Title: Defects, quantum trace map, and exact WKB

Speakers: Fei Yan

Collection: QFT for Mathematicians 2022

Date: June 27, 2022 - 12:00 PM

URL: https://pirsa.org/22060081

Abstract: I will continue the discussions on line defects and surface defects in class S theories, making connections to the construction of the quantum trace map, as well as to the exact WKB method for higher order ODEs.

# Defects, quantum UV-IR map, and exact WKB

Fei Yan

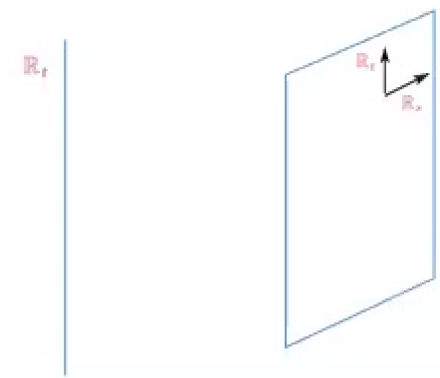
Rutgers University

QFT for mathematicians

Perimeter Institute for Theoretical Physics

June 27th, 2022

# Outlook



- Line defects in 4d N=2 theories:
  - The quantum UV-IR map and construction of link invariants
  - Line defect Schur indices and relation to 4d holomorphic topological twist and 2d VOAs
- Surface defects in 4d N=2 theories and the exact WKB method for higher order ODEs

# Line defects in 4d $N = 2$ theories

- Consider 4d $N = 2$ theory, with the insertion of a susy line defect extending along time direction, sitting at the origin of spacial $\mathbb{R}^3$. (susy Wilson-'t Hooft lines and generalizations)

  [Kapustin-Saulina],[Drukker-Morrison-Okuda],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Aharony-Seiberg-Tachikawa],[Gaiotto-Kapustin-Seiberg-Willett],[Gaiotto-Kapustin-Komargodski-Seiberg],[Ang-Roumpedakis-Seifnashri],[Agmon-Wang],[Bhardwaj,Huebner,Schafer-Nameki],...

- In abelian gauge theories, they are labeled by electromagnetic charge $\gamma$ and a parameter $\zeta \in \mathbb{C}^\times$ (preserved supercharges):
  Example: 4d $N = 2$ $U(1)$ theory, $\gamma$ purely electric:

$$\mathbb{L}(\gamma, \zeta) = \exp\left[ i\gamma \int_{\mathbb{R}_t} \left( A + \frac{1}{2}\left(\zeta^{-1}\phi + \zeta\bar{\phi}\right)\right)\right]$$

# The UV-IR map

4d $N = 2$ theories have a subspace of vacua called the Coulomb branch; the low energy effective field theory is $U(1)^r$ gauge theory. [Seiberg-Witten]
Starting with a susy line defect $\mathbb{L}$ in the UV, deform onto the CB,
$\to$ superposition of line defects in effective abelian theory. [Gaiotto-Moore-Neitzke]

An UV-IR map for line defects:

index for ground states of bulk-defect system

deform onto CB

$$\mathbb{L} \rightsquigarrow F(\mathbb{L}) := \sum_{\gamma} \underline{\overline{\Omega}}(\mathbb{L}, \gamma) X_{\gamma}$$

Defect Hilbert space $\mathcal{H}_{\mathbb{L},u} = \bigoplus_{\gamma} \mathcal{H}_{\mathbb{L},\gamma,u}$

$q \to -1$ IR Wilson-'t Hooft lines

$SO(3)$ rot.

spin refinement $\underline{\overline{\Omega}}(\mathbb{L}, \gamma, q) := \mathrm{Tr}_{\mathcal{H}_{\mathbb{L},\gamma,u}} (-q)^{2J_3} q^{2I_3} e^{-\beta\{\mathcal{Q},\mathcal{Q}^{+}\}} \in \mathbb{Z}[q, q^{-1}]$

$SU(2)_R$

---

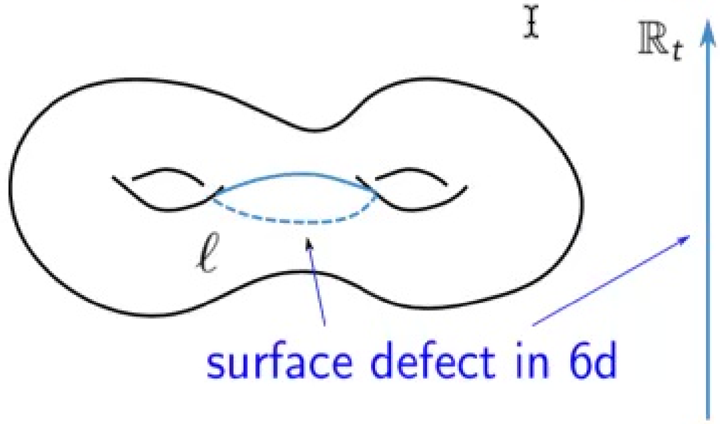
Example N=2 pure $SU(2)$ SYM weak-coupling region

$$F(\mathbb{L}_{\underline{2}}) = X_{(1,0)} + X_{(-1,0)} + X_{(1,1)}$$

fund. Wilson line $(\gamma_e, \gamma_m)$

# Line defects in 4d N=2 theories of class-$S$

Compactifying 6d $(2,0)$ theory of type $\mathfrak{gl}(N)$ on a Riemann surface $C$ with partial top. twist $\rightsquigarrow$ 4d N=2 theory of class-$S$ $T[C]$. [Gaiotto],[Gaiotto-Moore-Neitzke]




Line defects $\mathbb{L}$ in class-$S$ $\leftrightarrow$ "loops" $\ell$ on $C$ (junctions, laminations)

[Drukker-Morrison-Okuda],[Drukker-Gaiotto-Gomis],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke]...

