Title: Field theory in the 1/2 Omega-background

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Abstract: In this lecture I'll discuss various aspects of 4d N=2 and 5d N=1 supersymmetric QFT's in the 1/2 Omega-background (and along the way try to emphasize some relations to the 3d N=2 theories discussed in this workshop). Central to this story is the Nekrasov instanton partition function (or topological string partition function) in this background, which we will obtain through abelianization as an integral of a ratio of Wronskians of certain special solutions to the relevant Schrodinger equation. We will argue that a slight generalization of the above partition function solves an associated Riemann-Hilbert problem and defines a section of a distinguished line bundle over the moduli space of flat connections.

Field theory in $\frac{1}{2}\Omega$-background

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A few different set-ups:

6d $\mathcal{N}=1$ SYM

$\downarrow$ dimensionally reduce on $\mathbb{R}^4 \times T^2$, with $T^2 \to 0$

$T^\# \to \mathcal{M}_{Higgs}$

$\downarrow$

4d $\mathcal{N}=2$ SYM $\xrightarrow[IR]{}$ vacuum moduli space

* Coulomb $\mathcal{B}$
* Higgs

$\downarrow$ dimensionally reduce on $\mathbb{R}^3 \times S^1$, with $S^1 \to 0$

3d $\mathcal{N}=4$ SYM $\xrightarrow[IR]{}$ 3d σ-model into vacuum moduli space

* Coulomb
* Higgs $\mathbb{R}^{4\#}/G$

Also: 4d $\mathcal{N}=2$ SYM

$\downarrow$ reduce on $\mathbb{R}^3 \times S^1_R$

3d $\mathcal{N}=4$ SYM

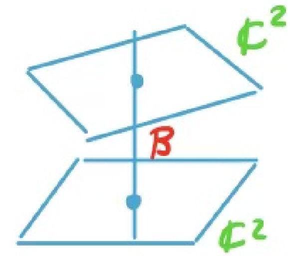
with $\mathcal{M}_C = \mathcal{M}^{\zeta}_{Hitchin}(R)$:

$$\begin{cases} F_A + R^2[\varphi, \bar{\varphi}] = 0 \\ \bar{\partial}_A \varphi = 0 \\ \partial_A \bar{\varphi} = 0 \end{cases}$$

A is a connection on $SU(k)$-bundle $E \to C$

and $\varphi \in \Omega^{1,0}(C, \mathrm{End}\,E)$ the Higgs field

Then: 5d $\mathcal{N}=1$ SYM



on $\mathbb{C}^2$-bundle over S^1_β:

$\beta \to 0$ $\downarrow$

$$(z_1, z_2, x_5) \to (z_1 e^{i\beta\epsilon_1}, z_2 e^{i\beta\epsilon_2}, x_5 + \beta)$$

also need $SU(2)_I$ rotation

$$\begin{pmatrix} e^{i\beta(\epsilon_1+\epsilon_2)/2} & 0 \\ 0 & e^{-i\beta(\epsilon_1+\epsilon_2)/2} \end{pmatrix}$$

to preserve susy

and gauge rotation $e^{i\beta \underline{a}} \in G$

4d $\mathcal{N}=2$ SYM

in the Ω-background

$\mathbb{C}_{\epsilon_1} \times \mathbb{C}_{\epsilon_2}$

$\rightsquigarrow$

$$\log Z^{Nek}(\epsilon_{1,2}, \tau, \underline{a}) = \frac{1}{\epsilon_1 \epsilon_2} \mathcal{F}_0(\tau, \underline{a}) + \text{terms less singular in } \epsilon_{1,2}$$

Twisting: 4d $\mathcal{N}=2$ theory has symmetry group

$$\underbrace{SU(2)_L \times SU(2)_R}_{SO(4)} \times SU(2)_I$$

R-symmetry

Poincare

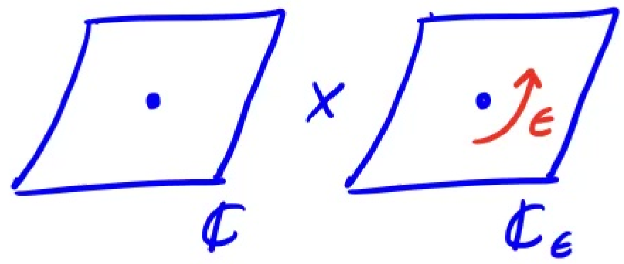
Ω-deformation is compatible with Donaldson twist:

new Poincaré group $SU(2)_L \times SU(2)_{diag}$

$$SU(2)_{diag} \subset SU(2)_R \times SU(2)_I$$

(in general 1 scalar supercharge $\overline{Q}$)

