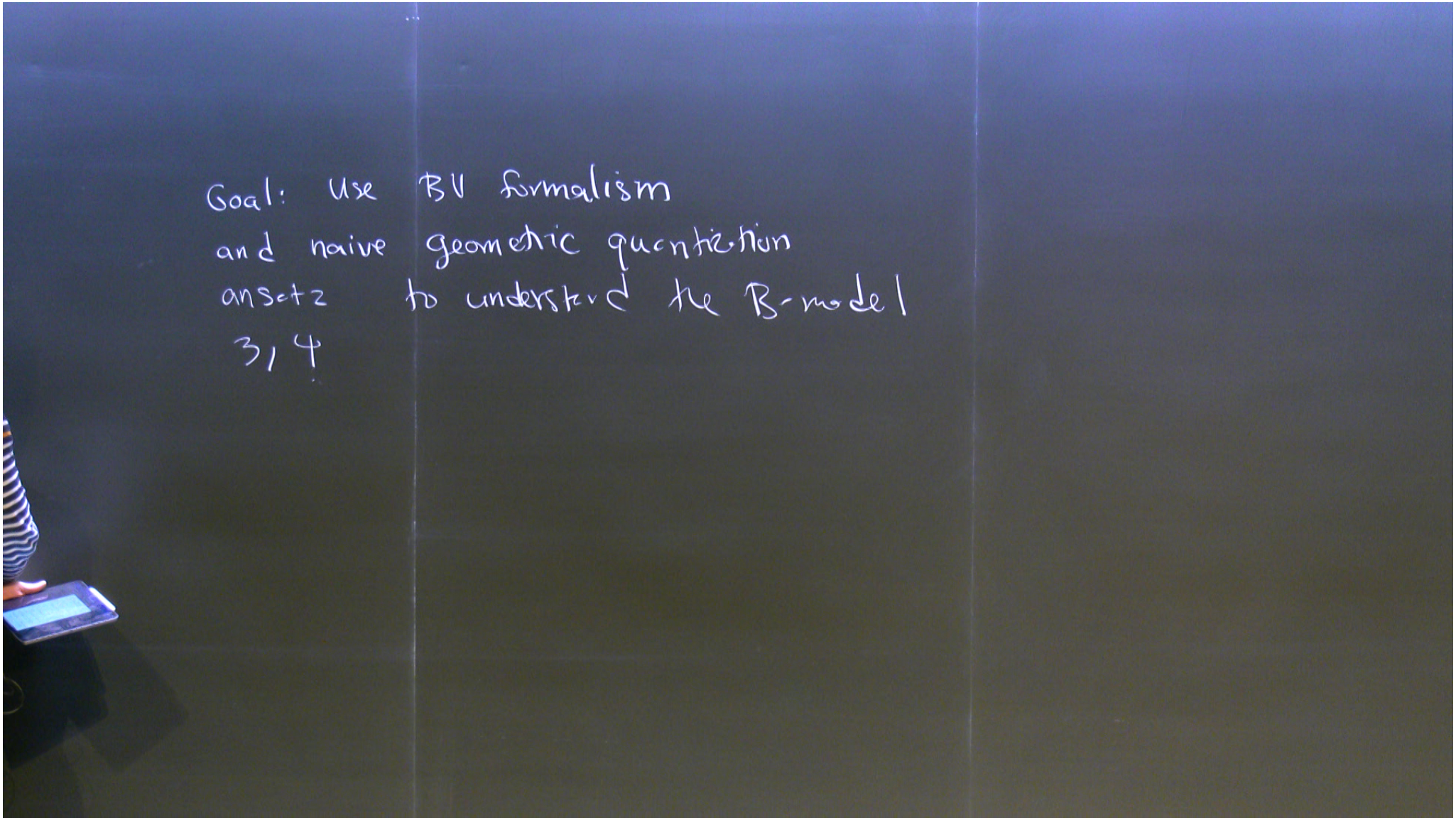
Title: A and B models in 3d and 4d I

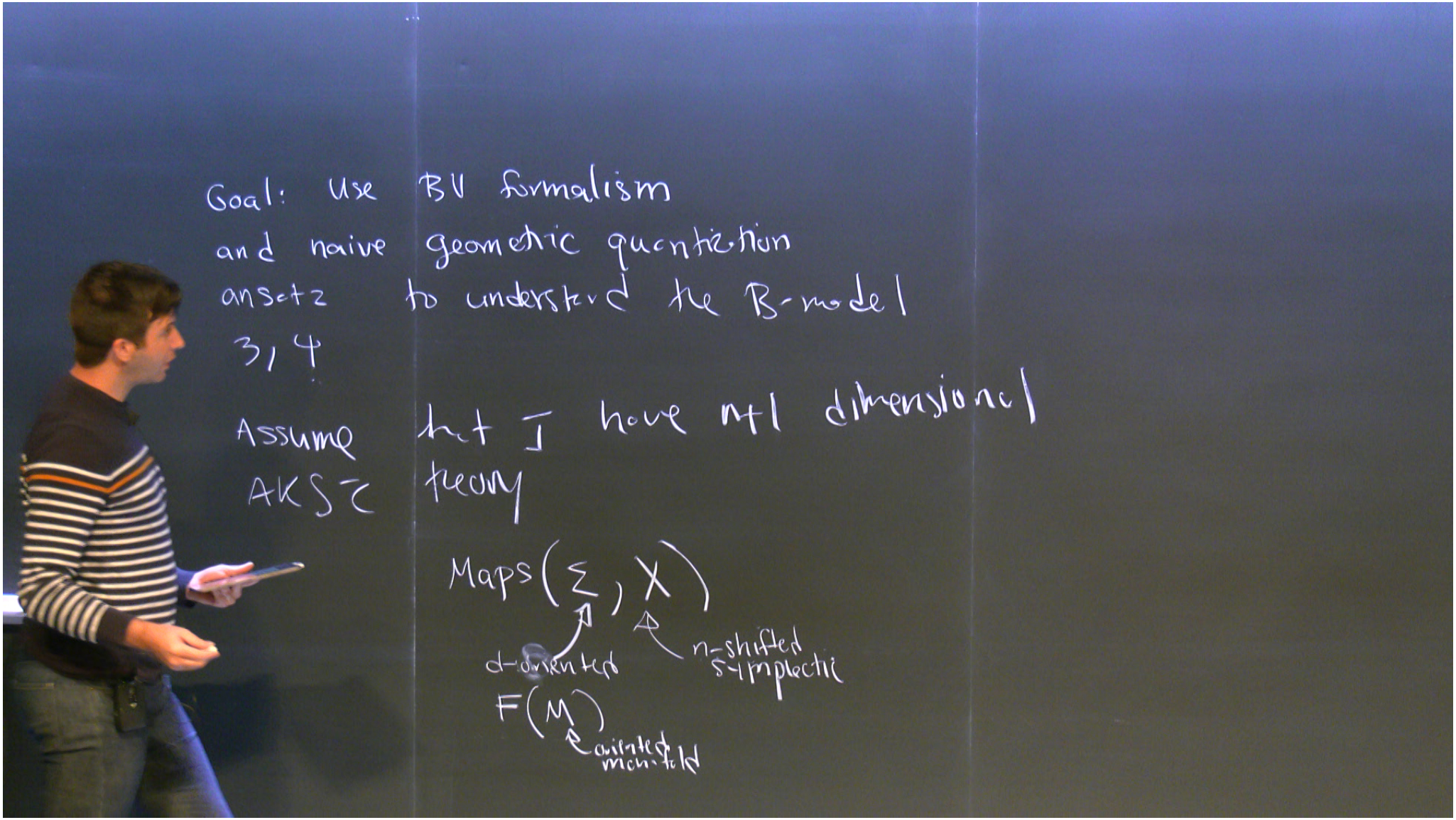
Speakers: Justin Hilburn

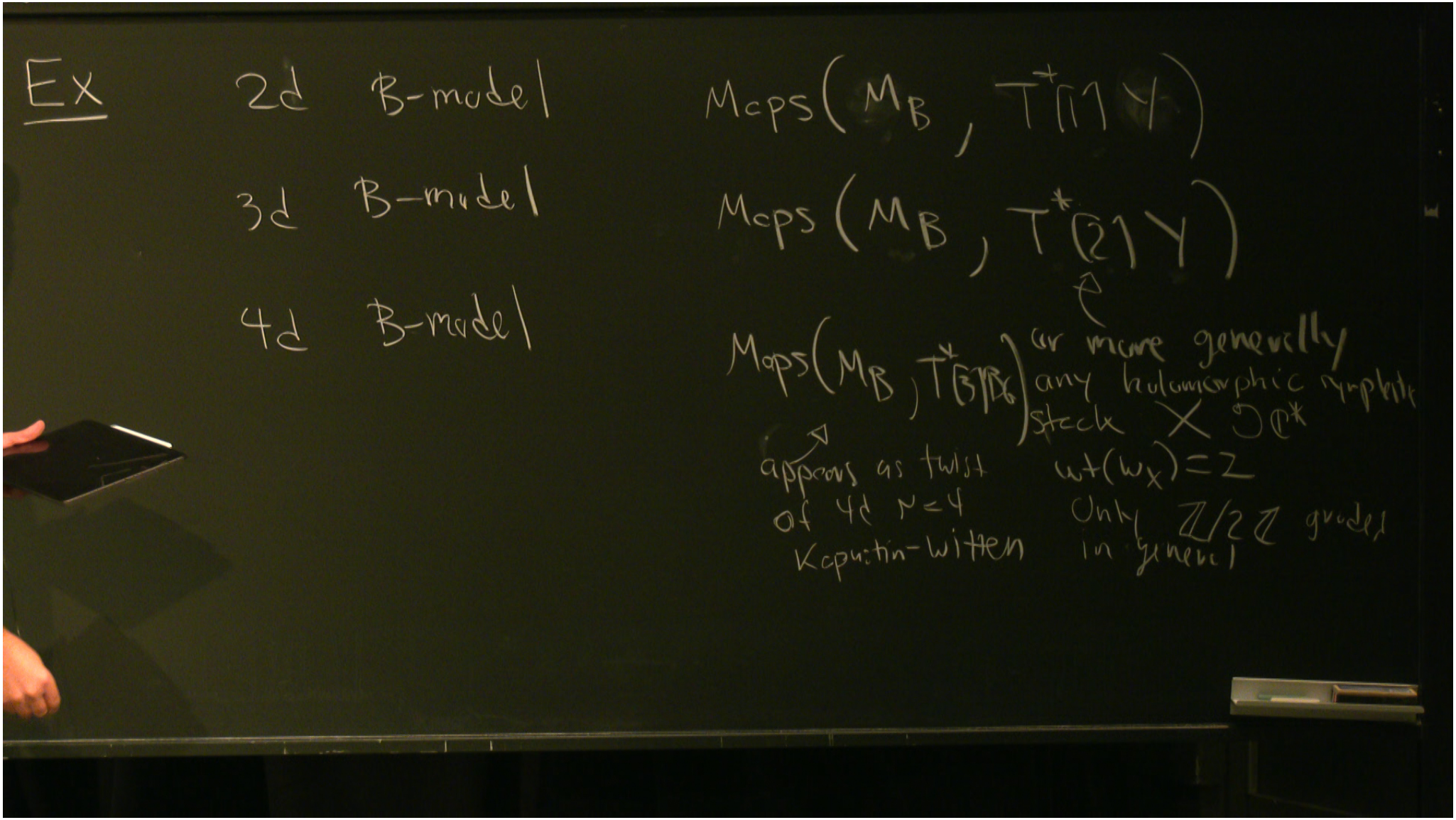
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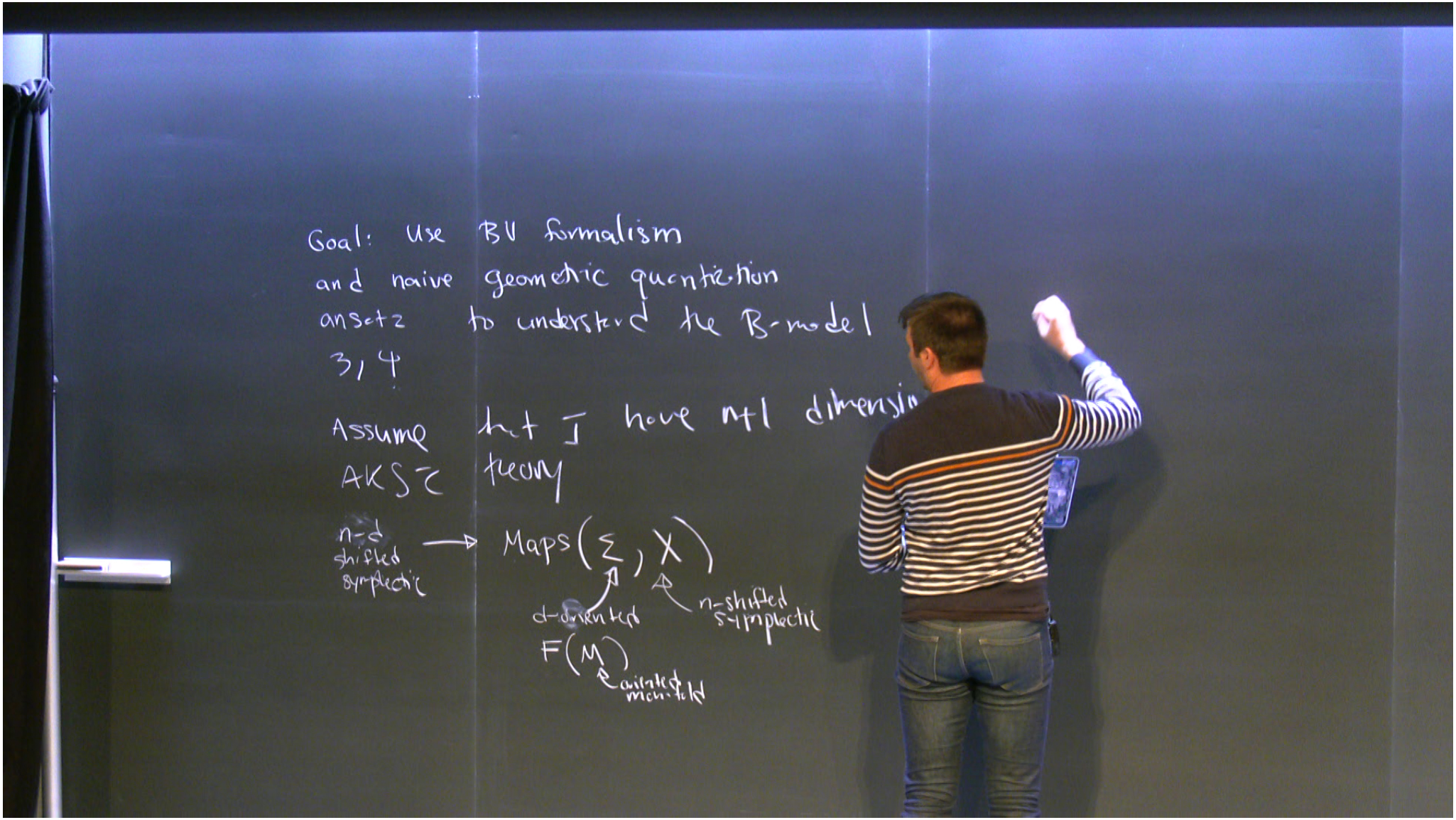
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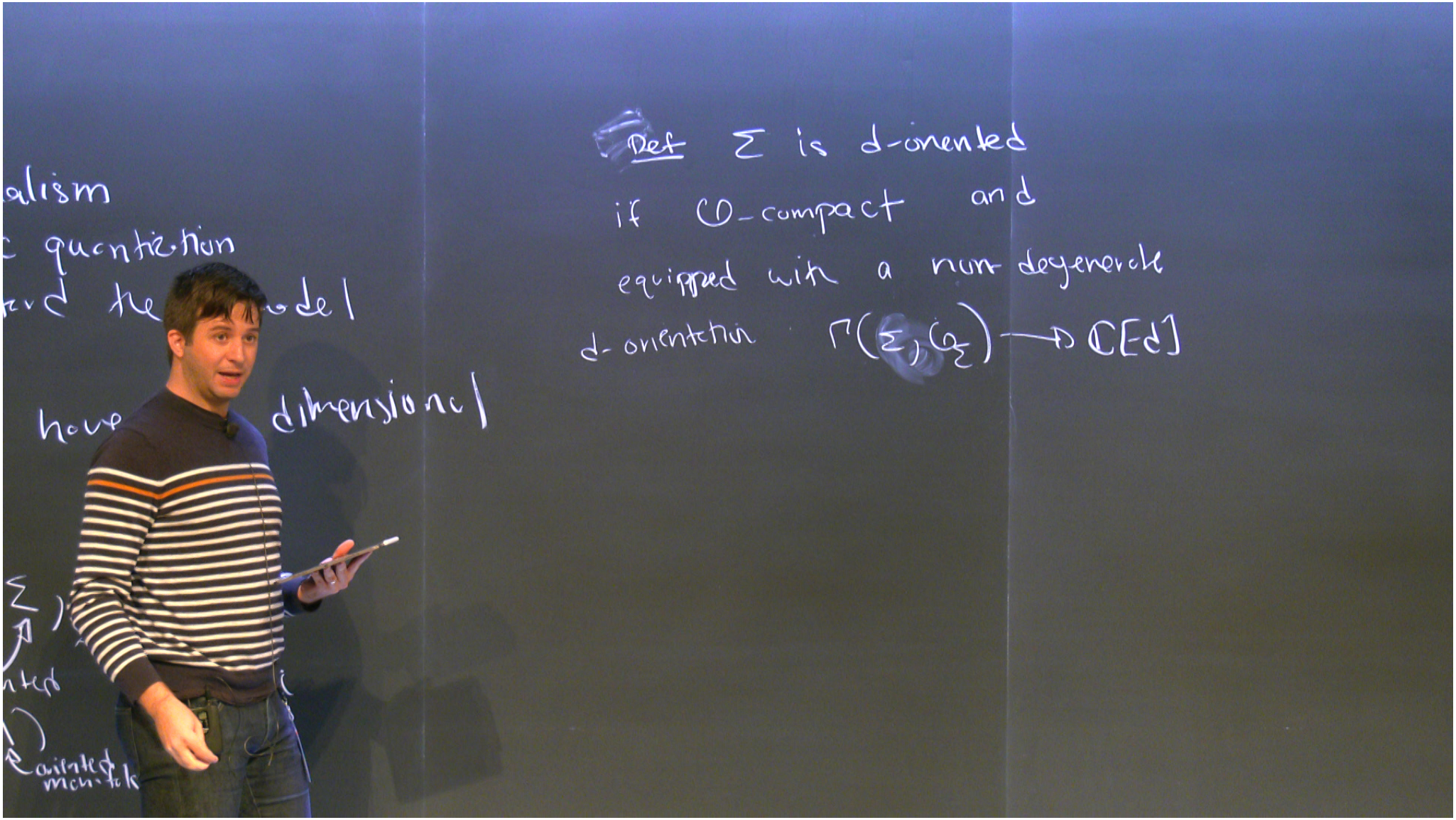
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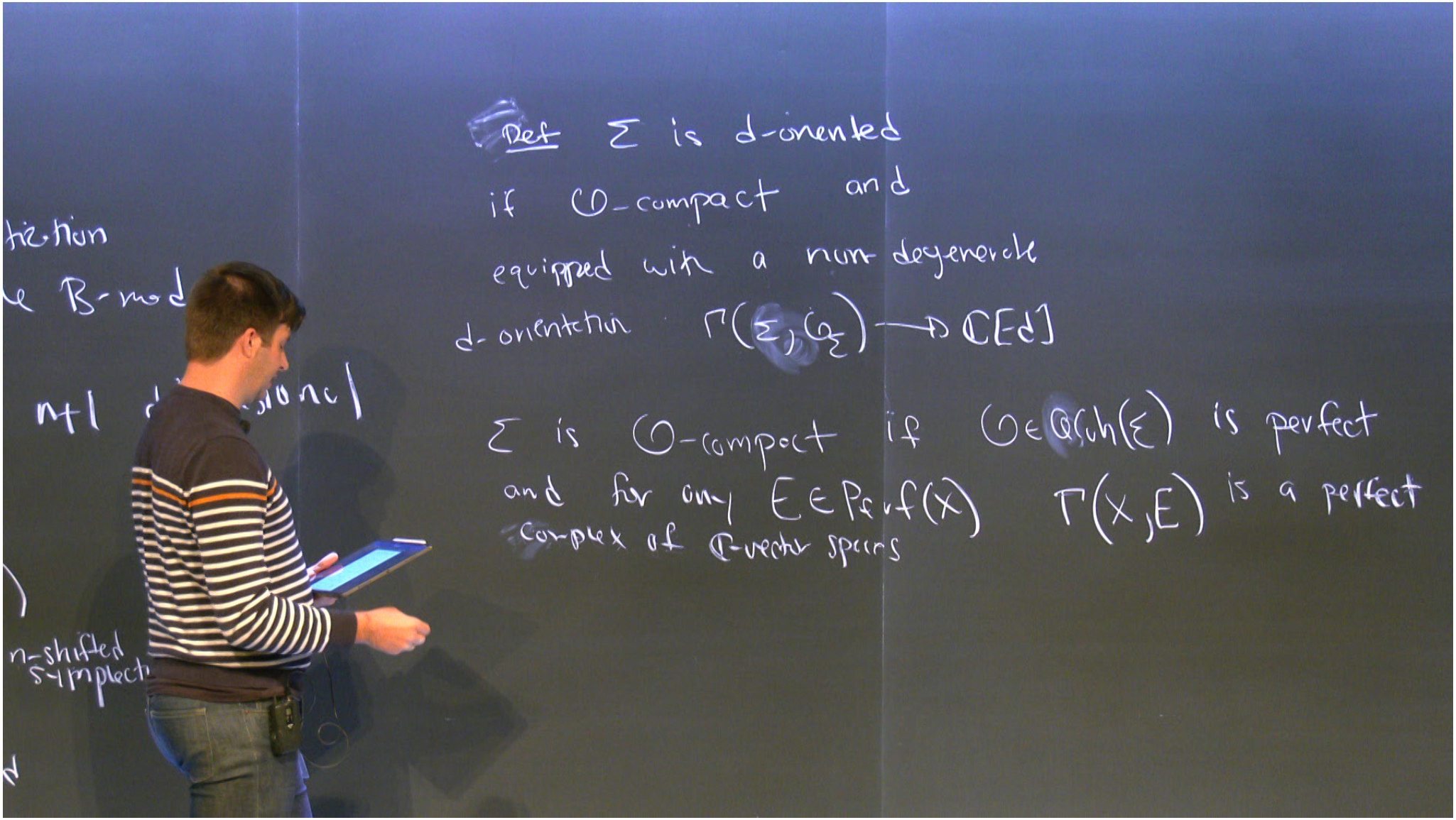

Goal: Use BV formalism
and naive geometric quantization
ansatz to understand the B-model
3, 4

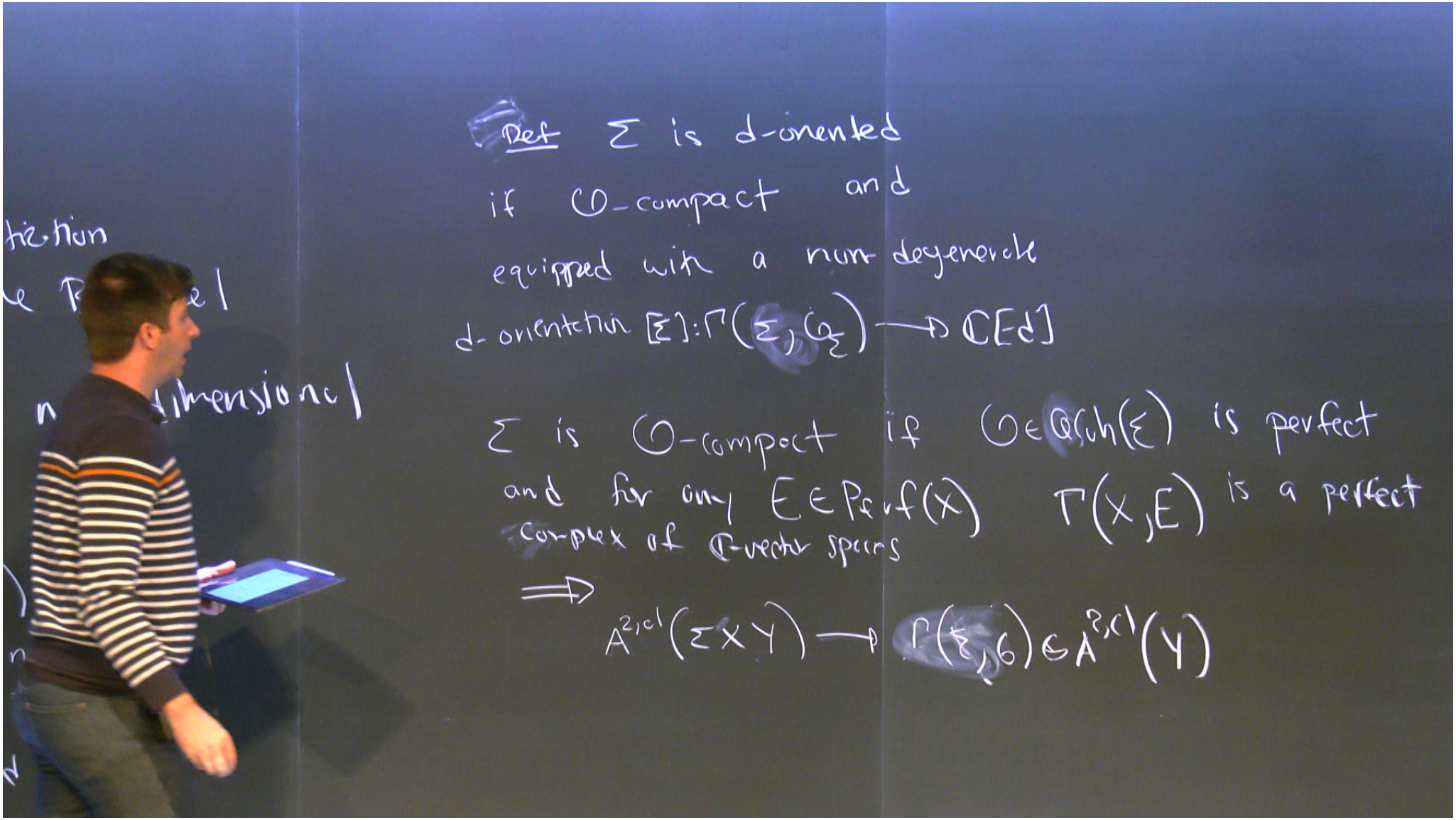

Goal: Use BV formalism
and naive geometric quantization
ansatz to understand the B-model
3, 4
Assume that I have n+1 dimensional
AKSZ theory
Maps(Σ, X)
d-oriented
n-shifted symplectic
F(M)
quasi-isomorphic


Ex
2d B-model
3d B-model
4d B-model
Maps(M_B, T*[1]Y)
Maps(M_B, T*[2]Y)
Maps(M_B, T*[3]B)
or more generally any holomorphic symplectic stack X
wt(ω_X) = 2
appears as twist of 4d N=4 Kapustin-Witten
Only ℤ/2ℤ graded in general


Goal: Use BV formalism
and naive geometric quantization
ansatz to understand the B-model
3, 4
Assume that I have n+1 dimensi
