

Title: Lie superalgebras and S-duality

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Abstract: We present a series of (partly proven) conjectures describing geometric realizations of categories of (finite-dimensional) representations of quantum super-groups $U_q(g)$ corresponding to Lie super-algebras g with reductive even part and a non-degenerate invariant form.

We shall also discuss the meaning of these conjectures from the point of view of local geometric Langlands correspondence as well as a connection to the work of Ben-Zvi, Sakellaridis and Venkatesh. Based on joint works with M.Finkelberg, V.Ginzburg and R.Travkin as well as the work of R.Travkin and R.Yang.



PI talk 2022

Thursday, June 30, 2022 11:56 AM

- ① Begin by recalling usual symplectic Satake equivalence and discuss why it is bad
- ② Replace it with something good (known)
- ③ Generalize part 3 (Lie superalgebras will appear)
- ④ Discuss relation to Do



Discussion relation to David's talk

G - reductive alg. group, connected / $\mathbb{C}$

$G^\vee$ - Langlands dual

$$K = \mathbb{C}((t)) \supset \mathcal{O} = \mathbb{C}[[t]]$$

$$Gr_G = G(K) / G(\mathcal{O})$$

Re

G - reductive alg. group, connected / $\downarrow$

$G^\vee$ - langlands dual

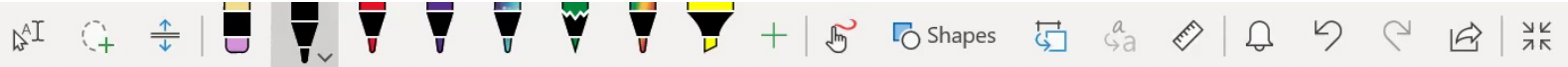
$$K = \mathbb{C}(t) \supset \mathcal{O} = \mathbb{C}[[t]]$$

$$\mathrm{Gr}_G = G(K) / G(\mathcal{O})$$

Theorem $\mathrm{Per}_{G(\mathcal{O})}^v(\mathrm{Gr}_G) =$

$$= \mathrm{D-mod}_{G(\mathcal{O})}(\mathrm{Gr}_G) \cong \mathrm{Rep } G^\vee$$

D.



$$\mathrm{Gr}_G = G(K)/G(\mathcal{O})$$

Theorem $\mathrm{Per}_{G(\mathcal{O})}(\mathrm{Gr}_G) =$
 $= \mathrm{D-mod}_{G(\mathcal{O})}(\mathrm{Gr}_G) \cong \mathrm{Rep } G^\vee$

Drawbacks of this statement:

- ① Doesn't hold as stated on the derived level

$$= D \bmod_{G(G)} (G \otimes_G) \cong \text{Rep } G$$

Drawbacks of this statement:

- ① Doesn't hold as stated on the derived level
- ② Doesn't extend to quantum level
 $\text{Rep } G^v$ has a natural deformation
 $\text{Rep}_q G^v$ - reps of the quantum



$\text{Rep } G^V$ has a natural deformation
 $\text{Rep}_q G^V$ - reps of the quantum
 (a form on $\mathfrak{g} = \text{Lie } G$ is fixed
 integral, even) $(\cdot, \cdot)$

$(\cdot, \cdot)$ defines some line bundle $\mathcal{L}$
 (determinant)
 on $G/\mathbb{G}$

(a form on $\mathfrak{g} = \text{Lie } G$ is fixed
integral, even) $(\cdot, \cdot)$

$(\cdot, \cdot)$ defines some line bundle $\mathcal{L}$
(determinant)
on Gr_G

Try to study $\text{Per}_{G(\mathbb{C})}(\mathcal{L})_q$

$$\text{Per}_{G(\mathbb{C})}(\mathcal{L})_q \simeq \text{Rep}_q G^\vee$$



on Gr_G

Try to study $\text{Per}_{G(\theta)}(\mathbb{Z})_q$

$$\text{Per}_{G(\theta)}(\mathbb{Z})_q \simeq \text{Rep}_q G^\vee$$

LHS is Vect is q is generic

Remark If $q^l = 1$ Then $\text{LHS} = \text{Rep } H$

H - some red. group $\text{rk } H = \text{rk } G$



H - some red. group $\text{rk } H = \text{rk } G$

Better statement FLE - fundamental
local equivalence
Lurie, Gaiitsgory, Gaiitsgory - Lysenko

