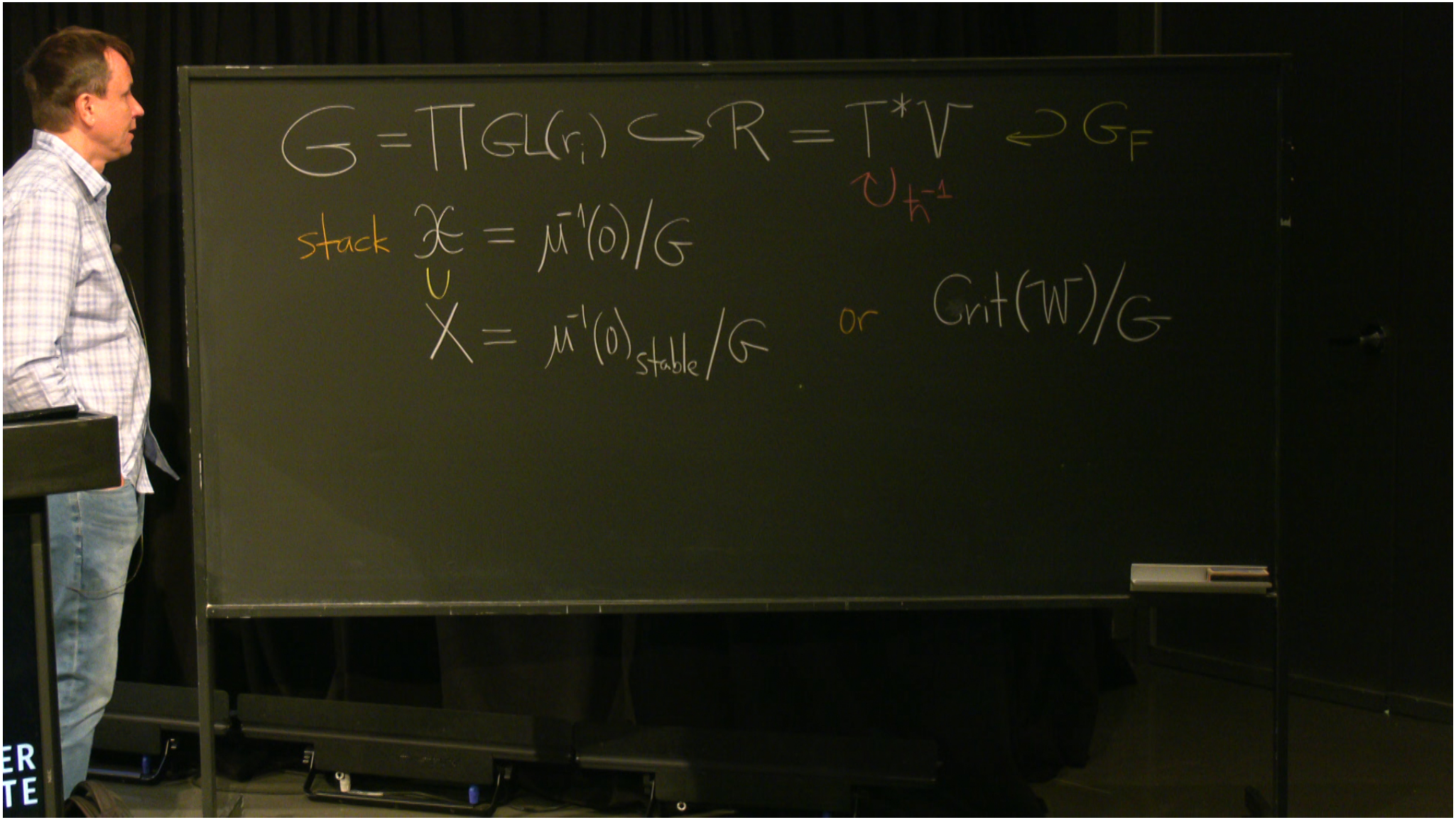
Title: 3d Gauge Theory and Elliptic Stable Envelopes III

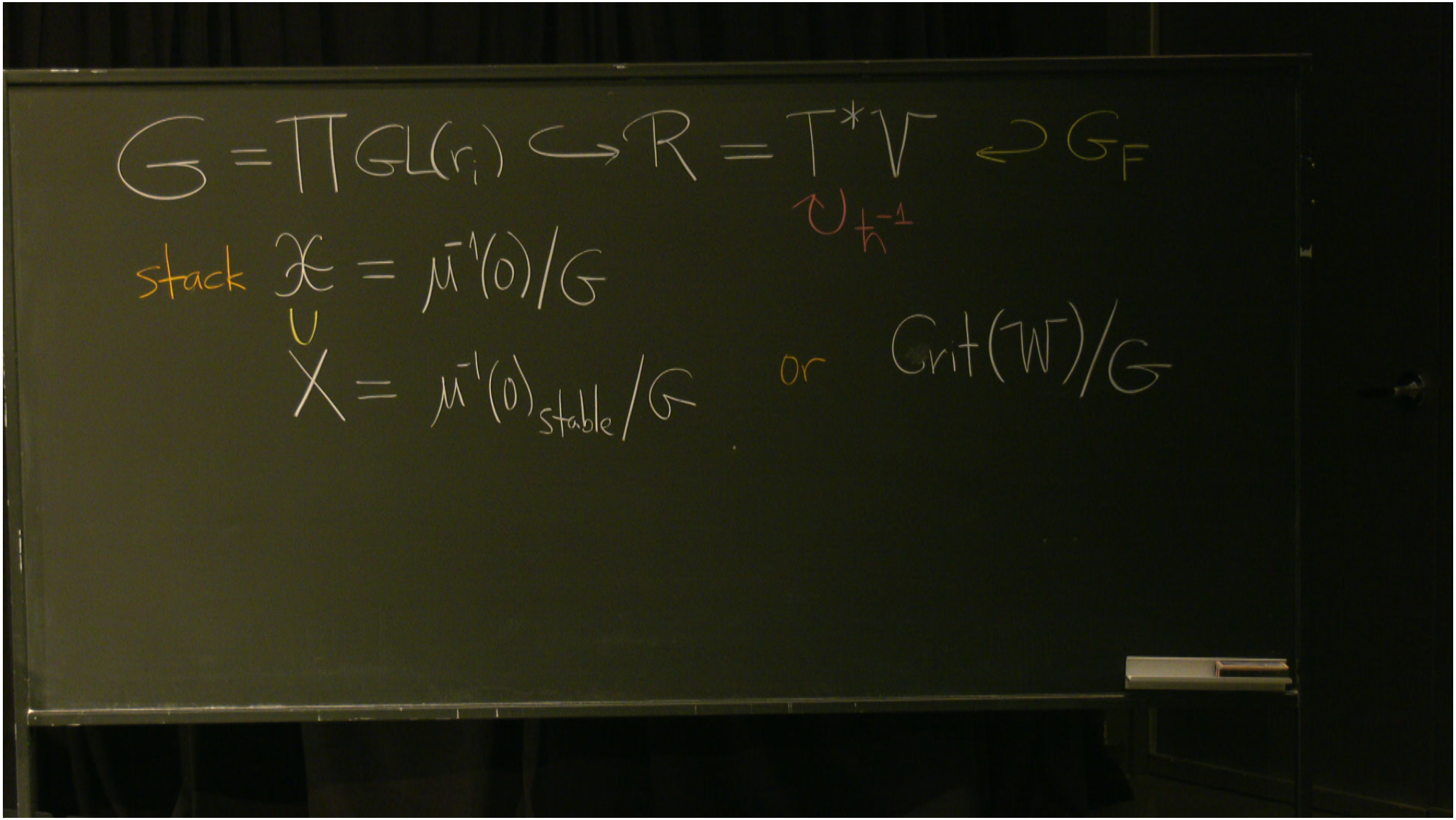
Speakers: Andrei Okounkov

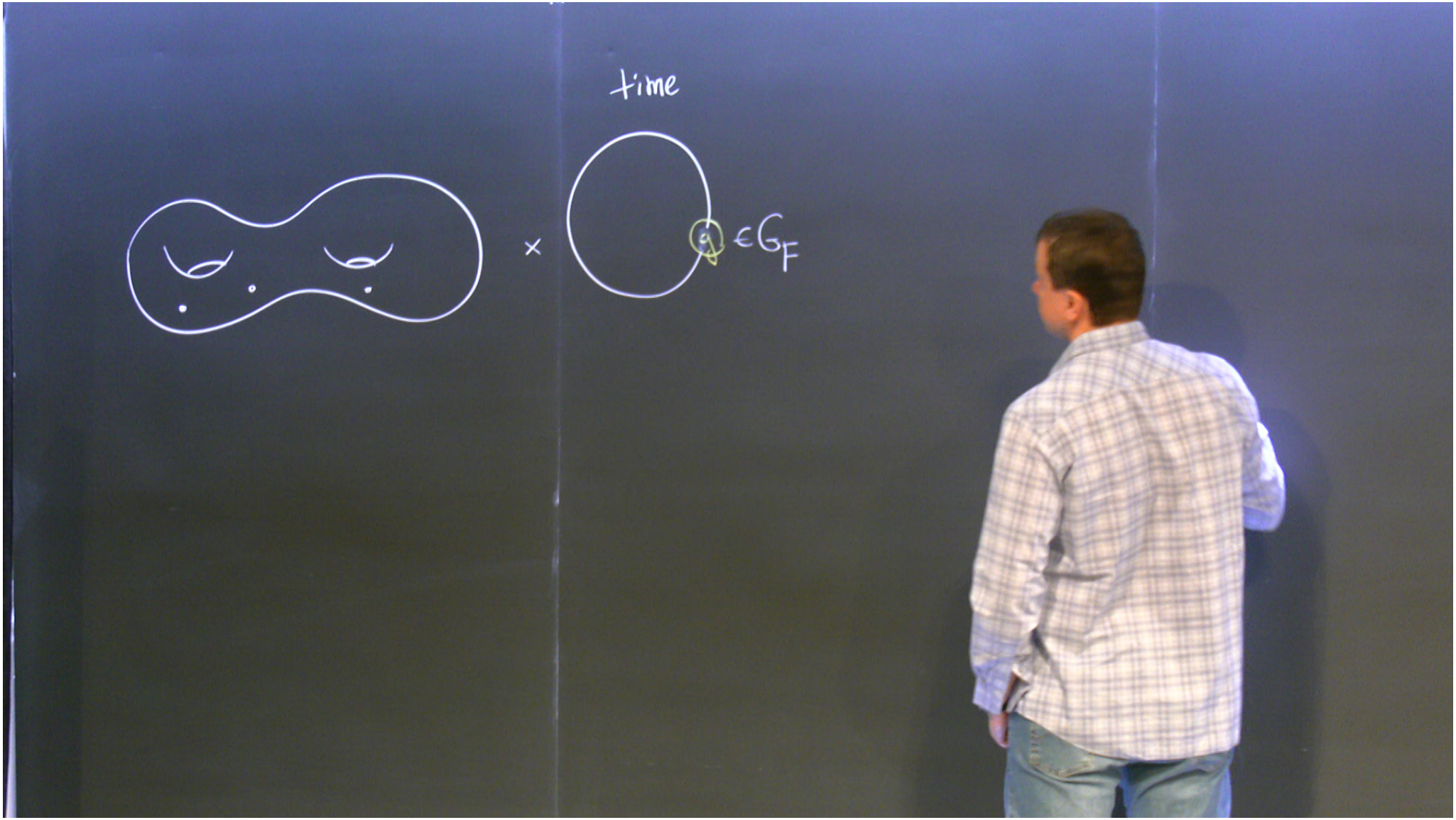
Collection: QFT for Mathematicians 2022

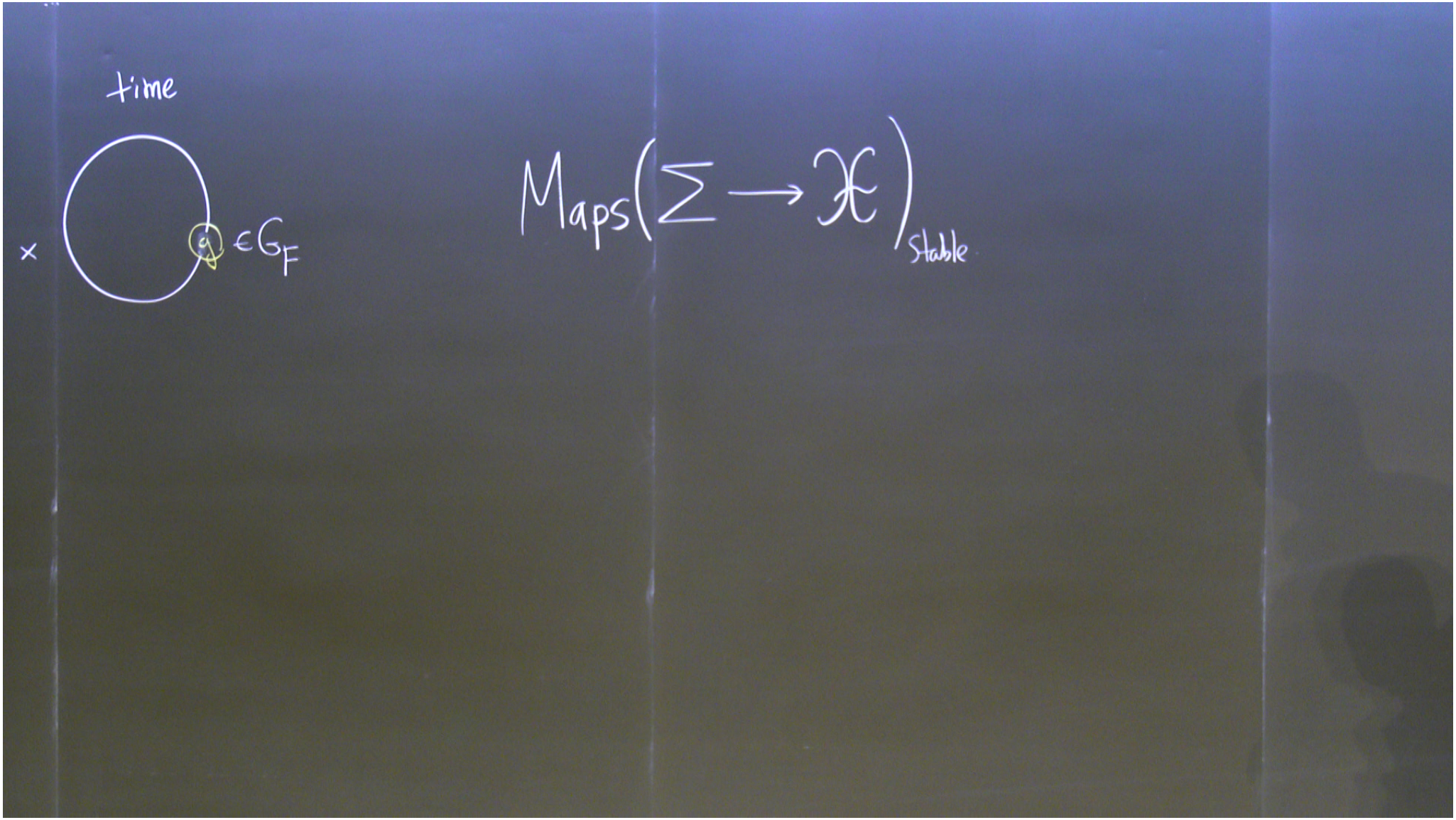
Date: June 23, 2022 - 9:00 AM

URL: https://pirsa.org/22060072


$G = \prod GL(r_i) \hookrightarrow R = T^*V \circlearrowleft G_F$
$\cup_{h^{-1}}$
stack $\mathcal{X} = \mu^{-1}(0)/G$
$\cup$
$X = \mu^{-1}(0)_{stable}/G$ or $Grit(W)/G$

