Title: Generalised Langlands, VOAs, and (generalised) tau-functions

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Abstract: In the first part of my talk I'll briefly review some aspects of the relations between N=4, d=4 SYM and vertex operator algebras (VOAs) discussed in recent work of Gaiotto and collaborators. The resulting picture predicts conjectural generalisations of the geometric Langlands correspondence. We will focus on a class of examples figuring prominently in recent work of Creutzig-Dimofte-Garner-Geer, labelled by parameters n (rank) and k. For the case k=1,n=2 we will point out that the conformal blocks of the relevant VOA, twisted by local systems, represent sections of natural holomorphic line bundles over the moduli spaces of local systems closely related to the isomonodromic tau functions. Observing the crucial role of (quantised) cluster algebras in the definition of the holomorphic line bundles suggests natural generalisations of this story to higher values of the parameters k and n.

Generalised Langlands, VOAs, and (generalised) tau-functions

Jörg Teschner

Talk in the Workshop QFT for Mathematicians 2022, Perimeter Institute

Based on work with M. Alim, I. Coman, P. Longhi, E. Pomoni, A. Saha, I. Tulli

University of Hamburg, Department of Mathematics
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Part I

Context:

Topologically twisted $N = 4$ QFT in $d = 3$ and $d = 4$

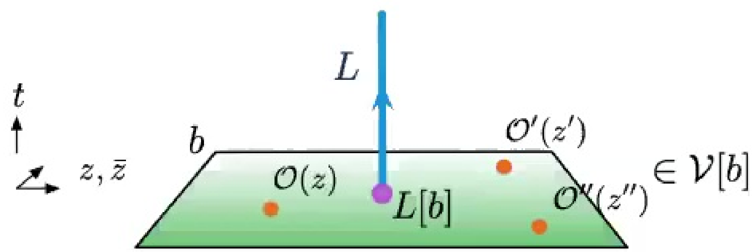
The following slides reflect my (limited) understanding of recent work of many colleagues, at the risk of errors and inaccuracies.

$N = 4$, $d = 3$ twisted SUSY QFT

$N = 4$, $d = 3$ SUSY QFT on manifolds with certain boundaries admit (A/B) twists which are **topological** in the 3d bulk, and **holomorphic** on the 2d boundary b.

Holomorphic boundary observables $\mathcal{O}(z)$ generate VOAs[1].

$\rightsquigarrow \cdots \rightsquigarrow$ expect rich generalisation of WZW-CS relationship, schematically[2],



assigning, in particular

- $T(C)$ – (dg) vector spaces (of VOA conformal blocks) to Riemann surfaces C,
- $T(D_x)$ – (dg) category (of line defects $\sim$ VOA representations),

where D_x is disc with puncture at x.

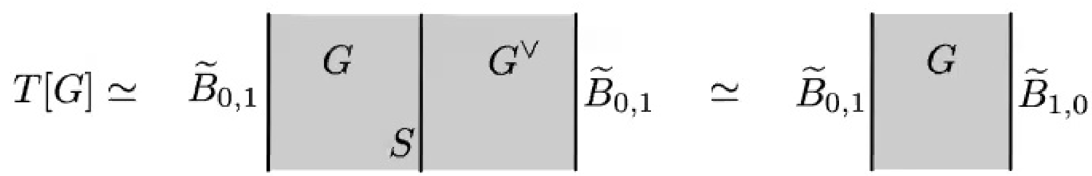
[1] Costello-Gaiotto, Costello-Creutzig-Gaiotto; closely related: important developments by S. Gukov and collaborators.
[2] Picture taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

$N = 4$, $d = 4$ SUSY Yang-Mills theory (SYM)

Topological twists of $N = 4$ SYM form the basis for the gauge-theoretic approach to the geometric Langlands correspondence initiated in the work of Kapustin and Witten.

Interesting relations to $N = 4$, $d = 3$ twisted SUSY QFT emerge by considering $N = 4$, $d = 4$ SYM on $M^4 = M^2 \times C$, with $M^2 = I \times \mathbb{R}$. Depending on the boundary conditions on the ends of I one gets various $N = 4$, $d = 3$ QFT in the IR.

Example:[3] Theory $T[G]$, G: compact Lie Group, mostly $SU(n)$.