$\text{Whit}(GR_G)$
 $\parallel$
 $(N(K), \chi)$ - equiva



$$\text{Whit}(GR_G)$$

$$\parallel$$

$$N \subset G$$

$$\text{max. unip.}$$

$$(N(K), \chi) - \text{equivariant D-module}$$

$$\chi - \text{gen. char.}$$

$$\text{Whit}(GR_G) \cong \text{Rep } G^V$$

$$\text{True on } \mathfrak{b}$$

Whit (GR_G)
 $\parallel$

$N \subset G$
 max. unip.

$(N(K), \chi)$ -equivariant D-module
 χ - gener. character

$$\text{Whit}(GR_G) \simeq \text{Rep } G^v$$

True on derived level

True with q .

$$\text{Whit}_q(GR_G) \simeq \text{Rep}_q G^v$$



True on derived level

True with q .

$$\text{Whit}_q(\text{GR}_G) \simeq \text{Rep}_q G^\vee$$

Discuss some series of generalizations
(we'll construct examples of dual
boundary cond. in generic k -twisted
4d YM)



Id YM mostly due to Gaiotto)

G - finite super-group

G_0 - reductive "more or less simple"

$\mathfrak{g} = \text{Lie } G$ $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$

$\mathfrak{g}$ has an invariant non-degenerate symmetric bilinear form

no recursive more or less simple

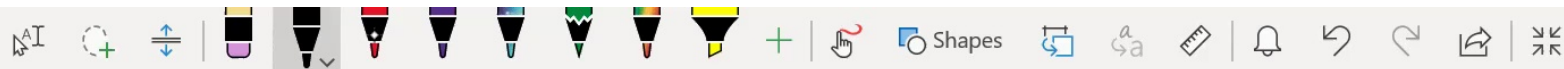
$$\mathfrak{g} = \text{Lie } G \quad \mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$\mathfrak{g}$ has an invariant non-degenerate symmetric bilinear form

$\mathfrak{g}_1$ becomes a symplectic representation of G_0

We can talk about $\text{Rep}_q G$

$$\underline{\text{Lie } G} = \underline{\mathfrak{g}}$$



$\mathfrak{g}$ has an invariant non-degenerate symmetric bilinear form

$\mathfrak{g}_1$ becomes a symplectic representation of G_0

We can talk about $\text{Rep}_g G$

$$\text{Lie } G = \underline{\mathfrak{g}}$$

$$\underline{\mathfrak{g}} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$$

$$\underline{[\cdot, \cdot]}: \mathfrak{g}_1 \times \mathfrak{g}_1 \rightarrow \mathfrak{g}_0 \text{ is } 0.$$

$L, -J : \mathcal{Y}_1 \wedge \mathcal{Y}_1 \rightarrow \mathcal{Y}_0$ is 0.

Goal To present a series of gen.
of the FLE statement

RHS $q=1$

Rep G

q is generic

Rep _{q} G

Rep G = G_0 -equivariant modules
over $\Lambda \mathcal{Y}_1$

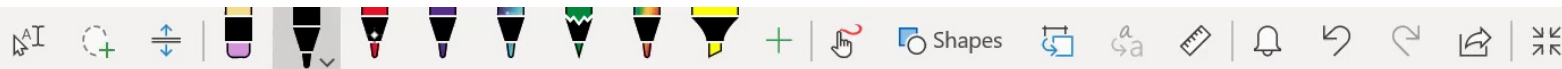


$\text{Rep } G = G_0\text{-equivariant modules}$
over Λg_1

—
What about CHS?