AKSZ theory
n-d shifted symplectic
Maps(Σ, X)
d-oriented
n-shifted symplectic
F(M)
oriented manifold


Def Σ is d-oriented
if CO-compact and
equipped with a non-degenerate
d-orientation Γ(Σ, O_Σ) → ℂ[-d]
have dimensional


Def Σ is d-oriented
if O-compact and
equipped with a non-degenerate
d-orientation Γ(Σ, O_Σ) → C[-d]
Σ is O-compact if O ∈ QCoh(Σ) is perfect
and for any E ∈ Perf(X) Γ(X, E) is a perfect
complex of C-vector spaces


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Σ is O-compact if O ∈ Qcoh(Σ) is perfect
and for any E ∈ Perf(X) Γ(X, E) is a perfect
complex of ℂ-vector spaces
⟹
A^{2,cl}(Σ × Y) → Γ(Σ, O) ⊗ A^{2,cl}(Y)

Def Σ is d-oriented if $\mathcal{O}$-compact and equipped with a non-degenerate d-orientation $[\Sigma]: \Gamma(\Sigma, \mathcal{O}_\Sigma) \longrightarrow \mathbb{C}[-d]$

Σ is $\mathcal{O}$-compact if $\mathcal{O} \in \mathrm{QCoh}(\Sigma)$ is perfect and for any $E \in \mathrm{Perf}(X)$ $\Gamma(X, E)$ is a perfect complex of $\mathbb{C}$-vector spaces

$\Longrightarrow$

$$\int_{(M)} : A^{2,cl}(\Sigma \times Y) \longrightarrow \Gamma(\Sigma, \mathcal{O}) \otimes A^{*,cl}(Y) \longrightarrow A^{*,cl}(Y, 2-d)$$

