Title: Geometrical structures in 4d N=2 class S theories

Speakers: Fei Yan

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Abstract: I will give an introduction to 4d N=2 class-S theories. I will describe the construction of such theories, the roles played by extended defects such as line defects and surface defects, as well as connections to Hitchin systems.

Geometric Structures in 4d N=2 class S theories

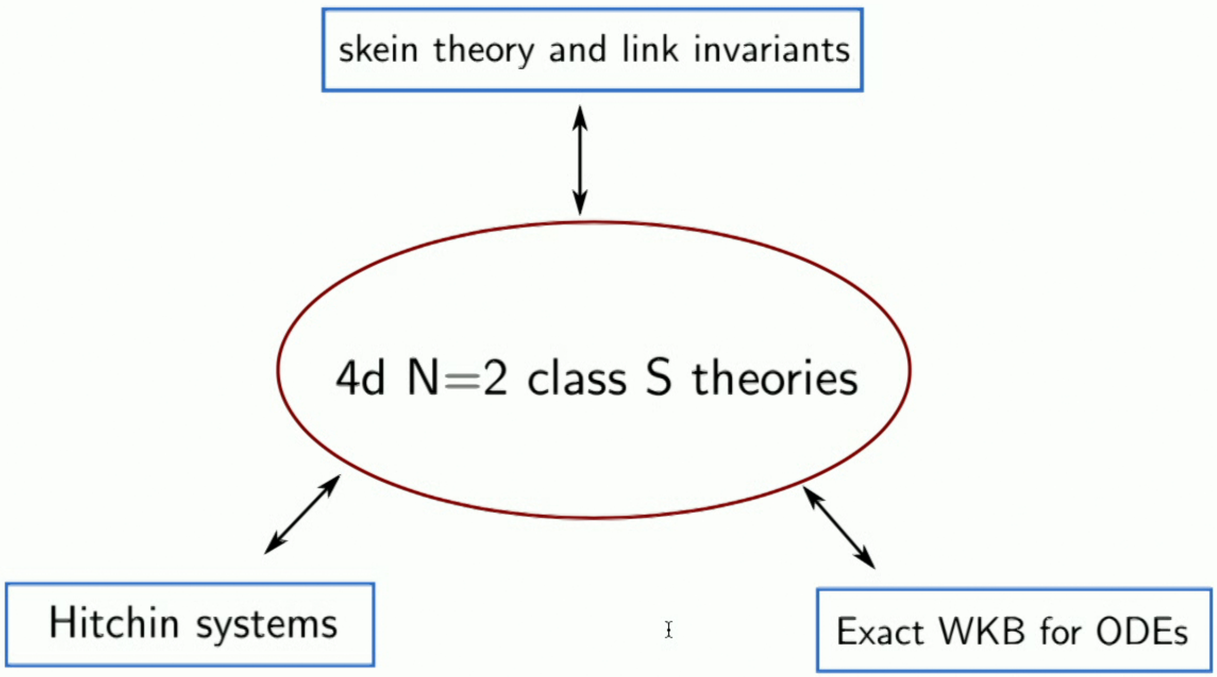
Fei Yan

Rutgers University

QFT for Mathematicians
Perimeter Institute

June 23th, 2021

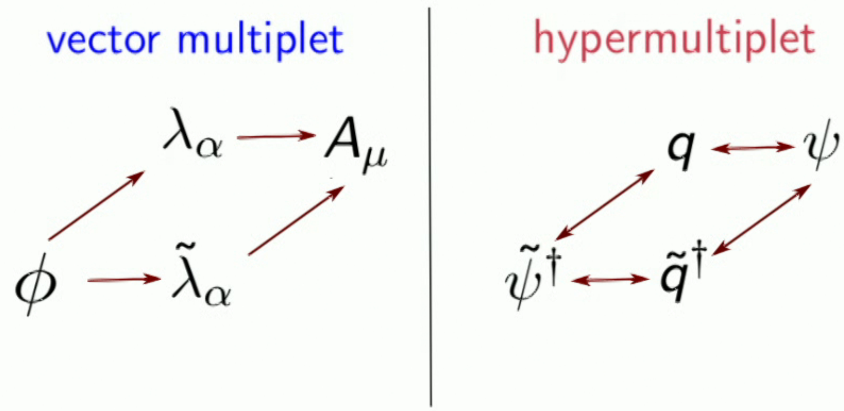
Outlook



skein theory and link invariants
4d N=2 class S theories
Hitchin systems
Exact WKB for ODEs

4d N=2 theories

- (3+1)-dim supersymmetric QFTs with 8 supercharges
- 4d N=2 supersymmetric multiplets:



- Lagrangian for G gauge theory with matter.
- Many 4d N=2 theories do not admit a Lagrangian description, with a geometric origin. [Katz-Klemm-Vafa],[Gaiotto],[Gaiotto-Moore-Neitzke],...

