Title: 3d B Models and Knot Homology I

Speakers: Lev Rozansky

Collection: QFT for Mathematicians 2022

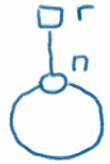
Date: June 21, 2022 - 11:00 AM

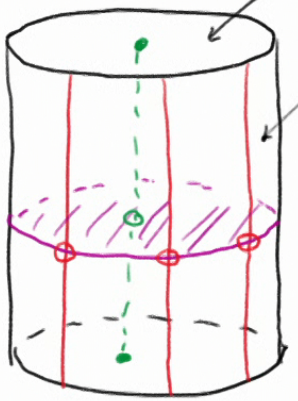
URL: https://pirsa.org/22060067

Matrix Factorizations v2

Sunday, June 19, 2022 10:30 AM

Setup (resolved conifold side)

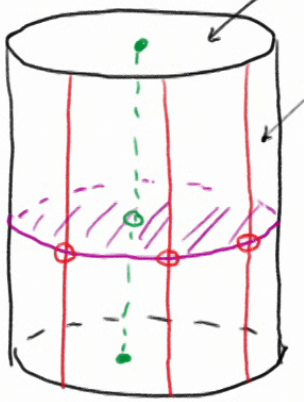
B-twisted 3d $N=4$ with target  on $\mathbb{D}^2 \times \mathbb{R}_{cob}$



boundary condition of NS-type

| – boundary line defects $\rightsquigarrow$ affine braids on n strands

| – bulk line defect carrying $\Lambda^\bullet T \leftarrow$ tautological bundle

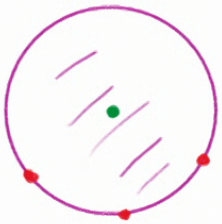


B-twisted 3d $N=4$ with target  on $D^2 \times \mathbb{R}_{cob}$

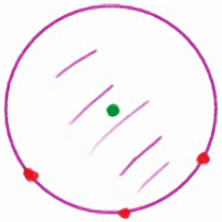
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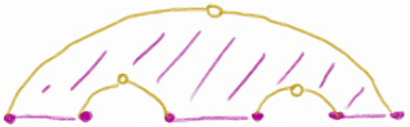
Goal: find the space of states of 

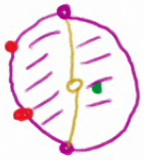
Method: cut and paste

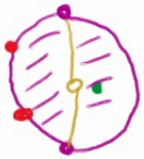
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Method: Cut and paste

 - a 2-category,  - an object,  - a cat. of morphisms (monoidal),

 - an object in the cat. of morphisms,  - monoidal structure

 - the space of homs



- the space of homs

Plan:

1. The category of matrix factorizations
2. The 2-category of $\mathfrak{X}^{sm} = T^*\mathfrak{X}$
3. Application: link homology from $Hilb_n(\mathbb{C}^2)$
4. Interfaces between quiver varieties

The category of matrix factorizations

if $\{a,b\} = 0$

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The category of matrix factorizations

Reminders • $(a+b)^2 = a^2 + b^2 + \underbrace{\{a,b\}}_{ab+ba} = a^2 + b^2$ if $\{a,b\} = 0$

• super-rules (a homological $\mathbb{Z}$ or only $\mathbb{Z}_2$ grading)

the super-commutator: $[a,b] = ab - (-1)^{\deg a \cdot \deg b} ba$

the super product rule: $d(ab) = (da)b + (-1)^{\deg a} a(db)$

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$Tr_V [A,B] = 0$

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A homotopy category $Kom(\mathcal{A})$ "DG-style"

$\mathcal{A}$ - an additive category, e.g. $\mathcal{A} = \mathbb{C}[\underline{x}]$-mod