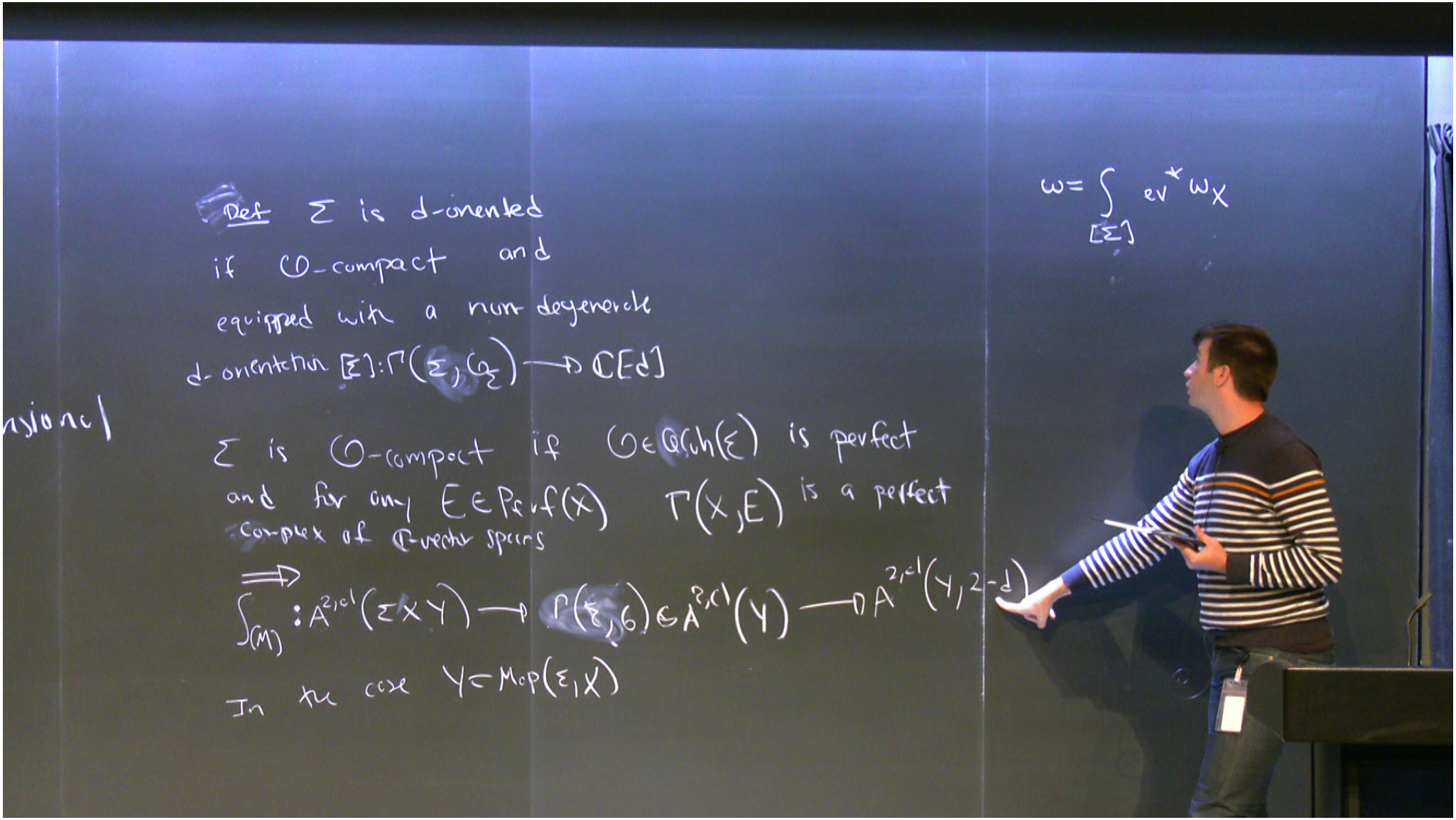
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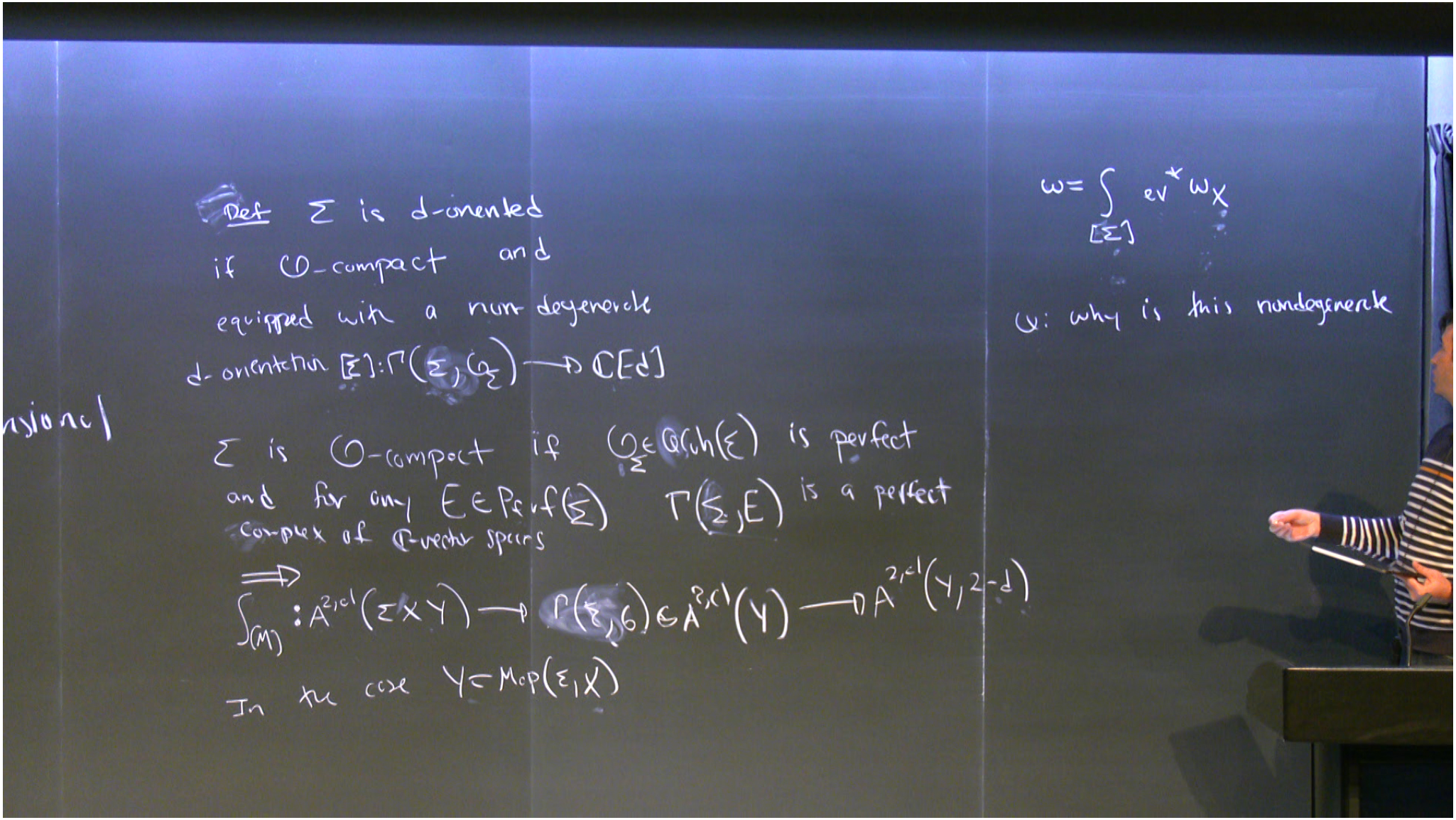
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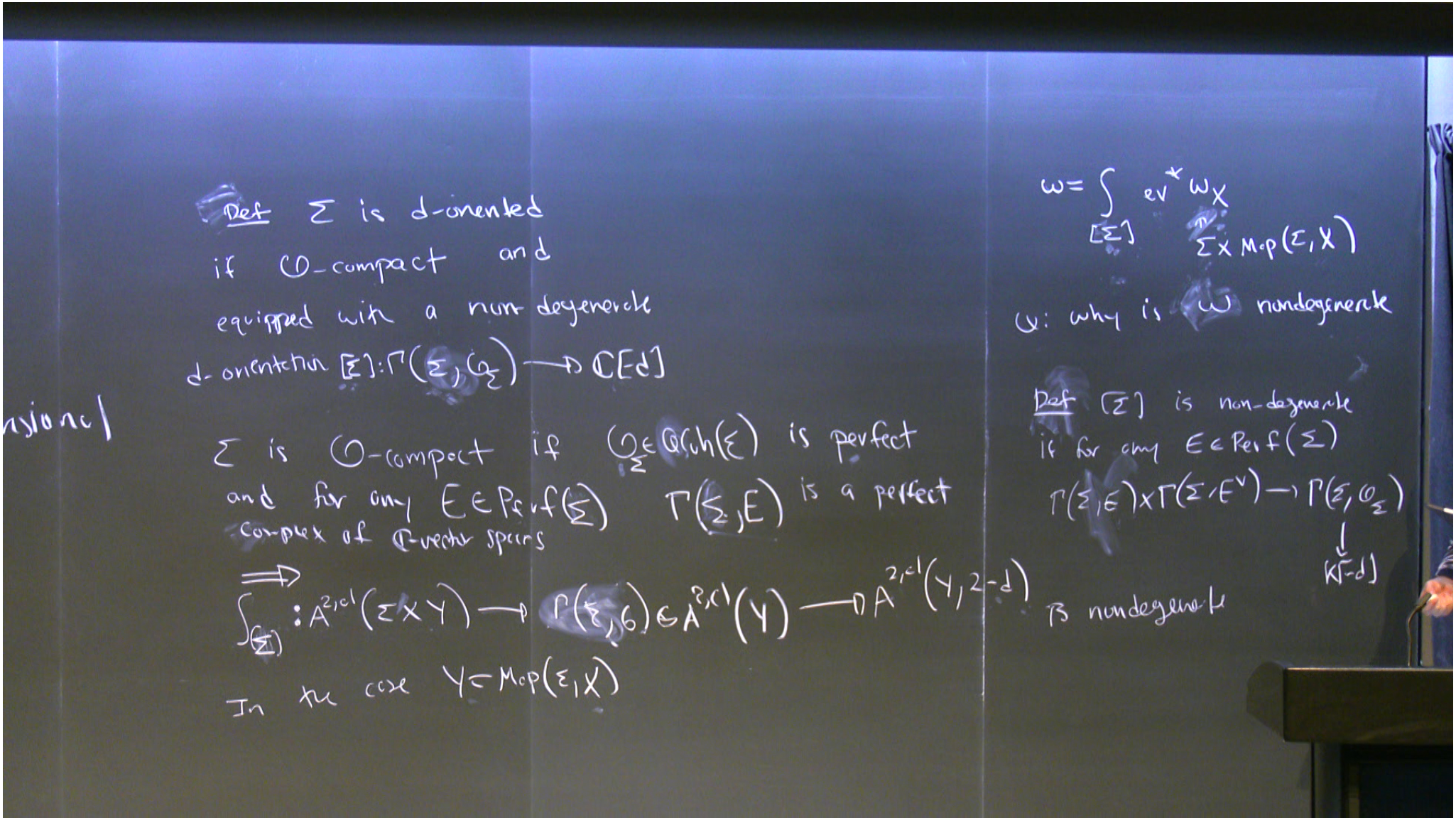
$\Longrightarrow$

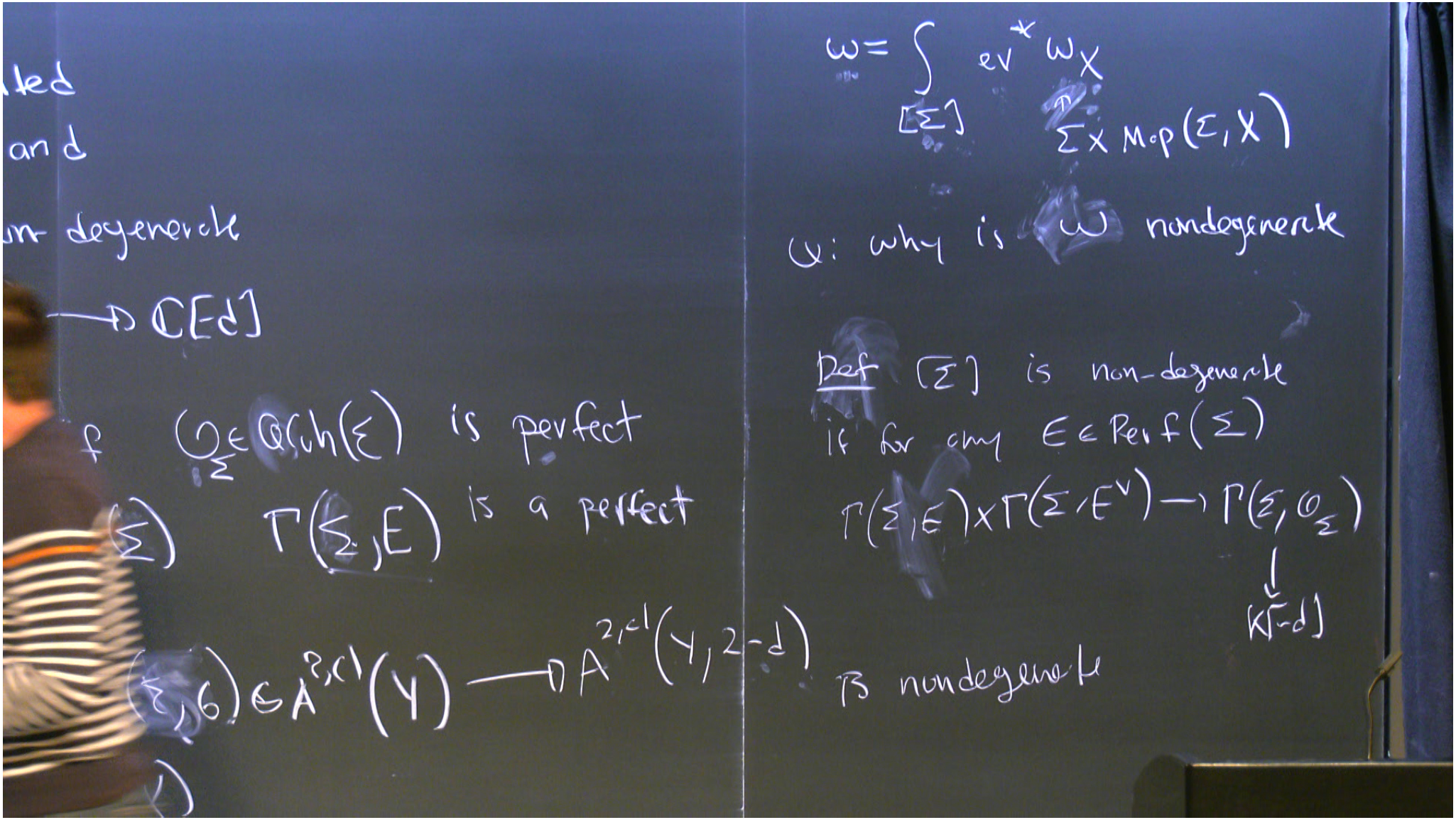
$$\int_{(M)} : A^{2,cl}(\Sigma \times Y) \longrightarrow \Gamma(\Sigma, \mathcal{O}) \otimes A^{2,cl}(Y) \longrightarrow A^{2,cl}(Y, 2-d)$$