[Fock-Goncharov],[Sikora],[Le],[Xie],[Saulina],[Coman-Gabella-Teschner],[Tachikawa-Watanabe],[Gabella]...

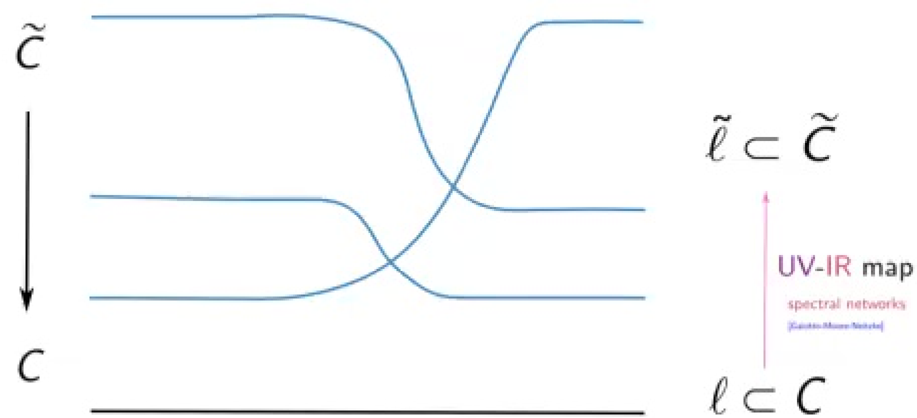
Rk: $\ell$ carries a $\mathfrak{gl}(N)$ representation, consider fundamental representation

# The UV-IR map: geometric picture

A pt. in Coulomb branch $\leftrightarrow$ a $N$-fold branched covering $\widetilde{C} \to C$,
$\widetilde{C} \subset T^*C$ is the Seiberg-Witten curve.

IR: bulk theory approx. by 6d $(2,0)$ theory of type $\mathfrak{gl}(1)$ on $\widetilde{C} \times \mathbb{R}^{3,1}$.

IR line defects $\leftrightarrow$ loops $\tilde{\ell}$ on $\widetilde{C}$



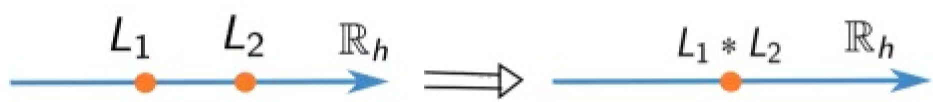
# Line defects OPE

The space of line defects equipped with an algebra structure:
line defects operator product $\mathbb{L}_1\mathbb{L}_2$
This algebra structure admits a quantization via skein algebras.

[Reshetikhin-Turaev],[Turaev],[Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],
[Drukker-Gomis-Okuda-Teschner],[Tachikawa-Watanabe],[Coman-Gabella-Teschner],[Gabella]...

Turning on $\Omega$-bkg on a $\mathbb{R}^2$-plane: non-commutative associative OPE $*$

[Nekrasov-Shatashvili],[Gaiotto-Moore-Neitzke],[Ito-Okuda-Taki],[Yagi],[Oh-Yagi],...



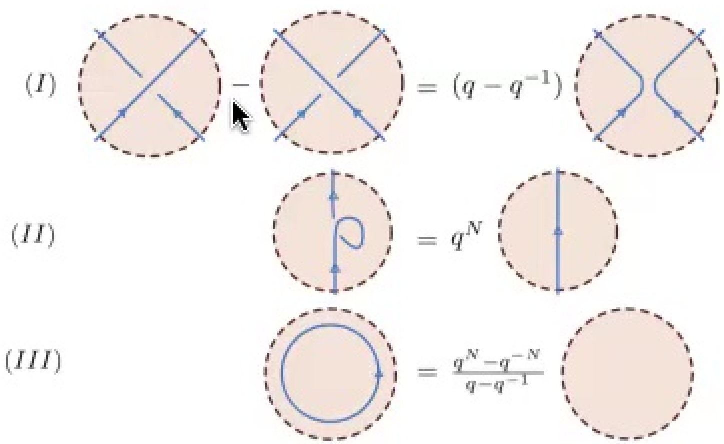

IR: quantum torus algebra $X_{\gamma_1} * X_{\gamma_2} = (-q)^{\langle \gamma_1, \gamma_2 \rangle} X_{\gamma_1+\gamma_2}$

quantum UV-IR map: UV skein algebra $\to$ quantum torus algebra

# The UV and IR skein algebras

UV skein algebra: $\mathfrak{gl}(N)$ (1-para) HOMFLY skein algebra of $M = C \times \mathbb{R}_h$
IR skein algebra: (twisted) $\mathfrak{gl}(1)$ skein algebra of $\widetilde{M} = \widetilde{C} \times \mathbb{R}_h$
algebra structure $\leftrightarrow$ stacking links along $\mathbb{R}_h$