Coulomb branch of 4d N=2 theories

- Coulomb branch: $\langle q \rangle = \langle \tilde{q} \rangle = 0$, Higgs branch: $\langle \phi \rangle = 0$
- At a generic point on the Coulomb branch $\mathcal{B}$, the low energy effective theory is 4d N=2 $U(1)^r$ gauge theory. [Seiberg-Witten] (here r=1)
- Central charge and BPS states

  $\mathcal{H}_u$: 1-particle Hilbert space on $\mathbb{R}^3$, vacua at $\infty \leftrightarrow u \in \mathcal{B}$

$$\mathcal{H}_u = \bigoplus_{\gamma \in \Gamma} \mathcal{H}_{u,\gamma} \qquad \gamma = (n_e, n_m, n_f)$$

electro-magnetic and flavor charge

$$E_\gamma \geqslant |Z_\gamma| \qquad Z_\gamma = n_e a + n_m a_D + n_f \mu$$

central charge

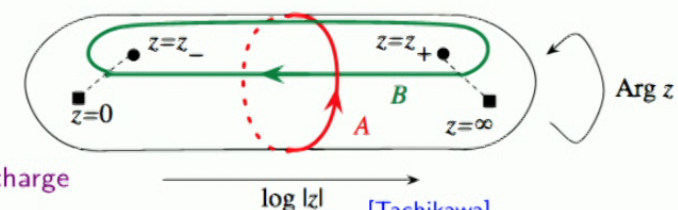
elec.period   mag.period   mass

The Seiberg-Witten curve

- The low energy dynamics is encoded in Seiberg-Witten curve.
- Example: SW curve for 4d $N = 2$ pure $SU(2)$ Yang-Mills is

$$\widetilde{C}_{\mathrm{SU(2)}} : \ \Lambda^2 z + \frac{\Lambda^2}{z} = x^2 - u,$$

with SW differential $\lambda = \frac{x}{z} dz$. ($u \in$ CB, Λ: strong scale)



electro-magnetic charge
$\gamma \in H_1(\widetilde{C}, \mathbb{Z})$

[Tachikawa]

Electro-magnetic periods: $a \sim \oint_A \lambda, \ a_D \sim \oint_B \lambda$

Gauge coupling $\tau(a) = \frac{\partial a_D}{\partial a}$

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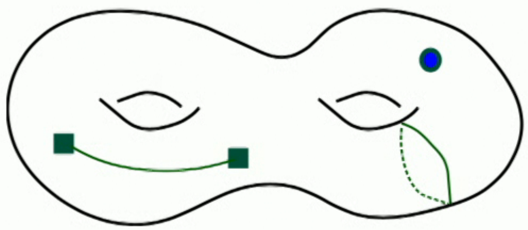
elec.period mag.period mass

4d N=2 theories of class S

- Class S theories $\mathcal{T}[\mathfrak{g}, C]$ are 4d N=2 supersymmetric theories originating from twisted compactification of a 6d $(2,0)$ theory of type $\mathfrak{g}$ ($\{A, D, E\}$) on a Riemann surface C with appropriate decorations (punctures or twisted lines) [Gaiotto],[Gaiotto-Moore-Neitzke]



$\times \mathbb{R}^{3,1}$

4d N=2 theories of class S: Coulomb branch

- 6d tensor branch → Coulomb branch $\mathcal{B}$ of class S theory. $\mathcal{B}$ is parameterized by meromorphic d_k-differentials on C

$$\mathcal{B} \subset \bigoplus_{k=1}^{r} H^0 \left(C, K_C^{\otimes d_k} \left(\sum_i p_{d_k}^{(i)} z_i \right) \right)$$

[Gaiotto],[Gaiotto-Moore-Neitzke],[Chacaltana-Distler-Tachikawa],[Chacaltana-Distler-Trimm-Zhu] ...

- The Seiberg-Witten curve $\widetilde{C}$ is a branched covering of C, embedded in T^*C.

4d N=2 theories of class S: an example

4d N=2 pure SU(2) Yang-Mills revisited:

- The Riemann surface $C_{SU(2)}$: $\mathbb{CP}^1$ with 2 irregular punctures
- The Seiberg-Witten curve

$$\widetilde{C}_{\mathrm{SU(2)}} : \lambda^2 - \phi_2(z) = 0,\ \phi_2(z) = \left(\frac{\Lambda^2}{z} + \frac{u}{z^2} + \frac{\Lambda^2}{z^3} \right) dz^2$$

The Seiberg-Witten differential $\lambda = ydz$
y: fiber coordinate of T^*C

Relation to Hitchin systems

Further compactifying $\mathcal{T}[\mathfrak{g}, C]$ on S^1_R, the low energy effective theory is a 3d $N = 4$ sigma model with target space $M_H(G, C)$.