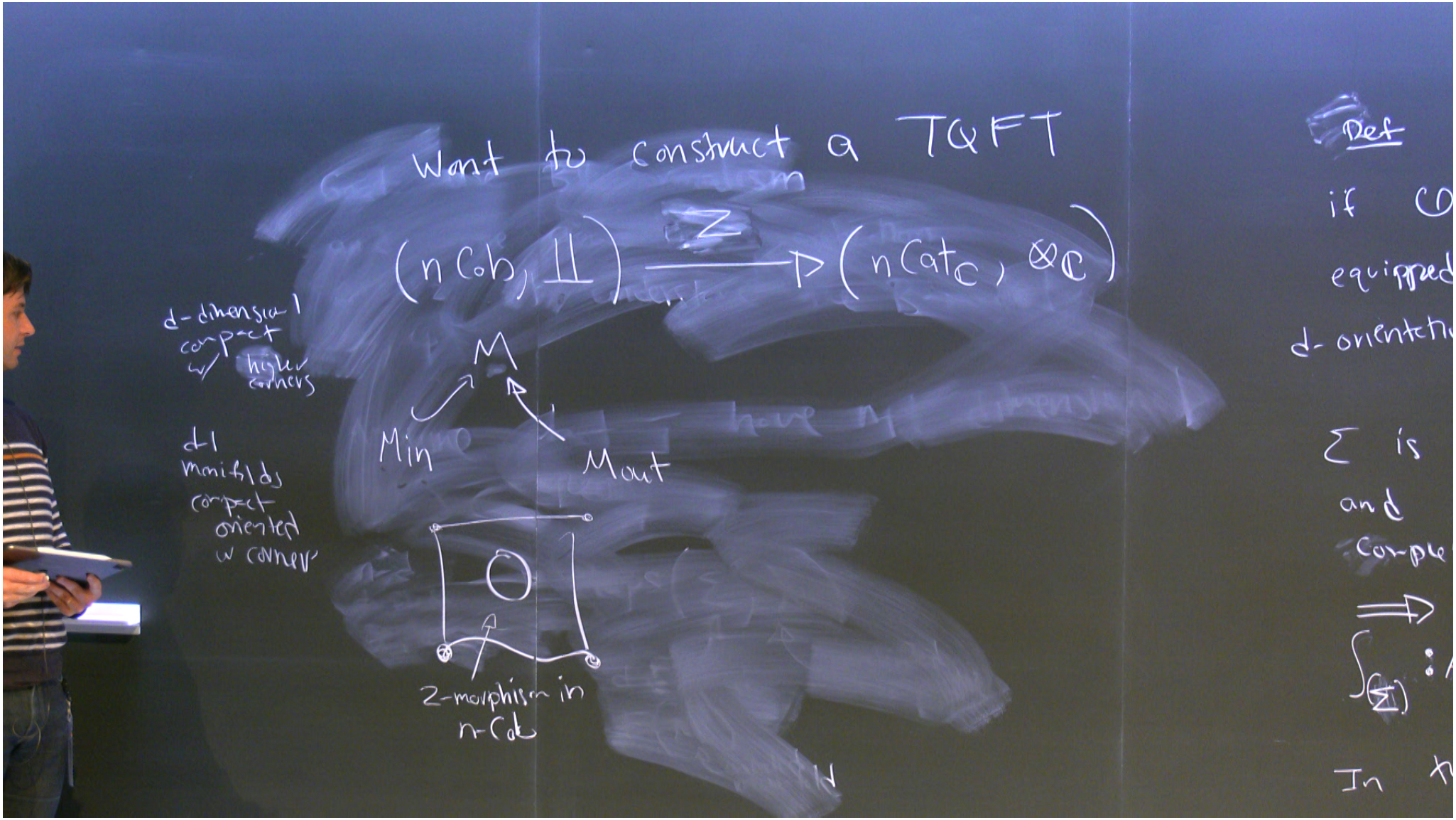
In the case $Y = \mathrm{Map}(\Sigma, X)$

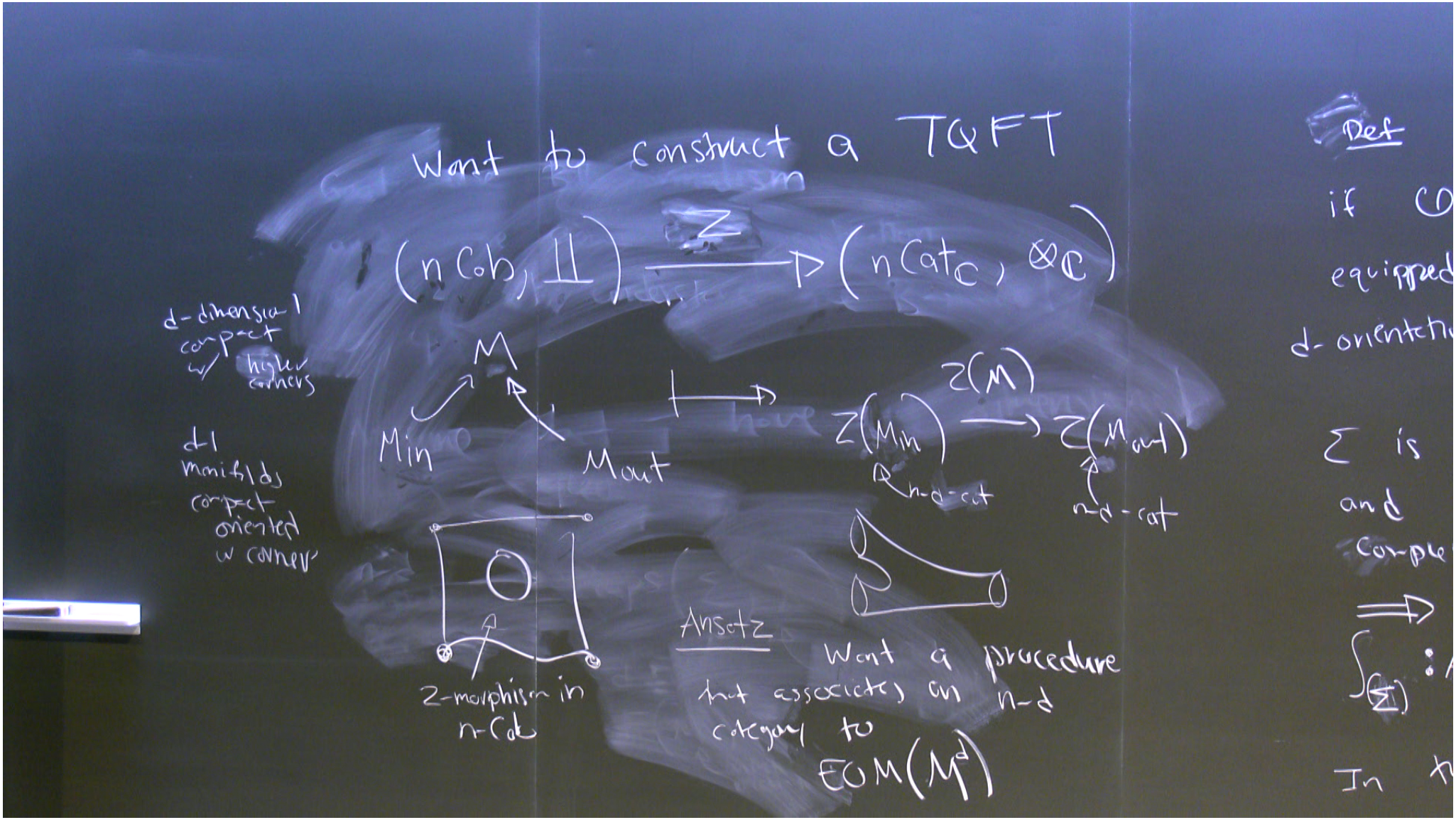

Def Σ is d-oriented
if O-compact and
equipped with a non-degenerate
d-orientation [Σ]: Γ(Σ, O_Σ) → ℂ[-d]
Σ is O-compact if O ∈ QCoh(Σ) is perfect
and for any E ∈ Perf(X) Γ(X, E) is a perfect
complex of ℂ-vector spaces
∫_(M) : A^{2,cl}(Σ × Y) → Γ(Σ, O) ⊗ A^{2,cl}(Y) → A^{2,cl}(Y, 2-d)
In the case Y = Map(Σ, X)
ω = ∫_[Σ] ev* ω_X

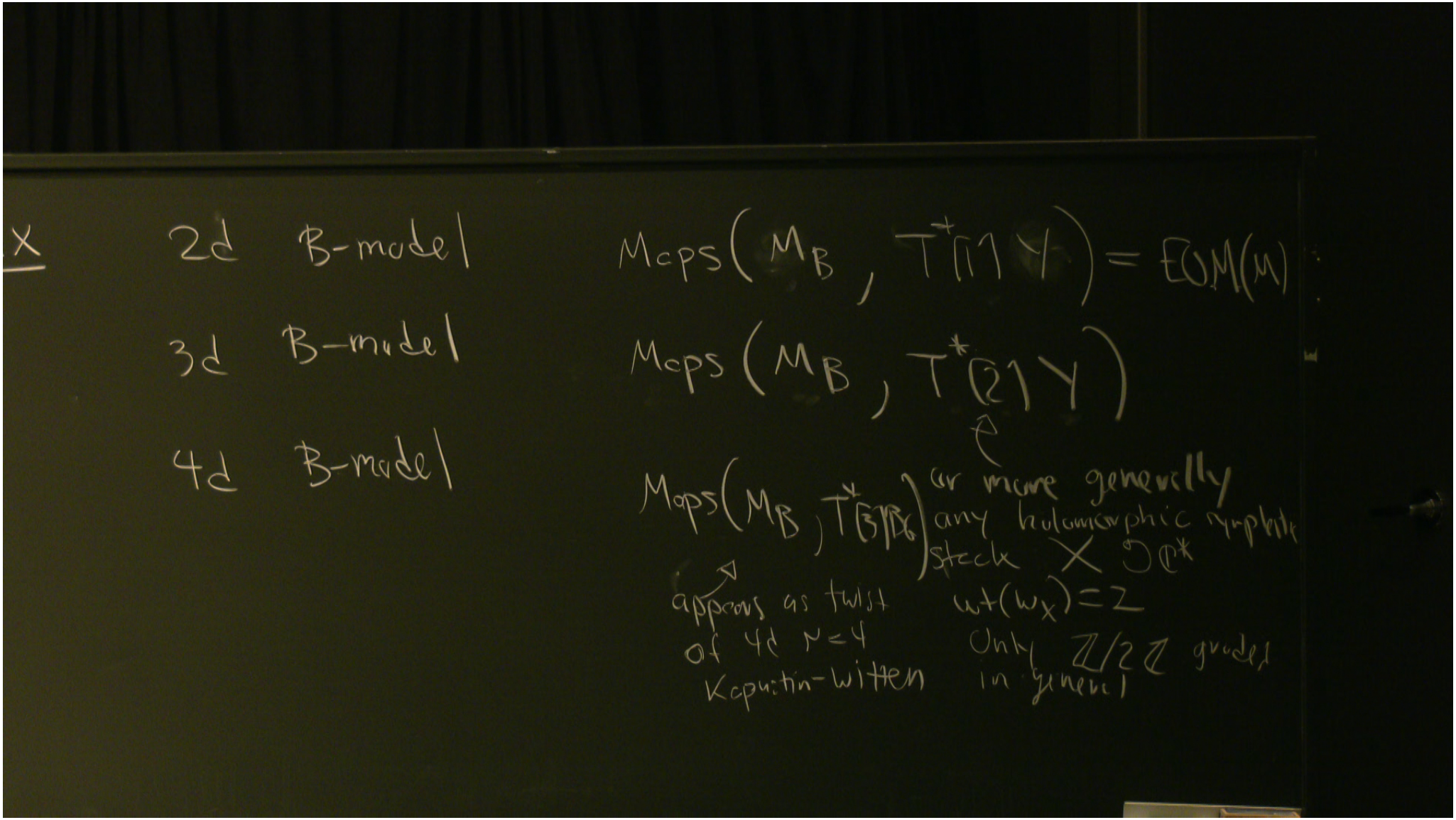

Def Σ is d-oriented
if ℂ-compact and
equipped with a non-degenerate
d-orientation [Σ]: Γ(Σ, ℂ_Σ) → ℂ[-d]
Σ is ℂ-compact if ℂ_Σ ∈ Qcoh(Σ) is perfect
and for any E ∈ Perf(Σ) Γ(Σ, E) is a perfect
complex of ℂ-vector spaces
∫_(M): A^{2,cl}(Σ × Y) → Γ(Σ, ℂ) ⊗ A^{2,cl}(Y) → A^{2,cl}(Y, 2-d)
In the case Y = Map(Σ, X)
ω = ∫_[Σ] ev* ω_X
ω: why is this nondegenerate

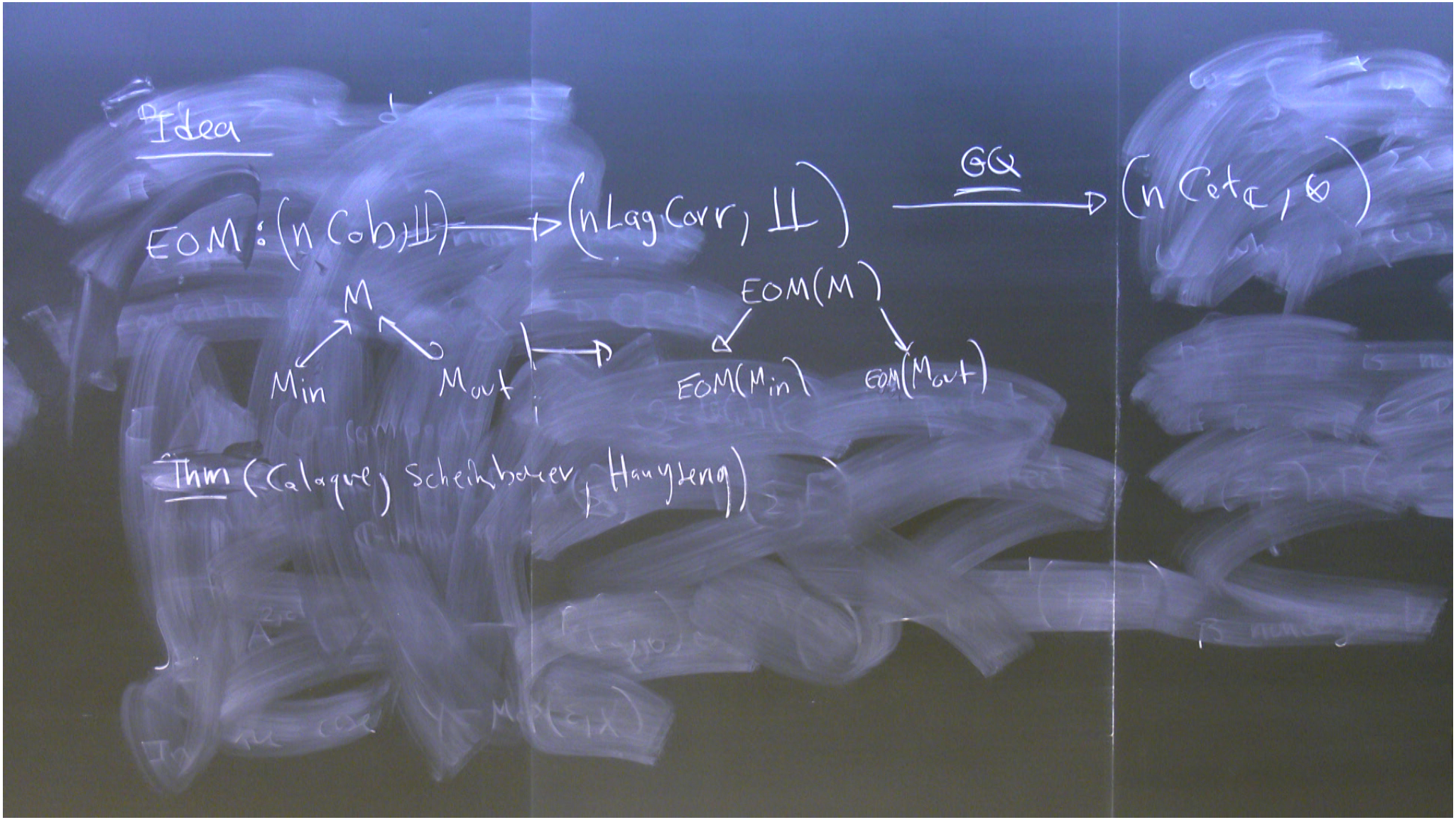

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∫_[Σ] : A^{0,•}(Σ × Y) → Γ(Σ, O) ⊗ A^{0,•}(Y) → A^{0,•}(Y, 2-d)
In the case Y = Map(Σ, X)
ω = ∫_[Σ] ev* ω_X
Σ × Map(Σ, X)
Q: why is ω nondegenerate
Def [Σ] is non-degenerate
if for any E ∈ Perf(Σ)
Γ(Σ, E) × Γ(Σ, E^∨) → Γ(Σ, O_Σ)
↓
K[-d]
is nondegenerate