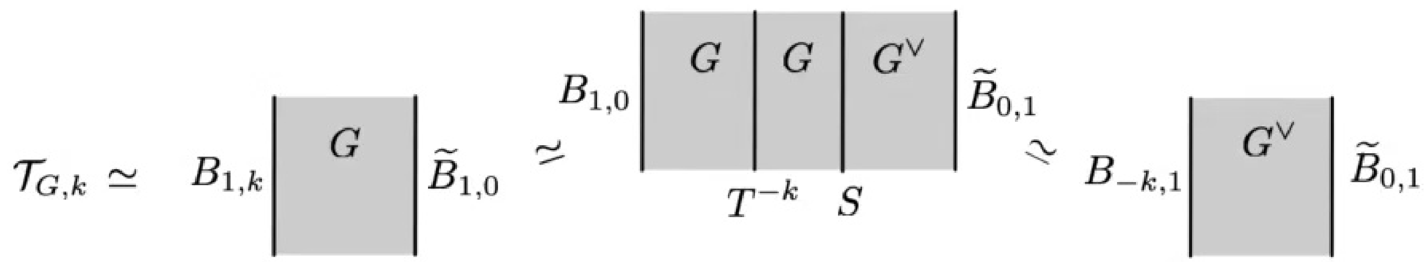
using S-duality interface, and boundary conditions for $N = 4$ SYM of following types:

- $\tilde{B}_{0,1}$ Dirichlet,
- $\tilde{B}_{1,0}$ S-dual of Dirichlet.

[3]Picture taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

$3d$ **Theories** $\mathcal{T}_{G,k}$ **from** $4d$

Another interesting example was recently studied in Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559, defining a family of $N = 4$, $d = 3$ QFT denoted $\mathcal{T}_{G,k}$, with[4]



The resulting $N = 4$, $d = 3$ QFT admit topological twists defining $\mathcal{T}^A_{G,k}$. Coming from $4d$: Induced by Kapustin-Witten's GL twist at $\Psi = 0$.

Creutzig-Dimofte-Garner-Geer argue that the boundary VOAs $\mathcal{D}_k(\mathfrak{g})$ for $\mathcal{T}^A_{G,k}$ are the Feigin-Tipunin logarithmic VOAs $\mathcal{FT}_k(\mathfrak{sl}_n)$, $\mathfrak{g}$: Lie algebra of G. Let $\mathcal{D}_{n,k} = \mathcal{D}_k(\mathfrak{sl}_n)$.

The argument is based on the idea of corner VOAs:

[4] Picture taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

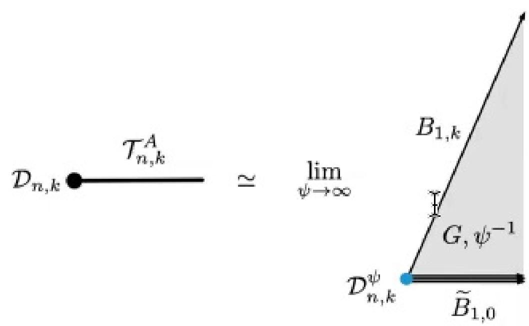
Corner VOAs

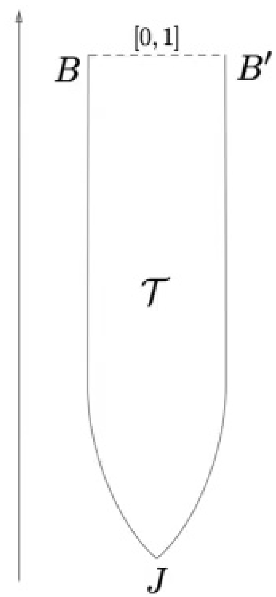
$N = 4$ SYM admits $3d$ boundaries with boundary conditions B. They can meet at $2d$ corners. Twist can make $4d$ bulk topological, leaving corners holomorphic. Holomorphic fields at corners $\rightsquigarrow$ corner VOA. (Gaiotto, Creutzig-Gaiotto, Rapcak-Gaiotto)

Expect effective $3d$ descriptions for corner configurations, with $3d$ theory determined by boundary conditions meeting at corner. Then corner VOA = boundary VOA.[5]

Right: Relation between conformal blocks in corner VOAs and states in $T(C) = \mathrm{Hom}(B, B')$.

Bottom: Example for relation of 3d boundary and 4d corner VOAs.



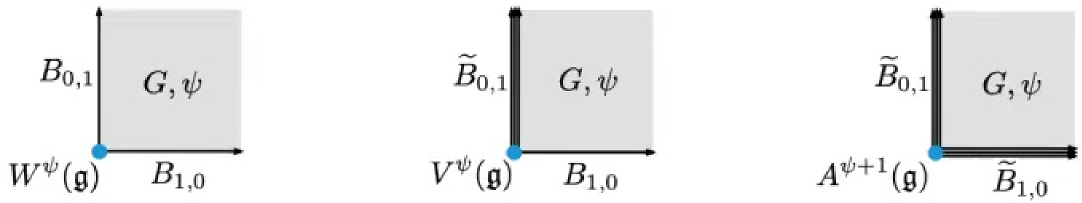


[5]Pictures taken from Frenkel-Gaiotto, arXiv:1805.00203 and Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