- Starting from 6d and changing the order of compactification on $C \times S^1_R$ [Gaiotto-Moore-Neitzke], $M_H(G, C)$ is identified with the moduli space of solutions to Hitchin's equations:

$$F_A + R^2 \left[\Phi, \bar{\Phi}\right] = 0,$$
$$\bar{\partial}_A \Phi = 0, \quad \partial_A \bar{\Phi} = 0.$$

  $\partial + A$ is a G-connection in a top. trivial G-bundle $V \to C$, $\Phi \in \Omega^{1,0}(\mathrm{End}\, V)$ is the Higgs field.

- Seiberg-Witten curve $\longleftrightarrow$ spectral curve, characteristic of Φ, 4d Coulomb branch $\longleftrightarrow$ Hitchin base (Casimirs of Φ)

Relation to Hitchin systems

$M_H(G,C)$ is hyperkähler, has a $\mathbb{CP}^1$-worth of complex structures J_ζ.
Different J_ζ expose different features of $M_H(G,C)$:

[Hitchin], [Simpson], [Biquard-Boalch], [Gaiotto-Moore-Neitzke]...

- $\zeta = 0$: (M_H, J_0) diff. to moduli space of Higgs bundles M_{Higgs}, which is a complex integrable system: $M_{\text{Higgs}} \to \mathcal{B}$ with generic fiber being compact tori.

  The Seiberg-Witten curve $\widetilde{C}$ identified with spectral curve:
  $$\widetilde{C} = \{(z \in C, \lambda \in T_z^* C) : \text{Det}\,(\Phi(z) - \lambda) = 0\} \subset T^* C$$

- $\zeta \in \mathbb{C}^\times$: Hitchin's equations indicate $\partial + \mathcal{A}$ is flat, with
  $$\mathcal{A} := \frac{R}{\zeta}\Phi + A + R\zeta\bar{\Phi}$$

  (M_H, J_ζ) diff. to a moduli space of flat $G_{\mathbb{C}}$-connections on C.

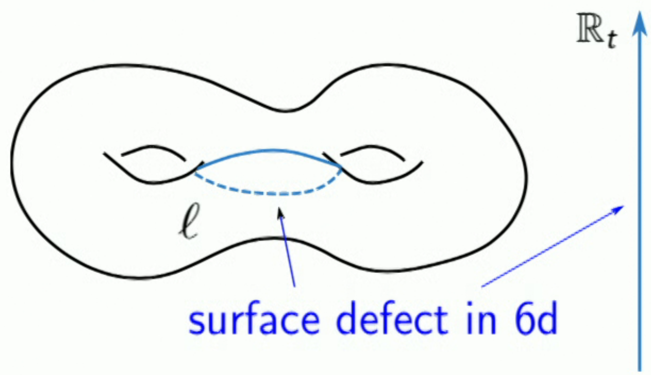
Line defects in class S theories

$\mathcal{T}[\mathfrak{g}, C]$ admits families of line defects $\mathbb{L}(\zeta)$ extending along $\mathbb{R}^t$-direction, where $\zeta \in \mathbb{C}^\times$ parametrizes preserved supercharges.

[Kapustin],[Kapustin-Saulina],[Drukker-Morrison-Okuda],[Drukker-Gaiotto-Gomis],[Drukker-Gomis-Okuda-Teschner] [Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Aharony-Seiberg-Tachikawa],[Moore-Royston-van den Bleeken]...

$\mathbb{L}(\zeta, \mathfrak{p}, \mathcal{R})$ depends on path $\mathfrak{p}$ on C (up to homotopy), carrying representation $\mathcal{R}$ of $\mathfrak{g}$.

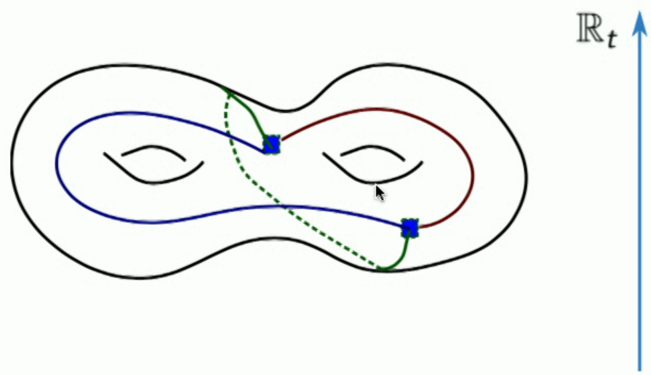
- $\mathfrak{g} = A_1$, C only has regular punctures:
  $\mathfrak{p}$ is a non-self-intersecting closed curve on C.