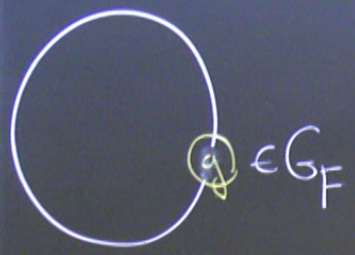

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time
× ∈ G_F

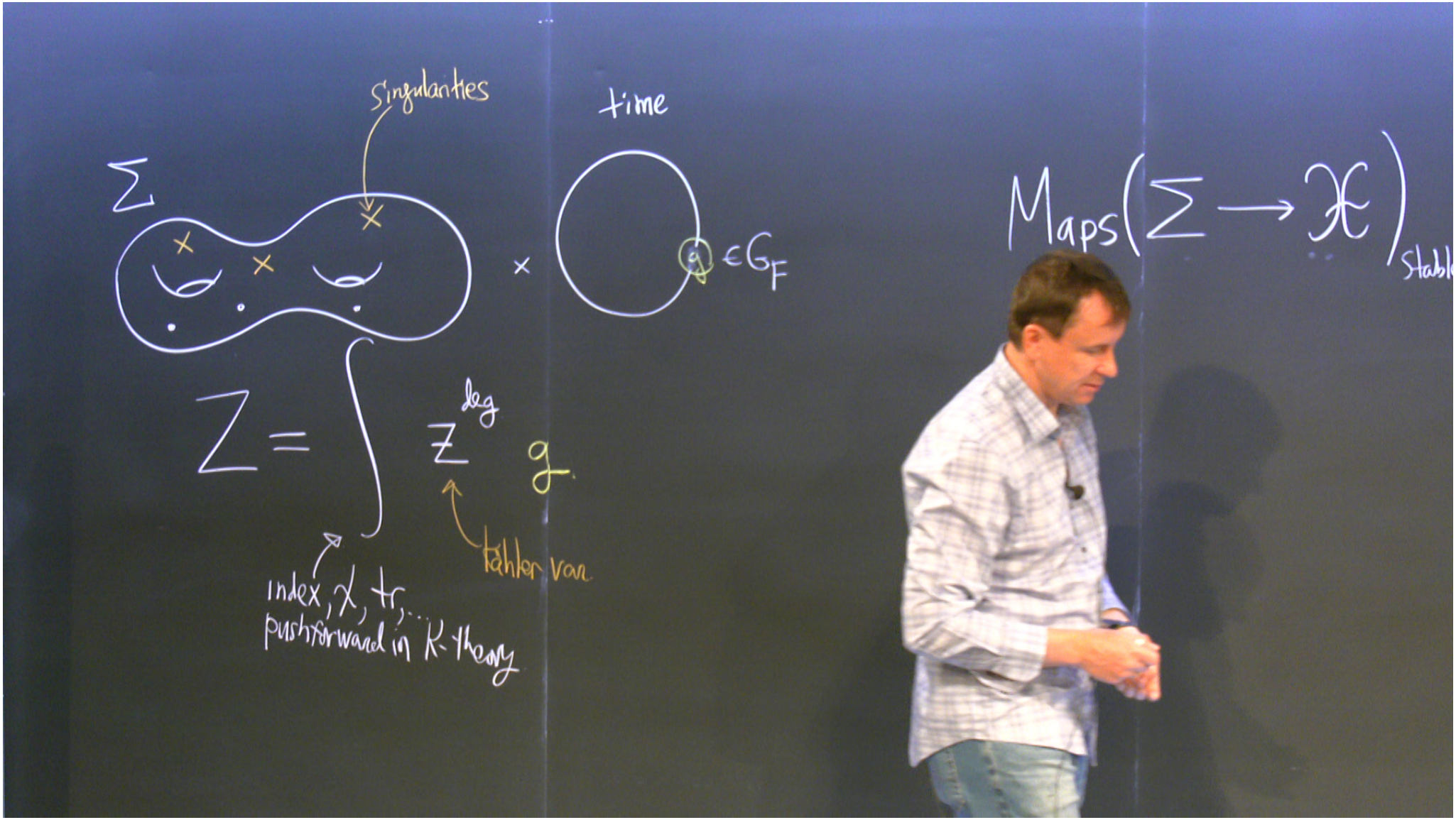

time
$\in G_F$
$\mathrm{Maps}(\Sigma \to \mathcal{X})_{\text{stable}}$

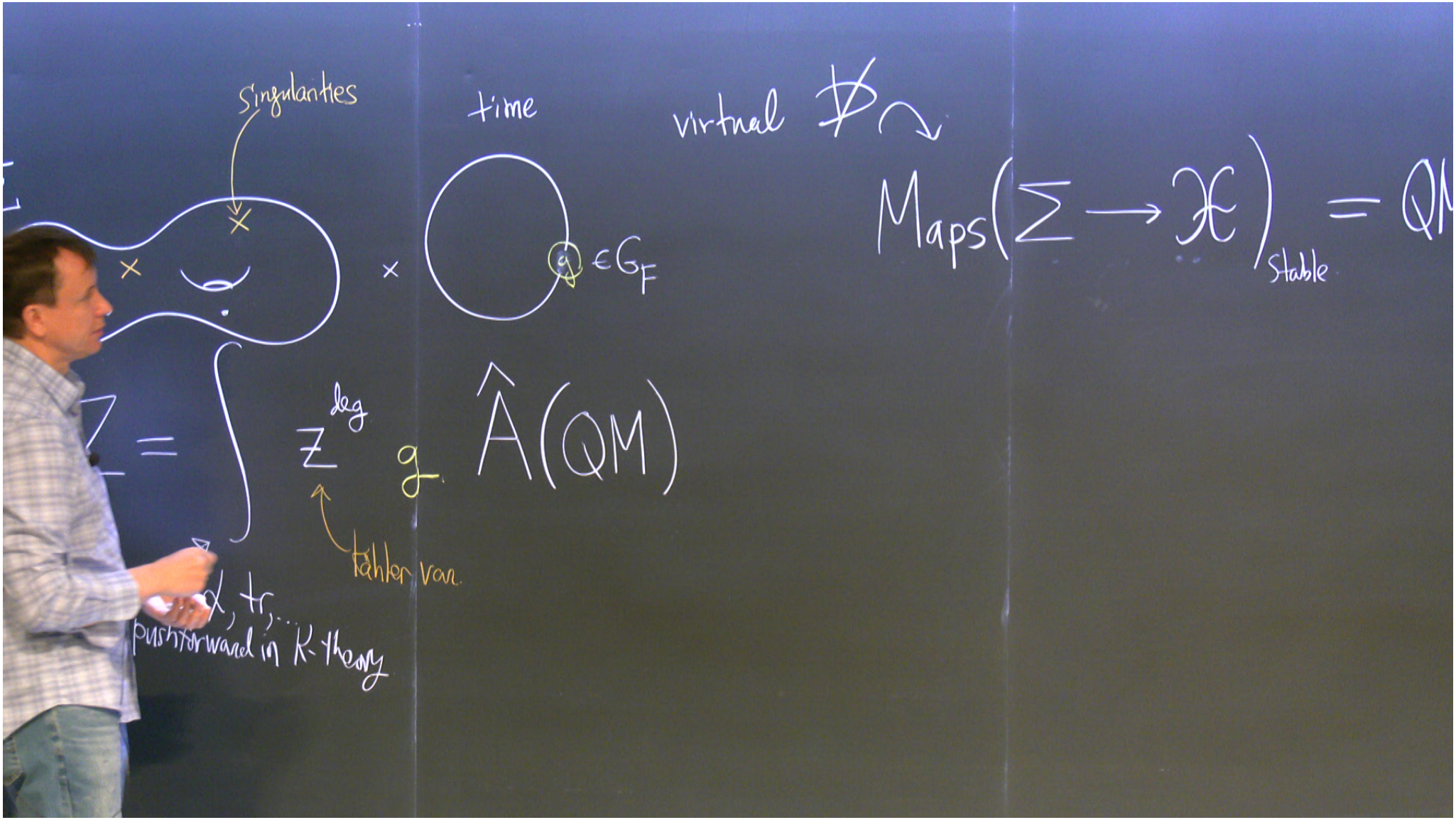

Singularities
time
Σ
$\times$
$\in G_F$
$\mathrm{Maps}\left(\Sigma \to \mathcal{X}\right)_{\mathrm{stable}}$

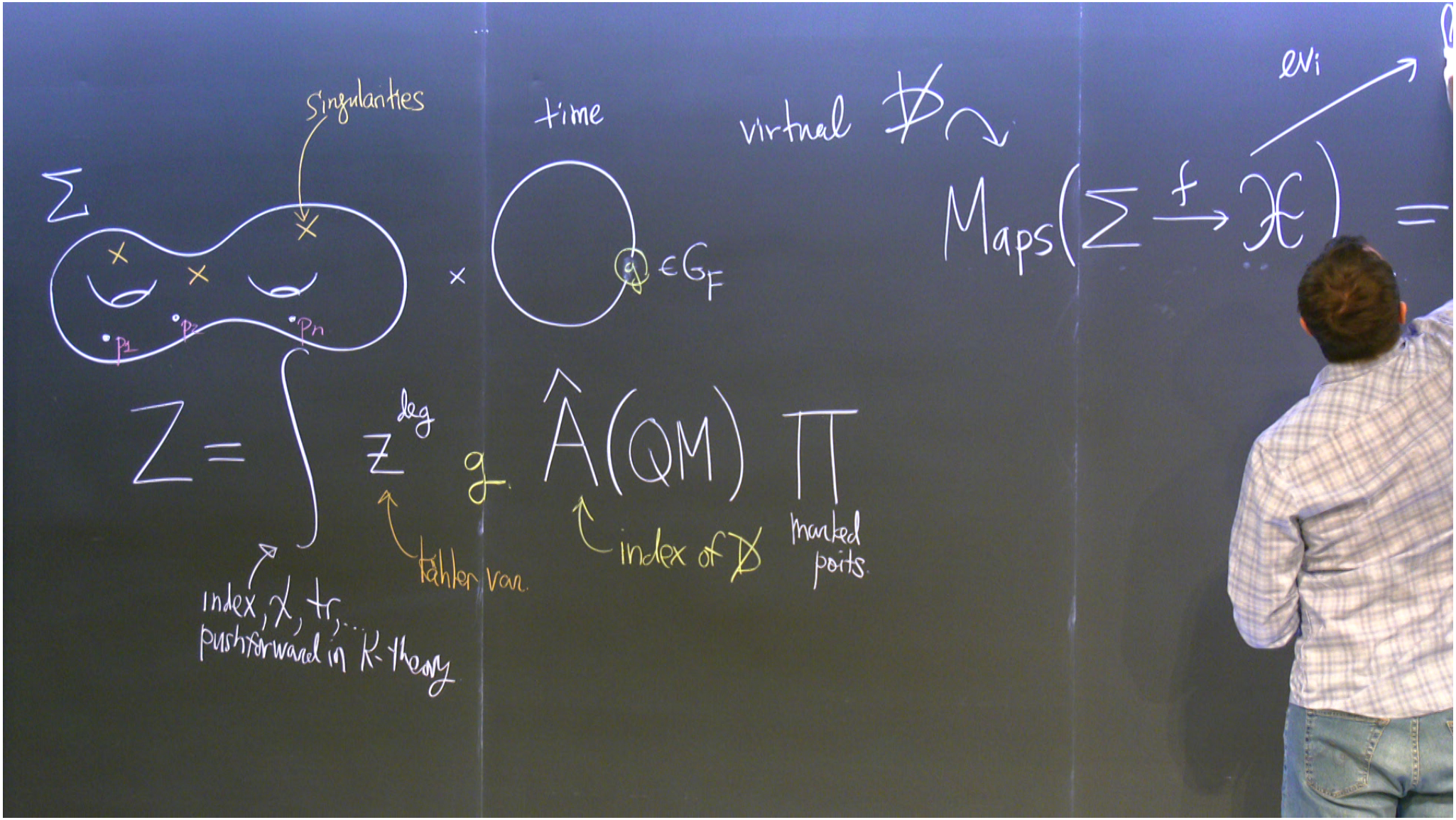
time

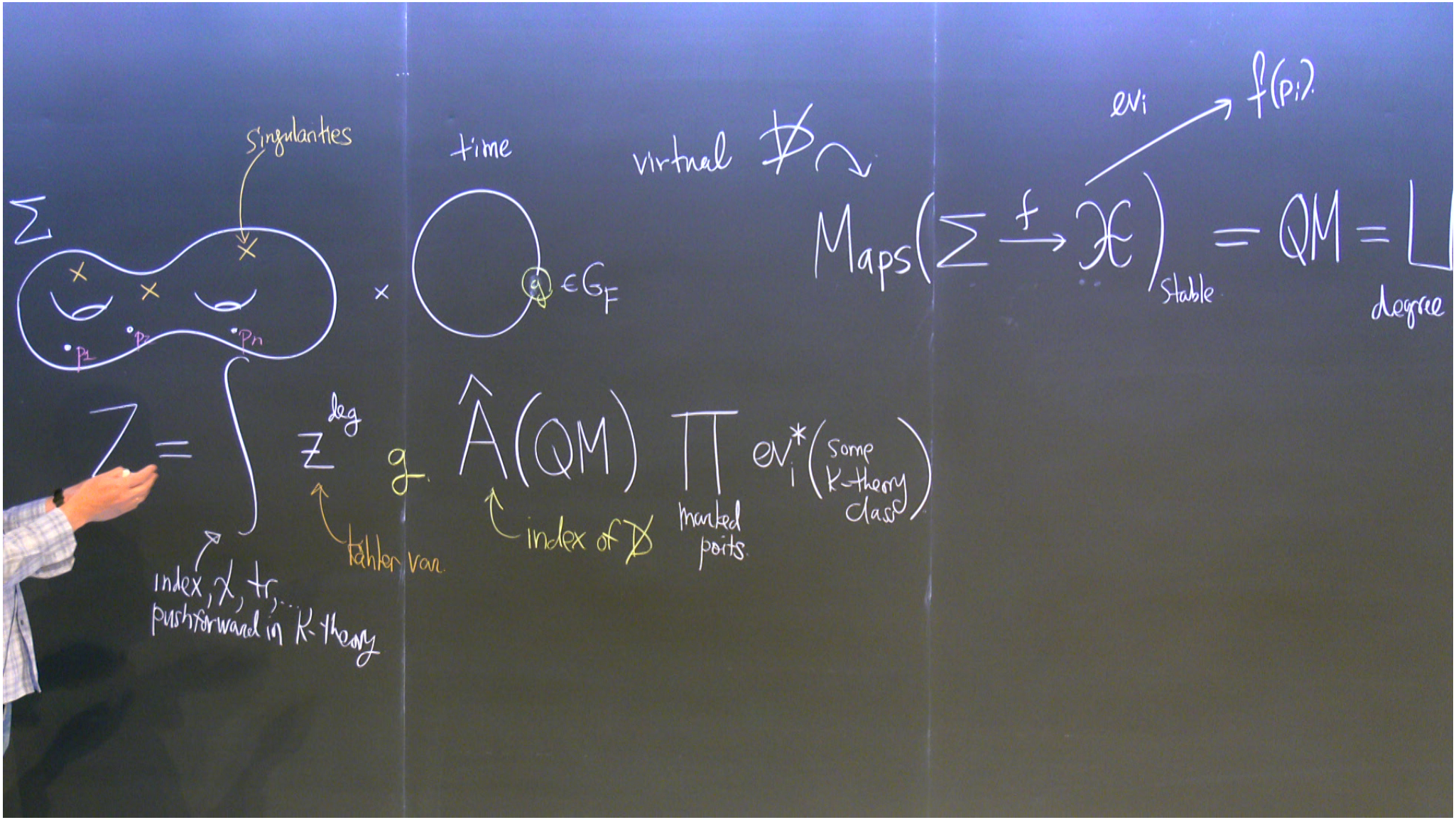


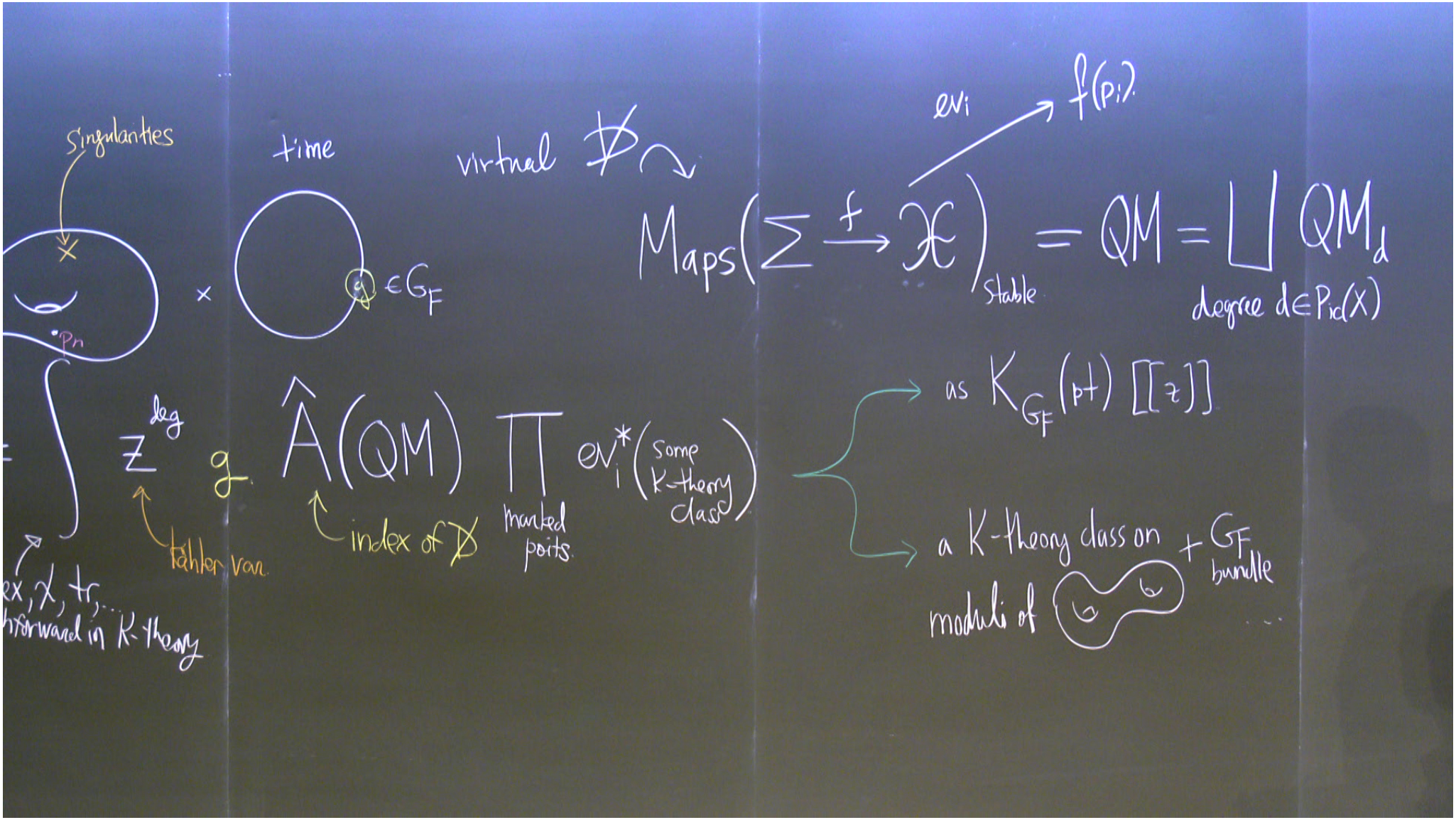
$$\mathrm{Maps}(\Sigma \to \mathcal{X})_{\text{stable}} = QM = \coprod_{\text{degree } d \in \mathrm{Pic}(X)} QM_d$$