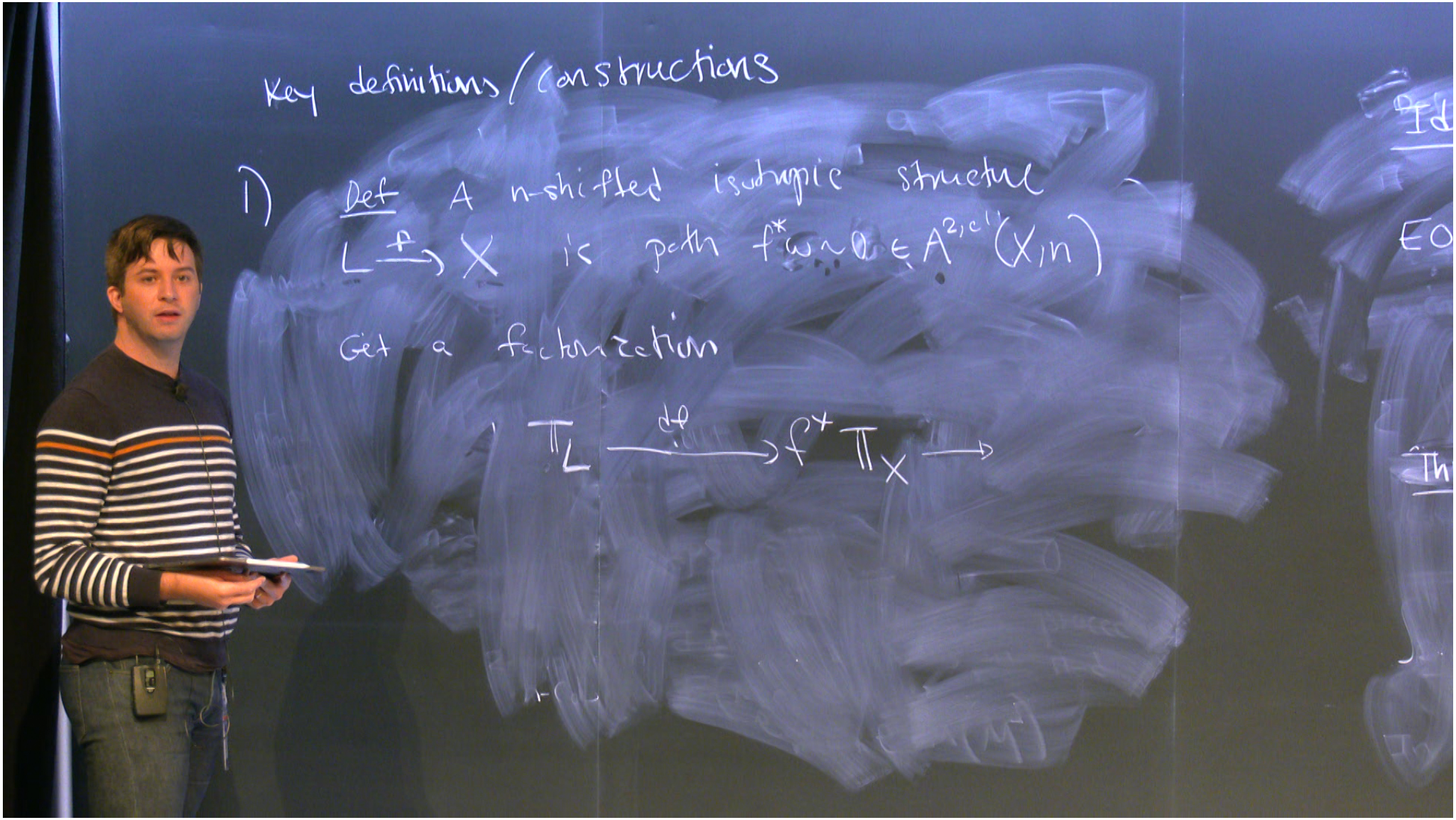

ted
and
un- degenerate
$\to \mathbb{C}[[d]]$
$\omega = \int_{[\Sigma]} ev^* \omega_X$
$\overline{\Sigma} \times Map(\Sigma, X)$
Q: why is ω nondegenerate
Def $[\Sigma]$ is non-degenerate
if for any $E \in Perf(\Sigma)$
$\Gamma(\Sigma, E) \times \Gamma(\Sigma, E^\vee) \to \Gamma(\Sigma, \mathcal{O}_\Sigma)$
$\downarrow$
$K[-d]$
$\mathcal{O}_\Sigma \in QCoh(\Sigma)$ is perfect
$\Gamma(\Sigma, E)$ is a perfect
$(\xi, \theta) \in A^{2,cl}(Y) \to A^{2,cl}(Y, 2-d)$
is nondegenerate

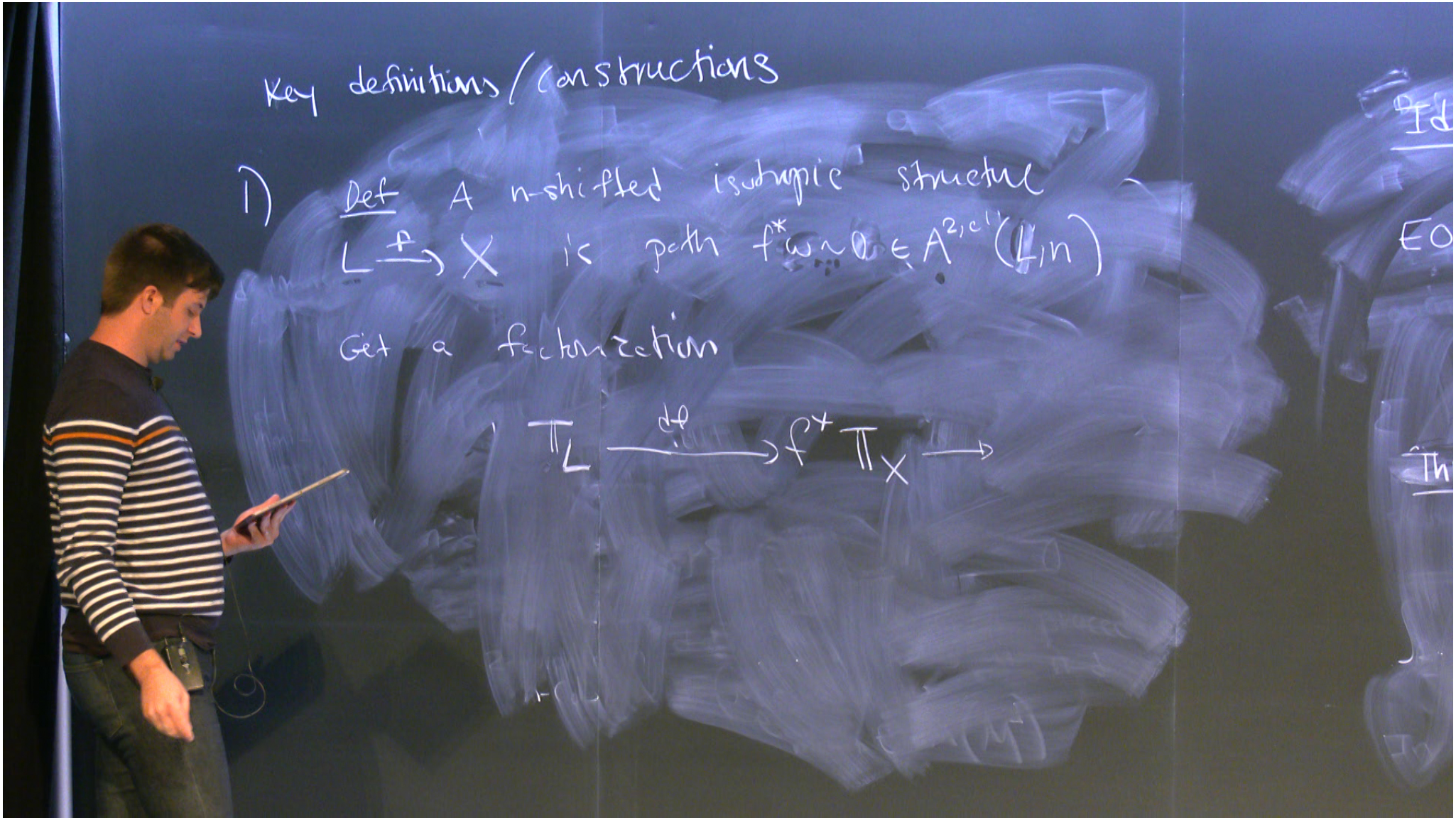

Want to construct a TQFT
$(n\mathrm{Cob}, \amalg) \xrightarrow{Z} (n\mathrm{Cat}_{\mathbb{C}}, \otimes_{\mathbb{C}})$
d-dimensional compact w/ higher corners
d-1 manifolds compact oriented w corners
M
Min
Mout
2-morphism in n-Cob
Def
if C
equipped
d- orientati
Σ is
and
Comple
In t

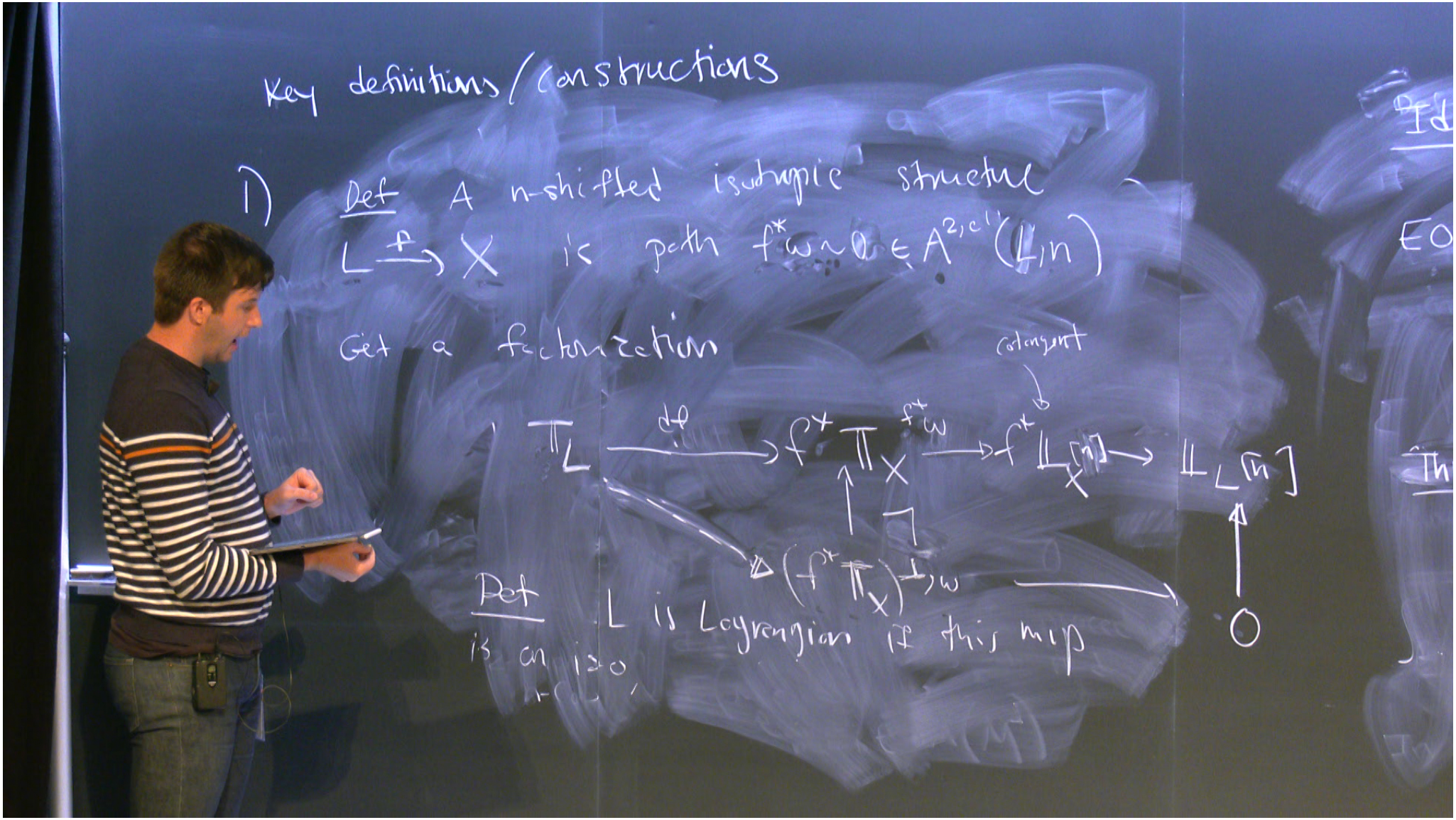

Want to construct a TQFT
$(n\text{Cob}, \amalg) \xrightarrow{Z} (n\text{Cat}_{\mathbb{C}}, \otimes_{\mathbb{C}})$
d-dimensional compact w/ higher corners
d-1 manifolds compact oriented w corners
M
M_{in}
M_{out}
$Z(M)$
$Z(M_{in})$
$Z(M_{out})$
n-d-cat
n-d-cat
2-morphism in n-Cob
Ansatz
Want a procedure that associates an n-d category to $\mathcal{E}OM(M^d)$
Def
if C
equipped
d-orientedn
Σ is
and
Com-pl
In t