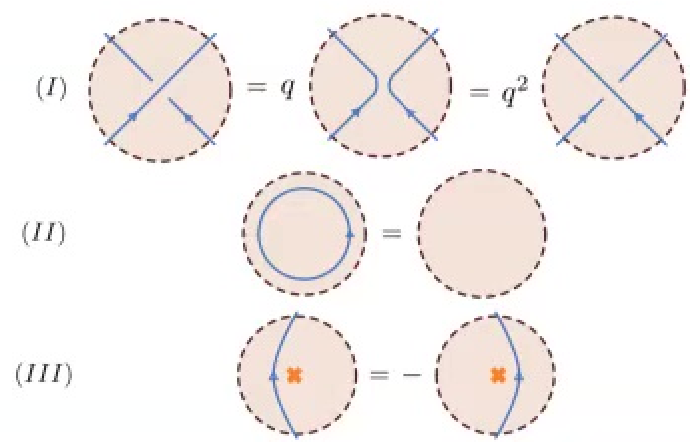
UV: $\mathbb{Z}[q^{\pm 1}]$-module of oriented framed links $L \subset M$




4d $\frac{1}{4}$-BPS line defects: $U(1)$ rot. & $U(1)_R \subset SU(2)_R$

[Witten], [Gaiotto-Witten], [Ooguri-Vafa], [Gukov-Schwarz-Vafa], [Dimofte-Gaiotto-Gukov], [Chun-Gukov-Roggenkamp], [Gukov-Putrov-Vafa], [Gukov-Pei-Putrov-Vafa] ...

IR: $\mathbb{Z}[q^{\pm 1}]$-module of oriented framed links $\widetilde{L} \subset \widetilde{M}$



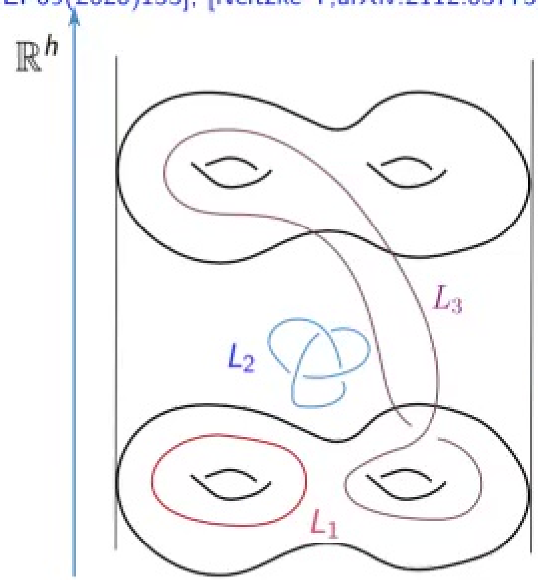

iso. to quantum torus

# The quantum UV-IR map

The quantum UV-IR map sends $L \subset M$ to combinations of $\widetilde{L} \subset \widetilde{M}$:

$$F(L) = \sum_{\widetilde{L}} \alpha(\widetilde{L})\widetilde{L}, \quad \alpha(\widetilde{L}) \in \mathbb{Z}[q^{\pm 1}]$$

[Neitzke-Y,JHEP09(2020)153], [Neitzke-Y,arXiv:2112.03775]




$F(L_1)$: gen. function for framed BPS index

$\underline{\Omega}(\mathbb{L}, \gamma, q)$

$$F(L_2) = q^{Nw(L_2)} P_{\text{HOMFLY}}(L_2, a = q^N, z = q - q^{-1})$$

self-linking number

N=2: vertex model

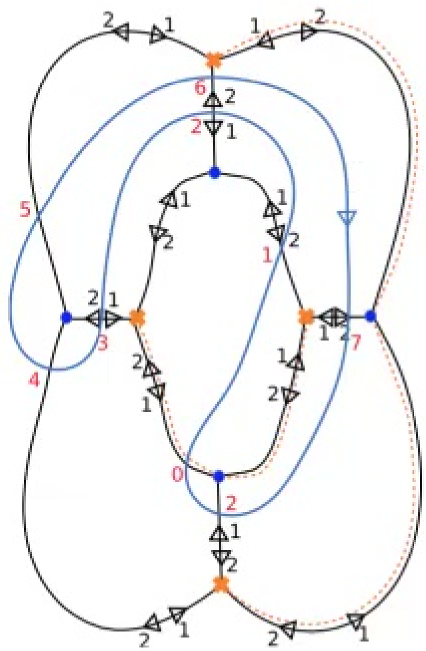
[Witten],[Gaiotto-Witten]

See also [Bonahon-Wong],[Goncharov-Shen],[Douglas-Sun],...