Roughly speaking we'll have some
kind of category of sheaves on Gr_{G_0}

—
What are the supergroups in question?
 $GL(m|n)$



$$SOsp(2k|2n)$$

$$F(4)$$

$$Spin(7) \times SL(2)$$

$$SOsp(2k+1|2n)$$

$$G(3)$$

$$G_2 \times SL(2)$$

$$G_0$$

Rough idea $G_0 \hookrightarrow \mathfrak{g}_1$ - symplectic
 $\parallel$
 M

dual $M^\vee$ with action of $G_0^\vee$

$$\text{If } M^\vee = T^*X^\vee \quad X^\vee \supset$$



$$\text{If } M^\vee = T^* X^\vee \quad X^\vee \hookrightarrow G_0^\vee$$

$$\text{Perv}_{G_0^\vee}(X^\vee(K)) \simeq \text{Rep } G + g\text{-version}$$

variant: $M^\vee$ is some kind of twisted cotangent bundle

$$Y^\vee \hookrightarrow H \quad X^\vee = T^* Y^\vee // (H, \chi)$$

$$\text{Perv}_H(Y^\vee(K))$$



$$GL(m/n)$$

$$m \leq n$$

$$g_1 = T^* \text{Mat}(n \times m) \hookrightarrow GL(n) \times GL(m)$$

$$\parallel \qquad \qquad \qquad \parallel$$

$$M \qquad \qquad \qquad G_0$$

$$m=n \quad M^V \subseteq T^*(GL(n) \times \mathbb{C}^n)$$

$$M^V = ?$$



$$GL(m/n) \quad m \leq n$$

$$g_1 = T^* \text{Mat}(n \times m) \supset GL(n) \times GL(m)$$

$\parallel$ $\parallel$
 M G_0

$$M^V = ? \quad m=n \quad M^V \subseteq T^* \left(GL(n) \times \mathbb{C}^n \right)$$

$\cup$
 $GL(n) \times GL(n)$

$$m = n-1 \quad M^V = T^* GL(n)$$



$\mathcal{M}$
 $\mathcal{M}$

$\mathcal{G}_0$

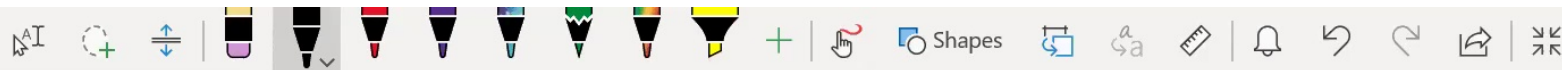
$$m=n \quad \mathcal{M}^V \subseteq T^*(GL(n) \times \mathbb{C}^n) \cup GL(n) \times GL(n)$$

$\mathcal{M}^V = ?$

$$m = n-1$$

$$\mathcal{M}^V = T^* GL(n)$$

$$\cup GL(n) \times GL(n-1)$$


 $M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$$GL(n) \times GL(n)$$

$$m = n-1$$

$$M^V = T^* GL(n)$$

$$GL(n) \times GL(n-1)$$

$$m < n \quad M^V = GL(n) \times S_{m,n}$$

$S_{m,n}$ = slow slice to

$$e_{m,n} = \begin{pmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 0 \\ 0 & 0 \end{pmatrix} \quad n-m$$

← symplectic
 $GL(n) \times GL(m)$
 action



$$= \text{Perr}_{GL(n, K)} (Gr_{GL(n) \times GL(n)})$$

$$\simeq \text{Rep } \underline{GL(n/n)}$$

Same thing with q

$$m=n-1 \quad \text{Perr}_{GL(n-1, 0)} (Gr_{GL(n)}) \simeq \text{Rep } \underline{GL(n-1/n)}$$

$$\text{Perr}_{GL(n-1, 0)} (Gr_{GL(n)})_q \simeq \text{Rep}(GL(n-1/n))$$



Rep_q GL(n|m)