Corner VOAs

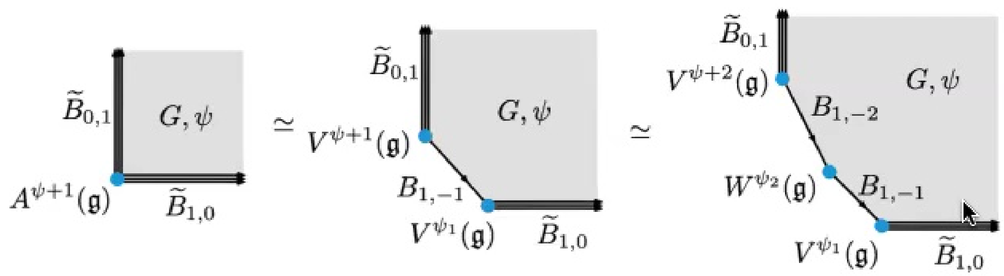
Useful toolkit for building corner VOAs: (Gaiotto, Creutzig-Gaiotto, Rapcak-Gaiotto)

Basic examples of corner VOAs:[6]



- $V^{\psi}(\mathfrak{g})$: affine VOA at level $\psi - h^{\vee}$,
- $W^{\psi}(\mathfrak{g}) \simeq W^{1/\psi}(\mathfrak{g})$: principal W-algebra of $\mathfrak{g}$.

Alternative representations from slicing:



[6]Pictures taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

Feigin-Tipunin algebras as extensions of W-algebras

Boundary VOAs $\mathcal{D}_{n,k}$ of $\mathcal{T}^{A}_{SU(n),k}$ predicted by Creutzig-Dimofte-Garner-Geer to be Feigin-Tipunin algebras $\mathcal{FT}_k(\mathfrak{sl}_n)$. These algebras are representable as (Sugimoto)

$$\mathcal{FT}_k(\mathfrak{sl}_n) = \bigoplus_{\lambda \in Q^+} R_\lambda \otimes W^{1/k}_{\lambda,0}, \qquad \text{(FT)}$$

- Q^+: positive roots in root lattice Q of $\mathfrak{g} = \mathfrak{sl}_n$,
- R_λ: finite-dimensonal irreducible $\mathfrak{sl}_n$-modules with weight λ,
- $W^{\psi}_{\lambda,\mu}$ simple quotient of the W-algebra $W^{\psi}(\mathfrak{sl}_n)$-module with weight $\lambda - \psi\mu$.

Key feature:

- VOAs $\mathcal{FT}_k(\mathfrak{sl}_n)$ admit group of automorphisms $G^{\vee}_{\mathbb{C}} = PGL(n,\mathbb{C})$.
- Conformal blocks of $\mathcal{FT}_k(\mathfrak{sl}_n)$ admit twisting with $G^{\vee}_{\mathbb{C}}$-local systems.

Spaces of conformal blocks for generic twist expected to be $2^g k^{3g-3}$-dimensional, related to "semi-simplification" of non-semisimple TQFT from $U_q(\mathfrak{sl}_2)$ at $q = e^{\frac{\pi i}{k}}$. [7]

[7] Costantino/Geer/Patureau-Mirand.

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A (trivial?) special case

Case $k = 1$: Right hand side makes perfect sense, but $\mathcal{D}_{n,1}$: lattice VOA V_Q.

However,

- V_Q has well-know super-VOA extension $\mathrm{FF}(n)$, containing V_Q via bosonisation.
- Relation (FT) for $k = 1$: Consequence of "bosonisation" formulae

$$\begin{aligned} \psi_s(z) &= e^{+\mathrm{i}\varphi_0(z)} V^{+s}_{1/2}(z) \\ \bar{\psi}_s(z) &= e^{-\mathrm{i}\varphi_0(z)} V^{-s}_{1/2}(z) \end{aligned} \quad \Rightarrow \quad \bar{\psi}_s(x)\psi_t(y) \sim \frac{\delta_{st}}{x-y},$$

where $V^{\pm}_{1/2}(z)$: Degenerate fields of $W^1(\mathfrak{sl}_2) = \mathrm{Vir}_{c=1}$, φ_0 auxilliary free boson.