$x_1, \dots, x_n$

An object of $Kom(\mathcal{A})$ is a pair $\hat{M} = (M, D_M)$

$M = \bigoplus_i M^i$, $D_M \in End_{\mathcal{A}}(M)$, $\deg D_M = 1$, $\boxed{D_M^2 = \mathbb{1}_M}$

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For $\hat{M}$ and $\hat{N}$ consider $Hom_{\mathcal{A}}(M, N) \circlearrowright d$

$df = D_N \circ f - (-1)^{\deg f} f \circ D_M = [\tilde{D}, \tilde{f}]$

if we present $\tilde{f} = \begin{bmatrix} M & N \\ 0 & 0 \\ f & 0 \end{bmatrix}$, $\tilde{D} = \begin{bmatrix} D_M & 0 \\ 0 & D_N \end{bmatrix}$

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Check: $\boxed{d^2 = 0}$, because $[\tilde{D}, [\tilde{D}, \tilde{f}]] = [\underbrace{\tilde{D}^2}_{0}, \tilde{f}] = 0$

Now define $\mathrm{Hom}_{\mathrm{kom}(\mathcal{A})}(\hat{M}, \hat{N}) = H_d(\mathrm{Hom}_{\mathcal{A}}(M, N))$

- $df = 0$ - f is a chain map
- $f = dg$ - f is homotopically trivial

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The derived category $D(\mathbb{C}[x]\text{-mod})$: require that M is a free $\mathbb{C}[x]$-module

Matrix factorizations (all you need to know)

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$MF(\mathcal{X}; W)$ is the right version of $D\,Coh(\underline{Crit\ W})$

$\mathcal{X}$: a variety

W: $\mathbb{C}[\mathcal{X}]$

$Crit\ W$: the critical locus: $dW = 0$

To simplify, take $\mathcal{X} = \mathbb{C}^n_{\underline{x}}$

$\hat{M} = (M, D_M)$ is the same, except:

- only $\mathbb{Z}_2$-grading: $M = M^{(0)} \oplus M^{(1)}$
- $\boxed{D_M^2 = W \cdot \mathbb{1}_M}$

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Finally require M to be free, then $\mathrm{rk}\, M^{(0)} = \mathrm{rk}\, M^{(1)}$

and D_{01}, D_{10} are square matrices.

Example $\mathbb{C}[x]$, $W = x^N$, $\mathbb{C}[x] \underset{x^{N-n}}{\overset{x^n}{\rightleftarrows}} \mathbb{C}[x]$

- $MF(\underline{x}, W)$ is a deformation of $D(\mathbb{C}[x]\text{-mod})$ by $W \in HH^0(\mathbb{C}[x]) = \mathbb{C}[x]$

D_{10}

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- Critical locus

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Apply $\partial/\partial x_i$ to $D^2 = W$:

$$[D, \partial D/\partial x_i] = \partial W/\partial x_i \leftarrow \text{homologically trivial}$$

Hence $MF(\underline{x}, W)$ is supported in the formal neighb. of $Crit(W)$

- The tensor product over $\mathbb{C}[x]$

$$MF(\underline{x}, W_1) \times MF(\underline{x}, W_2) \xrightarrow{\otimes} MF(\underline{x}, W_1 + W_2)$$

the super-potentials add

because $(M, D) \otimes (M, D) = (M \otimes M, D \otimes 1 + (-1)^F 1 \otimes D)$

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because $(M_1; D_1) \otimes (M_2; D_2) = (M_1 \otimes M_2;\ \underbrace{D_1 \otimes 1\!\!1}_{\tilde{D}_1} + \underbrace{(-1)^F 1\!\!1 \otimes D_2}_{\tilde{D}_2})$

$\{\tilde{D}_1, \tilde{D}_2\} = 0 \Rightarrow (\tilde{D}_1 + \tilde{D}_2)^2 = \tilde{D}_1^2 + \tilde{D}_2^2$