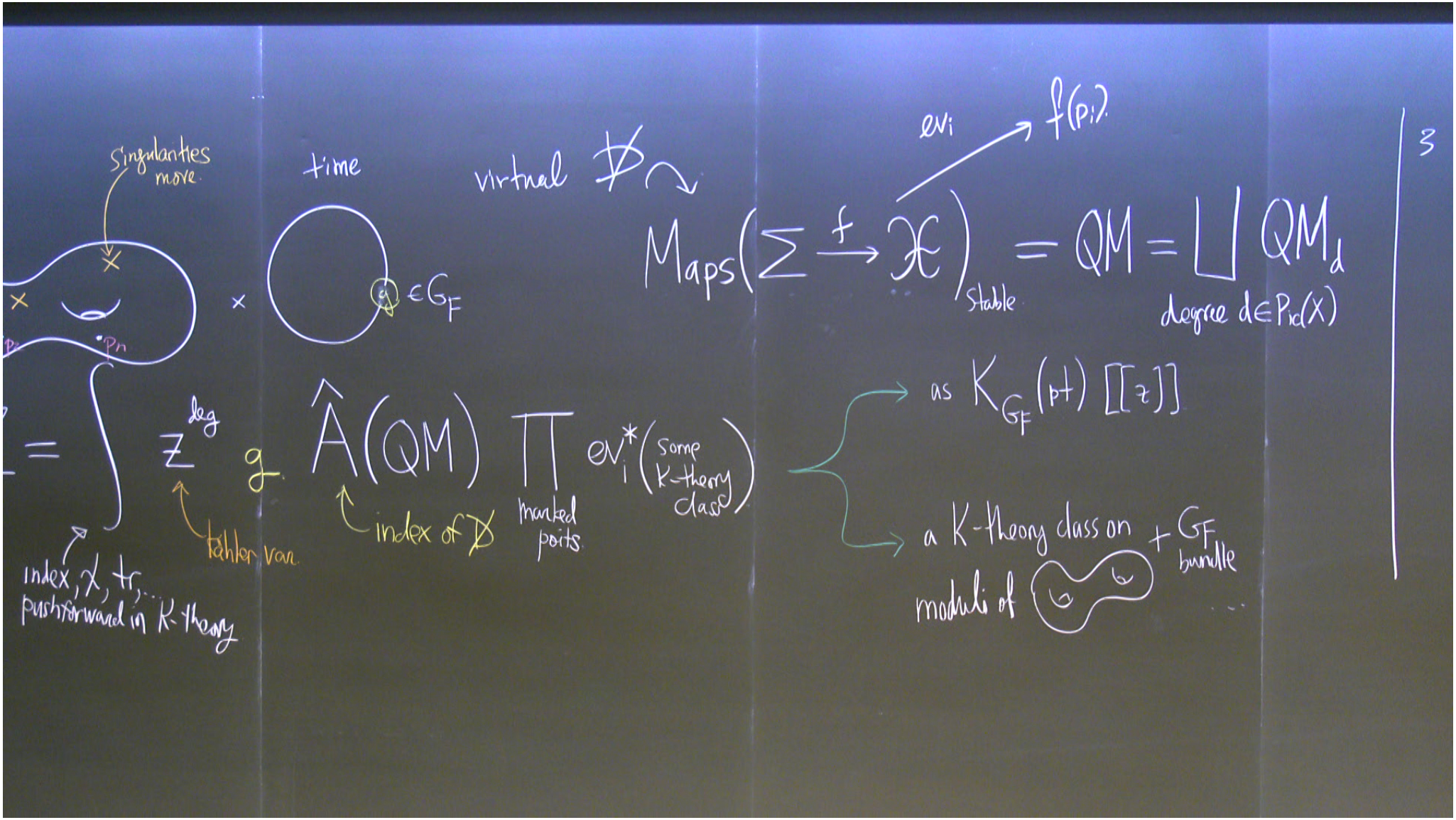

Singularities
time
Σ
× ∈ G_F
Maps(Σ → 𝔛)
Stable
Z =
Z^deg g
Kähler var.
index, χ, tr, ...
pushforward in K-theory


Singularities
time
virtual
$\in G_F$
$\hat{A}(QM)$
deg
Kähler var.
$\chi, tr, \ldots$
pushforward in K-theory
stable


Singularities
time
virtual
ev_i
Maps(Σ → X) =
Σ
p1
p2
pn
× G_F
Z = ∫
z^deg
g
Â(QM)
Π
index, χ, tr, ...
pushforward in K-theory
Kähler var.
index of D
marked points


Singularities
time
virtual
Σ
$\times$
$\in G_F$
p_1
p_2
p_n
$Z = \int z^{\deg g} \hat{A}(QM) \prod ev_i^*(\text{some K-theory class})$
index, χ, tr, ... pushforward in K-theory
Kähler var.
index of $\not{D}$
marked pts.
ev_i
$f(p_i)$
$\text{Maps}(\Sigma \xrightarrow{f} \mathcal{X})_{\text{stable}} = QM = L$
degree


Singularities
time
virtual
ev_i
$f(p_i)$
$M_{aps}(\Sigma \xrightarrow{f} \mathcal{X})_{stable} = QM = \coprod_{degree\ d \in Pic(X)} QM_d$
$\in G_F$
z^{deg}
$\hat{A}(QM)$
$\prod$
ev_i^*(some K-theory class)
as $K_{G_F}(pt)[[z]]$
a K-theory class on moduli of
+ G_F bundle
fiber var.
index of $\not{D}$
marked points
forward in K-theory


singularities move
time
virtual
$\mathcal{D}$
ev_i
$f(p_i)$
$\mathcal{M}aps\left(\Sigma \xrightarrow{f} \mathcal{X}\right)_{stable} = QM = \coprod_{\text{degree } d \in Pic(X)} QM_d$
$\in G_F$
p_1
p_n
$= \int$
z^{deg}
g
$\hat{A}(QM)$
$\prod$
ev_i^*(some K-theory class)
Kähler var.
index of $\mathcal{D}$
marked points
as $K_{G_F}(pt)[[z]]$
a K-theory class on moduli of
$+ G_F$ bundle
index, χ, tr, ... pushforward in K-theory
3

(p_i)

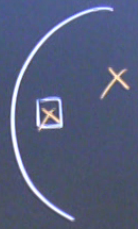
$$QM = \coprod_{\text{degree } d \in Pic(X)} QM_d$$

$[[z]]$

$+ G_F$ bundle

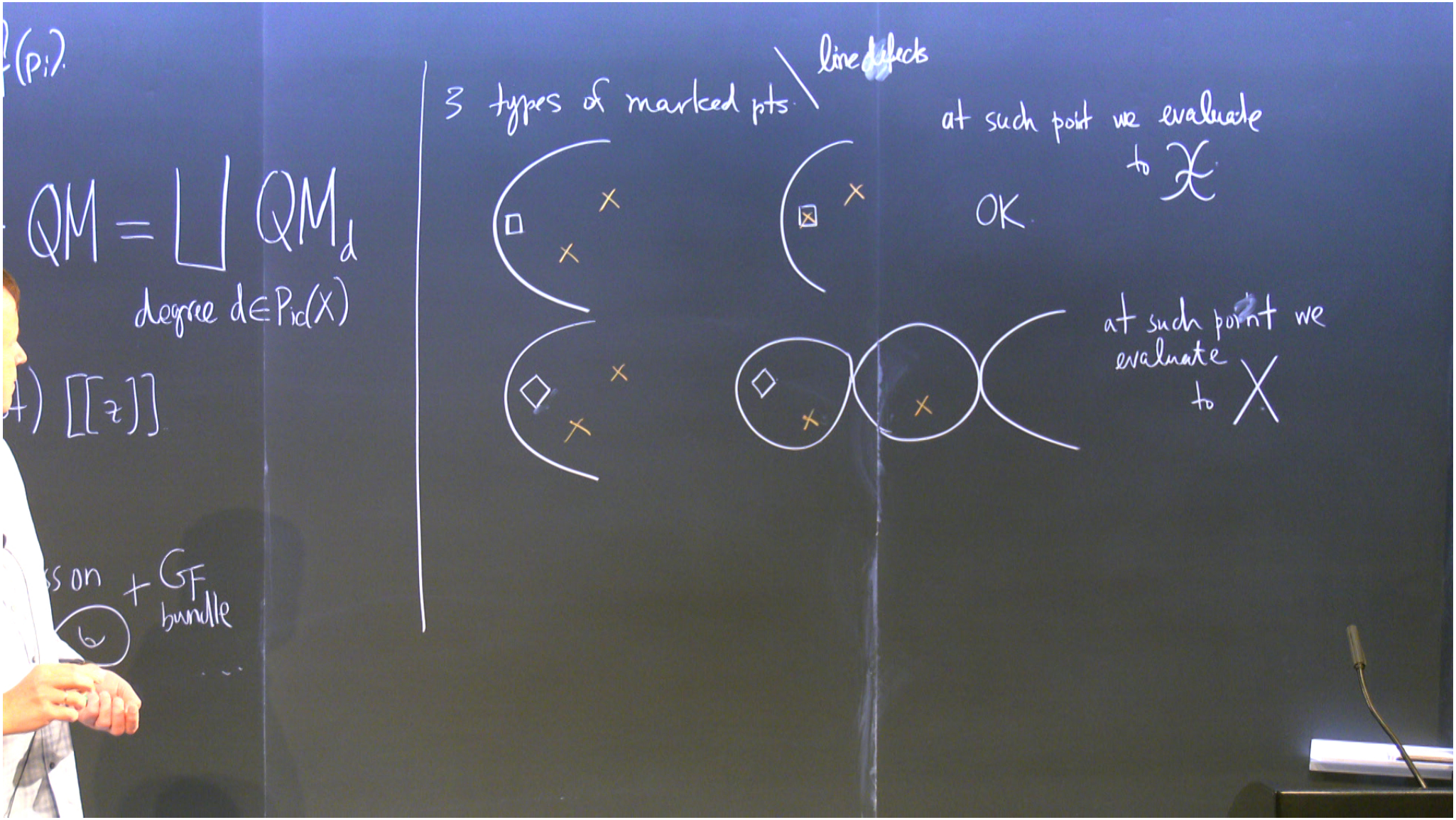
3 types of marked pts / line defects





at such point we evaluate to $\mathcal{X}$

OK


$\ell(p_i)$
$QM = \coprod QM_d$
degree $d \in Pic(X)$
$[[z]]$
3 types of marked pts
line defects
at such point we evaluate to $\mathcal{X}$
OK
at such point we evaluate to X
$+ G_F$ bundle

(p_i)