Feigin-Tipunin algebras as extensions of W-algebras

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Part II

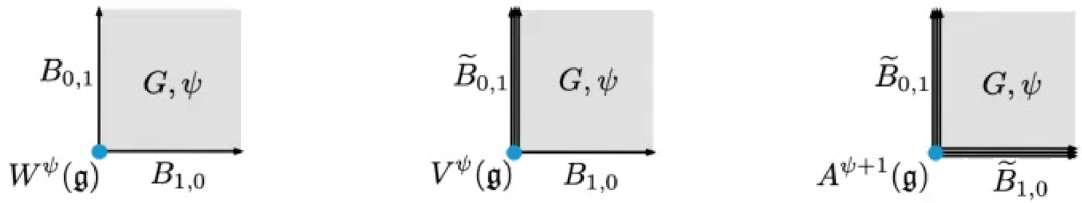
Twisted free fermion conformal blocks,

Virasoro algebra and tau-functions

Corner VOAs

Useful toolkit for building corner VOAs: (Gaiotto, Creutzig-Gaiotto, Rapcak-Gaiotto)

Basic examples of corner VOAs:[6]



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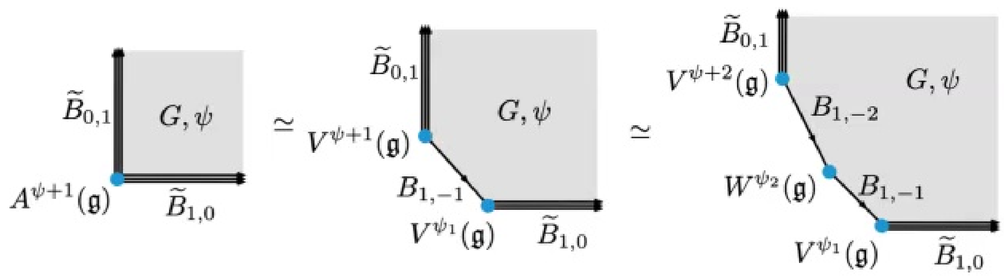

[6] Pictures taken from Creutzig-Dimofte-Garner-Geer, arXiv:2112.01559

Twisted free fermion conformal blocks

Free fermion CFT defined by VOA $\mathrm{FF}(n)$,

$$\bar{\psi}_s(x)\psi_t(y) \sim \frac{\delta_{st}}{x-y}, \qquad s,t = 1,\dots,n,$$

associates 2^g-dimensional spaces of conformal blocks to Riemann surfaces $C = C_{g,n}$, characterised by functionals

$$G_{st}(x,y) := \left\langle\, \bar{\psi}_s(x)\psi_t(y) \,\right\rangle_C^{\mathrm{FF}}.$$

$\mathrm{FF}(n)$ has $GL(n,\mathbb{C})$-automorphism $\rightsquigarrow$ can define twisted conformal blocks, characterised by functionals satisfying

$$G_\rho(x,\gamma.y) = G_\rho(x,y)\cdot\rho(\gamma)$$

for given representation $\rho : \pi_1(C) \to G = GL(n,\mathbb{C})$.

Upshot: **Sheaf** with stalks $\mathrm{CB}_{\mathrm{FF}}(C)$ over $\mathrm{LocSys}_{G^\vee}(C)$.

Twisted free fermion conformal blocks II – Main claim[8]

Away from singularities, $\exists$ structure of **holomorphic line bundle** over $\mathrm{LocSys}_{G^\vee}(C)$.

- Cover $\mathrm{LocSys}_{G^\vee}(C)$ with charts U_α, coordinates of Fock-Goncharov (FG) or Fenchel-Nielsen (FN) type.
- Pick Darboux coordinates $(\sigma_\alpha^r, \eta_r^\alpha)$, $r = 1, \ldots, h$, $h := 3g - 3 + n$, in U_α.
- Define transition functions on $U_\alpha \cap U_\beta$ as the difference **generating functions**

$$G_{\alpha\beta}(\sigma_\alpha + \delta_r, \sigma_\beta) = e^{2\pi i \eta_r^\alpha} G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta),$$

$$G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta + \delta_r) = e^{-2\pi i \eta_r^\beta} G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta).$$

$\rightsquigarrow \cdots \rightsquigarrow$ holomorphic line bundle $\mathcal{L}$ over $\mathrm{LocSys}_{G^\vee}(C) \times \mathcal{M}_{g,n}$.

Claim: Suitably normalised free fermion partition functions $\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau) := \langle \mathrm{id} \rangle_C^{\mathrm{FF}}$ represent **holomorphic sections**, satisfying

$$\frac{\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau)}{\mathcal{T}_\beta(\sigma_\beta, \eta_\beta; \tau)} = G_{\alpha\beta}(\sigma_\alpha, \sigma_\beta),$$

(i) $\mathcal{T}_\alpha(\sigma_\alpha + \delta_r, \eta_\alpha; \tau) = e^{2\pi i \eta_r^\alpha} \mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau)$

(ii) $\mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha + \delta_r; \tau) = \mathcal{T}_\alpha(\sigma_\alpha, \eta_\alpha; \tau)$.

[8] Coman-Longhi-J.T.