Today we are interested in $\frac{1}{2}\Omega$- background $\mathbb{C} \times \mathbb{C}_\epsilon$:



[Nekrasov-Shatashvili]

on $\mathbb{C}^2$-bundle over S^1_β: $(z_1, z_2, x_5) \to (z_1,\ z_2 e^{i\beta\epsilon},\ x_5 + \beta)$

with $\beta \to 0$

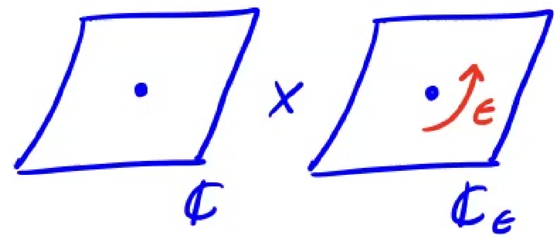
(note that the $\mathbb{C}_\epsilon$-bundle over S^1 is also known as $\mathbb{C} \times_q S^1$)

$\rightsquigarrow$ relation to 3d $\mathcal{N}=2$ theories

The $\frac{1}{2}\Omega$-background effectively defines

a 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{C}$

with 4 supercharges

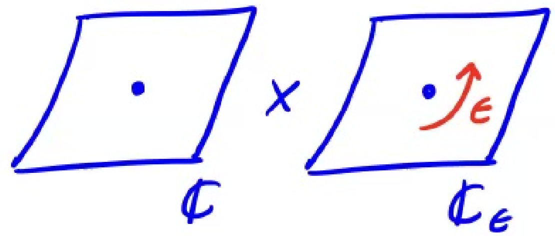


At low energies $E \ll |\epsilon|$, this theory is described by

r abelian vector multiplets coupled to an effective twisted superpotential

$$\frac{1}{\epsilon}\widetilde{W}^{\mathrm{eff}}(\epsilon,\tau,\underline{a}) = \lim_{\epsilon_2\to 0} \epsilon_2 \log Z(\epsilon_{1,2},\tau,\underline{a}) = \frac{1}{\epsilon}F_0 + \dots$$

The $\frac{1}{2}\Omega$-background effectively defines a 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{C}$.



Note: the Donaldson twist is <u>not</u> the appropriate twist here

consider $U(1)_1 \times U(1)_2 \times U(1)_I$

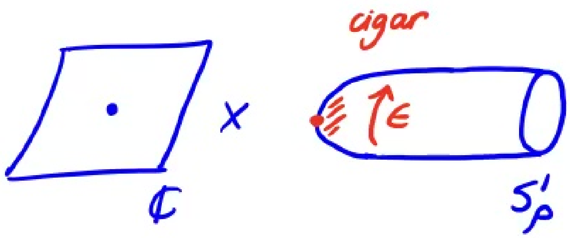
$\subset SU(2)_L \times SU(2)_R \times SU(2)_I$

define twist wrt $U(1)_1 \times \mathrm{diag}\,(U(1)_2 \times U(1)_I)$

- local operators?
- boundary conditions?
- defects?

More insights from duality chain:

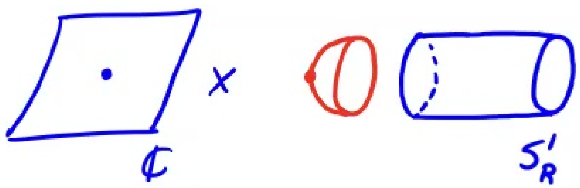
[Nekrasov-Witten]



$$ds^2 = dr^2 + f(r)\,d\varphi^2$$

$$f(r) \sim r^2 \quad r \to 0$$
$$\to \rho^2 \quad r \to \infty$$

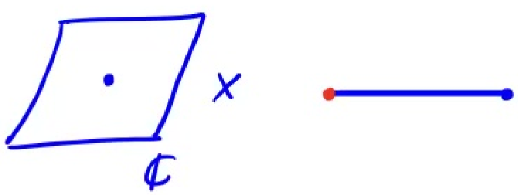
$\downarrow$ undo $\frac{1}{2}\Omega$-deformation away from origin cigar