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- The dual matrix factorization: for $M = M^{(0)} \underset{D_{01}}{\overset{D_{10}}{\rightleftarrows}} M^{(1)}$ in $MF(x; W)$

define $M^\vee = M^{(0)\vee} \underset{-D_{01}^\vee}{\overset{D_{10}^\vee}{\rightleftarrows}} M^{(1)\vee}$ in $MF(x; \underbrace{-W}_{\text{negative}})$

Then $\boxed{\mathrm{Hom}_{MF}(M, N) = H_{D_{tot}}(\underbrace{M^\vee \otimes N}_{W_{tot}=0})}$

- Koszul matrix factorizations

$-D_{0,1}^{\vee}$ negative

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- Koszul matrix factorizations

A Koszul complex: $\mathbb{C}[\underline{x}]/p(\underline{x}) \sim K(p) = \left(\mathbb{C}[\underline{x}] \xrightarrow{p} \mathbb{C}[\underline{x}]\right) = \left(\mathbb{C}[\underline{x},\theta], d = \theta\cdot p\right)$ (θ: odd, $\theta^2 = 0$)

More generally: $\mathbb{C}[\underline{x}]/(p_1,\dots,p_m) \rightsquigarrow \bigotimes_{i=1}^{m} K(p_i) = \left(\mathbb{C}[\underline{x},\underline{\theta}], d = \sum \theta_i p_i\right)$

For $W = pq$ a Koszul matrix factorization $\boxed{K(p;q) = \mathbb{C}[\underline{x}] \underset{q}{\overset{p}{\rightleftarrows}} \mathbb{C}[\underline{x}]}$

$= \left(\mathbb{C}[\underline{x},\theta], D = p\theta + q\frac{\partial}{\partial\theta}\right)$

- **Koszul matrix factorizations**

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Remark $K(p,q)[1] = K(q,p)$

For $W = \sum p_i q_i$

define $K(\underline{p};\underline{q}) = \bigotimes^{m} K(p_i;q_i) = \left(\mathbb{C}[\underline{x},\underline{\theta}]; D = \sum p_i\theta_i + q_i\frac{\partial}{\partial\theta_i}\right)$

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- **Row operations**

$$\begin{pmatrix} p_1 & q_1 \\ p_2 & q_2 \end{pmatrix} \cong \begin{pmatrix} p_1 & q_1 - r q_2 \\ p_2 + r p_1 & q_2 \end{pmatrix}, \quad r \in \mathbb{C}[x] \qquad \begin{cases} \theta_1 = \tilde{\theta}_1 + r\tilde{\theta}_2 \\ \theta_2 = \tilde{\theta}_2 \end{cases}$$

Example 1 In $MF(x; x^N)$, $n \le N/2$

$$End\left(K(x^n; x^{N-n})\right) \cong K\begin{pmatrix} x^n & x^{N-n} \\ x^{N-n} & -x^n \end{pmatrix} \cong K\begin{pmatrix} x^n & 0 \\ x^{N-n} & 0 \end{pmatrix} = \mathbb{C}[x]/(x^$$

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Example 2 Hochschild (co-) homology

For $D(\mathbb{C}[x]\text{-mod})$ identity bimodule: $B_1 = \left(\mathbb{C}[x,y] \xrightarrow{x-y} \mathbb{C}[x,y]\right)$

$$End(B_1) = K\begin{pmatrix} x-y \\ x-y \end{pmatrix} = K\begin{pmatrix} x-y \\ 0 \end{pmatrix} = \mathbb{C}[x,\theta]$$

For $MF(x; W)$ identity (bi-) MF $B_1 \in MF(x, y; W(y) - W(x))$

$\mathrm{End}\left(K\left(\begin{pmatrix} x^{N-n} & -x^n \end{pmatrix}\right)\right) \cong K\left(\begin{pmatrix} x^{N-n} \\ 0 \end{pmatrix}\right) = \mathbb{C}[x]/(x^n) \otimes \mathbb{C}[\theta$