# T[C]: $SU(2)$ with $N_f = 4$

Take $N = 2$, $M = C \times \mathbb{R}^h$ where $C$ is a four-punctured sphere,

$$\widetilde{C} = \{\lambda : \lambda^2 + \phi_2 = 0\} \subset T^*C, \quad \phi_2 = -\frac{z^4 + 2z^2 - 1}{2(z^4-1)^2} dz^2.$$

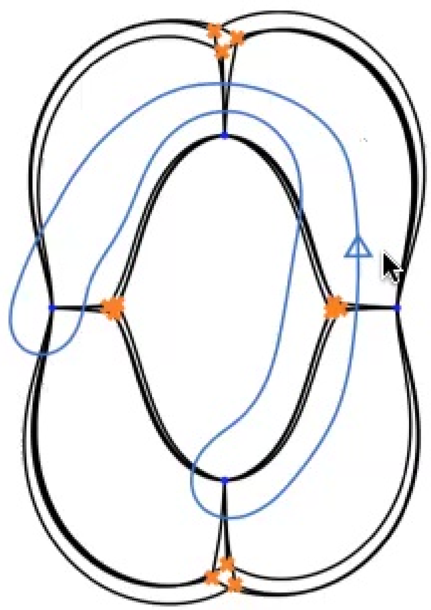


$$\begin{aligned}
& X_{\gamma_1+\mu_1-\mu_3} + X_{\gamma_2+\mu_1-\mu_3} + X_{\gamma_1+\gamma_2+\mu_1-\mu_3} + X_{-\gamma_2-\mu_1+\mu_3} + X_{\gamma_1+\mu_1+\mu_3} \\
& + X_{\gamma_1-\gamma_2+\mu_1+\mu_3} + X_{2\gamma_1-3\gamma_2-\mu_2+2\mu_3-3\mu_4} \underline{-(q+q^{-1})} X_{2\gamma_1-2\gamma_2-\mu_2+2\mu_3-3\mu_4} \\
& + X_{2\gamma_1-\gamma_2-\mu_2+2\mu_3-3\mu_4} + X_{\gamma_1-2\gamma_2-\mu_1+\mu_3-2\mu_4} + X_{\gamma_1-\gamma_2-\mu_1+\mu_3-2\mu_4} \\
& + X_{\gamma_1+\mu_1+\mu_3-2\mu_4} \underline{-(q+q^{-1})} X_{2\gamma_1+\mu_1+\mu_3-2\mu_4} \underline{-(q+q^{-1})} X_{2\gamma_1-2\gamma_2+\mu_1+\mu_3-2\mu_4} \\
& + X_{\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4} + \underline{(2+q^2+q^{-2})} X_{2\gamma_1-\gamma_2+\mu_1+\mu_3-2\mu_4} + X_{\gamma_1-\mu_2-\mu_4} \\
& + X_{\gamma_1-\gamma_2-\mu_2-\mu_4} + X_{\gamma_1+\mu_2-\mu_4} + X_{\gamma_1-\gamma_2+\mu_2-\mu_4} + X_{\gamma_1+2\mu_1+\mu_2-\mu_4} \\
& \underline{-(q+q^{-1})} X_{2\gamma_1+2\mu_1+\mu_2-\mu_4} + X_{2\gamma_1-\gamma_2+2\mu_1+\mu_2-\mu_4} + X_{\gamma_1+\gamma_2+2\mu_1+\mu_2-\mu_4} \\
& + X_{2\gamma_1+\gamma_2+2\mu_1+\mu_2-\mu_4} + X_{\gamma_1-2\gamma_2-\mu_2+2\mu_3-\mu_4} + X_{\gamma_1-\gamma_2-\mu_2+2\mu_3-\mu_4}.
\end{aligned}$$

# T[C]: $SU(3)$ gauging of 2 copies of $E_6$ MN theories

Take $N = 3$, $M = C \times \mathbb{R}^h$ with $C$ a sphere with four full punctures.

$$\widetilde{C} = \{\lambda : \lambda^3 + \phi_2\lambda + \phi_3 = 0\} \subset T^*C \ (\phi_3 \text{ very small})$$

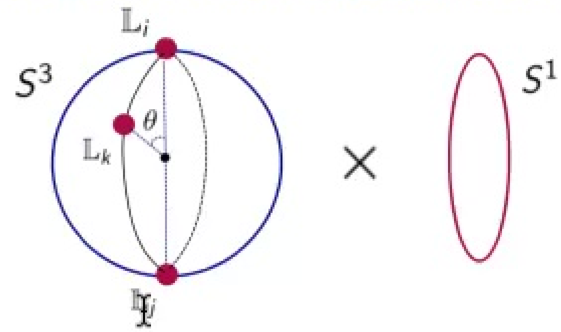


915 $X_\gamma$ appear in the UV-IR expansion:

- 707 $X_\gamma$ with coefficient 1
- 192 $X_\gamma$ with coefficient $-q - q^{-1}$
- 16 $X_\gamma$ with coefficient $q^2 + 2 + q^{-2}$

# Line defect Schur index in 4d N=2 theories

- The line defect Schur index is the $S^3 \times_q S^1$ partition function, with line defects $\mathbb{L}_i$ inserted along a great circle of $S^3$.
[Dimofte-Gaiotto-Gukov],[Gang-Koh-Lee],[Cordova-Gaiotto-Shao],[Dedushenko-Fluder]...