$q=1$ Proven B-Finkelberg-Ginzburg-Travkin
 $m=n$ or $m=n-1$

Travkin-Yang any $m < n$

q generic $m=n-1$ B-Finkelberg-Travkin

$m=0$ $u_{m,n} = \text{max}$



$$\text{Perv}_{GL(n-1,0)}(Gr_{GL(n)})_q \cong \text{Rep}_{GL(n-1/n)}$$

$$m < n$$

$$GL(n) \times S_{m,n} = T^* GL(n) / (U_{m,n}, \chi)$$

$$\text{Perv}_{GL(m,0)} \times (U_{m,n}(k), \chi) (Gr_{GL(n)})_q$$

◇ 12



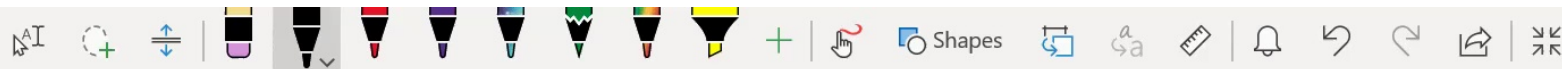
and we just get the TLT for $GL(n)$

On the level of formulation we have similar statements in all other cases.

Proven for $q=1$ in the $SOsp$ case

Lemma In the above case $\mathfrak{g}_1$ is always a hyperspherical symplectic representation





is always a hyperspherical symplectic representation of G_0

G - red. group

M - symplectic rep. of G

when does it define a boundary condition?

$\langle \cdot, \cdot \rangle_M$ - inv. form on $\mathfrak{g}$





If $\frac{1}{2} \langle \cdot, \cdot \rangle$ is still integral and even

$\Rightarrow M$ defines a boundary cond.

If not then it does not

Probably it defines a boundary cond. in some twisted version of

4d YM

$\text{Perv}_{G(\theta)}$





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$\text{Perv}_{G(\theta)}(GR_G)$



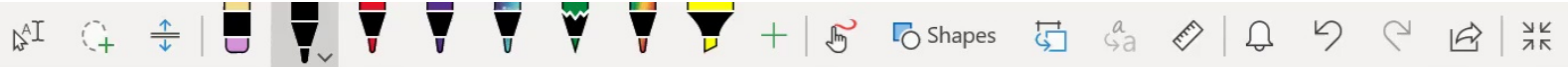


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$$\text{Perv}_{G(\mathcal{O})}(\mathbb{G}_G)_{\frac{1}{2}(\cdot, \cdot)_M} \cong \text{Rep } H$$

$$\text{Perv}_{\text{Sp}(2n, \mathcal{O})}(\mathbb{G}_{\text{Sp}(2n)})_{\frac{1}{2}} \cong \text{Rep } \text{Sp}(2n)$$

◇



$$SOsp(2k+1/2n)$$

$$\mathfrak{g}_1 = \mathbb{C}^{2k+1} \otimes \mathbb{C}^{2n} \quad \text{-- anomaly is present}$$

$$k < n$$

$$\text{dual group } Sp(2k) \times Sp(2n)$$



$$SOsp(2k+1/2n)$$

$$\mathfrak{g}_1 = \mathbb{C}^{2k+1} \otimes \mathbb{C}^{2n} \quad \text{-- anomaly is present}$$

$$k < n$$

$$\text{dual group } Sp(2k) \times Sp(2n)$$

$$M^\vee = Sp(2n) \times \text{slodowny slice}$$



$$m < n$$

$$GL(n) \times S_{m,n} = T^* GL(n) / (U_{m,n}, \chi)$$

$$\text{Perv}_{GL(m, \mathbb{C}) \times (U_{m,n}(k), \chi)} (Gr_{GL(n)})_q$$

12

$$\text{Rep}_q GL(n|m)$$

$q=1$ Proven by Finkelberg-Ginzburg-Travkin