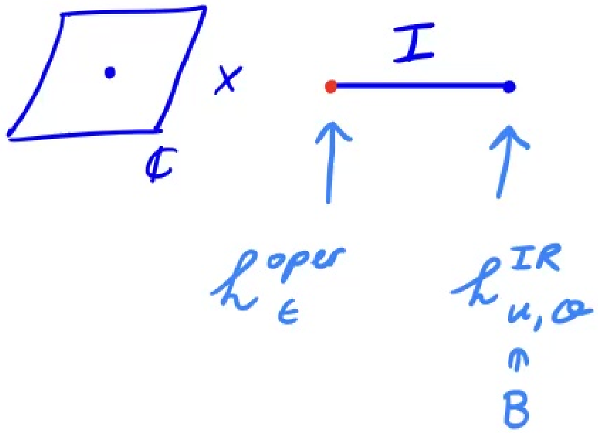
in return for a field definition

[Nekrasov-Witten]

$\downarrow$ compactify on S^1_R



in IR 3d sigma model with target $\mathcal{M}^{\zeta}_{Hitchin}(R) \leftarrow \zeta/R = \epsilon$



$\arg(\epsilon) = \vartheta \rightsquigarrow$ 4 supercharges preserved

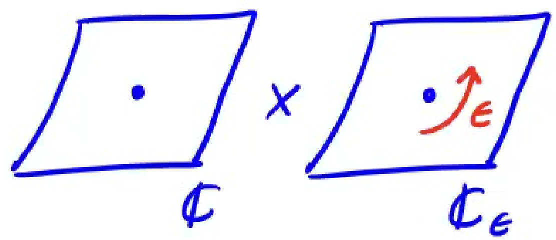
Note that boundary condition at ∞ in $\mathbb{C} \times \mathbb{C}_\epsilon$ becomes very explicit

$I \to 0 \ \Big\downarrow$ [Kapustin - Rozansky - Saulina]

2d $\mathcal{N}=(2,2)$ sigma model into B with superpotential $W_{u,\vartheta}(\epsilon, \tau, \underline{a})$

= generating function for $\mathcal{L}^{oper}_{\epsilon} - \mathcal{L}^{IR}_{u,\vartheta}$

Thus, we find that the 2d $\mathcal{N}=(2,2)$ theory on $\mathbb{C}\times\mathbb{C}_\epsilon$ is governed in the IR by an object $W_{u,\mathcal{O}}(\epsilon,\tau,\underline{a})$



- How is $W_{u,\mathcal{O}}(\epsilon,\tau,\underline{a})$ defined?
- How is it related to $\frac{1}{\epsilon}\widetilde{W}^{\text{eff}}(\epsilon,\tau,\underline{a}) = \lim_{\epsilon_2\to 0}\epsilon_2 \log Z(\epsilon_{1,2},\tau,\underline{a})$?
- What are UV boundary conditions?

in a weakly coupled region

- $W_{u,\vartheta}(\epsilon, \tau, \underline{a})$ may be obtained* as the Borel resummation of the ϵ-expansion of $\widetilde{W}^{eff}(\epsilon, \tau, \underline{a})$ in the direction ϑ:

divergent $f(\epsilon) = \sum_{n=0}^{\infty} c_n \epsilon^n$ where $c_n \sim n!$

$\rightsquigarrow$ Borel transform $\mathcal{T}f(s) = \frac{c_n}{n!} s^n$

$\rightsquigarrow$ Borel sum $B_\vartheta f(\epsilon) = \int_0^{\infty e^{i\vartheta}} e^{-s}\, \mathcal{T}f(\epsilon s)\, ds$

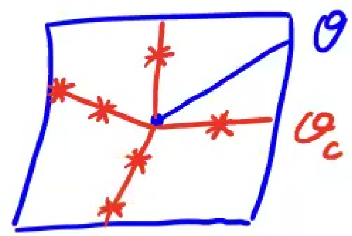
Might think of Borel sum as "transseries":

zero modes from inst + anti-inst

$$B_\vartheta f(t) = \sum\sum c_{k,\ell,n}\, t^n e^{-kc/t} \ln\left(\pm \frac{1}{t}\right)^\ell$$

non-pert instantons

- The Borel sum is not defined along "critical" directions ϑ_c where the Borel transform has singularities