- The index counts the operators living at the junction between different line defects.

$$\mathcal{I}_{\{\mathbb{L}_i\}}(q, x) = \sum_{\text{junc. ops}} (-1)^{2R} q^{R-J_\perp} x^f$$

$R$: $SU(2)_R$ Cartan, $J_\perp$: perpendicular rotation

# Relation to holomorphic topological twist in 4d N=2

The setup is compatible with the holomorphic twist wrt $Q = Q^1_- + \tilde{Q}_{2\dot{-}}$

[Kaputsin],[Costello-Dimofte-Gaiotto],[Oh-Yagi],[Butson],[Cautis-Williams],[Niu],...

- The space of local operators in $Q$-cohomology corresponds to the vacuum module of a Poisson vertex algebra $\mathcal{V}$.

$$\chi[\mathcal{V}] := \mathrm{Tr}_{\mathcal{V}}(-1)^F q^J = \mathcal{I}_{\mathrm{Schur}}(q),$$

  where $F = 2R$, $J = R - J_\perp$.

- The space of operators at the junction of lines gives rise to other modules of the Poisson vertex algebra $\mathcal{V}$, whose graded character coincides with the line defect Schur index.

# Relation to 2d VOAs

- If the 4d N=2 theory is also conformal, the Poisson vertex algebra $\mathcal{V}$ can be further quantized by turning on an Omega background $\rightsquigarrow$ 2d VOA introduced by [Beem-Lemos-Liendo-Peelaers-Rastelli-Van Rees] [Oh-Yagi],[Jeong],[Butson]
- The line defect Schur index can be expanded in terms of the VOA characters. Line defects OPE $\leftrightarrow$ Verlinde algebra [Cordova-Gaiotto-Shao],[Neitzke-Y]
  Example: $(A_1, A_2)$ AD theory $\leftrightarrow$ $(2,5)$ minimal model

$$\mathcal{I}_L(q) = q^{-1/2}\left(\chi_{1,1}(q) \mp \chi_{1,2}(q)\right)$$

# Relation to 2d VOAs

- If the 4d N=2 theory is also conformal, the Poisson vertex algebra $\mathcal{V}$ can be further quantized by turning on an Omega background $\rightsquigarrow$ 2d VOA introduced by [Beem-Lemos-Liendo-Peelaers-Rastelli-Van Rees] [Oh-Yagi],[Jeong],[Butson]
- The line defect Schur index can be expanded in terms of the VOA characters. Line defects OPE $\leftrightarrow$ Verlinde algebra [Cordova-Gaiotto-Shao],[Neitzke-Y]
  Example: $(A_1, A_2)$ AD theory $\leftrightarrow$ $(2,5)$ minimal model

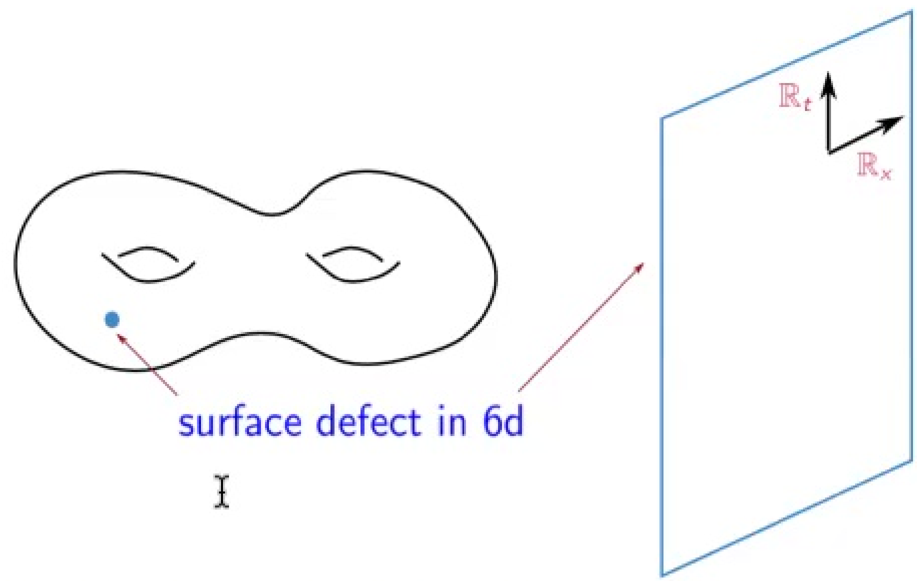
$$\mathcal{I}_L(q) = q^{-1/2}\left(\chi_{1,1}(q) - \chi_{1,2}(q)\right)$$

- Relations to $U(1)_r$-fixed locus in the corres. Hitchin moduli space [Fredrickson-Pei-W.Yan-Ye],[Neitkze-Y],[Dedushenko-Gukov-Nakajima-Pei-Ye]

# Intermediate summary

- The quantum UV-IR map for line defects in class $S$ theories, counting the ground states with spin for bulk-defect system, unified with a new computation of HOMFLY polynomials.
- Line defect Schur indices and relation to 4d holomorphic topological twist and 2d VOAs

# Surface defects in 4d $N = 2$ class-S theories




[Gukov-Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto],[Gaiotto-Moore-Neitzke], [Gaiotto-Gukov-Seiberg],...