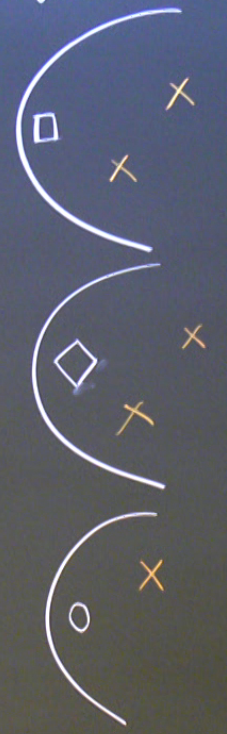
$QM = \coprod QM_d$

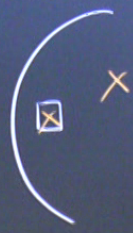
degree $d \in Pic(X)$

$t)$ $[[z]]$

class on + GF bundle

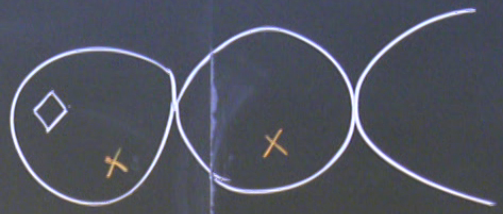
3 types of marked pts / line defects





at such point we evaluate to $\mathcal{X}$

OK

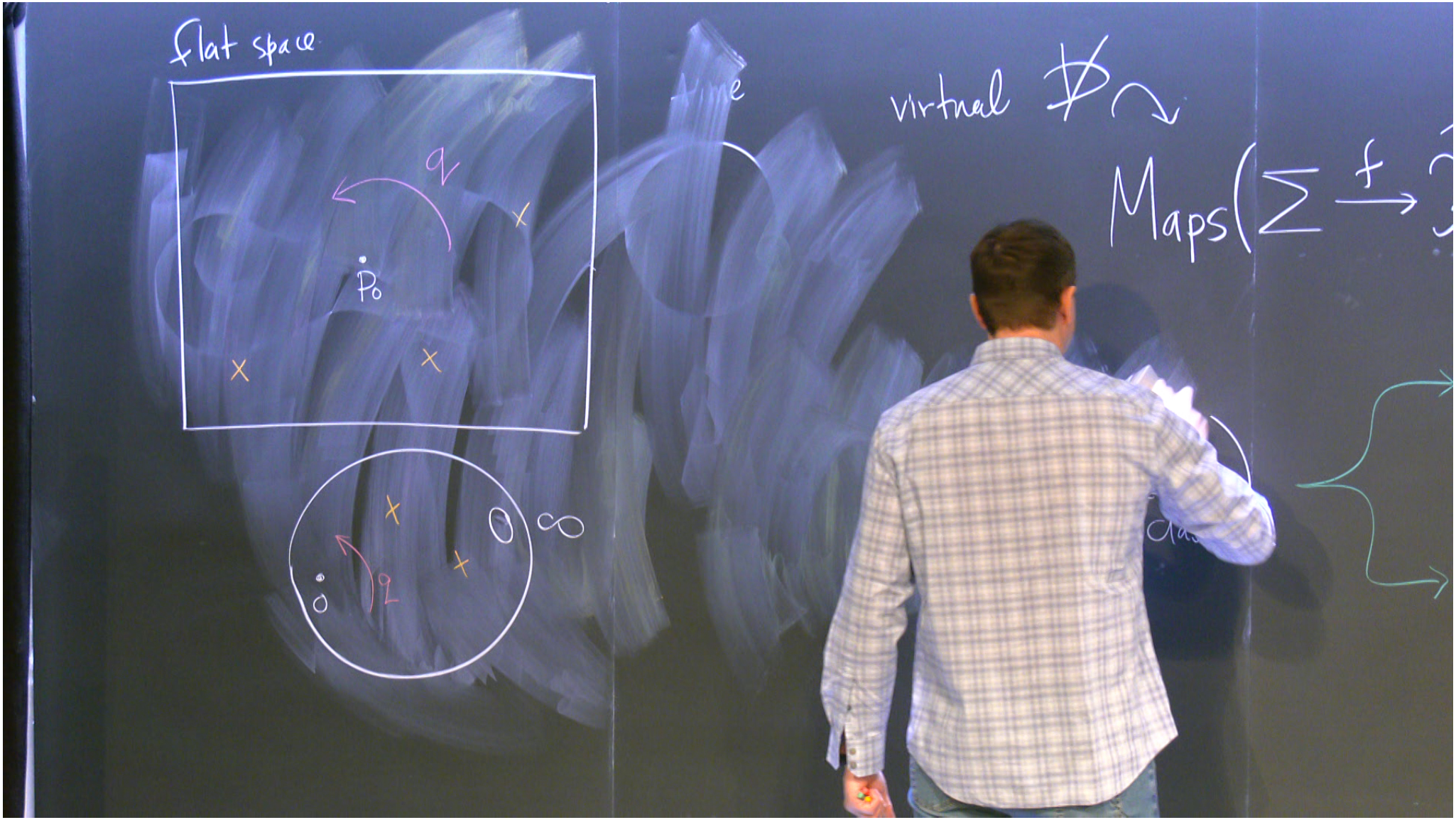


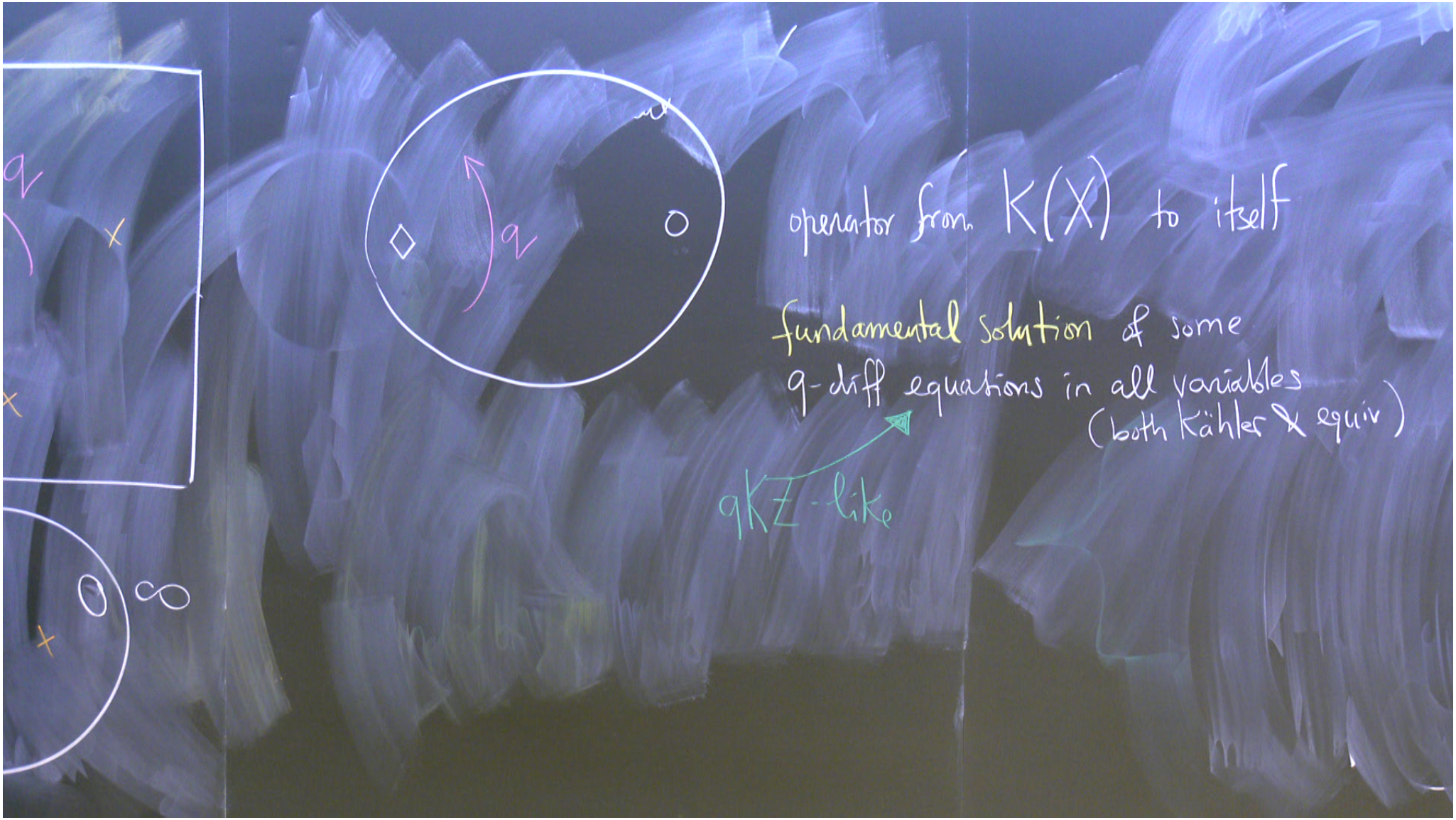
at such point we evaluate to X

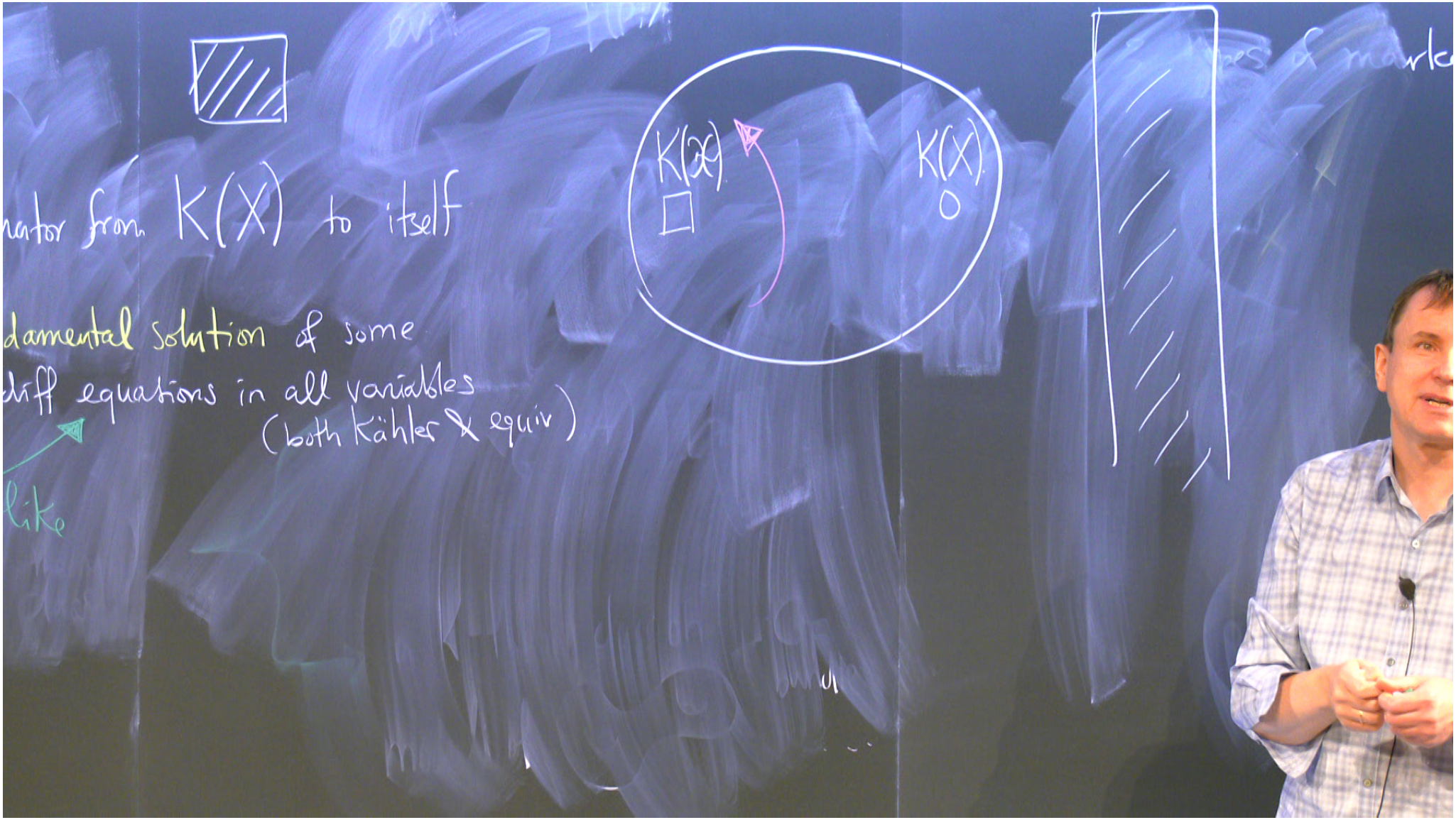


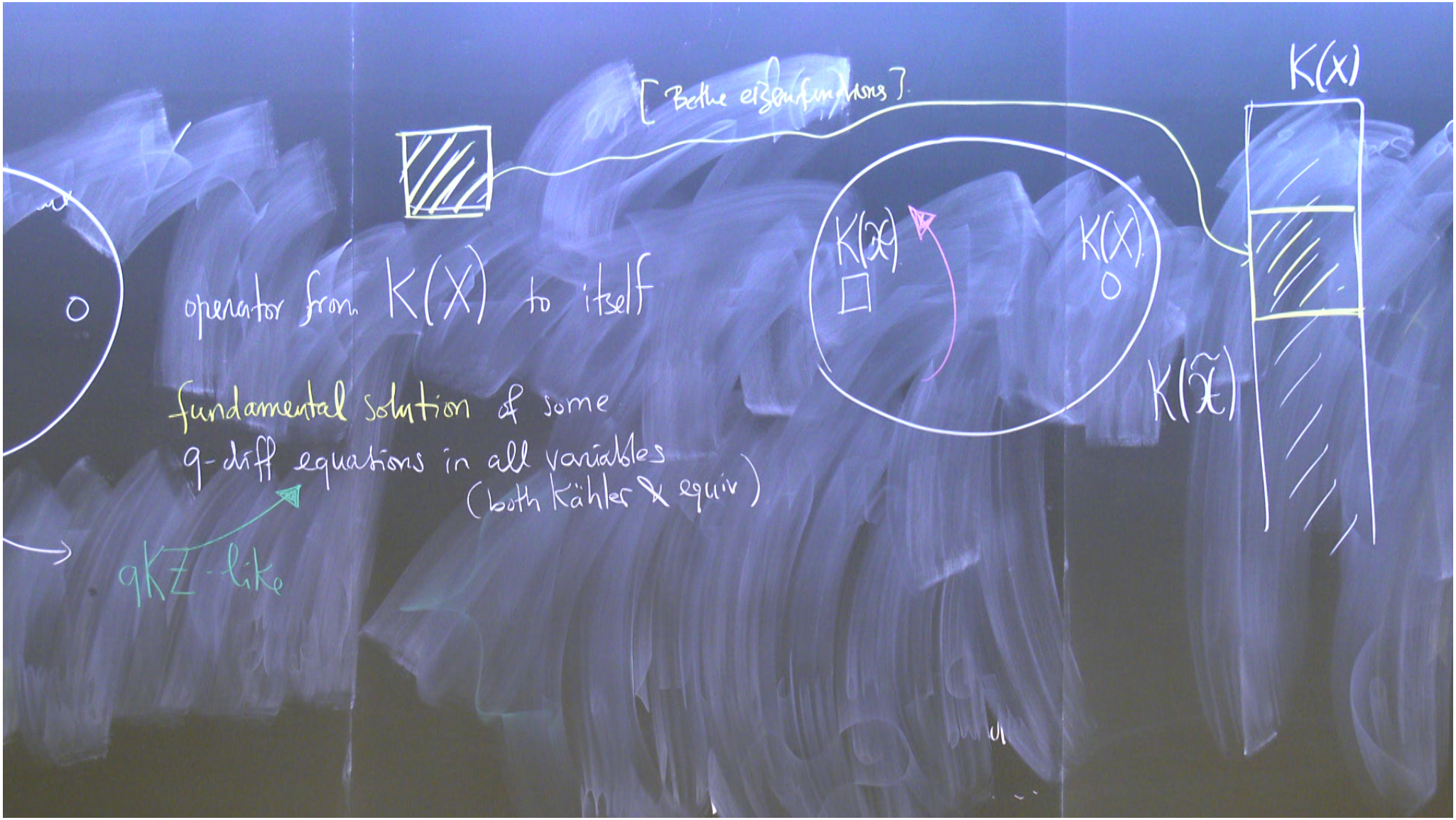
NOT OK

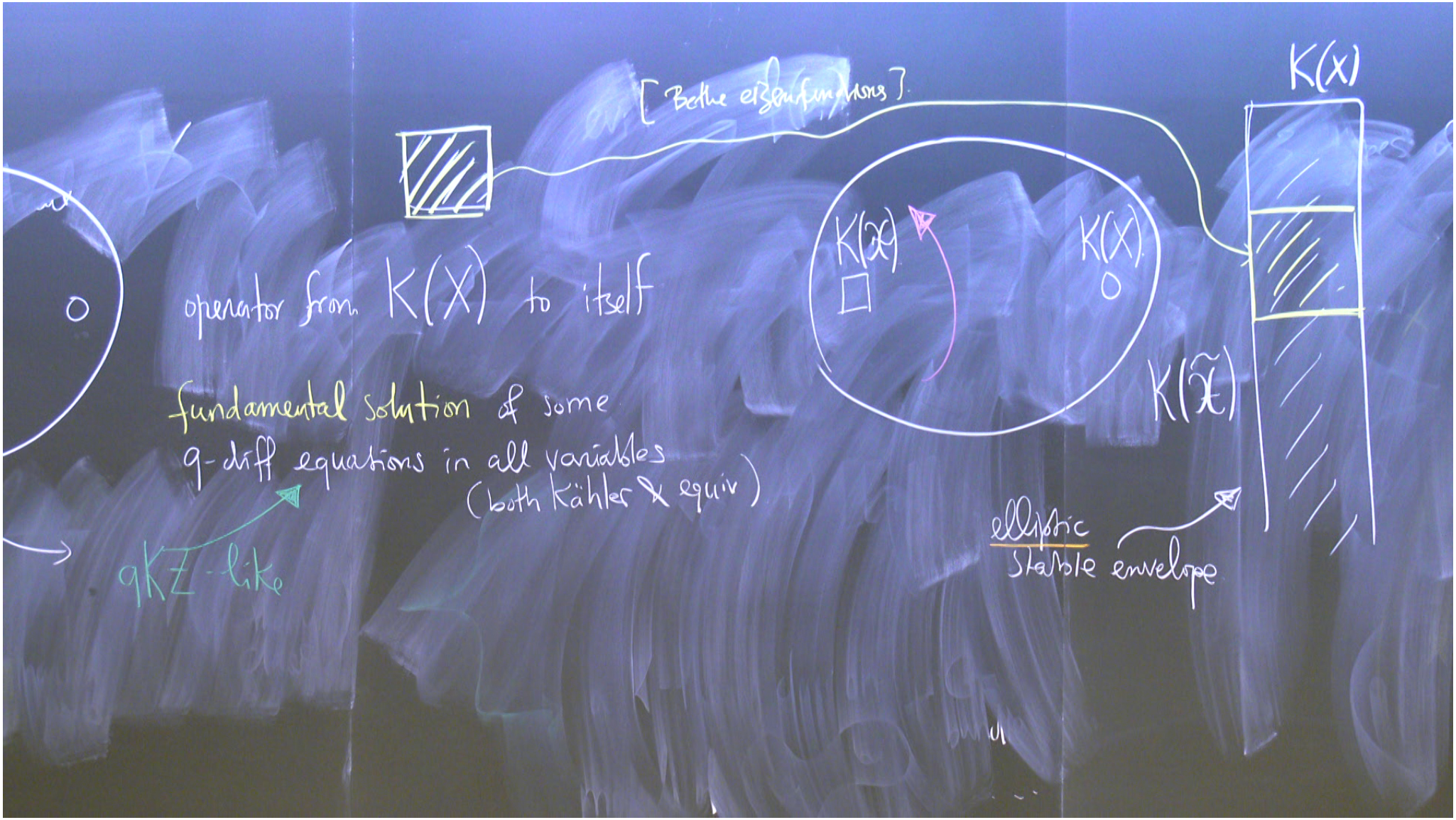
open condition

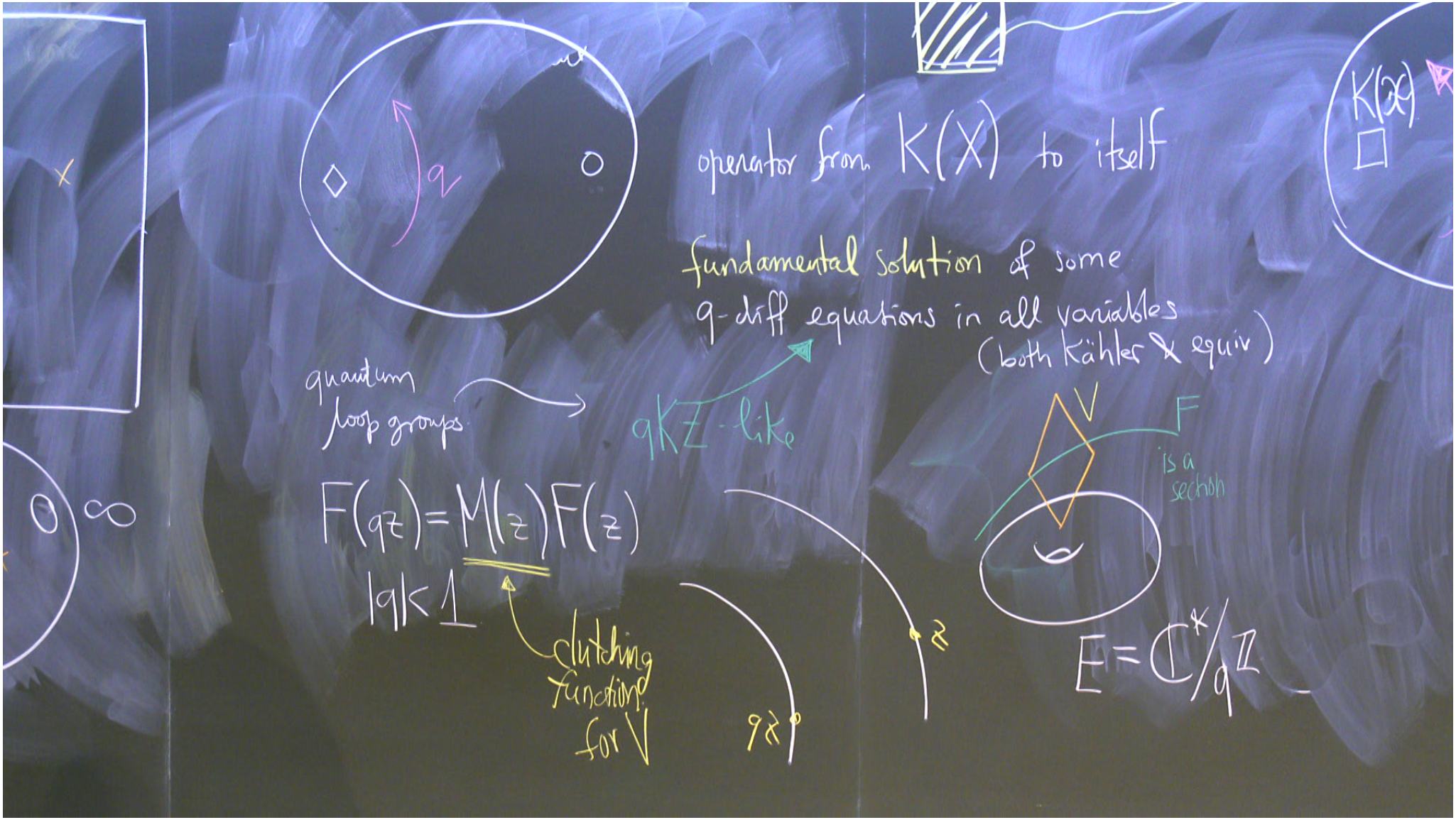

flat space
q
P_0
virtual
Maps($\Sigma \xrightarrow{f}$

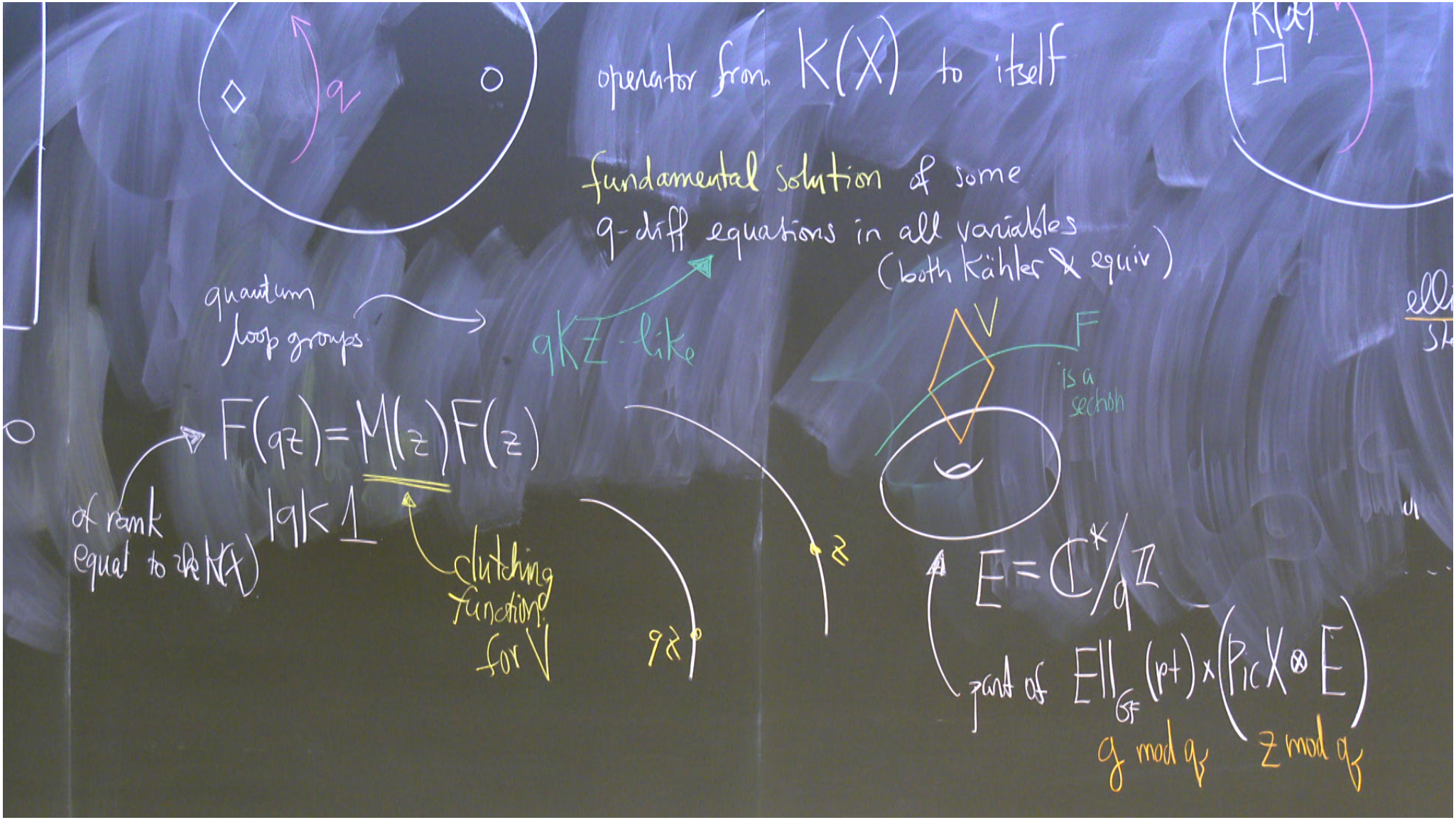

q
q
0
∞
operator from K(X) to itself
fundamental solution of some
q-diff equations in all variables
(both Kähler & equiv)
qKZ-like

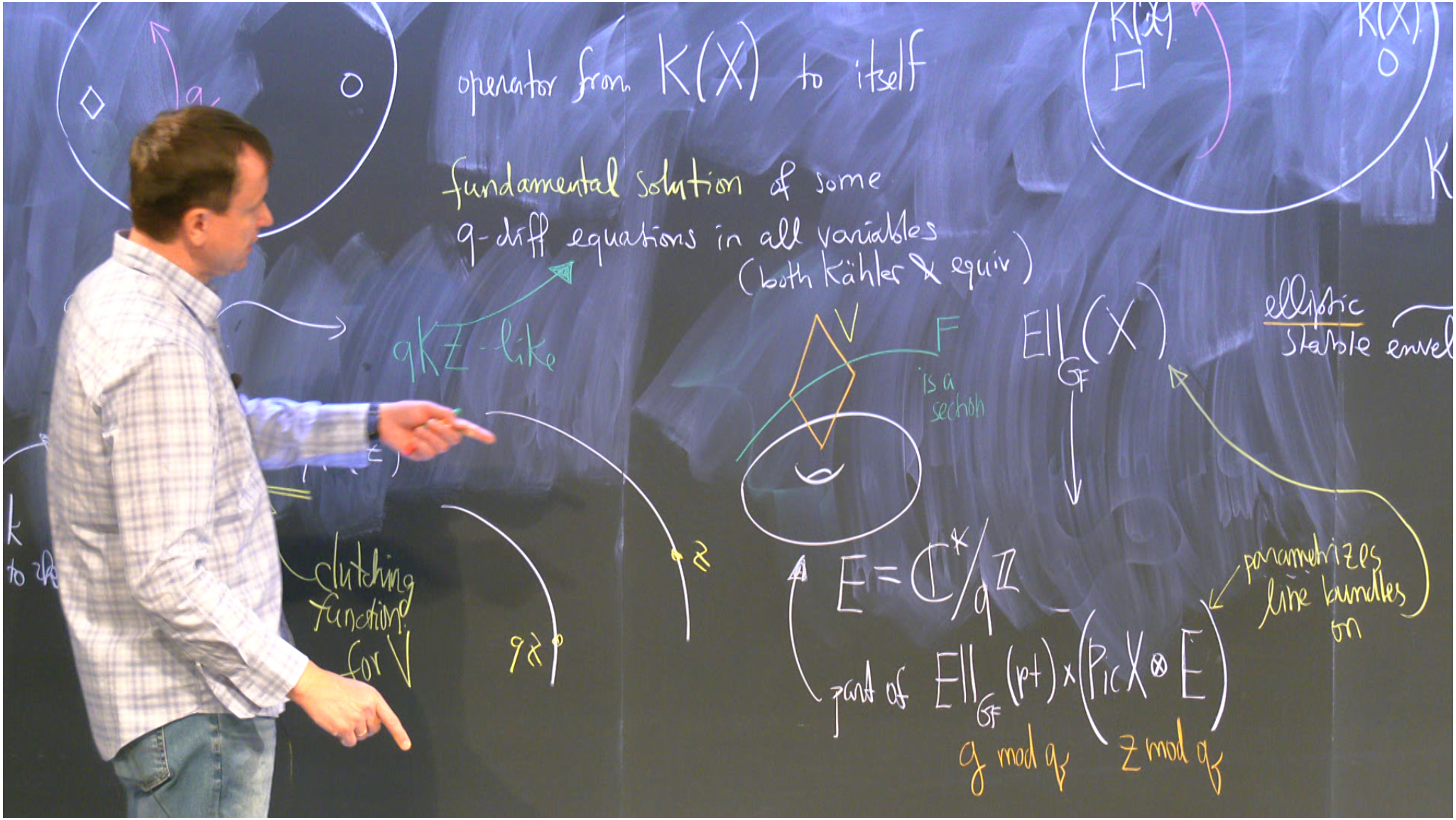