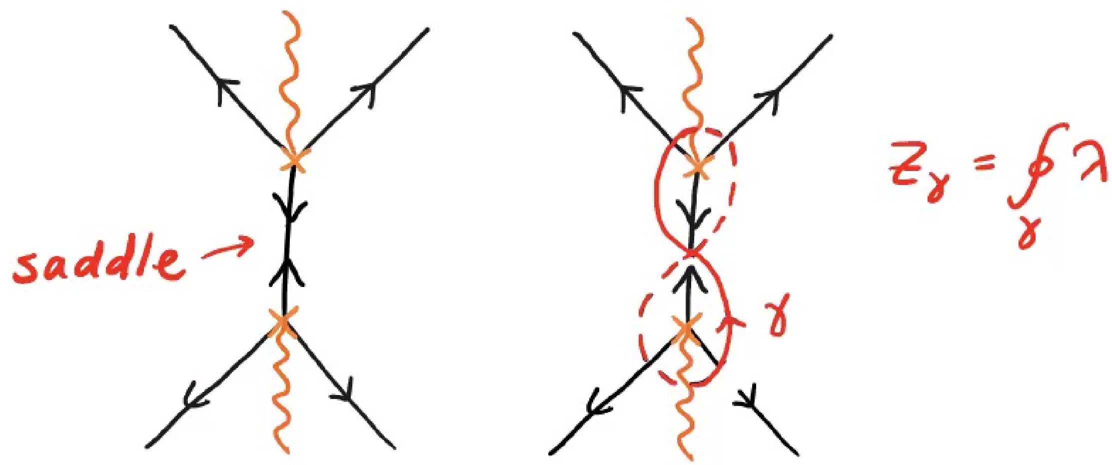
Borel plane

$\rightsquigarrow$ $W_{u,\vartheta}(\epsilon,\tau,\underline{a})$ is piece-wise constant in ϑ and "jumps" across the critical rays.

- The critical directions ϑ_c correspond to the phases

  $\vartheta_c = \arg Z_\gamma$ of BPS particles in the 4d $\mathcal{N}=2$ thy

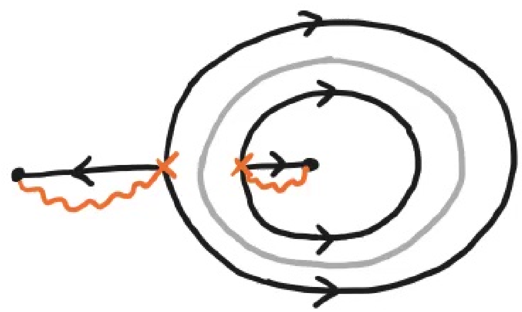
These particles may be visualized as saddle trajectories in the spectral network $\mathcal{W}^{\vartheta}(u)$:



Hence $W_{u,\vartheta}$ encodes the 4d particle spectrum!

- If the 4d $\mathcal{N}=2$ theory has a weakly coupled description, there is a special phase ϑ_{FN} with $\vartheta_{FN} = \arg(W\text{-boson})$

- At this phase the spectral network $\mathcal{W}_{FN}$ includes a family of compact trajectories:

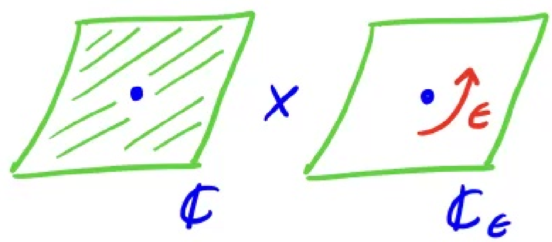


- We have $W_{u,\vartheta_{FN}}(\epsilon,\tau,\underline{a}) = \widetilde{W}^{\text{eff}}(\epsilon,\tau,\underline{a})$

- So far, we have implicitly assumed that the 4d $\mathcal{N}=2$ theory has a weakly coupled description as a gauge thy with gauge gp G coupled to some matter.

- We have also implicitly chosen a "standard Neumann" UV boundary condition at ∞ of $\mathbb{C} \times \mathbb{C}_\epsilon$.

  Neumann $A_{\parallel}$
  Dirichlet $A_{\perp}$



[Nekrasov-Witten]

- In the IR the boundary condition is then specified by a point $u \in \mathcal{B}$, a polarization of Σ_u and a phase ϑ.

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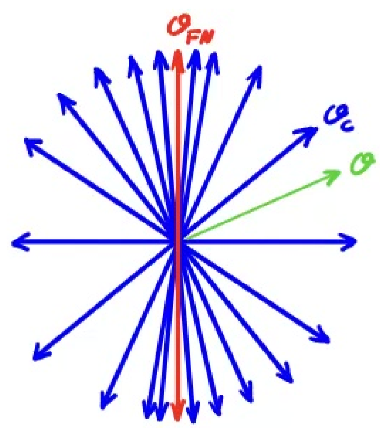
- The corresponding complex Lagrangian $\mathcal{L}^{IR}_{u,\vartheta} \subset \mathcal{M}_{Hitchin}$ is given by $\mathcal{X}^{\vartheta}_{B_j} = 1$, for the corresponding set of B-cycles B_j, where $\mathcal{X}^{\vartheta}_{\gamma_j}$ is a set of complex Darboux coordinates on $\mathcal{M}_{Hitchin}$

  these coordinates may be found from $\mathcal{W}^{\vartheta}_u$ using $\mathcal{W}$-abelianization [Gaiotto-Moore-Neitzke], [Hollands-Neitzke], [Nikolaev]