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For $D(\mathbb{C}[x]\text{-mod})$ identity bimodule: $B_{\mathbb{1}} = \left(\mathbb{C}[x,y] \xrightarrow{x-y} \mathbb{C}[x,y]\right)$

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$$B_{\mathbb{1}} = \left(\mathbb{C}[x,y] \underset{\frac{W(x)-W(y)}{x-y}}{\overset{x-y}{\rightleftarrows}} \mathbb{C}[x-y]\right) \quad \frac{W(x)-W(y)}{x-y} = W(x,y)$$

$$\mathrm{End}(B_{\mathbb{1}}) = \begin{pmatrix} x-y & W(x,y) \\ & \end{pmatrix} \cong \begin{pmatrix} x-y & 0 \\ & \end{pmatrix} \sim \begin{pmatrix} W'(x) & 0 \end{pmatrix} \sim \mathbb{C}[x]/(W')$$

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Prop If $\underline{p} = (p_1, \dots, p_k)$ form a regular sequence and $\sum p_i q_i = \sum p_i \tilde{q}_i$

then $K(\underline{p}; \underline{q}) \cong K(\underline{p}; \underline{\tilde{q}})$ (by row ops)

A linear super-potential vs. a derived scheme

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$$M \rightsquigarrow M \otimes \left(\mathbb{C}[a,\theta],\ d = \pm a\cdot\theta + f(\underline{x})\frac{\partial}{\partial\theta}\right)$$

$$d^2 = \pm a\cdot f(\underline{x})$$

- Knörrer periodicity $MF(\underline{x}, a, b;\ W(\underline{x}) + ab) \cong MF(\underline{x};\ W(\underline{x}))$

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$$D(\mathbb{C}[\underline{x}]/f(\underline{x})\text{-mod})$$

$$M \rightsquigarrow M \otimes \left(\mathbb{C}[a,\theta],\ d = \pm a\cdot\theta + f(\underline{x})\frac{\partial}{\partial\theta}\right)$$

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- Knörrer periodicity $MF(\underline{x}, a, b;\ W(\underline{x}) + ab) \cong MF(\underline{x};\ W(\underline{x}))$

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- $MF(\underline{x}; W)$ vs $D\,Coh^{(2)}(Crit(W))$ (2): homological $\mathbb{Z}_2$-grading

equivalent* if $Crit\ W$ is smooth and $\dfrac{\partial^2 W}{\partial x_i \partial x_j}$ is non-degenerate on $N\ Crit(W)$

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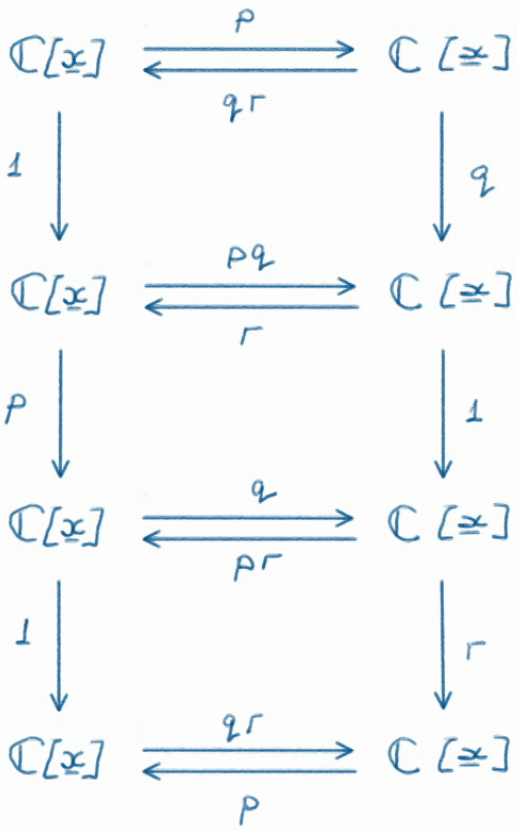
$$M \xrightarrow{f} N \rightsquigarrow (M[1] \xrightarrow{f} N),\ D = D_M + D_N + f$$