rator from K(X) to itself
damental solution of some
diff equations in all variables
(both Kähler & equiv)
like
K(x)
K(X)

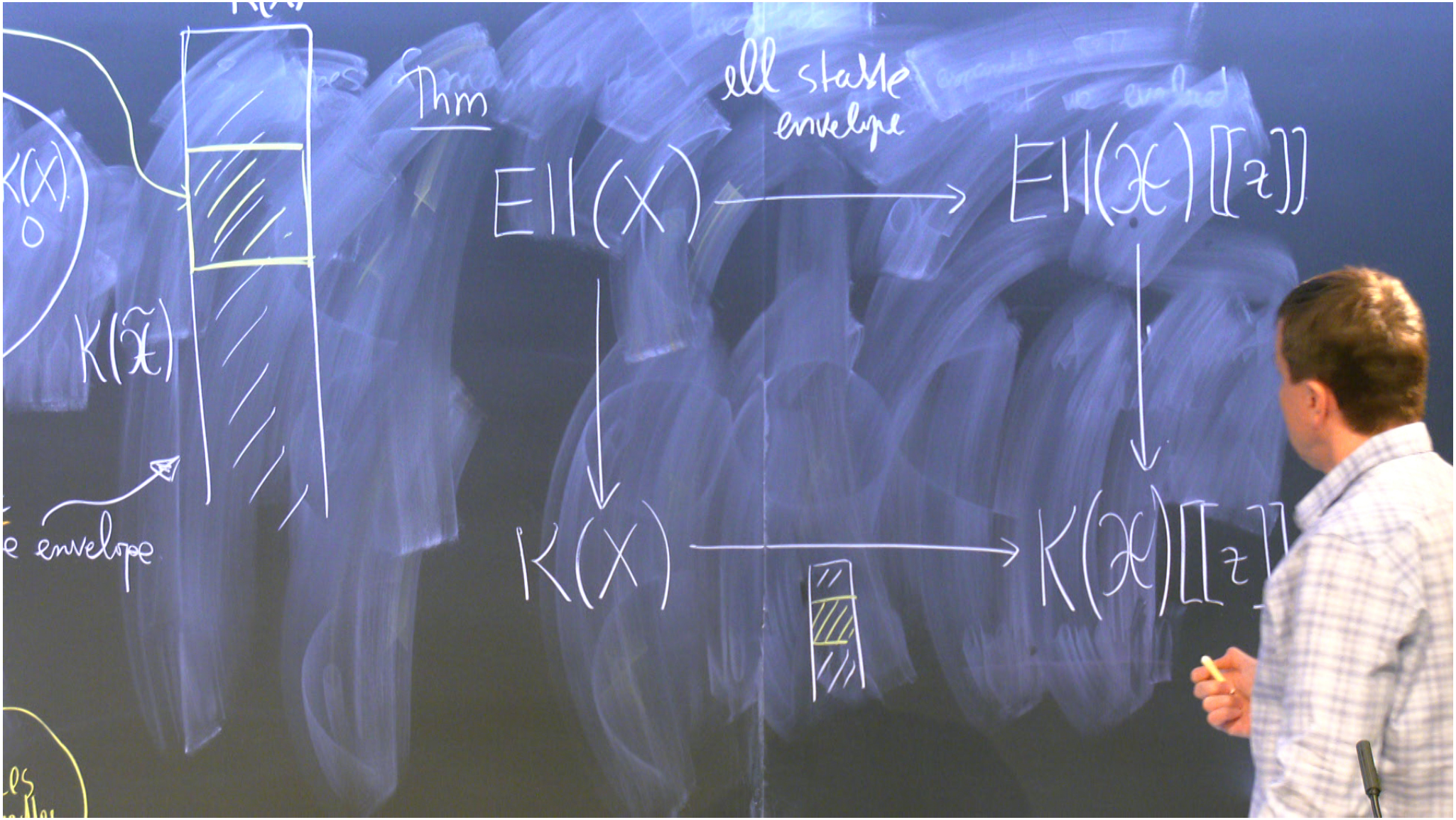

[Bethe eigenfunctions]
K(X)
operator from K(X) to itself
K(X)
K(X)
K(X̃)
fundamental solution of some
q-diff equations in all variables
(both Kähler & equiv)
qKZ-like

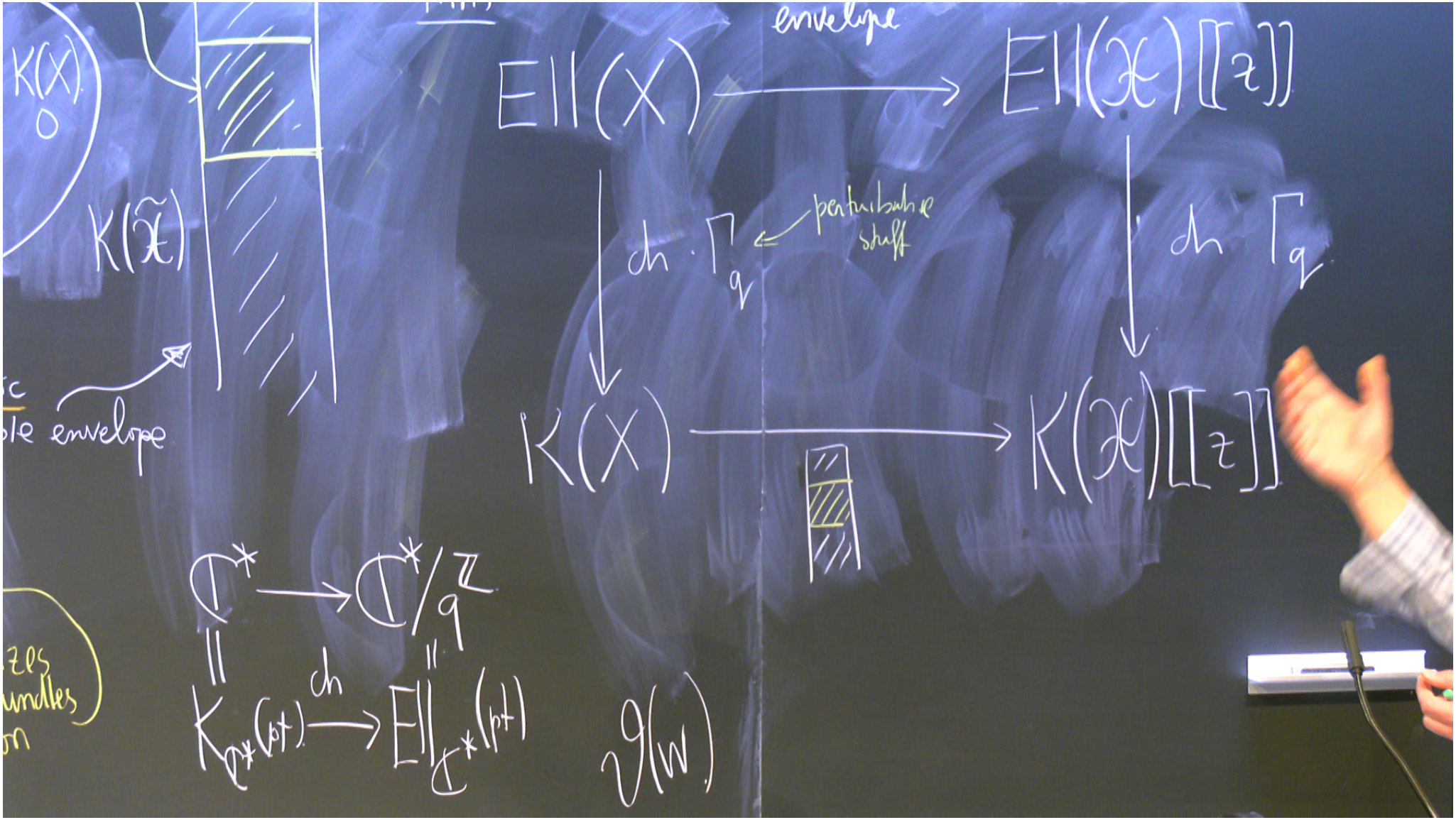

[Bethe eigenfunctions]
K(X)
operator from K(X) to itself
K(x)
K(X)
K(x̃)
fundamental solution of some
q-diff equations in all variables
(both Kähler & equiv)
qKZ-like
elliptic
Stable envelope

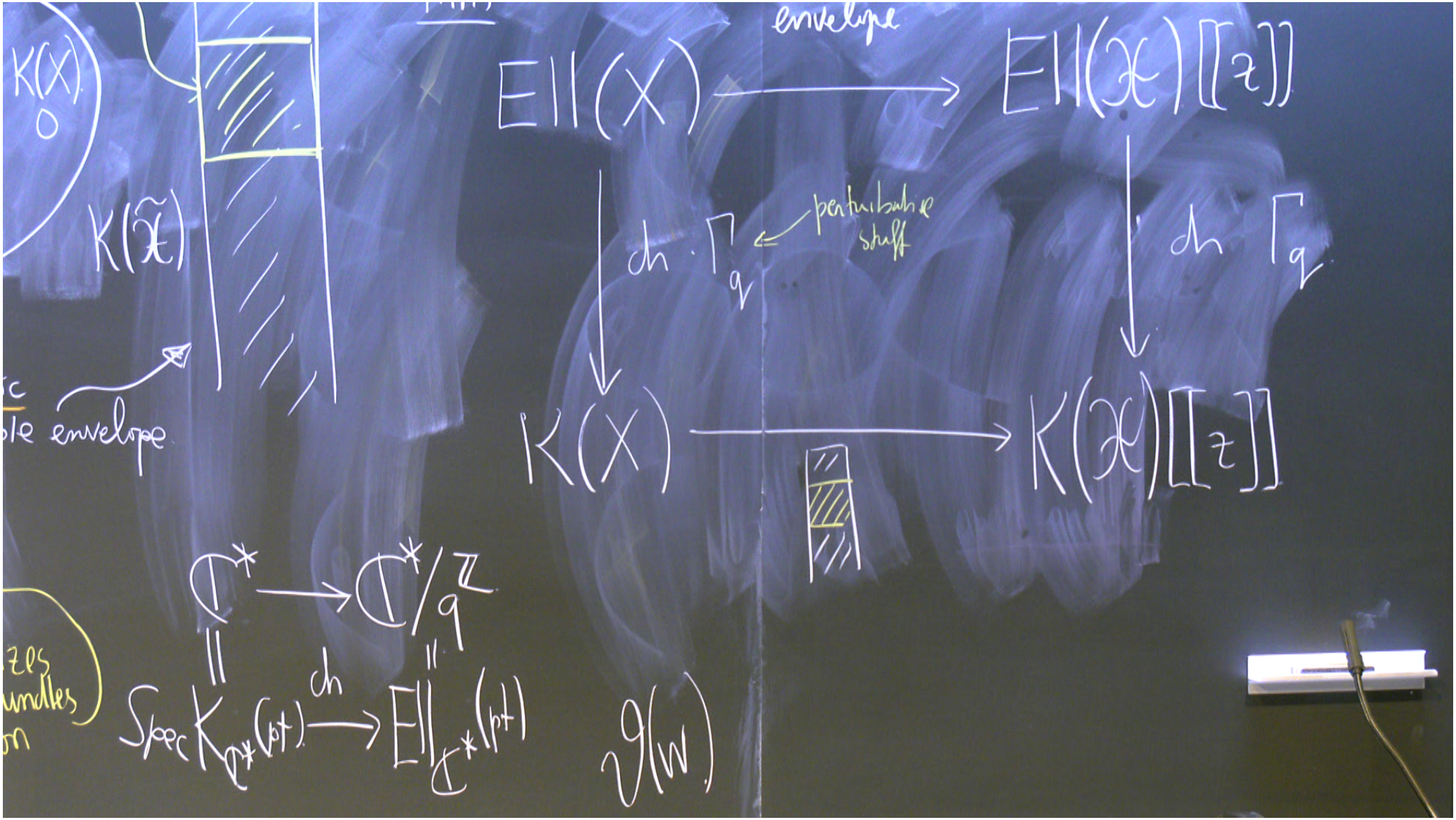

operator from K(X) to itself
fundamental solution of some
q-diff equations in all variables
(both Kähler & equiv)
quantum
loop groups
qKZ-like
F(qz) = M(z) F(z)
|q| < 1
clutching
function
for V
V
F
is a
section
E = C*/qZ