- canonical surface defects preserve 2d $(2,2)$ susy $\subset$ 4d N=2 susy
- $z \in C \leftrightarrow$ marginal chiral deformation parameter for the defect theory

# Surface defects and Schrödinger equations

- Consider the Seiberg-Witten curve of certain 4d $N = 2$ theory:

$$\widetilde{C}: \; x^2 + P(z) = 0,$$

Promoting $x$ (momentum) and $z$ (position) to Heisenberg operators $\rightsquigarrow$ Schrödinger equation:

$$\left[\partial_z^2 + \hbar^{-2} P(z, \hbar)\right] \psi(z) = 0$$

- Turning on the Nekrasov-Shatashvili limit of Ω-background along the surface defect quantizes the Seiberg-Witten curve into Schrödinger equations or higher rank analogue, in a canonical way. [Nekrasov-Shatashvili],[Nekrasov],[Jeong],[Jeong-Nekrasov],[Jeong-Lee-Nekrasov],...

- This can also be derived through the conformal limit [Gaiotto] of the Hitchin moduli space or from the AGT-correspondence. [Alday-Gaiotto-Tachikawa],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],...

# Exact WKB for Schrödinger equations

WKB ansatz: $\psi(z) = \exp\left(\hbar^{-1}\int_{z_0}^{z}\lambda(z')dz'\right) \;\rightarrow\; [\partial_z^2 + \hbar^{-2}P(z)]\psi(z) = 0$

$\lambda(z)$ obeys the Ricatti equation

$$\lambda(z)^2 + P(z) + \hbar\partial_z\lambda(z) = 0.$$

Build a formal series solution $\lambda^{\text{formal}}$ in powers of $\hbar$,

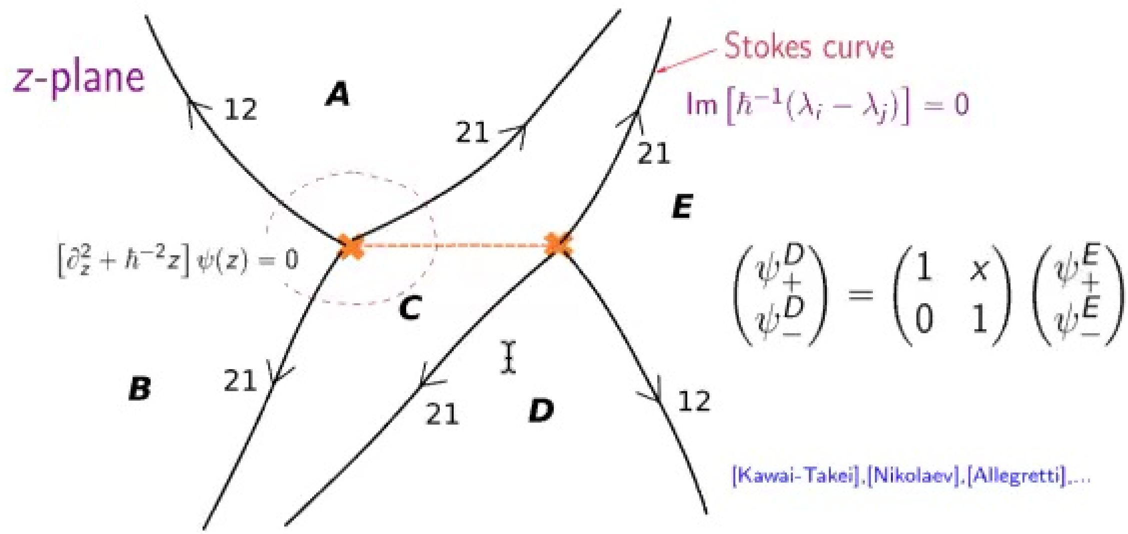
$$\text{order-}\hbar^0 : \; y^2 + p(z) = 0, \quad \text{classical SW curve}$$

Choose a branch labeled by $i \in \{\pm\}$:

$$\lambda_i^{\text{formal}} = y_i - \hbar\frac{P'}{4P} + \hbar^2 y_i \frac{5P'^2 - 4PP''}{32P^3} + \ldots$$

$\longrightarrow$ Two formal solutions $\psi_\pm^{\text{formal}}(z, \hbar)$ as series in $\hbar$.

# Exact WKB for Schrödinger equations




- Borel resummation gives two actual solutions $\psi_\pm(z)$ within each region, where the solutions jump across a Stokes curve.
- Stokes curves $\leftrightarrow$ soliton spectrum of surface defects [Gaiotto-Moore-Neitzke]
  - ⋆ geometrical way for solving $\psi$, exact quantization conditions etc

# The Voros symbol

The Voros symbol: $\mathcal{X}_\gamma(\hbar) \in \mathbb{C}^\times$, $\gamma \leftrightarrow$ 1-cycles of Seiberg-Witten curve

- $\mathcal{X}_\gamma(\hbar)$ captures the Borel resummed WKB periods:

$$\Pi_\gamma(\hbar) := \oint_\gamma \lambda^{\text{formal}}(\hbar) dz = \sum_{n=0}^{\infty} \Pi_\gamma^{(n)} \hbar^n$$

- $\mathcal{X}_\gamma(\hbar)$ expressed as Wronskians of distinguished local solutions:
  - ⋆ asymptotically decaying solutions as z approaches a singularity
  - ⋆ eigenvectors of the monodromy around a loop
- $\mathcal{X}_\gamma(\hbar)$ encodes exact quantization conditions for spectral problems.