An important example: $\mathbb{C}[x]/(p) \xrightarrow{q} \mathbb{C}[x]/(pq) \xrightarrow{1} \mathbb{C}[x]/(q)$

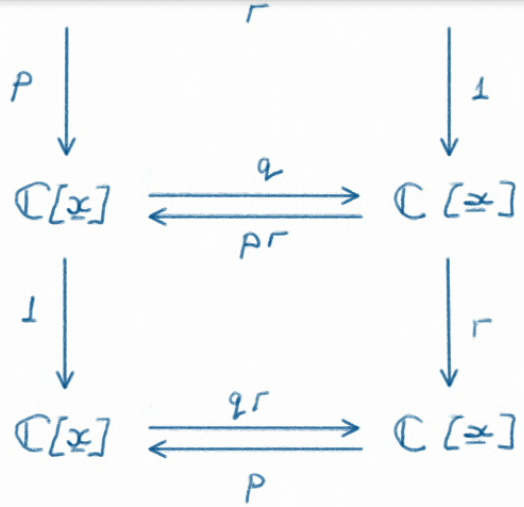
becomes in $MF(\mathbb{C}[x]; p \cdot q \cdot r)$

$$K(p, qr) \longrightarrow K(pq, r)$$
$$\nwarrow^{[1]} \quad \swarrow$$
$$K(q, pr)$$

$$\mathbb{C}[x] \xrightarrow{p} \mathbb{C}[x]$$



- How to restore the homological $\mathbb{Z}$-grading to $MF(x; W)$?



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mplectic algebraic geometry

day, June 20, 2022 9:20 PM

$\mathcal{X}^{sm} \rightsquigarrow \ddot{B}(\mathcal{X}^{sm})$ ← the 2-category of boundary conds of a 3d B-model (A. Kapustin, LR, N. Saulina)

$\mathcal{X}^{sm}$: symplectic variety

$\ddot{B}(\mathcal{X}^{sm})$: a 2-category

Naive definition

An object: $\mathcal{L} \to L \hookrightarrow \mathcal{X}^{sm}$

L: lagrangian subvariety

$L \hookrightarrow \mathcal{X}^{sm}$

lagrangian subvariety

A category of morphisms:

$$\mathrm{Hom}_{\ddot{B}}(\mathcal{I}_1, \mathcal{I}_2) = D\mathrm{Coh}\left(\mathcal{I}_1\big|_{L_{12}} \times_{L_{12}} \mathcal{I}_2\big|_{L_{12}}\right), \quad L_{12} = L_1 \cap L_2$$

This is true* if L_1, L_2 are smooth and have a "good" intersection

locally isom. to $T_p L_1 \cap T_p L_2$

But in some interesting cases the base L is singular and $\mathcal{I} \to L$ is not a bundle

$\mathcal{X}^{sm}$

subvariety

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Rigorous definition for $\mathcal{X}^{sm} = T^*\mathcal{X}$

A "simple" object is a function $W \in \mathbb{C}[\mathcal{X}]$

$$\underset{B}{Hom}(W_1, W_2) := MF(\mathcal{X}; W_2 - W_1)$$

A composition of morphisms is the tensor product:

$$MF(\mathcal{X}; W_2 - W_1) \times MF(\mathcal{X}; W_3 - W_2) \xrightarrow{\otimes} MF(\mathcal{X}; W_3 - W_1)$$

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Symplectic geometry

W determines a Lagrangian subvariety $L_W = \{\text{graph of } dW\} \subset T^*\mathcal{X}$

Ex For $\mathcal{X} = \mathbb{C}_x$, $L_W = \{y = \frac{dW}{dx}\} \subset \mathbb{C}_x \times \mathbb{C}_y$