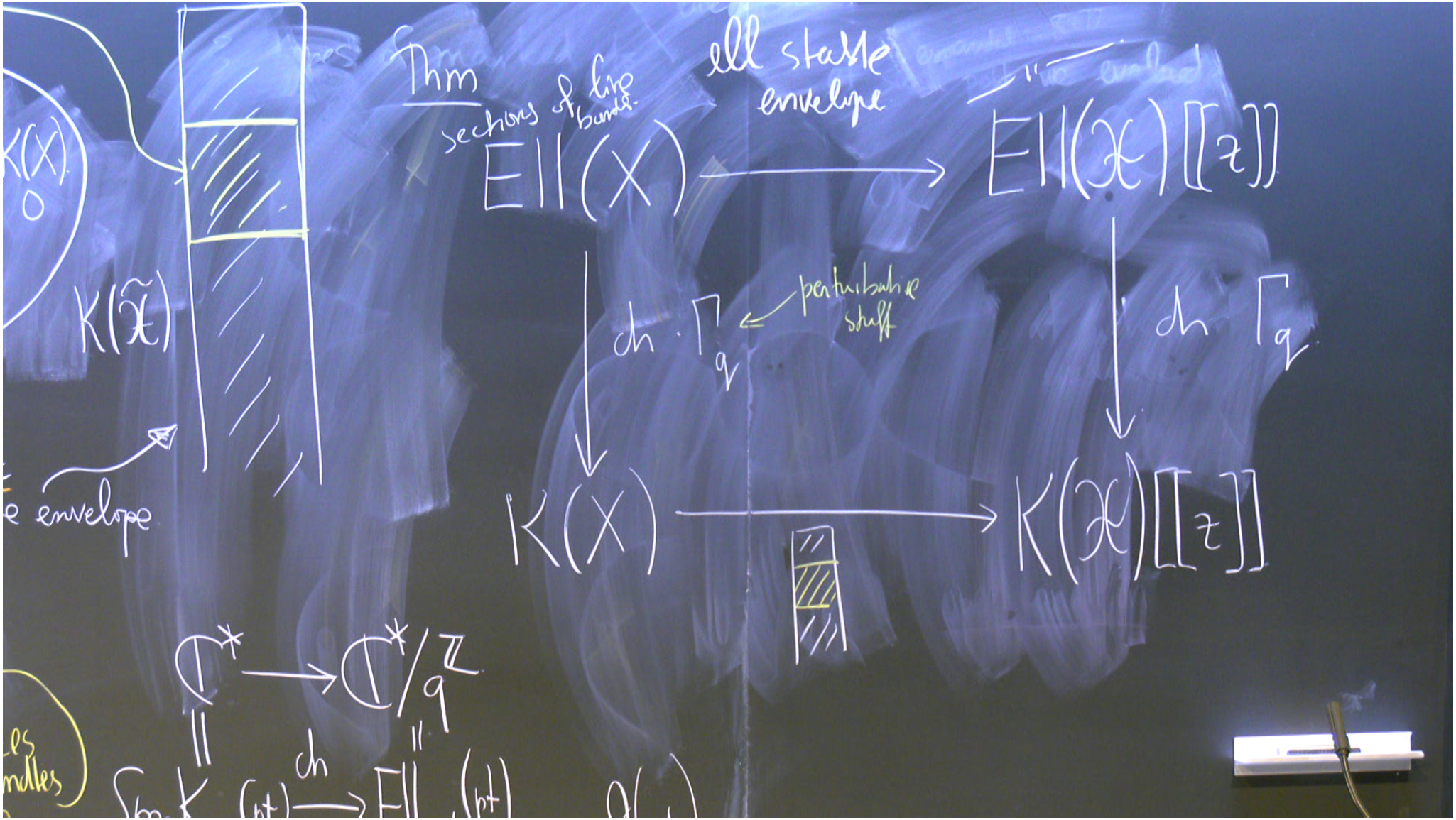

q
operator from K(X) to itself
fundamental solution of some
q-diff equations in all variables
(both Kähler & equiv)
quantum
loop groups
qKZ-like
V
F
is a
section
F(qz) = M(z)F(z)
|q|<1
of rank
equal to the K(X)
clutching
function
for V
qz
z
E = ℂ*/qℤ
part of Ell_Gr(pt) × (Pic X ⊗ E)
q mod q_f
z mod q_f


operator from K(X) to itself
fundamental solution of some
q-diff equations in all variables
(both Kähler & equiv)
qKZ-like
V
F
is a section
Ell_GF(X)
elliptic
stable envel
E = C*/qZ
clutching
functions
for V
qz
parametrizes
line bundles
on
part of Ell_GF(pt) × (Pic X ⊗ E)
q mod q_F
Z mod q_F

