundary cond. in generic KW-twisted 4d YM)

4d YM — mostly due to Gaiotto)

G · finite super-group

G_0 - reductive "more or less simple"

$\mathfrak{g} = \mathrm{Lie}\, G \qquad \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

$\mathfrak{g}$ has an invariant non-degenerate symmetric bilinear form is reductive more or less simple

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$\mathfrak{g}$ has an invariant non-degenerate symmetric bilinear form

$\mathfrak{g}_1$ becomes a symplectic representation of G_0

We can talk about $\mathrm{Rep}_q\, G$

$\mathrm{Lie}\, \underline{G} = \underline{\underline{\mathfrak{g}}}$

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Lie $\underline{G} = \underline{\mathfrak{g}}$

$\underline{\mathfrak{g}} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

$\underline{[\cdot,\cdot]} : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ is 0.

$[\cdot,\cdot] : \mathfrak{g}_1 \times \mathfrak{g}_1 \to \mathfrak{g}_0$ is 0.

Goal To present a series of gen. of the FLE statement

RHS $q=1$ $\qquad$ $\operatorname{Rep} \underline{G}$

q is generic $\qquad$ $\operatorname{Rep}_q G$

$\operatorname{Rep} \underline{G} = G_0$-equivariant modules over $\Lambda \mathfrak{g}_1$

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$\operatorname{Rep} \underline{G} = G_0$-equivariant modules over $\Lambda \mathfrak{g}_1$

What about CHS?

Roughly speaking we'll have some kind of category of sheaves on Gr_{G_0}

What are the supergroups in question?

$GL(m|n)$

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$SOSp(2k|2n)$ $\qquad$ $SOSp(2k+1|2n)$

$F(4)$ $\qquad$ $G(3)$

$Spin(7) \times SL(2)$ $\qquad$ $G_2 \times SL(2)$ $\qquad$ G_0

Rough idea $\quad G_0 \circlearrowright \mathfrak{g}_1$ – symplectic

$$\mathfrak{g}_1 = M$$

dual $M^\vee$ with action of $G_0^\vee$

If $M^\vee = T^* X^\vee \qquad X^\vee \circlearrowleft$

... with action of G_0

If $M^\vee = T^* X^\vee$ $\quad X^\vee \circlearrowleft G_0^\vee$

$\mathrm{Perv}_{G^\vee(\mathcal{O})}(X^\vee(K)) \simeq \underline{\mathrm{Rep}\, G}$ + q-version

<u>variant</u>: $M^\vee$ is some kind of twisted cotangent bundle

$Y^\vee \circlearrowleft H$ $\quad X^\vee = T^* Y^\vee /\!/ (H, \chi)$

$\mathrm{Perv}_H(Y^\vee(K))$

$$GL(m|n) \qquad m \le n$$

$$\mathfrak{g}_1 = T^* \, Mat(n \times m) \circlearrowleft GL(n) \times GL(m)$$

$$\mathfrak{g}_1 = M, \qquad GL(n) \times GL(m) \supseteq G_0$$

$$m = n \quad M^\vee = T^*(GL(n) \times \mathbb{C}^n)$$

$$M^\vee = ?$$

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$$GL(m|n) \qquad m \leq n$$

$$\mathfrak{g}_1 = T^* \mathrm{Mat}(n \times m) \circlearrowleft GL(n) \times GL(m)$$

$$\begin{array}{c} \mathfrak{g}_1 \\ \shortparallel \\ M \end{array} \qquad \begin{array}{c} GL(n) \times GL(m) \\ \shortparallel \\ G_0 \end{array}$$

$$M^\vee = ?$$

$$m = n \quad M^\vee = T^*(GL(n) \times \mathbb{C}^n) \supset GL(n) \times GL(n)$$

$$m = n-1 \quad M^\vee = T^* GL(n)$$

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$$\mathcal{M}$$

$$\shortparallel$$

$$M$$

$$\begin{array}{c} \shortparallel \\ G_0 \end{array}$$

$$M^\vee = ?$$

$$m = n \qquad M^\vee = T^*(GL(n) \times \mathbb{C}^n)$$

$$\cup$$

$$GL(n) \times GL(n)$$

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$$GL(n) \times GL(n)$$

$$m = n-1 \qquad M^\vee = T^* GL(n)$$

$$\cup$$

$$GL(n) \times GL(n-1)$$

$$m < n \qquad M^\vee = GL(n) \times S_{m,n} \quad \leftarrow \text{symplectic}$$