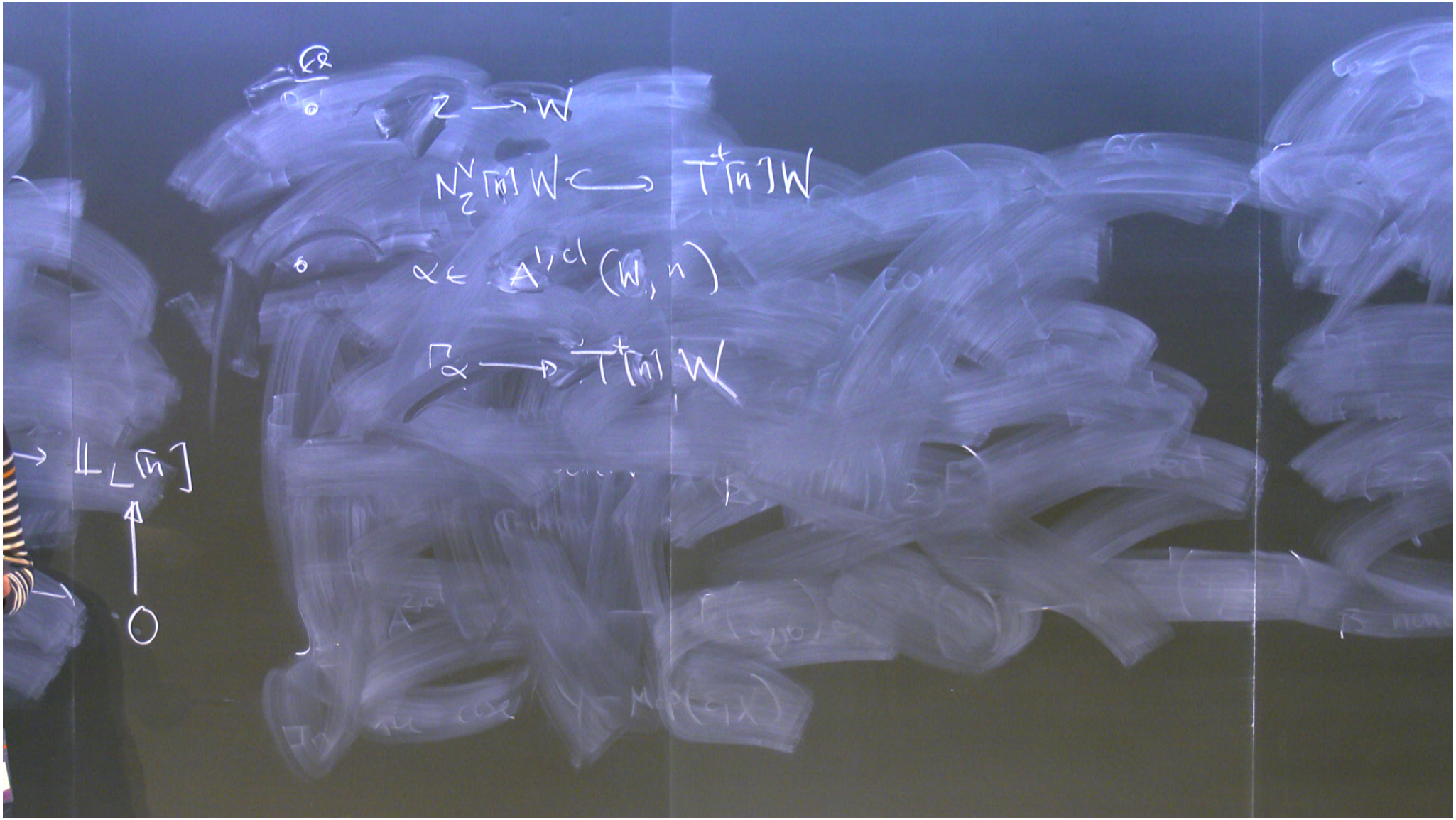

X
2d B-model
Maps(M_B, T*[1]Y) = EOM(M)
3d B-model
Maps(M_B, T*[2]Y)
4d B-model
Maps(M_B, T*[3]B_G)
or more generally any holomorphic symplectic stack X ↻ C*
appears as twist of 4d N=4 Kapustin-Witten
wt(ω_X) = 2
Only Z/2Z graded in general

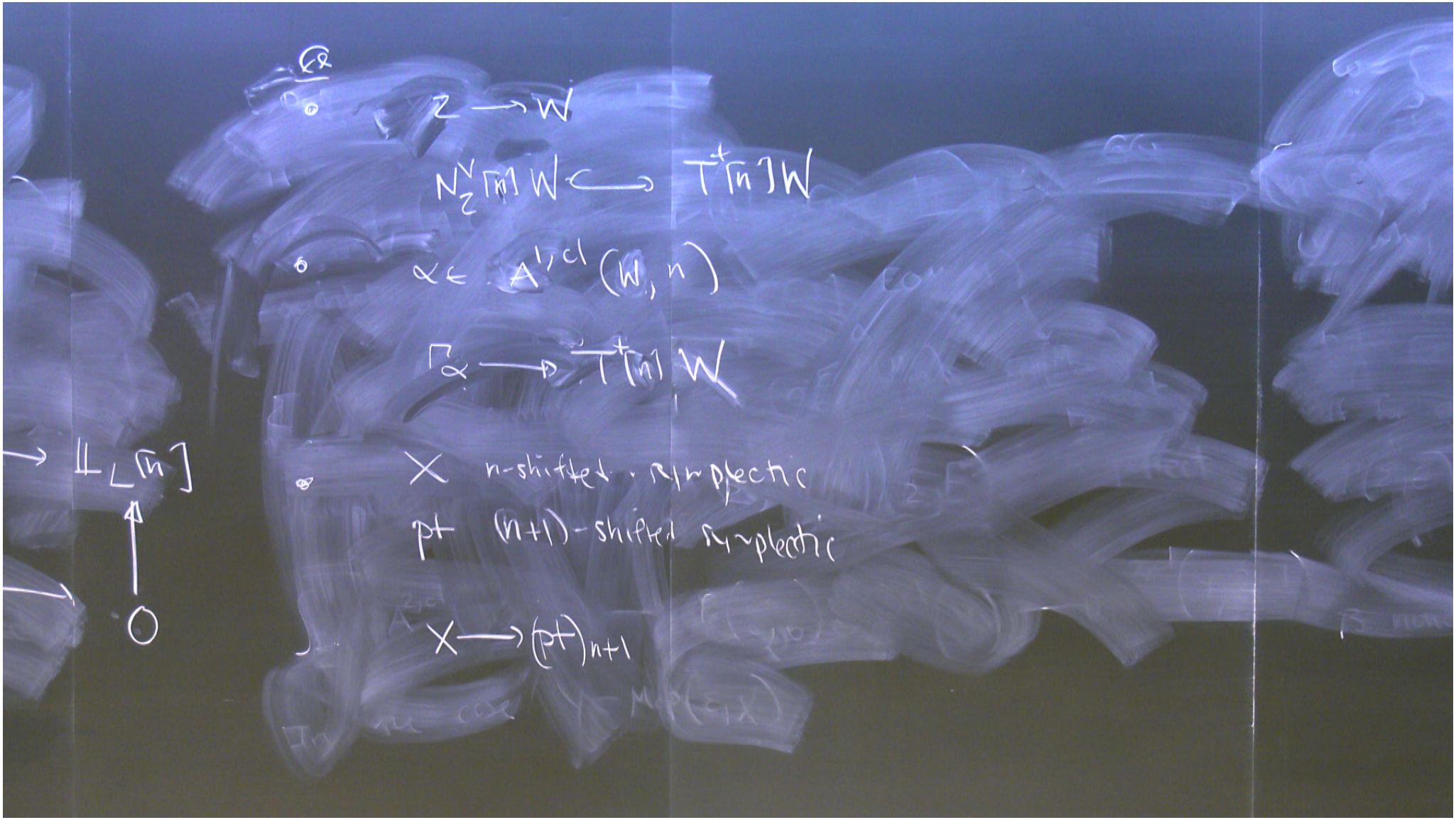

Idea
EOM : (nCob, ⊔) → (nLagCorr, ⊔)
GQ
(nCat_C, ⊗)
M
M_in
M_out
EOM(M)
EOM(M_in)
EOM(M_out)
Thm (Calaque, Scheimbauer, Haugseng)

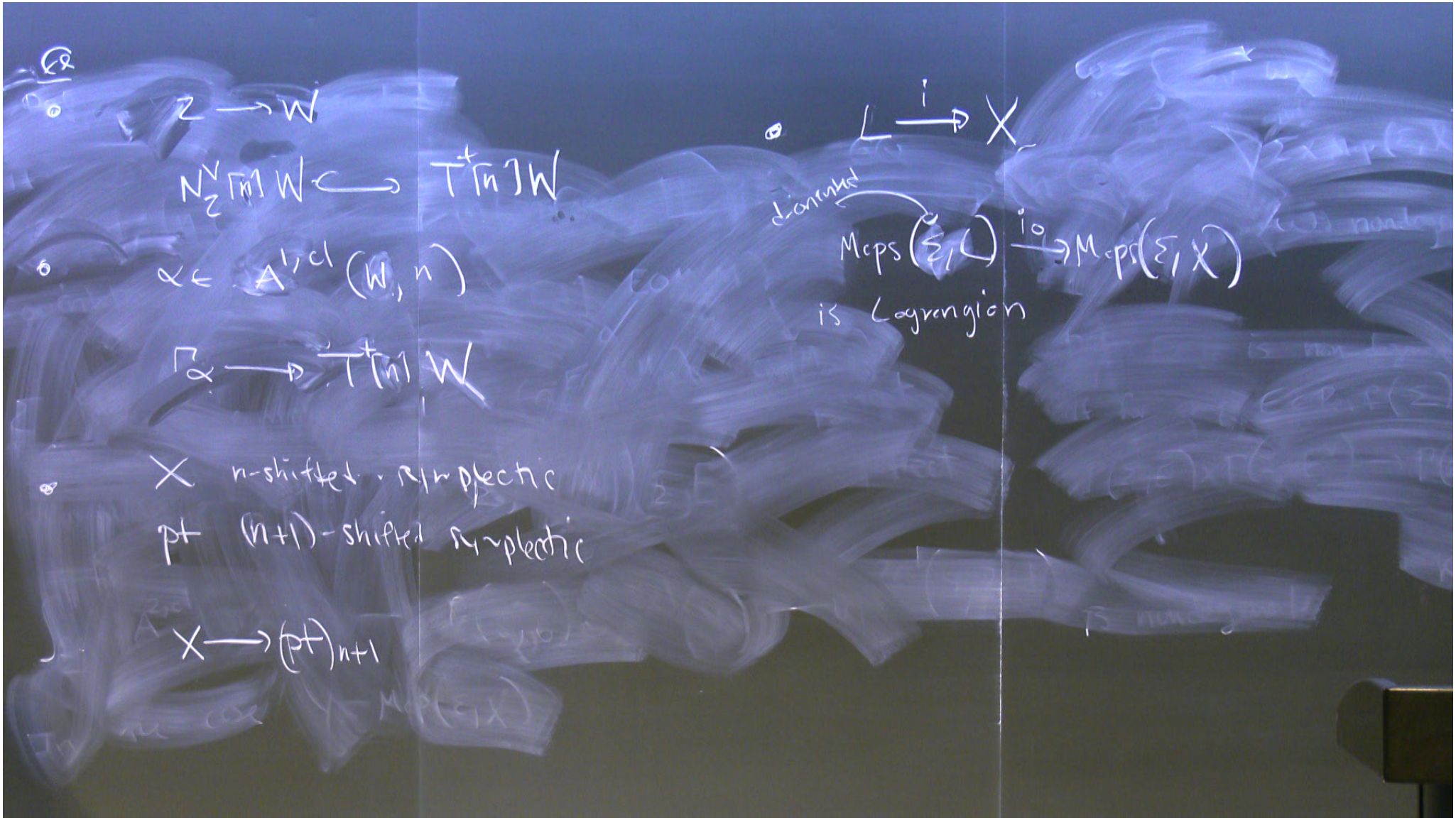

Key definitions / constructions
1) Def A n-shifted isotropic structure
$L \xrightarrow{f} X$ is path $f^*\omega \sim 0 \in A^{2,cl}(X, n)$
Get a factorization
$T_L \xrightarrow{df} f^* T_X \longrightarrow$


Key definitions / constructions
1) Def A n-shifted isotropic structure
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Key definitions / constructions
1) Def A n-shifted isotropic structure
$L \xrightarrow{f} X$ is path $f^*\omega \sim 0 \in A^{2,cl}(L,n)$
Get a factorization
cotangent
$\mathbb{T}_L \xrightarrow{df} f^*\mathbb{T}_X \xrightarrow{f^*\omega} f^*\mathbb{L}_X[n] \to \mathbb{L}_L[n]$
$(f^*\mathbb{T}_X)^{\perp}/\omega$
Def L is Lagrangian if this map
is an
0


Z → W
$N_Z^V[n]W \hookrightarrow T^+[n]W$
$\alpha \in A^{1,cl}(W,n)$
$\Gamma\alpha \longrightarrow T^+[n]W$
$\mathbb{L}_{\mathcal{L}}[n]$


Z → W
$N_Z^V[n]W \hookrightarrow T^*[n]W$
$\alpha \in A^{1,cl}(W, n)$
$\Gamma_\alpha \to T^*[n]W$
X n-shifted symplectic
pt (n+1)-shifted symplectic
$X \to (pt)_{n+1}$
$\mathbb{L}_{\mathcal{L}[n]}$
0


Ex
$Z \to W$
$N_Z^V[n]W \hookrightarrow T^*[n]W$
$\alpha \in A^{1,cl}(W, n)$
$\Gamma_\alpha \to T^*[n]W$
X n-shifted symplectic
pt (n+1)-shifted symplectic
$X \to (pt)_{n+1}$
$L \xrightarrow{i} X$
d-oriented
$Maps(\Sigma, L) \xrightarrow{i\circ} Maps(\Sigma, X)$
is Lagrangian