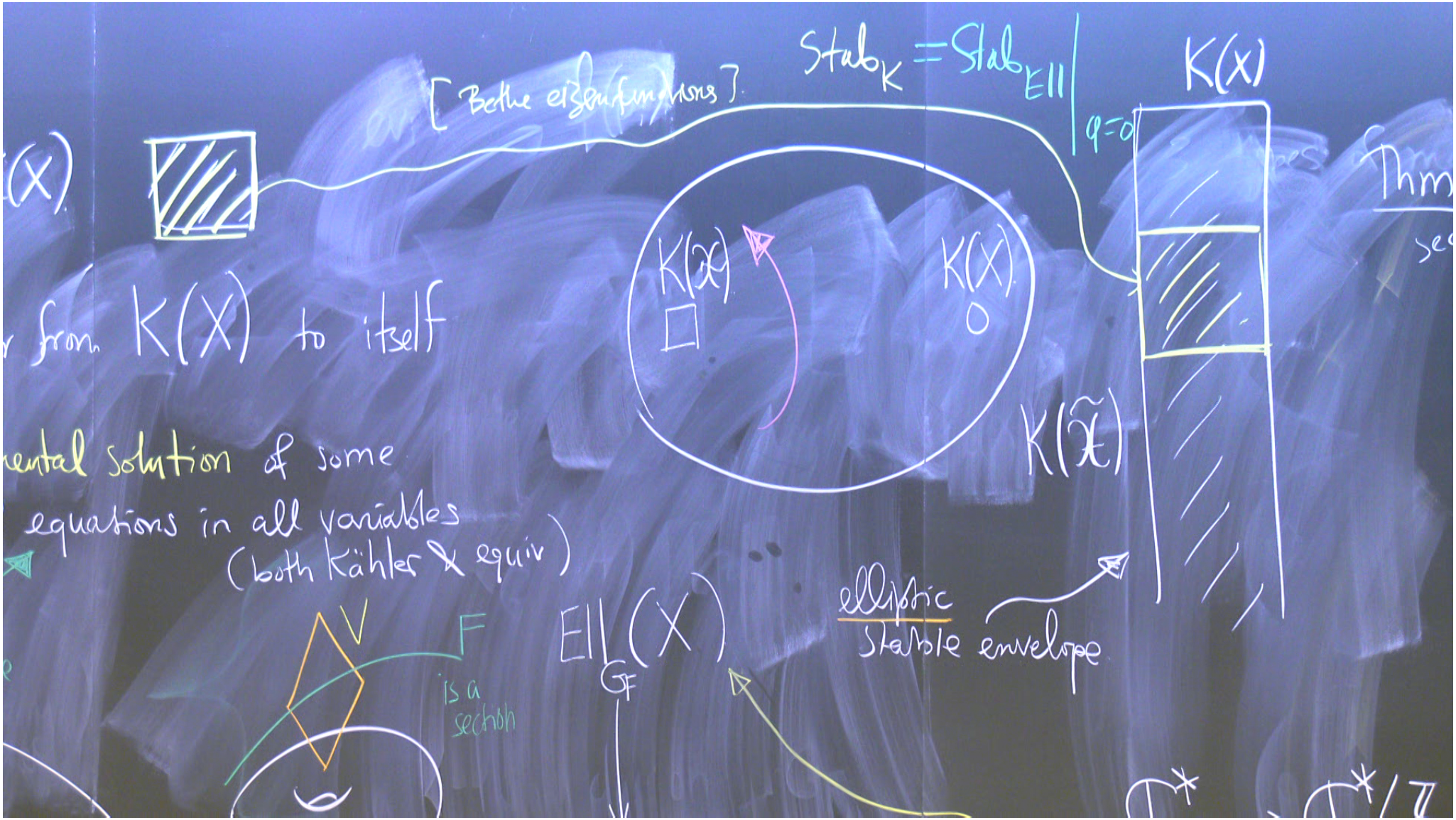

Thm
ell stable envelope
Ell(X)
Ell(X)[[z]]
K(X)
K(X)[[z]]
K(X̃)
envelope

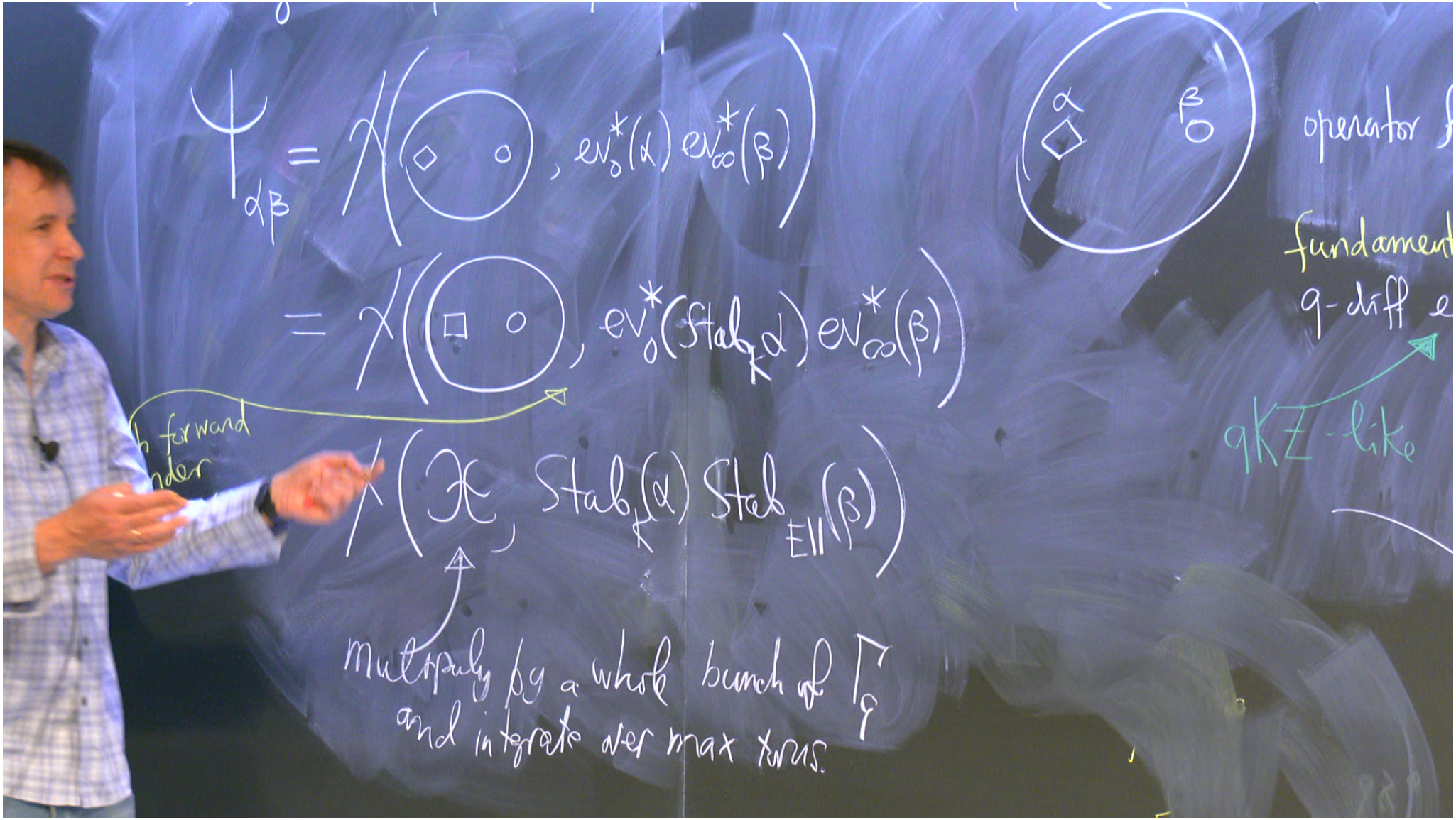

envelope
perturbative shift
ch
envelope

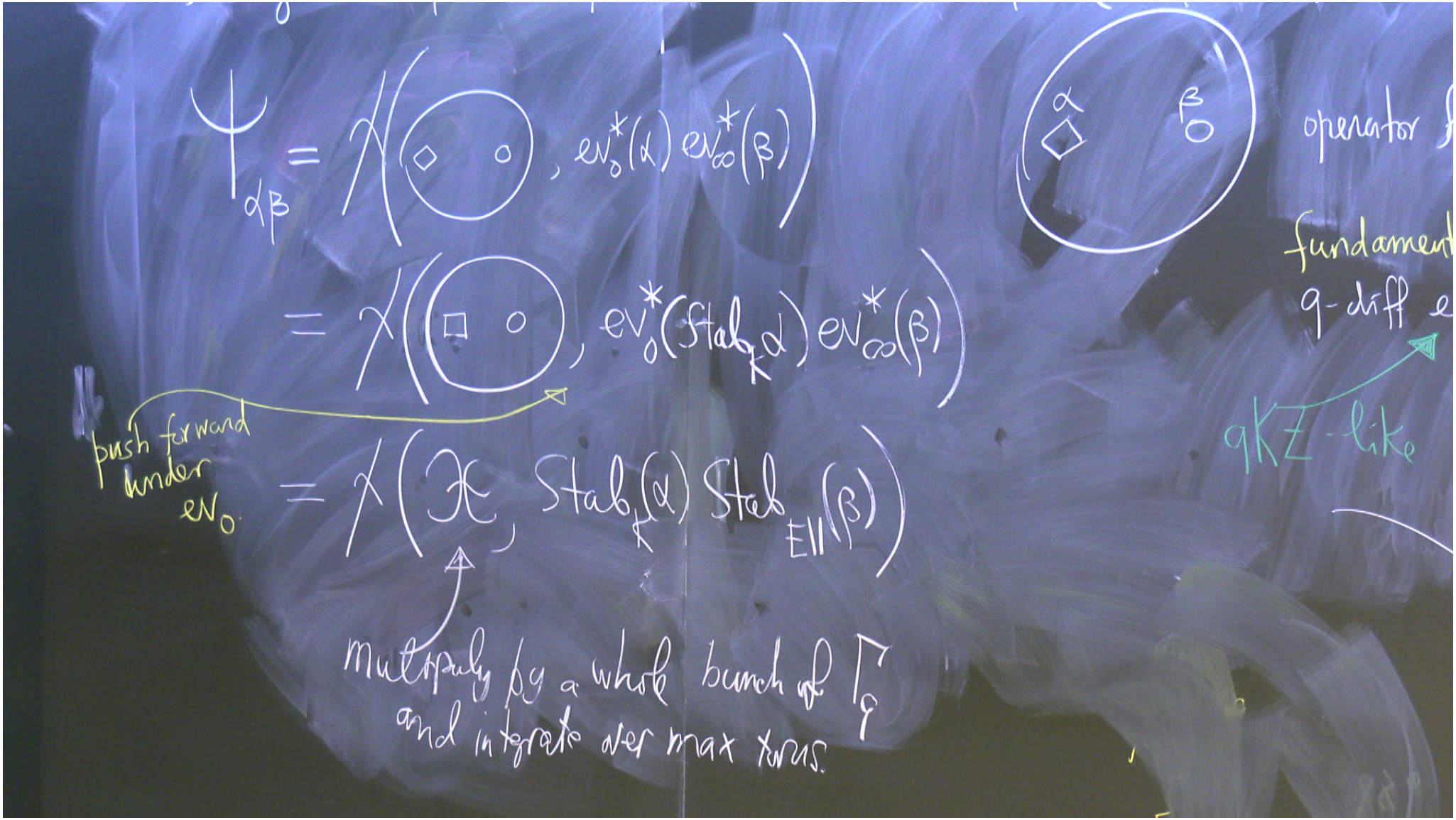

envelope
perturbative shift
ch
envelope

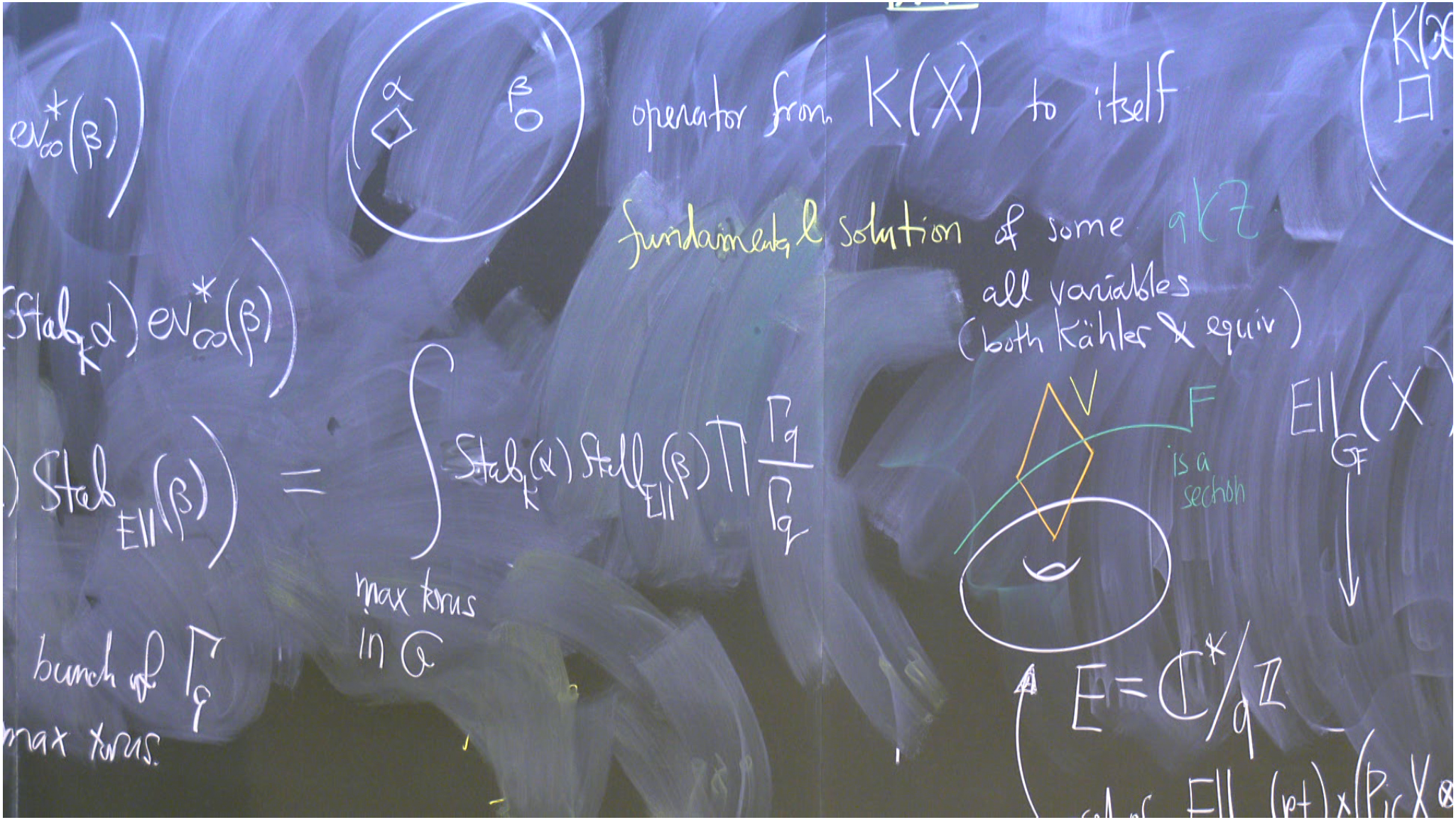

Thm
sections of line bundl.
ell stable envelope
Ell(X)
Ell(X)[[z]]
ch · Γ_q
perturbative stuff
ch Γ_q
K(X)
K(X)[[z]]
K(X̃)
envelope
ℂ*
ℂ*/q^ℤ
ch

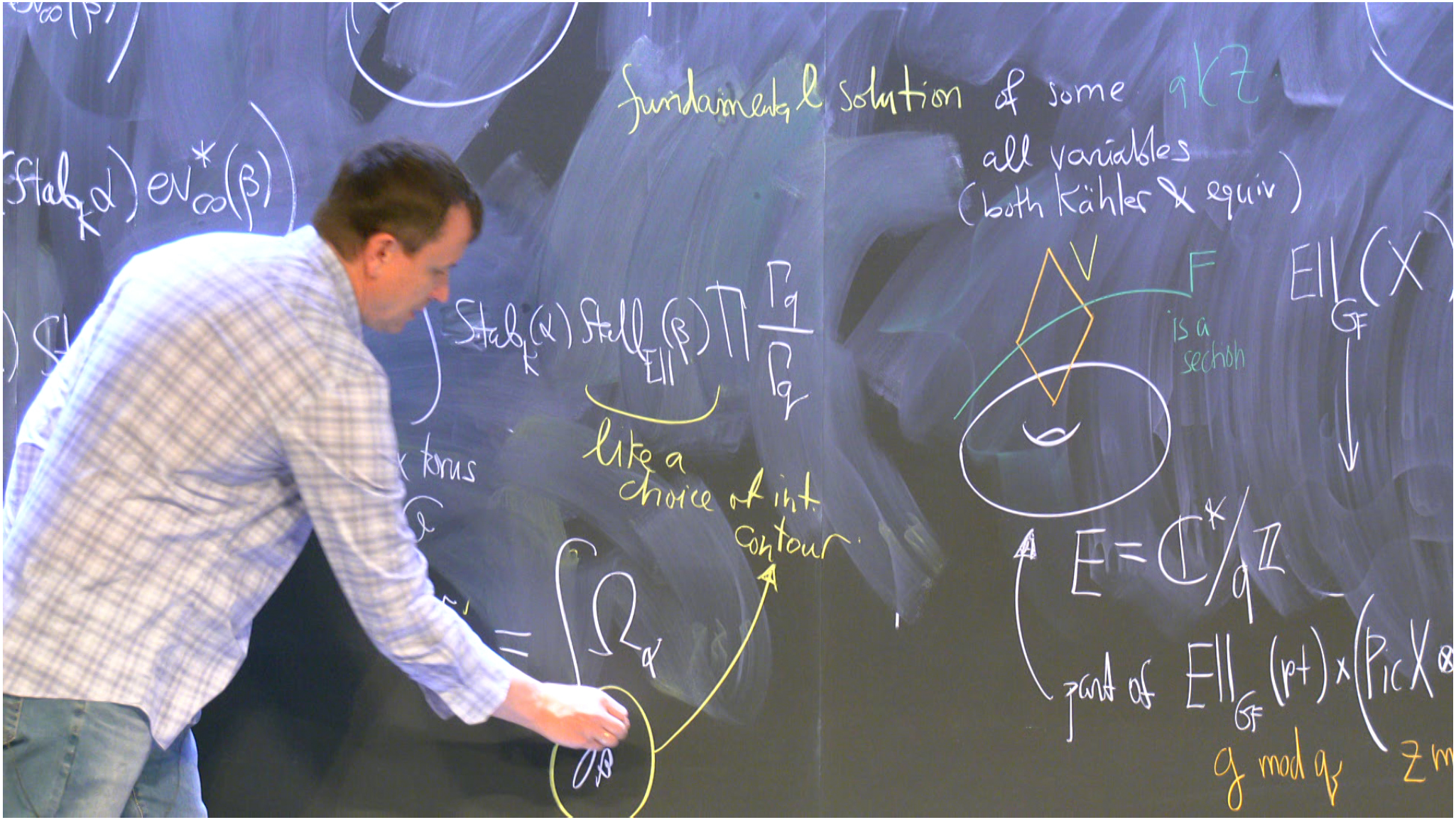

Solving the qKZ eq
$\Psi = (\Psi_{\alpha\beta})$ $\alpha,\beta \in K(X)$
$\Psi_{\alpha\beta} = \gamma($, $ev_0^*(\alpha)\, ev_\infty^*(\beta))$
α
β
operator from K(
fundamental soluti
q-diff equations
qKZ-like

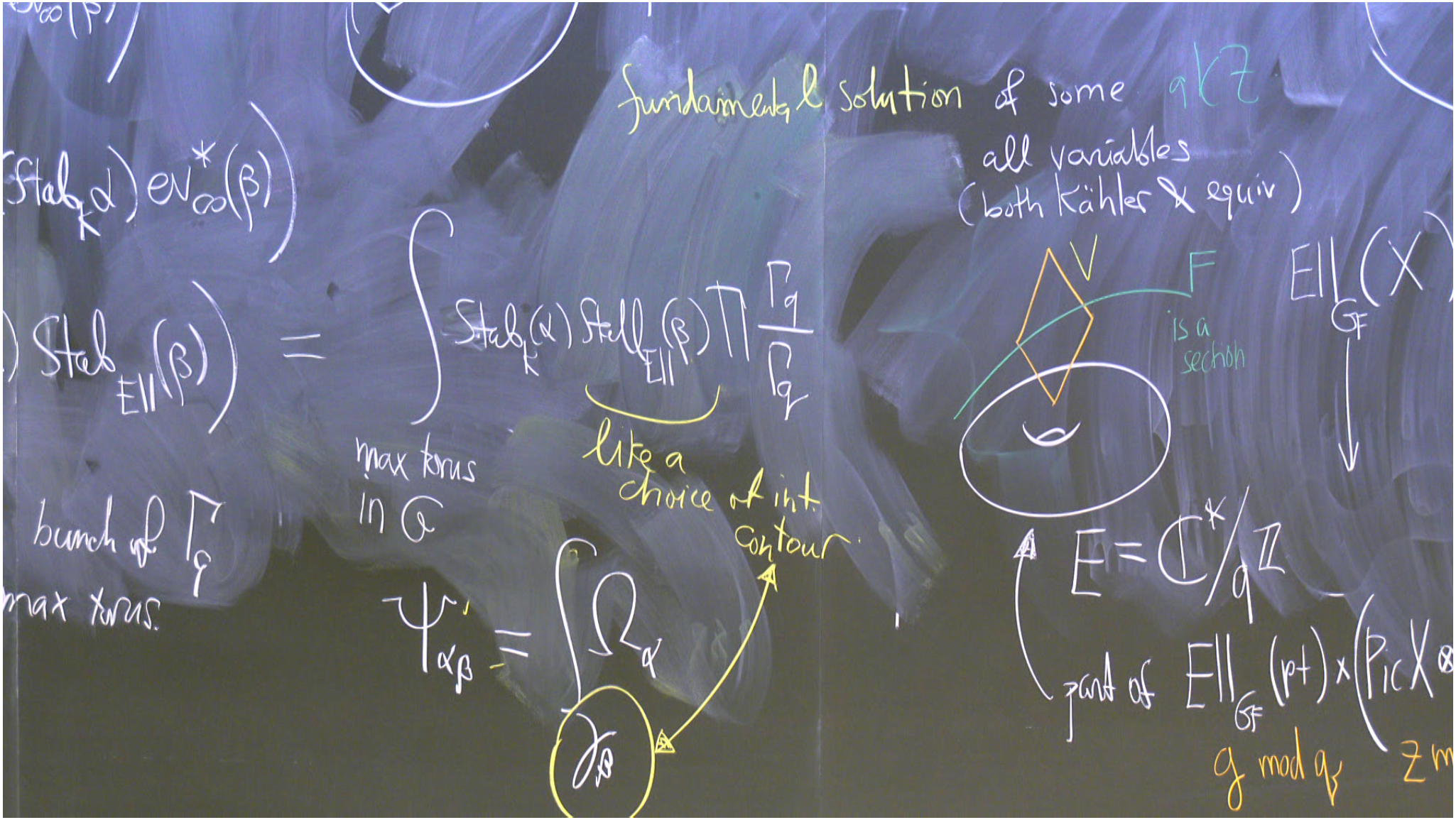

[Bethe eigenfunctions]
$Stab_K = Stab_{Ell}$
$q=0$
$K(X)$
$K(\mathcal{X})$
$K(\tilde{\mathcal{X}})$
from $K(X)$ to itself
equations in all variables
(both Kähler & equiv)
V
F
is a section
$Ell_G(X)$
elliptic
Stable envelope

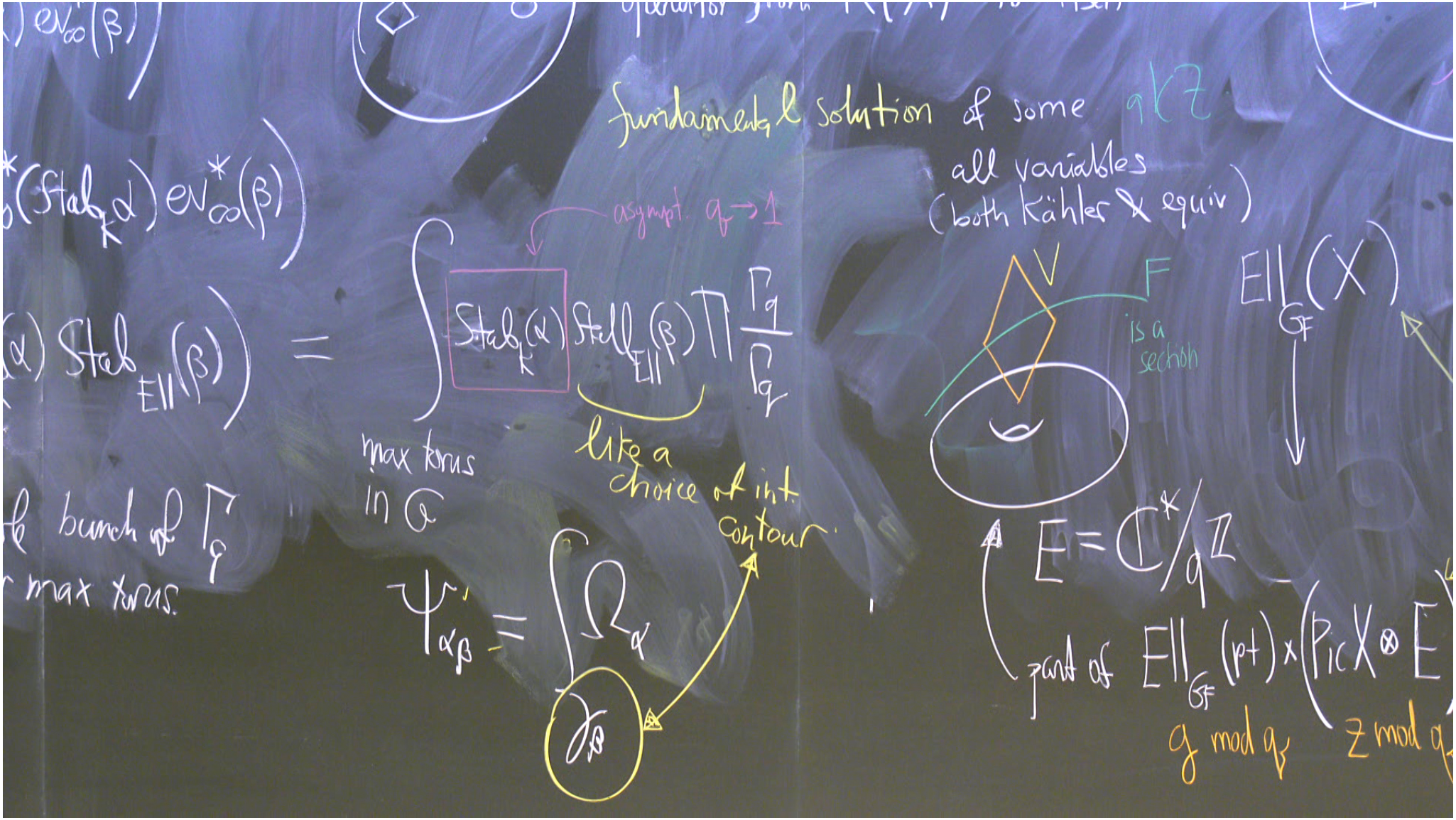

operator
fundamental
q-diff
qKZ-like
multiply by a whole bunch of Γ
and integrate over max torus.