Prop. The projection $T^*\mathcal{X} \xrightarrow{\pi} \mathcal{X}$ establishes a bijection

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Proof: $(x,y) \in L_{W_1} \cap L_{W_2} \iff W_1'(x) = W_2'(x)$

Cor: If L_1 and L_2 have a good intersection, then $Hom(W_1, W_2) \cong DCoh^{(2)}(L_1 \cap L_2)$

Example $(W = 0)$ determines $\mathfrak{X} \subset T^*\mathfrak{X}$

zero-section

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$$End(W=0) = MF(\mathcal{X}; 0) = DCoh^{(2)}(\mathcal{X}) \leftarrow \text{a monoidal category}$$

A general object of $\ddot{B}(T^*\mathcal{X})$ is a pair $(\underbrace{\mathcal{Z}}_{\text{auxiliary}}; W)$, $W \in \mathbb{C}[\mathcal{X} \times \mathcal{Z}]$

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($\mathcal{Z}$: auxiliary variety)

The category of morphisms:

$$Hom((\mathcal{Z}_1; W_1), (\mathcal{Z}_2; W_2)) = MF(\mathcal{X} \times \mathcal{Z}_1 \times \mathcal{Z}_2; W_2 - W_1)$$

Symplectic geometry

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$$\mathcal{X} \longleftarrow L$$

Example $(a;\, W(a,x))$ determines $\mathcal{L} = \{(a,x) \in \mathbb{C}_a \times T^*\mathbb{C}_x \mid y = \frac{\partial W}{\partial x},\ \frac{\partial W}{\partial a} = 0\}$

e.g. $W(a,x) = ax \rightsquigarrow \{y = a,\ x = 0\} = \underbrace{\{x=0\}}_{T^*_0\mathbb{C}_x} \subset T^*\mathbb{C}_x$

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"Symplectic" vs. "derived" algebraic geometry

A standard trick in additive cats:

for any $A \in Cat$, there is a functor $\mathcal{F}_A : Cat \longrightarrow End(A)\text{-mod}$

$B \longmapsto Hom(A,B)$

The same trick for $\ddot{B}(T^*\mathcal{X})$

$A = (W=0)$, $End(A) = (DCoh(\mathcal{X}), \otimes)$,

any $B \longmapsto Hom(A,B)$ - a $DCoh(\mathcal{X}), \otimes$-module

This functor is an equivalence of categories *

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Example $\ddot{B}(T^*\mathbb{C})$

$(a; ax) \rightsquigarrow \{x=0\} \subset T^*\mathcal{X}$

$End(a; ax) = MF(x, a, b; (b-a)x) \cong MF(a; 0) \cong D(\mathbb{C}[a]\text{-mod})$

$Hom_{\ddot{B}}((w=a); ax) = MF(x, a; ax) \cong Vect\text{-}\mathbb{C}$ ← but a module category over $D(\mathbb{C}[x]\text{-mod})$!

$\{y=0\} \cap \{x=0\} \subset \{y=0\}$ is a sky-scraper

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replace by a resolution $(\mathbb{C}[x, \theta]; d = x \cdot \partial/\partial\theta)$

Now $End(Vect\text{-}\mathbb{C}) = D\left((\mathbb{C}[x, \theta]; d = x\frac{\partial}{\partial\theta}) \otimes_{\mathbb{C}[x]} \mathbb{C}[x]/(x)\right)$

$= D(\mathbb{C}[\theta]\text{-mod}) \cong D(\mathbb{C}[a]\text{-mod})$

↑ Koszul duality

$\mathrm{Hom}_{\ddot{B}}\big((w=0); ax\big) = MF(x, a; ax) \cong Vect_{\mathbb{C}}$ ← but a module category over $D(\mathbb{C}[x]\text{-mod})$!