Line defects in class S theories

- C contains irregular punctures:
  $\mathfrak{p}$ corresponds to integral laminations [Fock-Goncharov],[Gaiotto-Moore-Neitzke], collection of paths either closed or open with ends on marked points corres. to Stokes directions at irregular punctures.
- In general $\mathfrak{p}$ could contain junctions, where paths carrying different $\mathcal{R}_i$ meet, associated with certain $\mathfrak{g}$-invariant tensor.

[Sikora],[Le],[Xie],[Saulina],[Coman-Gabella-Teschner],[Tachikawa-Watanabe],[Gabella]...



Line defects and Hitchin system

Upon circle compactification, the vacuum expectation values of $\mathbb{L}(\zeta)$ wrapping S^1_R are J_ζ-holomorphic functions on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.

[Gaiotto-Moore-Neitzke]

- $\mathfrak{g} = A_1$, C has only regular punctures:

$$\langle \mathbb{L}(\zeta, \mathfrak{p}, \mathcal{R}) \rangle = \mathrm{Tr}_{\mathcal{R}} \mathrm{Hol}_{\mathfrak{p}} \left(\frac{R\Phi}{\zeta} + A + R\zeta\bar{\Phi} \right)$$
$$= \mathrm{Tr}_{\mathcal{R}} \mathrm{Hol}_{\mathfrak{p}} \mathcal{A}(\zeta).$$

- In general, compute parallel transport of $\mathcal{A}(\zeta)$ along paths, contract together via $\mathfrak{g}$-invariant tensors.

The UV-IR map for line defects

A useful way to study $\mathbb{L}(\zeta)$ in class S theories, is deforming to a point u on the Coulomb branch $\mathcal{B}$ and follow the defect into IR.

The IR limit of $\mathbb{L}(\zeta)$ is a superposition of supersymmetric line defects in the abelian theory, with integer coefficients in this superposition given by framed BPS index $\overline{\underline{\Omega}}(\mathbb{L}(\zeta), \gamma, u)$. [Gaiotto-Moore-Neitzke],[Córdova-Neitzke],[Cirafici-Del Zotto], [Coman-Gabella-Teschner],[Moore-Royston-van den Bleeken],[Ito-Okuda-Taki],[Galakhov-Longhi-Moore],[Brennan-Dey-Moore],...

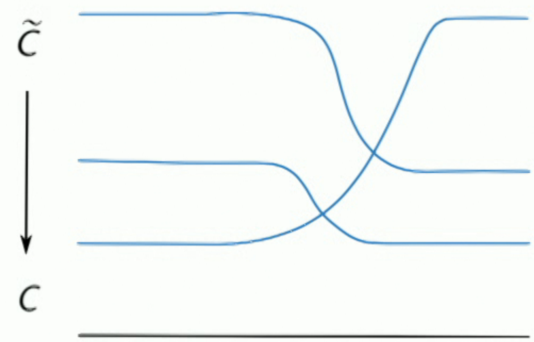
The UV-IR map for line defects:

$$\mathbb{L}(\zeta) \rightsquigarrow \sum_{\gamma} \overline{\underline{\Omega}}(\mathbb{L}(\zeta), \gamma, u) X_{\gamma}(\zeta)$$

$X_{\gamma}(\zeta)$ represent IR Wilson-'t Hooft lines with charge γ.

The IR line defects

Recall that, a point u on the Coulomb branch $\mathcal{B}$ corresponds to a branched covering $\widetilde{C} \to C$ (spectral curve/Seiberg-Witten curve):



Geometrically the IR line defect $X_\gamma(\zeta)$ correspond to loops $\tilde{\mathfrak{p}} \subset \widetilde{C}$ in class $\gamma \in H_1(\widetilde{C}, \mathbb{Z})$ (γ: IR charge).

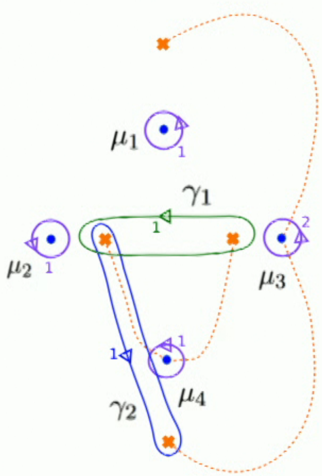
The UV-IR map $\leftrightarrow$ Uplift of $\mathfrak{p} \subset C$ to combinations of $\tilde{\mathfrak{p}} \subset \widetilde{C}$.