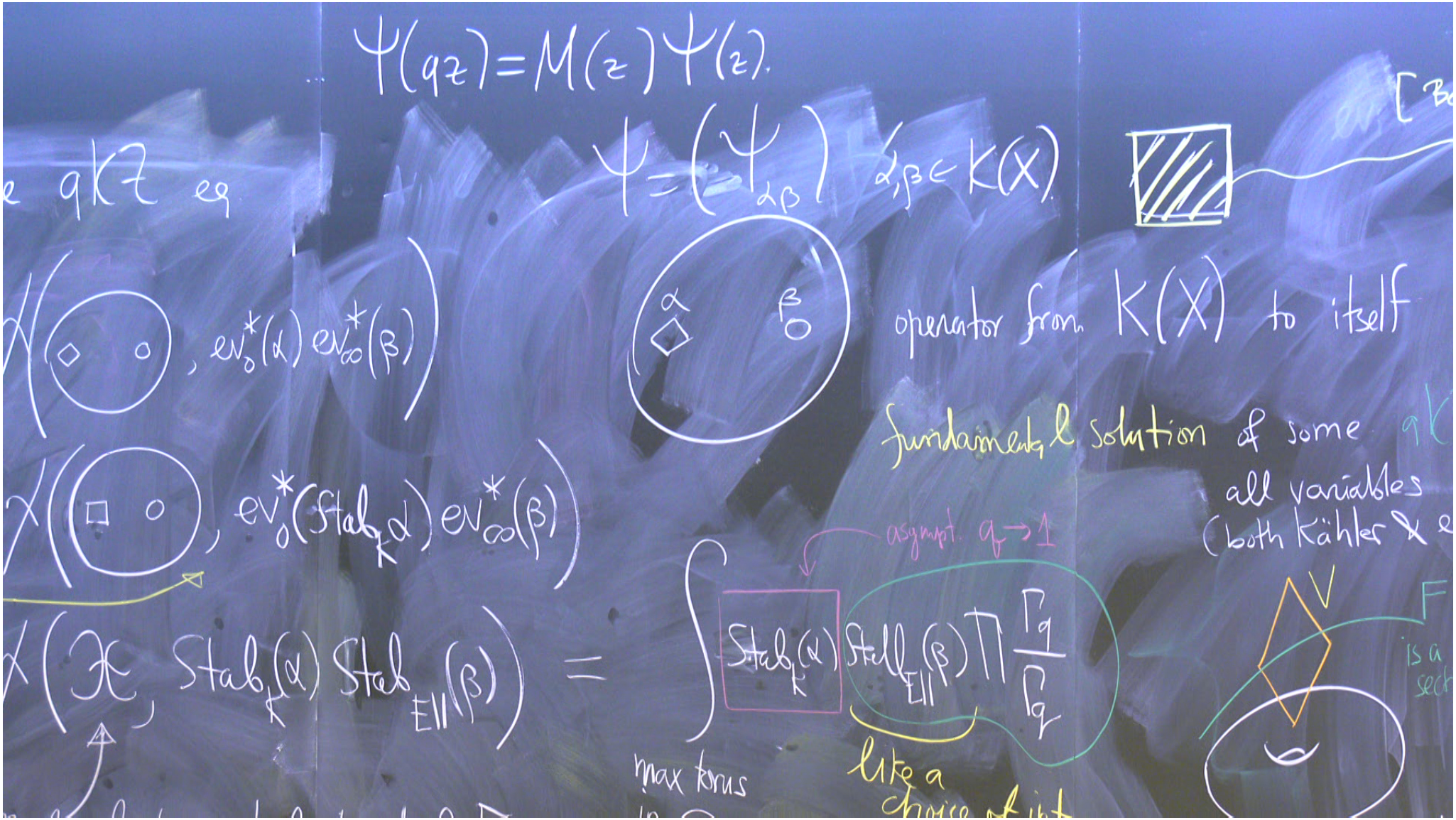

$\Psi_{\alpha\beta} = \chi\left(\diamond \; \circ , ev_0^*(\alpha)\, ev_\infty^*(\beta) \right)$
$= \chi\left(\square \; \circ , ev_0^*(\mathrm{Stab}_K \alpha)\, ev_\infty^*(\beta) \right)$
$= \chi\left(\mathcal{X}, \mathrm{Stab}_K(\alpha)\, \mathrm{Stab}_{E\Pi}(\beta) \right)$
push forward under ev_0
multiply by a whole bunch of Γ_ξ and integrate over max torus
α
β
operator
fundament
q-diff e
qKZ-like

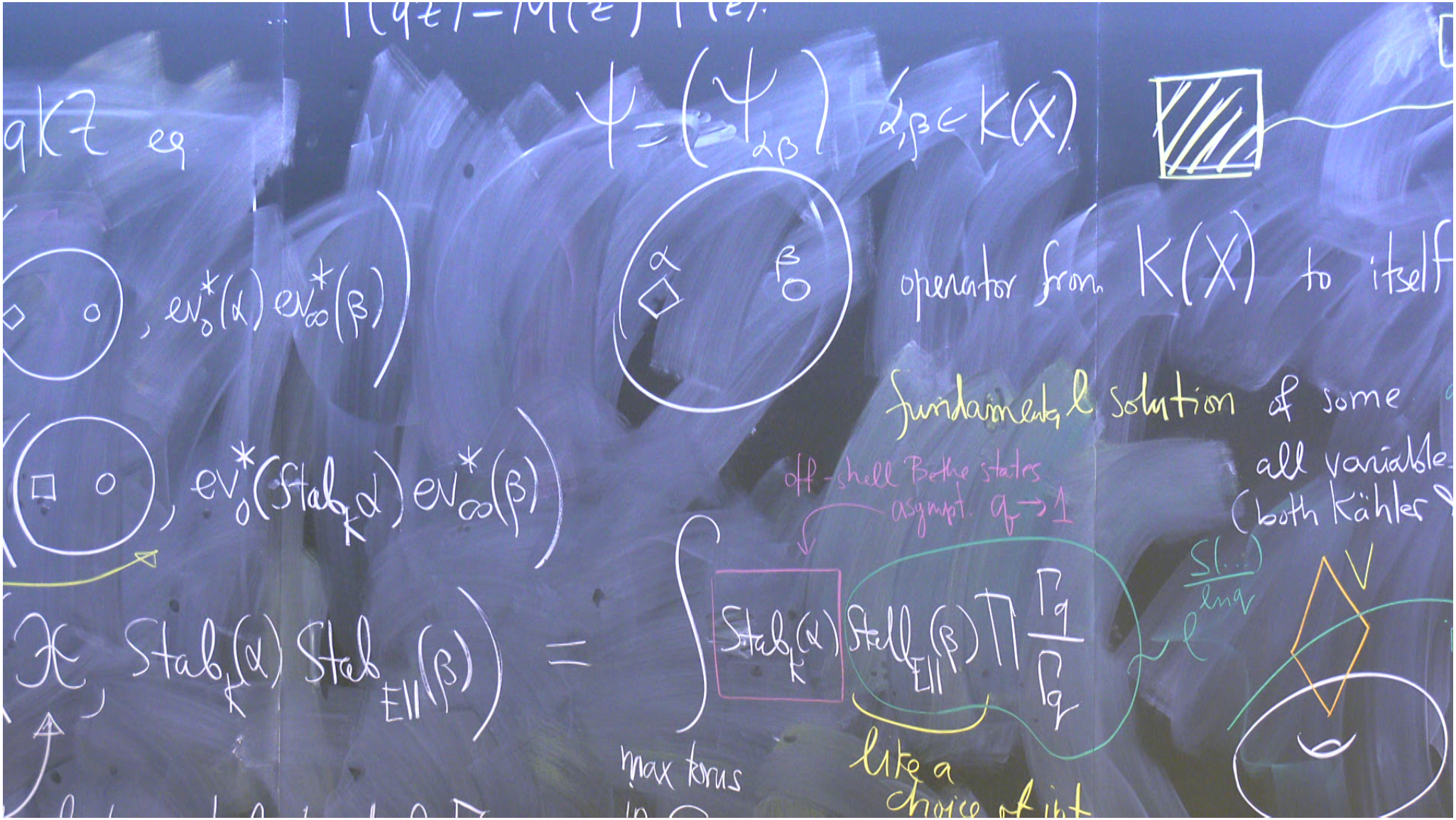

operator from K(X) to itself
fundamental solution of some qKZ
all variables
(both Kähler & equiv)
is a section
max torus
in G
bunch of Γ_ξ
max torus.
E = C*/qZ

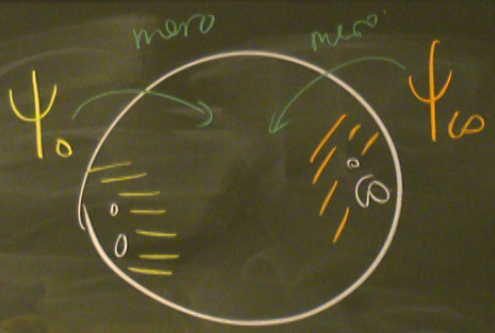

fundamental solution of some qKZ
all variables
(both Kähler & equiv)
V
F
is a section
like a choice of int. contour
E = ℂ*/qℤ
q mod q


fundamental solution of some
all variables
(both Kähler & equiv)
max torus
in G
like a
choice of int.
contour
bunch of
max torus
is a
section
V
F
E
is a point of
mod


fundamental solution of some qKZ
all variables
(both Kähler & equiv)
asympt. $q_f \to 1$
max torus in G
like a choice of int. contour
V
F
is a section
$E = \mathbb{C}^*/q^{\mathbb{Z}}$
part of
g mod q_f
Z mod q_f


$\Psi(qz) = M(z)\Psi(z)$
$\Psi = (\Psi_{\alpha\beta})$ $\alpha,\beta \in K(X)$
operator from $K(X)$ to itself
fundamental solution of some
all variables
(both Kähler
$\chi(\mathcal{X}, ev_0^*(\alpha)\, ev_\infty^*(\beta))$
$\chi(\mathcal{X}, ev_0^*(Stab_K\alpha)\, ev_\infty^*(\beta))$
$\chi(\mathcal{X}, Stab_K(\alpha)\, Stab_{E\|}(\beta)) = \int Stab_K(\alpha)\, Stab_{E\|}(\beta) \prod \frac{\Gamma_q}{\Gamma_{qr}}$
asympt. $q \to 1$
max torus
like a
V
F
is a


qKZ eq
operator from K(X) to itself
fundamental solution of some
off-shell Bethe states
asympt. $q_r \to 1$
all variable
(both Kähler
max torus
like a

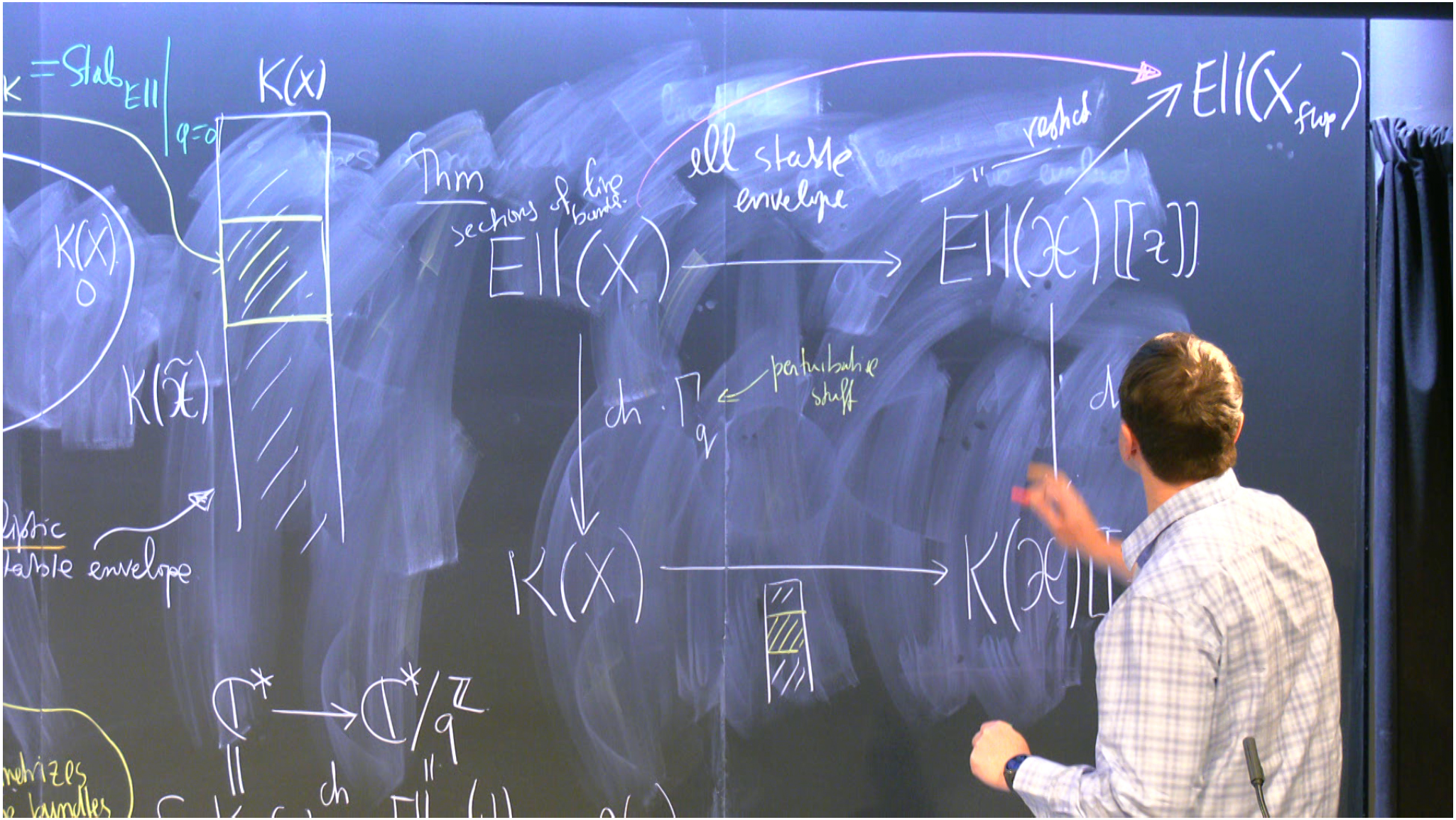


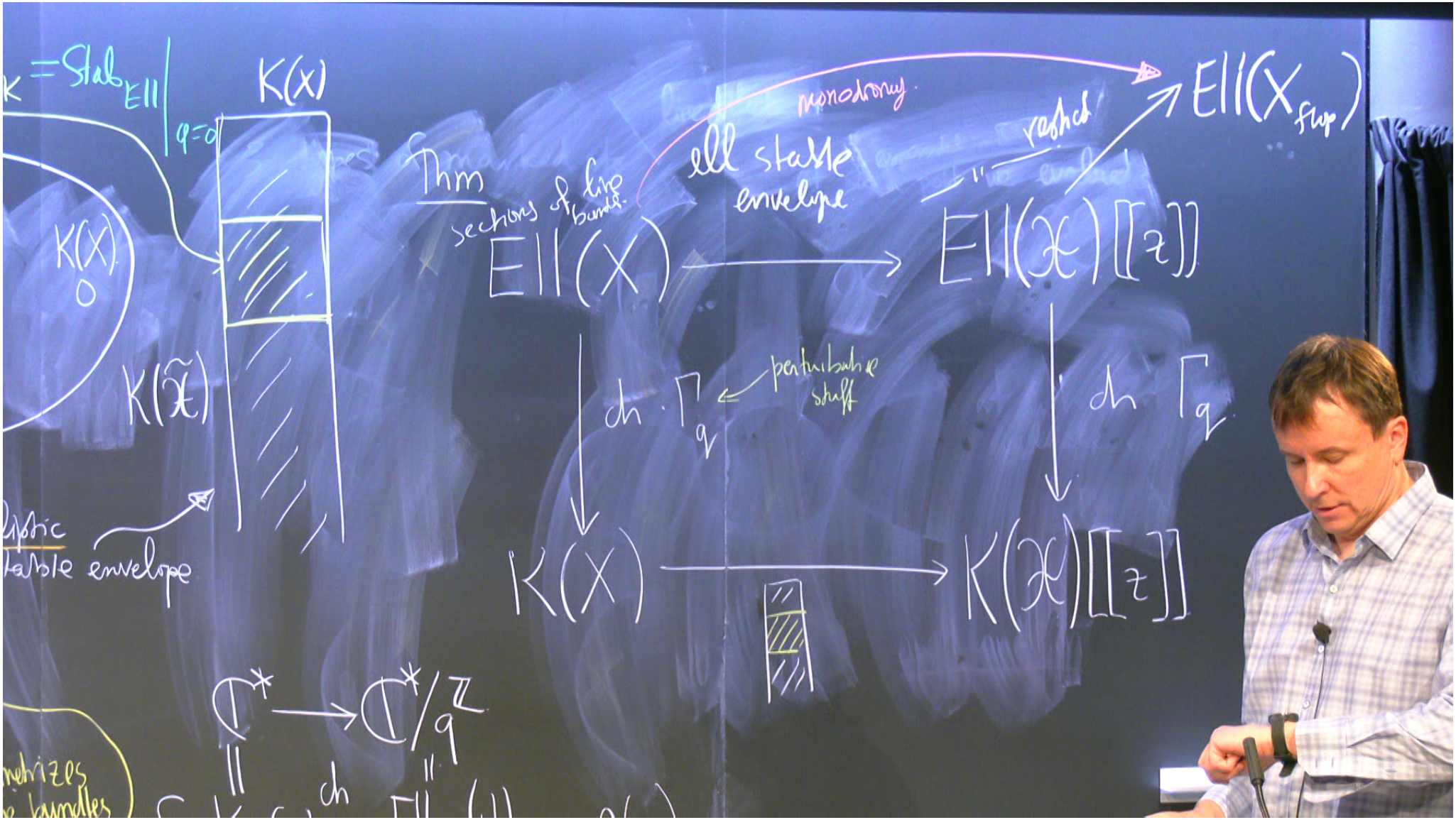
$$\Psi(qz) = M(z)\Psi(z)$$

$$M(0), M(\infty) \in GL(n)$$

$$\text{Monodromy} = \Psi_\infty^{-1}\Psi_0$$

q-periodic in z


K = Stab_Ell
K(X)
q=0
K(X)
0
K(X̃)
Thm
sections of line bundle
Ell(X)
ell stable envelope
reduced
Ell(X_flop)
Ell(X)[[z]]
ch · Γ_q
perturbative stuff
elliptic
stable envelope
K(X)
K(X)[
Γ*
Γ*/q^Z
ch
Γ
metrizes
e bundles


K = Stab_Ell
q=0
K(X)
K(X)
K(X̃)
Thm
sections of line bundle
Ell(X)
ell stable envelope
monodromy
reshd
Ell(X_flop)
Ell(X)[[z]]
ch · Γ_q
perturbative stuff
ch Γ_q
K(X)
K(X)[[z]]
elliptic
stable envelope
Γ*
Γ*/q^Z
ch
Γ^II