$Z \to W$

$N_Z^V[n]W \hookrightarrow T^*[n]W$

$\alpha \in A^{1,cl}(W, n)$

$\Gamma_\alpha \to T^*[n]W$

X n-shifted symplectic

pt $(n+1)$-shifted symplectic

$X \to (pt)_{n+1}$

$L \xrightarrow{i} X$

L oriented

$Maps(\Sigma, L) \xrightarrow{i_0} Maps(\Sigma, X)$

is Lagrangian

A Lagrangian correspondence is

$L \to \bar{X} \times Y$

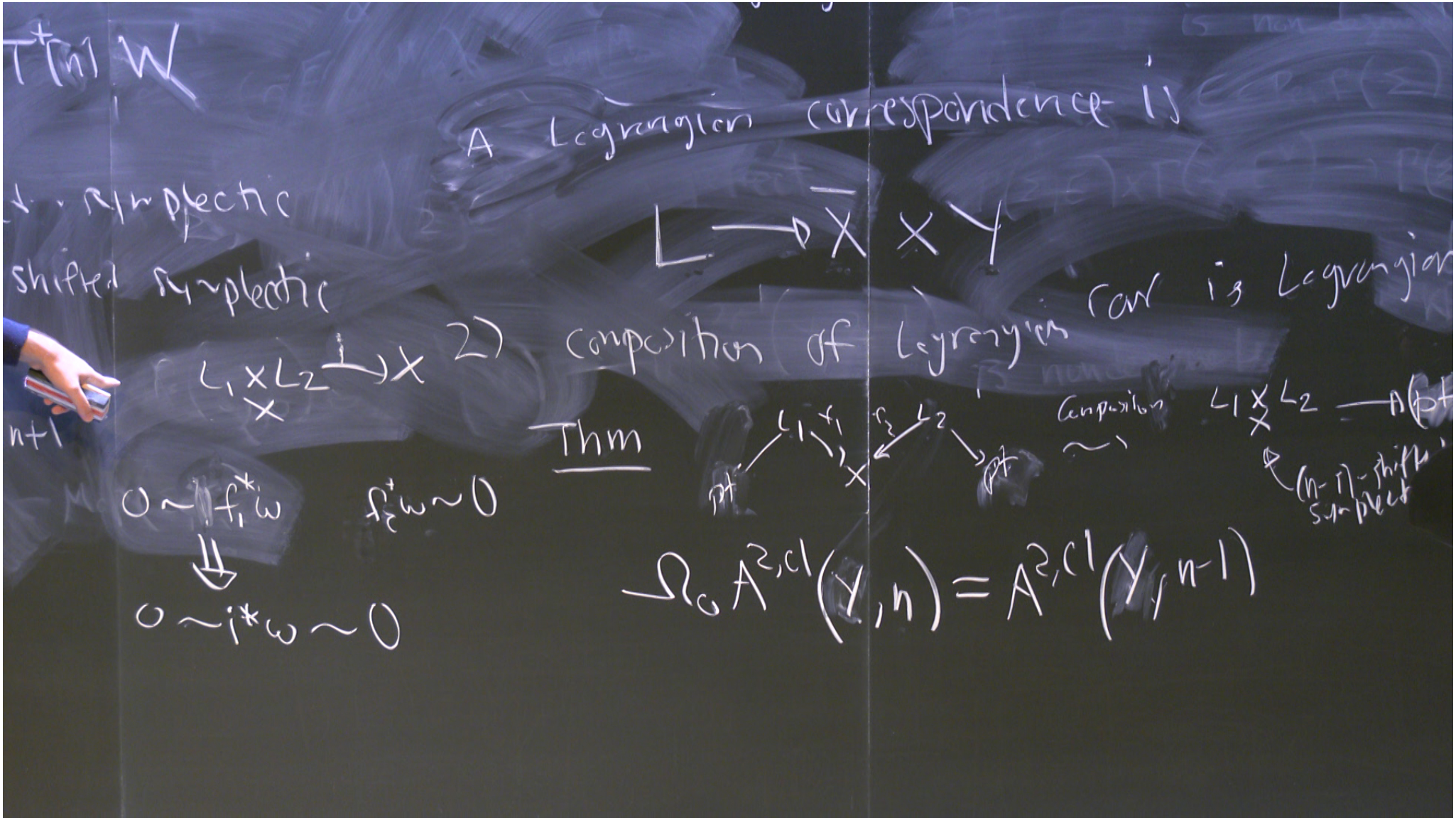
2) composition of Lagrangian corr is Lagrangian