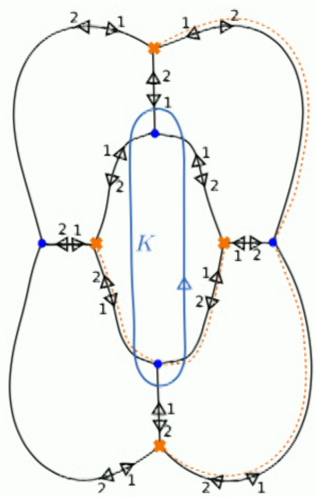
[Gaiotto-Moore-Neitzke]

The UV-IR map for line defects: an example

4d N=2 $SU(2)$ gauge theory with $N_f = 4$ fundamental hypermultiplets.
C is a four-punctured sphere. (punctures: blue, branch points: orange)

$$\widetilde{C} = \{\lambda : \lambda^2 + \phi_2 = 0\} \subset T^*C, \quad \phi_2 = -\frac{z^4 + 2z^2 - 1}{2(z^4-1)^2} dz^2.$$





The UV line defect labeled by $K \subset C$

7 IR line defects

$$X_{-\gamma_2-\mu_2+\mu_3} + X_{-\gamma_2-\mu_1-\mu_4} + X_{\gamma_1+\mu_1-\mu_4} + X_{-\gamma_1-\mu_1+\mu_4} + X_{\gamma_1-\gamma_2+\mu_1-\mu_4} + X_{\gamma_1-\gamma_2-\mu_2+\mu_3-2\mu_4} + X_{\gamma_1-2\gamma_2-\mu_2+\mu_3-2\mu_4},$$

The IR line defects and Hitchin system

Upon circle compactification, the VEV $\mathcal{X}_\gamma(\zeta)$ of $X_{\mathfrak{r}\gamma}(\zeta)$ wrapping S^1_R are local Darboux coordinates on $M_{\text{flat}}(G_{\mathbb{C}}, C)$: Fock-Goncharov, complexified Fenchel-Nielsen, or more general spectral coordinates.

[Fock-Goncharov],[Fenchel-Nielsen],[Gaiotto-Moore-Neitzke],[Nekrasov-Rosly-Shatashvili],

[Hollands-Neitzke],[Hollands-Kidwai],[Allegretti],[Nikolaev],[Jeong-Nekrasov],[Coman-Longhi-Teschner] ...

- $\mathcal{X}_\gamma$ has distinguished asymptotic behavior as $\zeta \to 0$ and $\zeta \to \infty$; has discontinuities across "BPS walls" controlled by Kontsevich-Soibelman symplectomorphisms.
  $\mathcal{X}_\gamma$ are solutions to a Riemann-Hilbert problem.

  [Gaiotto-Moore-Neitzke],[Gaiotto],[Bridgeland],[Barbieri],[Bridgeland-Barbieri-Stoppa]...

The UV-IR map as the trace map

- UV: $\langle \mathbb{L}(\zeta) \rangle$ are J_ζ-holomorphic trace functions on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.
- IR: $\mathcal{X}_\gamma(\zeta) := \langle X_\gamma(\zeta) \rangle$ are Darboux-coordinates on $M_{\text{flat}}(G_{\mathbb{C}}, C)$.
- The UV-IR map

$$\mathbb{L}(\zeta) \rightsquigarrow \sum_\gamma \overline{\underline{\Omega}}(\mathbb{L}(\zeta), \gamma) X_\gamma(\zeta)$$

then implies the trace map:

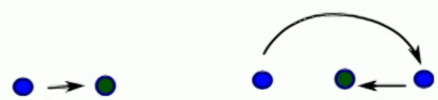
$$\mathrm{Tr}_{\mathcal{R}} \mathrm{Hol}_{\mathfrak{p}} \mathcal{A}(\zeta) = \sum_\gamma \overline{\underline{\Omega}}(\mathbb{L}(\zeta), \gamma) \mathcal{X}_\gamma(\zeta)$$

$\mathcal{A}(\zeta)$: flat $G_{\mathbb{C}}$-connection on C, $\mathfrak{p}$: path on C.

Line defects OPE

- The algebra structure on the space of J_ζ-holomorphic functions corresponds to line defects operator products (OPE):

$$\langle \mathbb{L}_1(\zeta)\mathbb{L}_2(\zeta)\rangle = \langle \mathbb{L}_1(\zeta)\rangle\langle \mathbb{L}_2(\zeta)\rangle$$



- This algebra structure admits a quantization via skein algebras.

  [Reshetikhin-Turaev],[Turaev],[Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto-Moore-Neitzke],
  [Drukker-Gomis-Okuda-Teschner],[Tachikawa-Watanabe],[Coman-Gabella-Teschner],[Gabella]...