- generic ϑ $\longrightarrow$ FG-type coordinates
- $\vartheta = \vartheta_{FN}$ $\longrightarrow$ FN-type coordinates

- The superpotential $W_{u,\vartheta}(\epsilon,\tau;\underline{a})$ may be formulated as

$$\mathcal{X}^{\vartheta}_{A_j}(\nabla^{oper}) \equiv -e^{\pi i a_j^{\vartheta}/\epsilon}$$

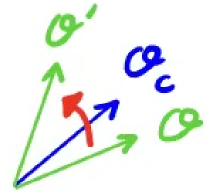
$$\mathcal{X}^{\vartheta}_{B_j}(\nabla^{oper}) \equiv e^{2\epsilon\,\partial W_{u,\vartheta}(\epsilon,\tau,\underline{a}^{\vartheta})/\partial a_j^{\vartheta}} \equiv e^{2\epsilon a^{\vartheta}_{D,j}}$$

- That is, $W_{u,\vartheta}(\epsilon,\tau,\underline{a})$ is the generating function of opers in the complex Darboux coord $\mathcal{X}^{\vartheta}_{\gamma}$ for $\vartheta=\vartheta_{FN}$ [Nekrasov-Rosly-Shatashvili] [H-Kidwai], [Nekrasov-Jeong]

- a_j^{ϑ} and $a^{\vartheta}_{D,j}$ are the Borel sums in the direction ϑ of the quantum periods $\frac{1}{\epsilon}\oint_{A_j/B_j} S(z,\epsilon)\,dz$ $\rightsquigarrow$ exact WKB method [Hollands-Neitzke]

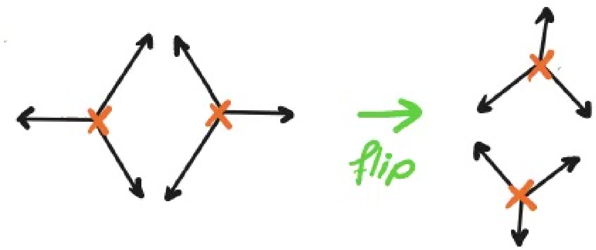
sol'n Ricatti eqn

- Change of boundary condition, e.g. across isolated critical ray:



$$a^{\vartheta'} = a^{\vartheta}$$
$$a_D^{\vartheta'} = a_D^{\vartheta} + \log\left(1 - e^{\pi i a^{\vartheta}/\epsilon}\right)$$

$$\Delta W = -\frac{\epsilon}{2\pi i}\, Li_2\left(e^{\pi i a^{\vartheta}/\epsilon}\right)$$



$\rightsquigarrow$ coupling the 3d $\mathcal{N}=2$  theory to the boundary

- Similar for any ϑ [Dimofte - Gaiotto - Veen], §5.7 of 2109.14699

note: change polarization $\Sigma \rightsquigarrow$ e.m. duality transformation
$\rightsquigarrow$ generalized Legendre transformation on W

e.g. $W \to W' = \epsilon\, a' \cdot a + W$

- So far, we focussed on a weakly coupled region $\subset \mathcal{B}$
However, the definition of $W_{u,\wp}(\epsilon, \tau, \underline{a})$ as a generating function of opers may be extended to all $\mathcal{B}$ and even to 4d $\mathcal{N}=2$ thys without a weakly coupled description

- $W_{u,\wp}(\epsilon, \tau, \underline{a})$ is also the solution to a Riemann-Hilbert problem specified by the corresponding BPS structure [Bridgeland], [Alim-Saha-Tulli-Teschner]

- $\exp W_{u,\wp}(\epsilon, \tau, \underline{a})$ is then a section of a "classical Chern-Simons" line bundle over the moduli space of flat connections

[Neitzke], [Alexandrov-Persson-Pioline], [Coman-Longhi-Teschner], [Alim-Saha-Tulli-Teschner]

the line bundle is defined by gluing between cluster charts using dilogs

• example: pure $SU(2)$

$W_{u,\vartheta} \in \mathfrak{h}$

$\downarrow$

$M_{Hitchin} \times \mathbb{C}^*_\epsilon \longrightarrow$

$\cup \quad \cup$

$\mathcal{U}_\vartheta \quad \mathcal{U}_{\vartheta'}$