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An equivariant version

$\mathcal{X} \circlearrowright G \rightsquigarrow \ddot{B}(T^*\mathcal{X} /\!/\!/ G)$ as $\ddot{B}^G(T^*\mathcal{X})$

An object of $\ddot{B}^G(T^*\mathcal{X})$ is $(Z; W)$, $W \in \mathbb{C}[\mathcal{X} \times Z]^G$

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$\mathrm{Hom}_{\ddot{B}}((\mathcal{Z}_1; W_1), (\mathcal{Z}_2; W_2)) = MF^G(\mathcal{X}\times\mathcal{Z}_1\times\mathcal{Z}_2; W_2 - W_1)$

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$$(d_{\mathcal{X}} W)^{-1}\left(T^*\mathcal{X}\right)^{stab}$$

(*) 

$T^* gl_n /\!/\!/ GL_n$ $\quad \mathcal{X} = gl_n \circlearrowright GL_n$

(*) 

$T^* gl_n /\!/\!/ GL_n \qquad \mathfrak{X} = gl_n \circlearrowright GL_n$

$T^* Fl_n \xrightarrow{\mu} \mathcal{N} \subset gl_n \subset gl_n$

$A = (T^* Fl_n ; W = Tr(\mu \cdot X))$

$End(A) = DCoh^{GL_n}(\underbrace{dSt_n}_{\text{derived Steinberg variety}})$

Bezrukavnikov – Riche : $Br_n^{aff} \to End(A)$

A. Oblomkov, L. R.

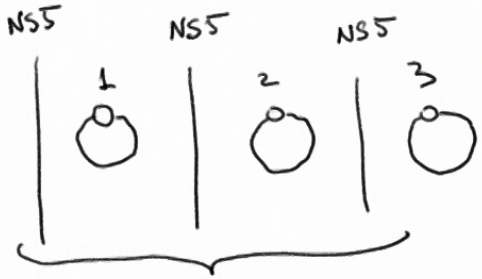
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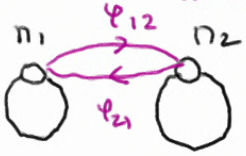
A. Oblomkov, L. R.



$T^* FL_3, W$



an edge of another quiver



$Tr\, X_1 \varphi_{21} \varphi_{12} - Tr\, X_2 \varphi_{12} \varphi_{21}$

(*) n $T^* gl_n /// GL_n$ $\mathfrak{X} = gl_n \circlearrowright GL_n$

$T^* Fl_n \xrightarrow{\mu} \mathcal{N} \subset gl_n \cong gl_n$

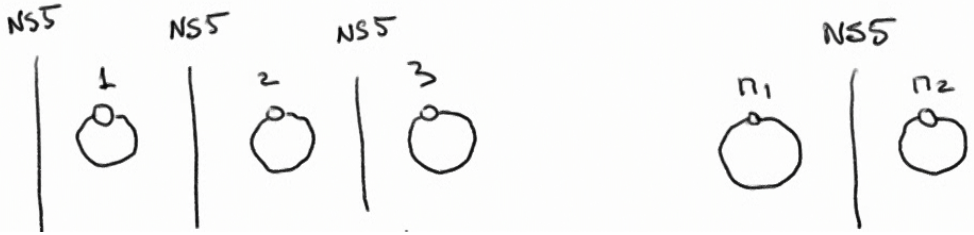
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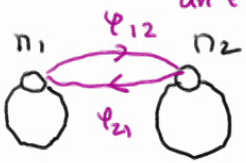
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$T^* FL_3, W$



← an edge of another quiver

$\mathrm{Tr}\, X_1 \varphi_{21} \varphi_{12} - \mathrm{Tr}\, X_2 \varphi_{12} \varphi_{21}$

$$B_{r_n}^{alt} = MF^{GL_n}\left(gl_n \times T^* FL_n \times T^* FL_n; W = \mathrm{Tr}\, X(\mu_2 - \mu_1)\right)$$

X