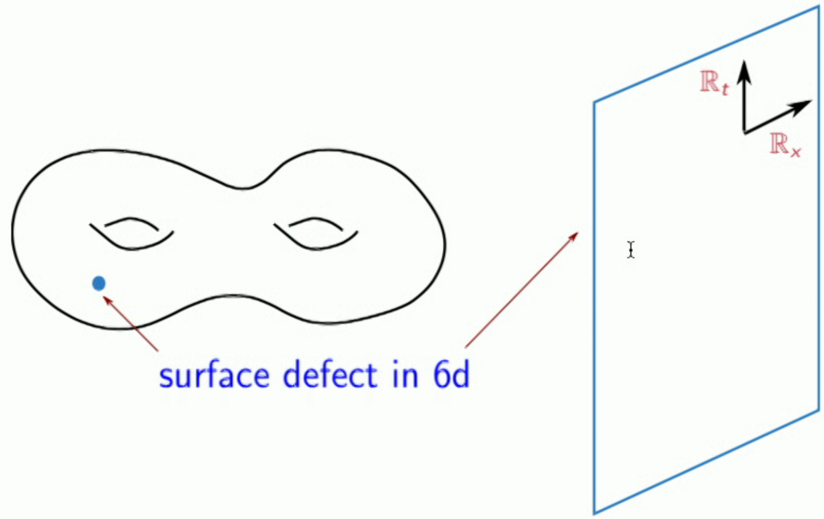
- This leads to a quantization of the UV-IR map (trace map).

  [c.f. my talk next Monday]

Intermediate Summary

- Introduction to 4d N=2 class S theories
- Relation to Hitchin systems
- Line defects in class S theories and the UV-IR map / trace map

Surface defects in 4d $N = 2$ class-S theories



[Gukov-Witten],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],[Gaiotto],[Gaiotto-Moore-Neitzke], [Gaiotto-Gukov-Seiberg],...

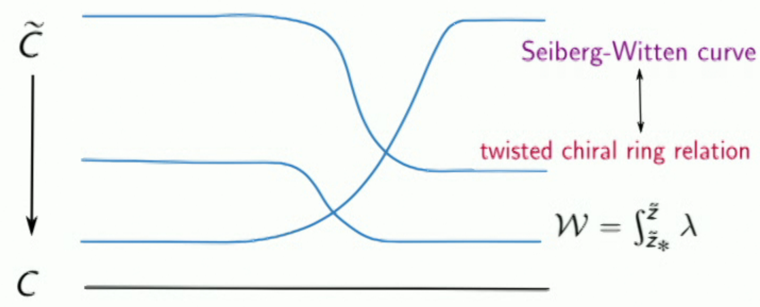
- canonical surface defects preserve 2d $(2, 2)$ susy $\subset$ 4d N=2 susy
- $z \in C \leftrightarrow$ marginal chiral deformation parameter for the defect theory

Surface defects in class-S theories: IR picture

Deforming into Coulomb branch (CB) of the 4d $N = 2$ theory:

- The 4d bulk is described by an effective abelian theory.
- The surface defect has a set of massive vacua, fibered over C to form a space of vacua $\widetilde{C} \to C$:
  1-form $\lambda = xdz$, $x \leftrightarrow$ VEV of twisted chiral ring operator

The vacua structure encodes the bulk Seiberg-Witten geometry.



Surface defects and Schrödinger equations

- Consider the Seiberg-Witten curve of certain 4d $N = 2$ theory:

$$\widetilde{C}: \; x^2 + P(z) = 0,$$

Promoting x (momentum) and z (position) to Heisenberg operators $\rightsquigarrow$ Schrödinger equation:

$$\left[\partial_z^2 + \hbar^{-2} P(z,\hbar)\right] \psi(z) = 0$$

- Turning on the Nekrasov-Shatashvili limit of Ω-background along the surface defect quantizes the Seiberg-Witten curve into Schrödinger equations or higher rank analogue, in a canonical way.
[Nekrasov-Shatashvili],[Nekrasov],[Jeong],[Jeong-Nekrasov],[Jeong-Lee-Nekrasov],...

- This can also be derived through the conformal limit [Gaiotto] of the Hitchin moduli space or from the AGT-correspondence.
[Alday-Gaiotto-Tachikawa],[Alday-Gaiotto-Gukov-Tachikawa-Verlinde],...