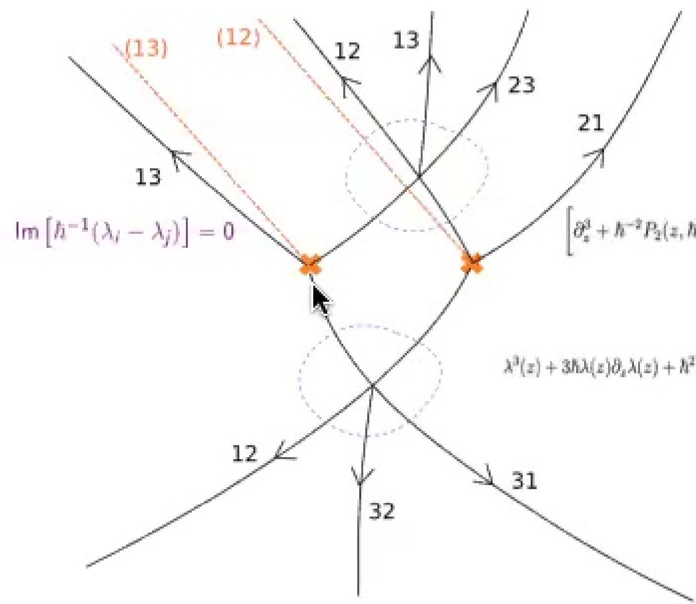
# Higher rank generalization

How do we generalize the story to higher order Schrödinger-like equations?

[Aoki-Kawai-Takei],[Dumitrescu-Fredrickson-Kydonakis-Mazzeo-Mulase-Neitzke],[Hollands-Neitzke], [Yan], ...

$$\left[\partial_z^N + P_2(z,\hbar)\partial_z^{N-2} + \dots P_N(z,\hbar)\right]\psi(z) = 0.$$

Structure of Stokes curves becomes complicated:




$$\left[\partial_z^3 + \hbar^{-2}P_2(z,\hbar)\partial_z + \left(\hbar^{-3}P_3(z,\hbar) + \frac{1}{2}\hbar^{-2}P_2'(z,\hbar)\right)\right]\psi(z) = 0$$

$$\psi(z) = \exp\left(\frac{1}{\hbar}\int_{z_0}^{z}\lambda(z)dz\right)$$

$$\lambda^3(z) + 3\hbar\lambda(z)\partial_z\lambda(z) + \hbar^2\partial_z^2\lambda(z) + P_2(z,\hbar)\lambda(z) + P_3(z,\hbar) + \frac{1}{2}\hbar P_2'(z,\hbar) = 0$$