$S_{m,n}$ = Slodowy slice to GL(n) × GL(m) action

$$e_{m,n} = \left(\boxed{\begin{matrix} 0 & 1 & & \\ & 0 & 1 & \\ & & 0 & 1 \\ & & & 0 \end{matrix}} \qquad \right) \; n-m$$

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$$= \mathrm{Perv}_{GL(n,K)}\left(Gr_{GL(n)\times GL(n)} \times \ldots \right)$$

$$\simeq \mathrm{Rep}\ \underline{GL}(n|n)$$

Same thing with q

$m = n-1$ $\quad \mathrm{Perv}_{GL(n-1,\mathcal{O})}\left(Gr_{GL(n)} \right) \simeq \mathrm{Rep}\ \underline{GL}(n-1|n)$

$$\mathrm{Perv}_{GL(n-1,\mathcal{O})}\left(Gr_{GL(n)} \right)_q \simeq \mathrm{Rep}\ GL(n-1|n)$$

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$$\mathrm{Rep}_q\, GL(n|m)$$

$q=1$ Proven B-Finkelberg-Ginzburg-Travkin $m=n$ or $m=n-1$

Travkin-Yang any $m<n$

q generic $m=n-1$ B-Finkelberg-Travkin

$m=0$ $\mathcal{U}_{m,n} =$ ma

$m=n+1$ $\quad \mathrm{Perv}_{GL(n-1,\mathcal{O})}\left(\mathrm{Gr}_{GL(n)}\right) \simeq \mathrm{Rep}\ \underline{GL(n-1|n)}$

$$\mathrm{Perv}_{GL(n-1,\mathcal{O})}\left(\mathrm{Gr}_{GL(n)}\right)_q \simeq \mathrm{Rep}_q\left(GL(n-1|n)\right)$$

$m<n$

$$GL(n)\times S_{m,n} = T^* GL(n) /\!/ (\mathcal{U}_{m,n},\chi)$$

$$\mathrm{Perv}_{GL(m,\mathcal{O})\ltimes(\mathcal{U}_{m,n}(K),\chi)}\left(\mathrm{Gr}_{GL(n)}\right)_q$$

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and we just get the FLE for $GL(n)$

On the level of formulation we have similar statements in all other cases.

Proven for $q=1$ in the SOSp case

Lemma In the above case $\mathcal{J}_1$ is always a hyperspherical symplectic represen is always a hyperspherical symplectic representation of G_0

G - red. group

M - symplectic rep. of G

When does it define a boundary condition?

$\langle \cdot, \cdot \rangle_M$ - inv. form on $\mathfrak{g}$

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If $\frac{1}{2}\langle\cdot,\cdot\rangle$ is still integral and even

$\Rightarrow$ M defines a boundary cond.

If not then it does not

Probably it defines a boundary cond. in some twisted version of 4d YM

$\mathrm{Perv}_{G(\mathcal{O})}$

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$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)$$

If not then it does not

Probably it defines a boundary cond. in some twisted version of 4d YM

$$\mathrm{Perv}_{G(\mathcal{O})}(\mathrm{Gr}_G)_{\frac{1}{2}\langle\cdot,\cdot\rangle_M} \simeq \mathrm{Rep}\, H$$

$$\mathrm{Perv}_{Sp(2n,\mathcal{O})}(\mathrm{Gr}_{Sp(2n)})_{\frac{1}{2}} \simeq \mathrm{Rep}\, Sp(2n)$$

$SOSp(2k+1|2n)$

$\mathfrak{g}_1 = \mathbb{C}^{2k+1} \otimes \mathbb{C}^{2n}$ — anomaly is present

$k < n$

dual group $Sp(2k) \times Sp(2n)$

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$SOSp(2k+1|2n)$

$\mathfrak{g}_1 = \mathbb{C}^{2k+1} \otimes \mathbb{C}^{2n}$ – anomaly is present

$k < n$

dual group $Sp(2k) \times Sp(2n)$

$M^\vee = Sp(2n) \times$ Slodowy slice

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$m < n$

$$GL(n) \times S_{m,n} = T^* \, GL(n) /\!/ (U_{m,n}, \chi)$$

$$\mathrm{Perv}_{GL(m,\mathcal{O}) \ltimes (U_{m,n}(K), \chi)} (Gr_{GL(n)})_q$$

$$\cong$$

$$\mathrm{Rep}_q \, GL(n|m)$$

$\bar{q} = 1$ Proven B-Finkelberg-Ginzburg-Travkin