Exact WKB for Schrödinger equations

WKB ansatz: $\psi(z) = \exp\left(\hbar^{-1}\int_{z_0}^{z}\lambda(z')dz'\right) \;\to\; [\partial_z^2 + \hbar^{-2}P(z)]\psi(z) = 0$

$\lambda(z)$ obeys the Ricatti equation

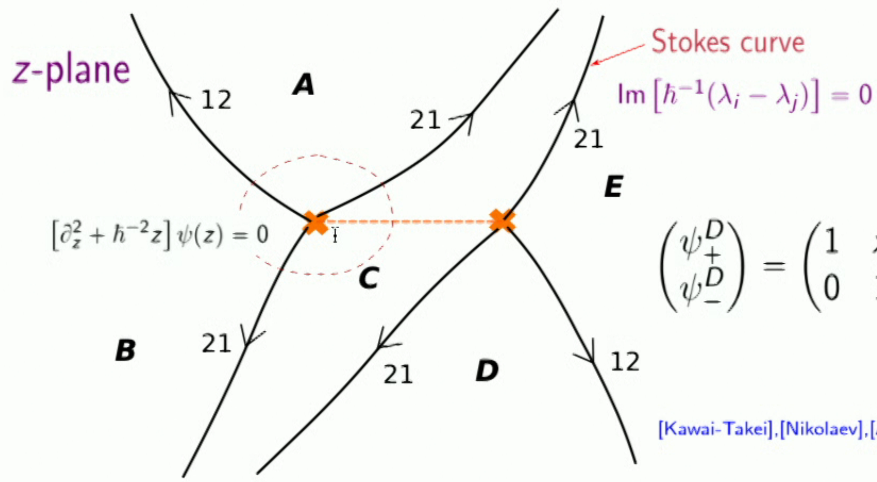
$$\lambda(z)^2 + P(z) + \hbar\partial_z\lambda(z) = 0.$$

Build a formal series solution λ^{formal} in powers of $\hbar$,

$$\text{order-}\hbar^0: \; (\lambda^{(0)})^2 + p(z) = 0, \quad \text{classical SW curve}$$

Choose a branch labeled by $i \in \{\pm\} \rightsquigarrow$ 2 formal solutions $\lambda_{\pm}^{\text{formal}}$
$\longrightarrow$ Two formal solutions $\psi_{\pm}^{\text{formal}}(z, \hbar)$ as series in $\hbar$.

Exact WKB for Schrödinger equations



$$\begin{pmatrix} \psi_+^D \\ \psi_-^D \end{pmatrix} = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_+^E \\ \psi_-^E \end{pmatrix}$$

[Kawai-Takei],[Nikolaev],[Allegretti],...

- Borel resummation gives two actual solutions $\psi_\pm(z)$ within each region, where the solutions jump across a Stokes curve.
- Stokes curves $\leftrightarrow$ soliton spectrum of surface defects [Gaiotto-Moore-Neitzke]

The Voros symbol

The Voros symbol: $\mathcal{X}_\gamma(\hbar) \in \mathbb{C}^\times$, $\gamma \leftrightarrow$ 1-cycles of Seiberg-Witten curve

- $\mathcal{X}_\gamma(\hbar)$ captures the Borel resummed WKB periods:

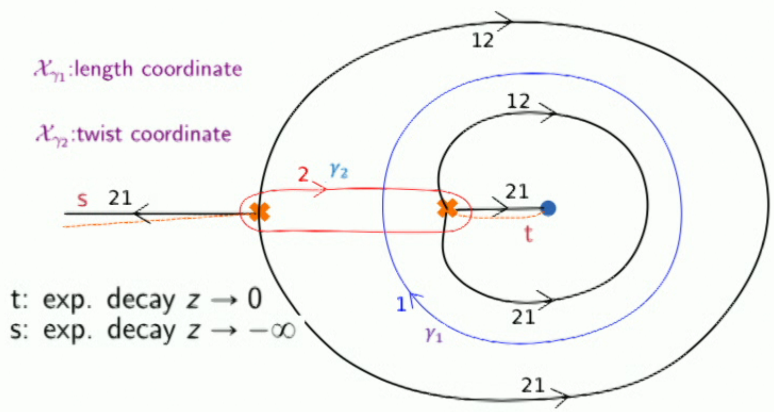
$$\Pi_\gamma(\hbar) := \oint_\gamma \lambda^{\text{formal}}(\hbar) dz = \sum_{n=0}^{\infty} \Pi_\gamma^{(n)} \hbar^n$$

- $\mathcal{X}_\gamma(\hbar)$ expressed as Wronskians of distinguished local solutions:
  - ⋆ asymptotically decaying solutions as z approaches a singularity
  - ⋆ eigenvectors of the monodromy around a loop
- $\mathcal{X}_\gamma(\hbar)$ encodes exact quantization conditions for spectral problems.