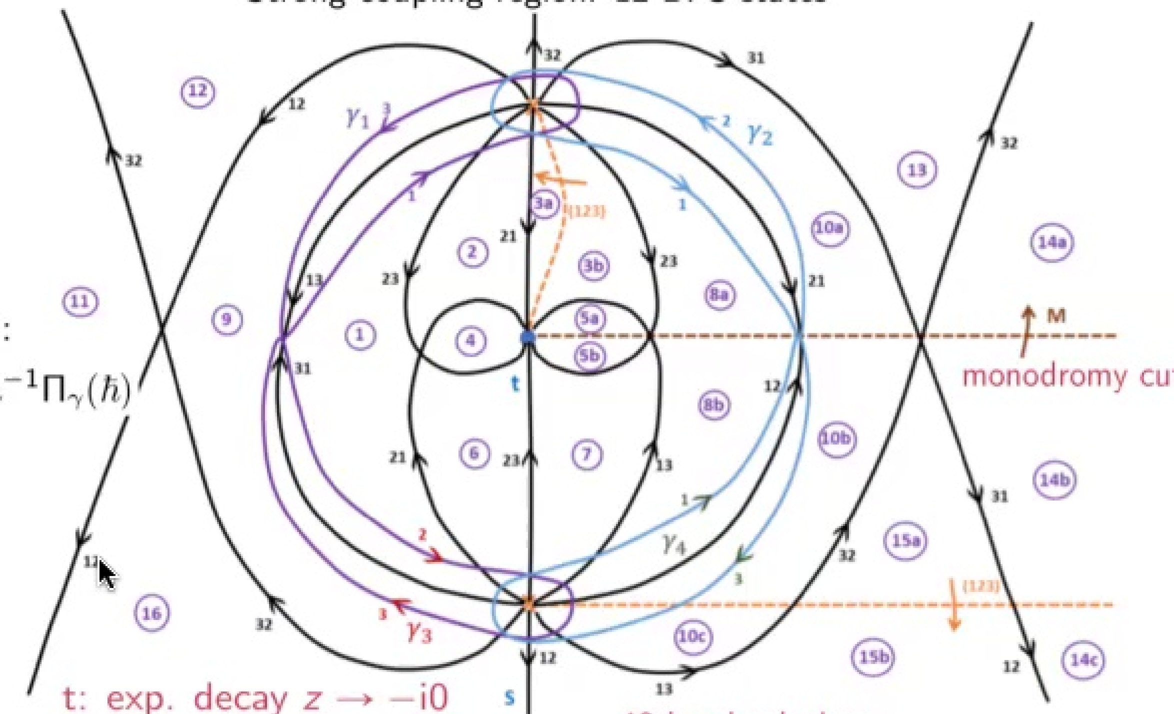
# A third-order ODE: pure $SU(3)$ SYM

$$\left[\partial_z^3 + \hbar^{-2}\frac{u_1+\hbar^2}{z^2}\partial_z + \left(\hbar^{-3}\left(\frac{\Lambda}{z^4} + \frac{u_2}{z^3} + \frac{\Lambda}{z^2} - \hbar^{-2}\frac{u_1+\hbar^2}{z^3}\right)\right)\right]\psi(z) = 0$$ [Yan]

Strong-coupling region: 12 BPS states

$\hbar \to 0$ :

$\log \mathcal{X}_\gamma(\hbar) \sim \hbar^{-1}\Pi_\gamma(\hbar)$




t: exp. decay $z \to -i0$

s: exp. decay $z \to -i\infty$

48 local solutions

# Numerical checks: Voros symbols

The Voros symbols $\mathcal{X}_\gamma(\hbar)$ expressed via special solutions $s, t$.

$\hbar \to 0$ :

$$\log\left(\mathcal{X}_\gamma(\hbar)\right) \sim \frac{1}{\hbar}\Pi_\gamma(\hbar)$$

quantum periods

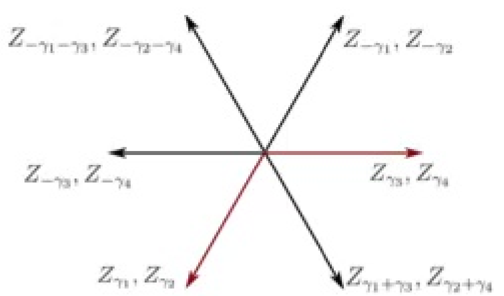
$$\Pi_\gamma(\hbar) := \oint_\gamma \lambda^{\text{formal}}(\hbar)dz = \sum_{n=0}^{\infty} \Pi_\gamma^{(n)}\hbar^n$$

| | $\hbar = \frac{1}{2}e^{i\pi/3}$ | |
|---|---|---|
| | Wronskians $(s,t)$ | $\frac{1}{\hbar}\Pi_\gamma(\hbar)$ at $o(\hbar^6)$ |
| $\log\mathcal{X}_{\gamma_1}$ | $-11.21119$ | $-11.21120$ |
| $\log\mathcal{X}_{\gamma_2}$ | $-11.21119$ | $-11.21120$ |
| $\log\mathcal{X}_{\gamma_3}$ | $5.60559 + 2.71805i$ | $5.60560 + 2.71808i$ |
| $\log\mathcal{X}_{\gamma_4}$ | $5.60559 + 2.71805i$ | $5.60560 + 2.71808i$ |

$\mathcal{X}_\gamma(\hbar)$ also computable via integral equations [Gaiotto],[Gaiotto-Moore-Neitzke]

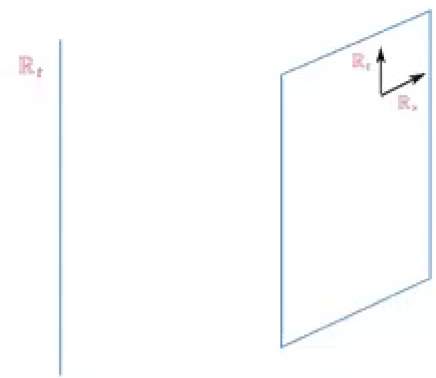
$$\mathcal{X}_\gamma(\hbar) = \exp\left[\frac{Z_\gamma}{\hbar} + \frac{1}{4\pi i}\sum_\mu \Omega(\mu)\langle\gamma,\mu\rangle \int_{\hbar'\in\mathbb{R}_- Z_\mu} \frac{d\hbar'}{\hbar'}\frac{\hbar'+\hbar}{\hbar'-\hbar}\log(1+\mathcal{X}_\mu(\hbar'))\right]$$

Classical period ($Z_\gamma$); BPS index ($\Omega(\mu)$)




| | $\hbar = e^{i\pi/3}$ | |
|---|---|---|
| | Wronskians $(s,t)$ | integral equation |
| $\log\mathcal{X}_{\gamma_1}$ | $-5.48645$ | $-5.48650$ |
| $\log\mathcal{X}_{\gamma_2}$ | $-5.48645$ | $-5.48650$ |
| $\log\mathcal{X}_{\gamma_3}$ | $2.74328 + 1.25232i$ | $2.74325 + 1.25238i$ |
| $\log\mathcal{X}_{\gamma_4}$ | $2.74328 + 1.25232i$ | $2.74325 + 1.25238i$ |

# Summary



- Line defects in 4d N=2 theories:
  - The quantum UV-IR map and construction of link invariants
  - Line defect Schur indices and relation to 4d holomorphic topological twist and 2d VOAs
- Surface defects in 4d N=2 theories and the exact WKB method for higher order ODEs