The Voros symbol: modified Mathieu operator

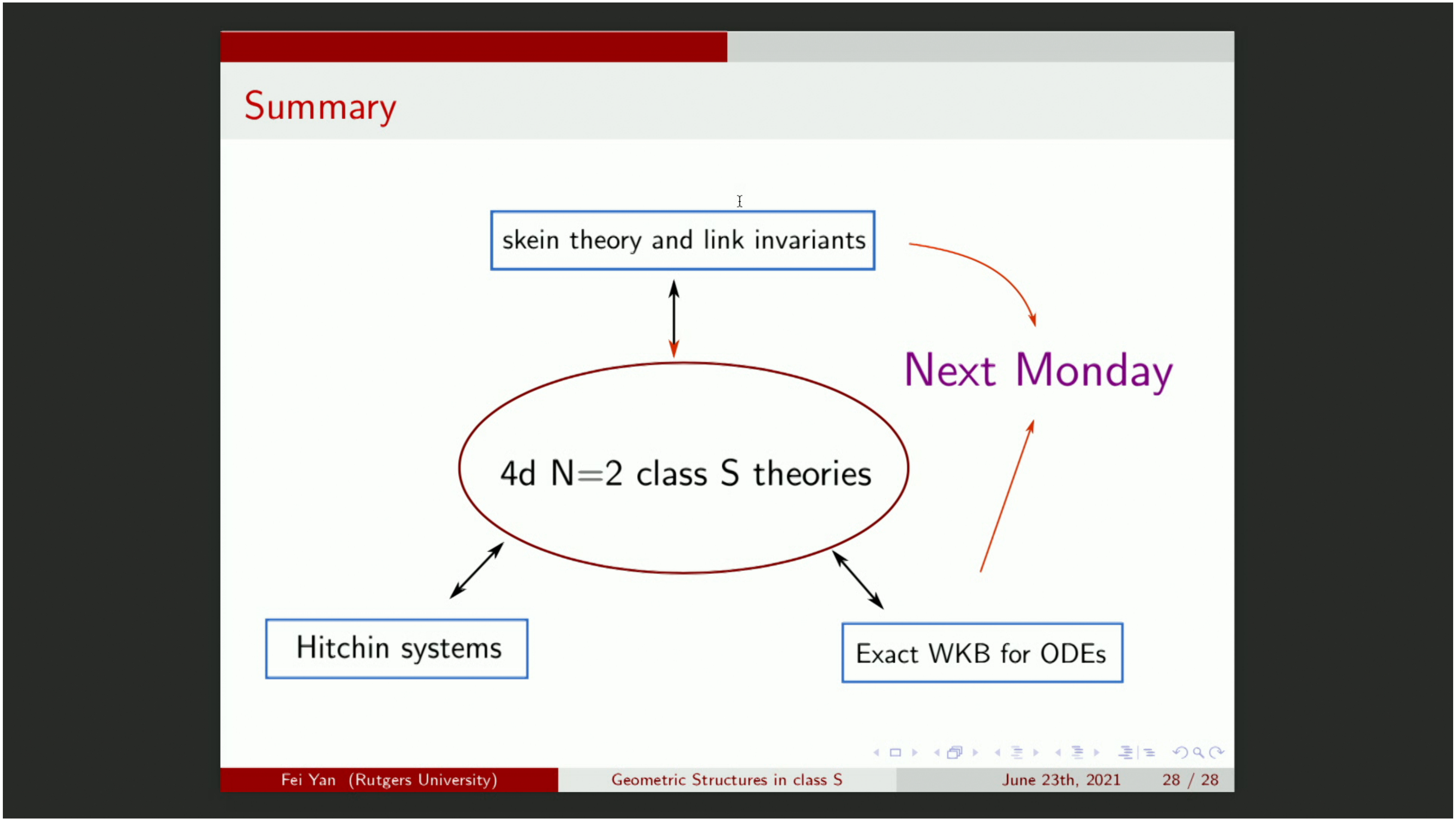
$$[-\hbar^2\partial_x^2 + 2\cosh(x) - 2E]\psi(x) = 0 \quad (E > 1)$$

$$z = -e^{-x} \to \left[\hbar^2\partial_z^2 + \left(\frac{1}{z^3} + \frac{1}{z} + \frac{2E + 0.25\hbar^2}{z^2}\right)\right]\tilde{\psi}(z) = 0.$$ SU(2) SYM



bound states: s prop. to $t \to \mathcal{X}_{\gamma_2} = 1$ (exact quantization condition)

[Mironov-Morozov],[He-Miao],[Basar-Dunne],[Dunne-Ünsal],[Codesido-Marino-Schiappa],[Hollands-Neitzke],...


Summary
skein theory and link invariants
Next Monday
4d N=2 class S theories
Hitchin systems
Exact WKB for ODEs
Fei Yan (Rutgers University)
Geometric Structures in class S
June 23th, 2021
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Thank You and Stay Safe!