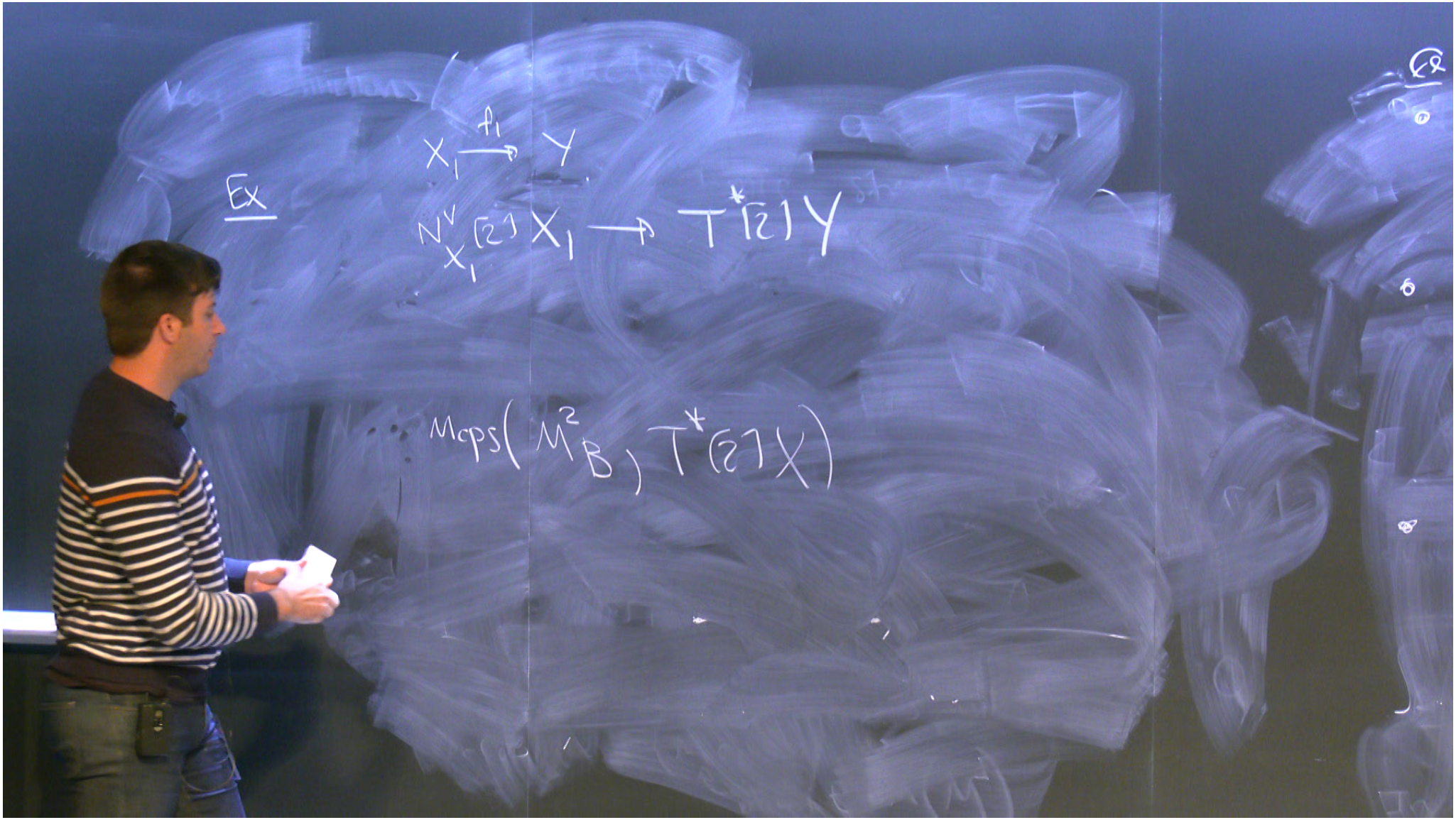
Thm

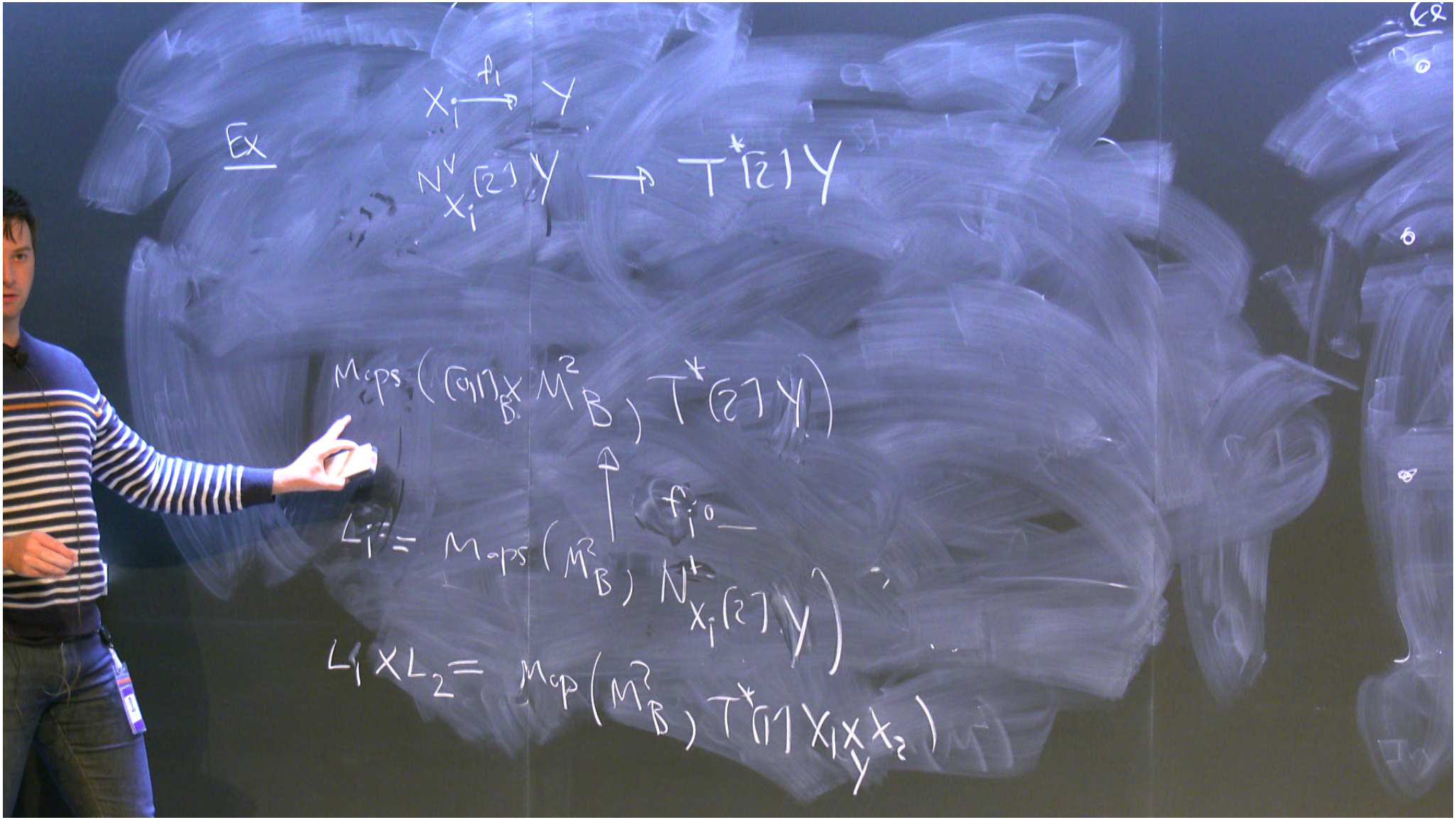


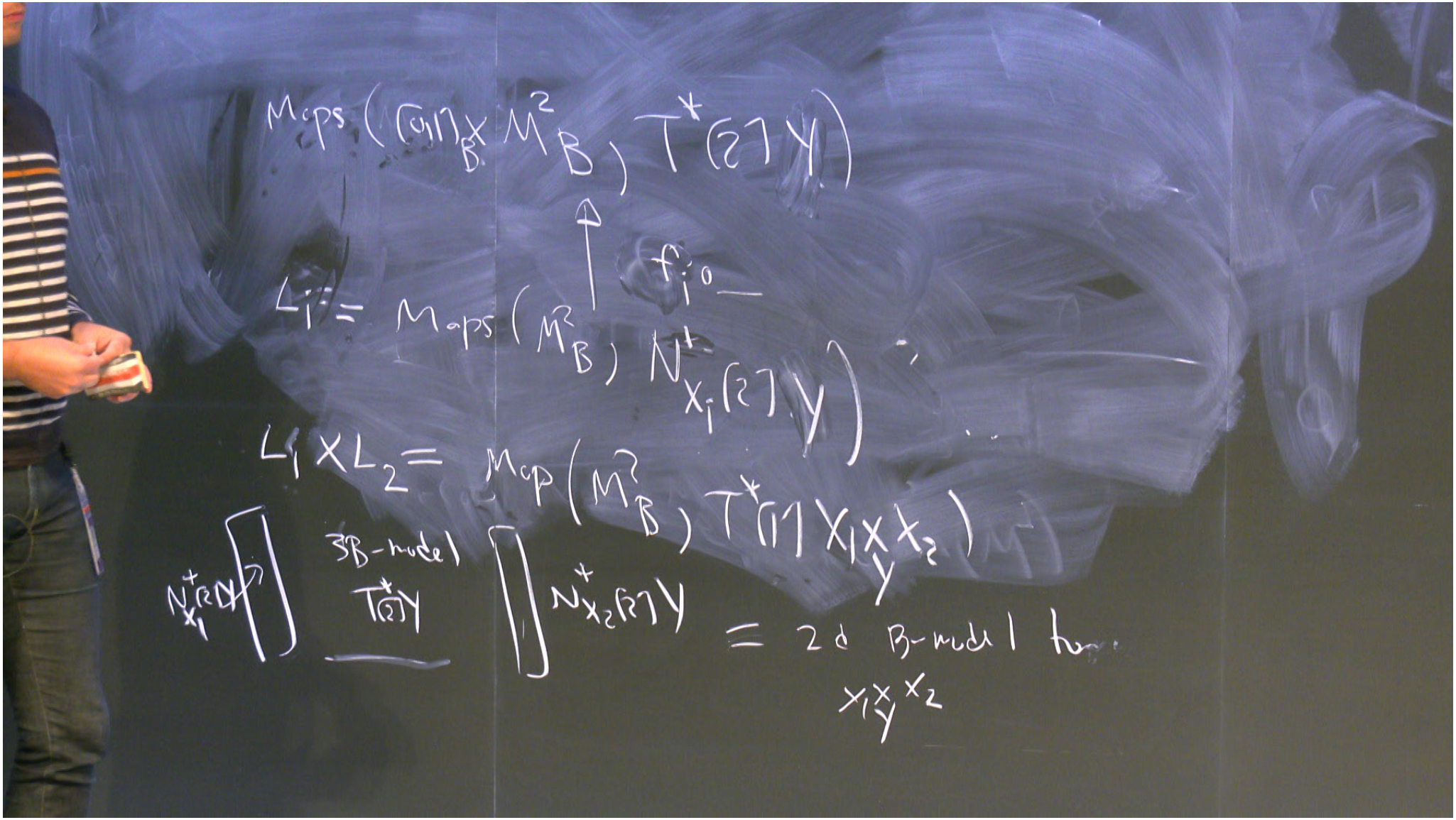
Composition

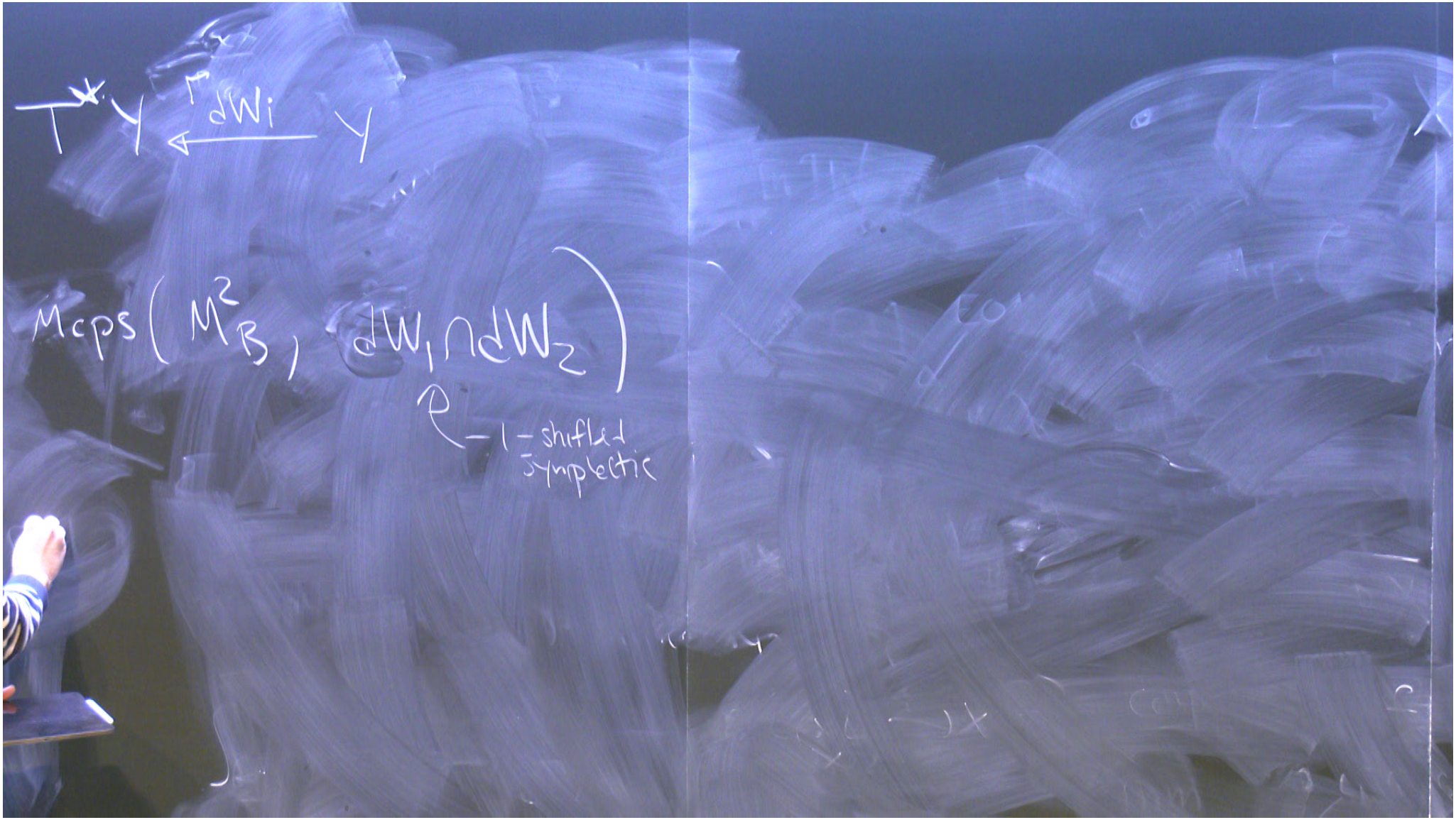
$L_1 \times_X L_2 \to (pt)_n$

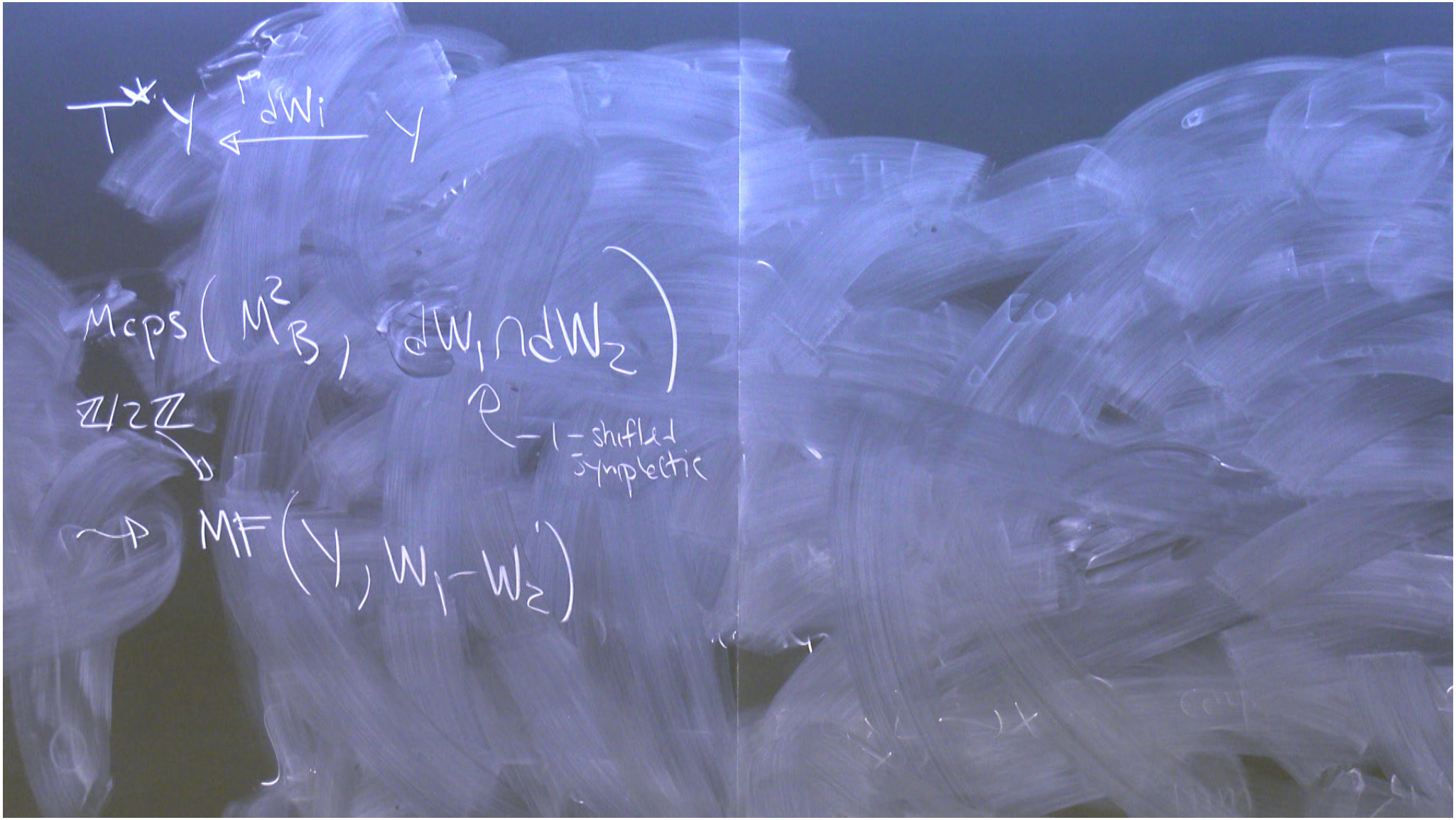

T*[n] W
A Lagrangian correspondence is
L → X × Y
shifted symplectic
2) composition of Lagrangian corr is Lagrangian
n+1
Thm
Composition
(n-1)-shifted symplectic
0 ~ f1*ω
f2*ω ~ 0
0 ~ i*ω ~ 0
Ω0 A^{2,cl}(Y, n) = A^{2,cl}(Y, n-1)

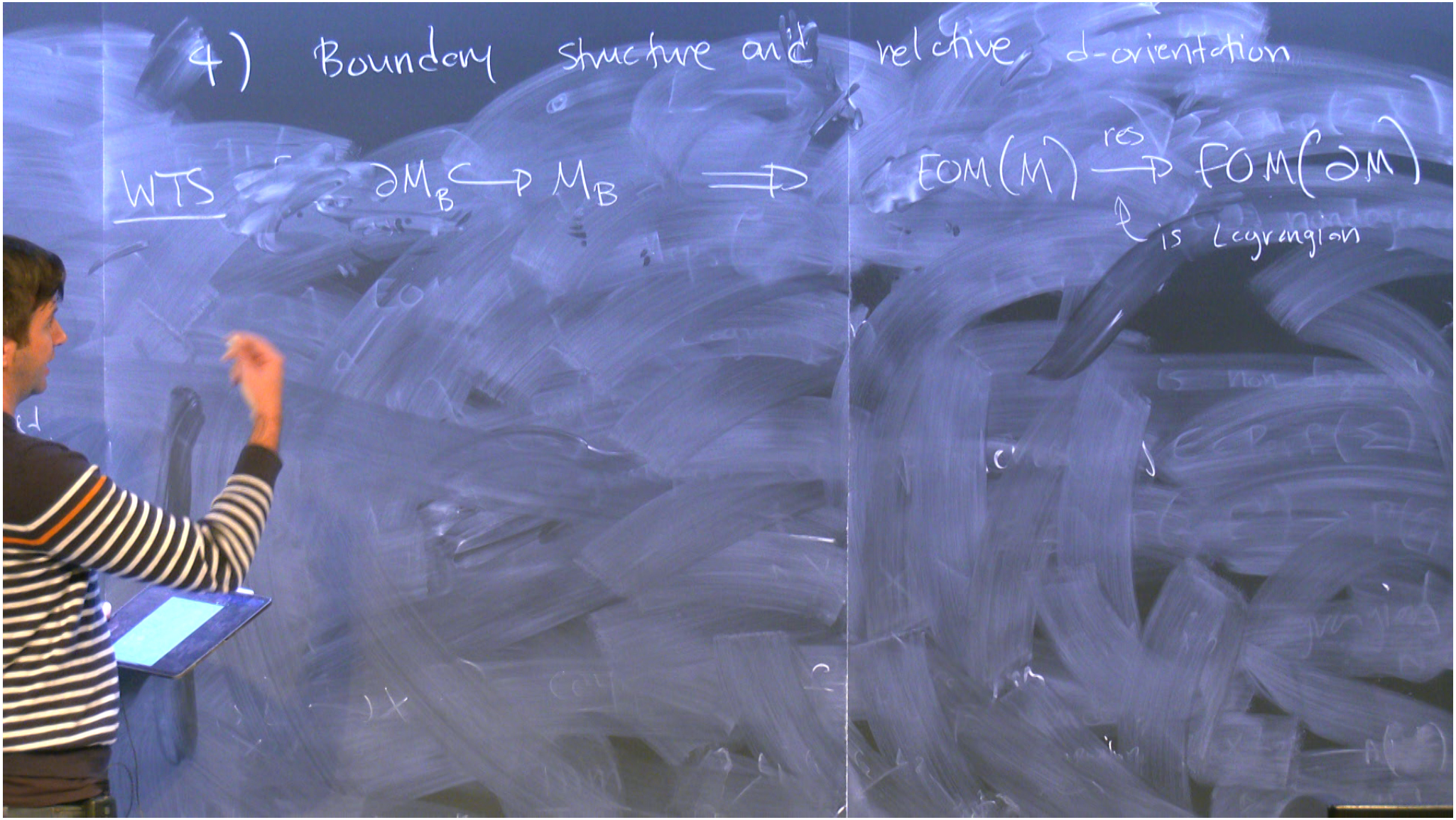

Ex
$X_1 \xrightarrow{f_1} Y$
$N^V_{X_1}(2) X_1 \to T^*(2) Y$
$\mathrm{Maps}(M^2_B, T^*(2) X)$

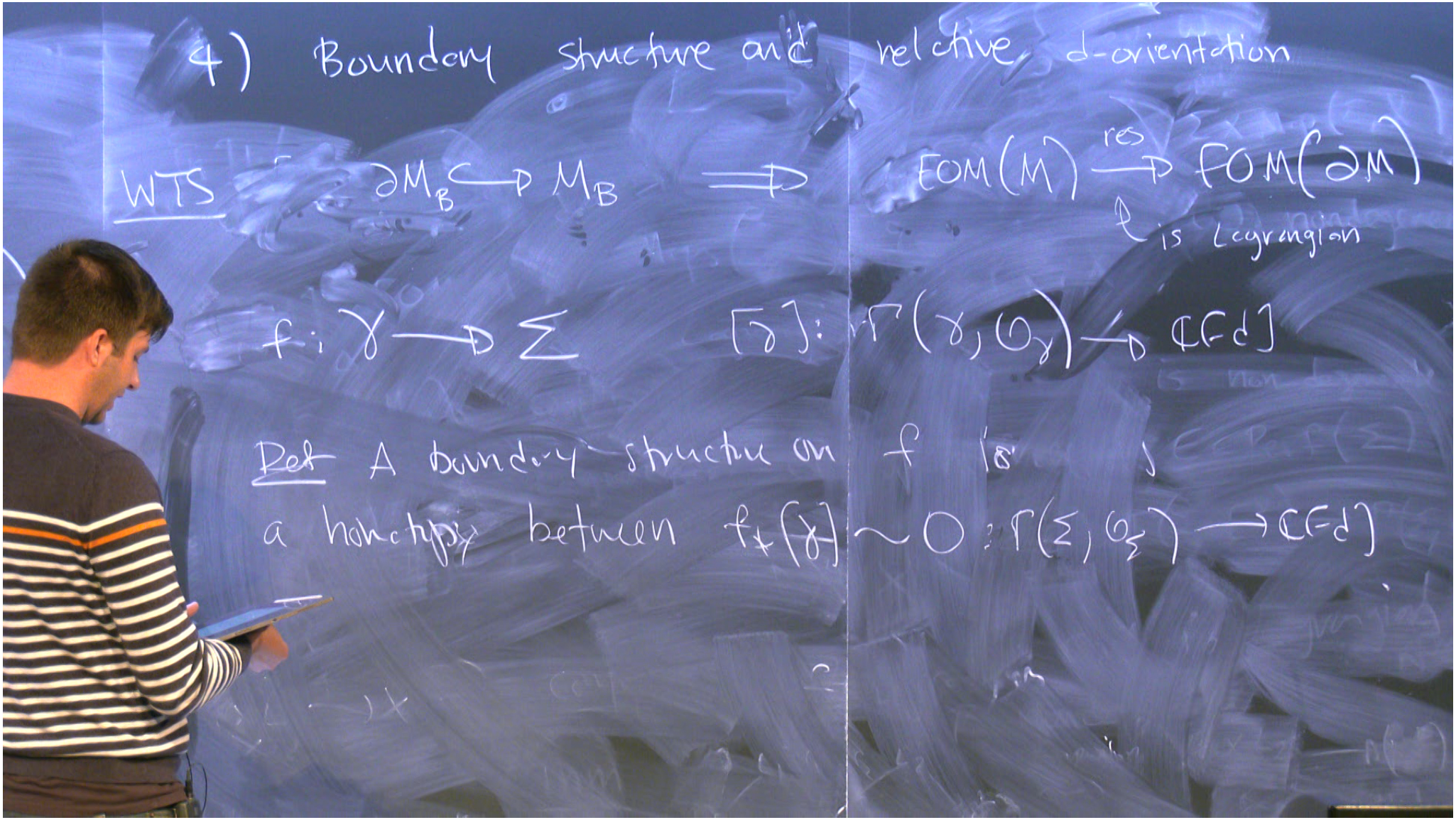

Ex
$X_i \xrightarrow{f_1} Y$
$N^V_{X_i}[2]Y \to T^*[2]Y$
$\mathrm{Maps}(\Gamma(\Sigma T_B^* \times M_B^2), T^*[2]Y)$
$L_i = \mathrm{Maps}(M_B^2, N^*_{X_i}[2]Y)$
f_{i0}
$L_1 \times L_2 = \mathrm{Map}(M_B^2, T^*[1]X_1 \times_Y X_2)$

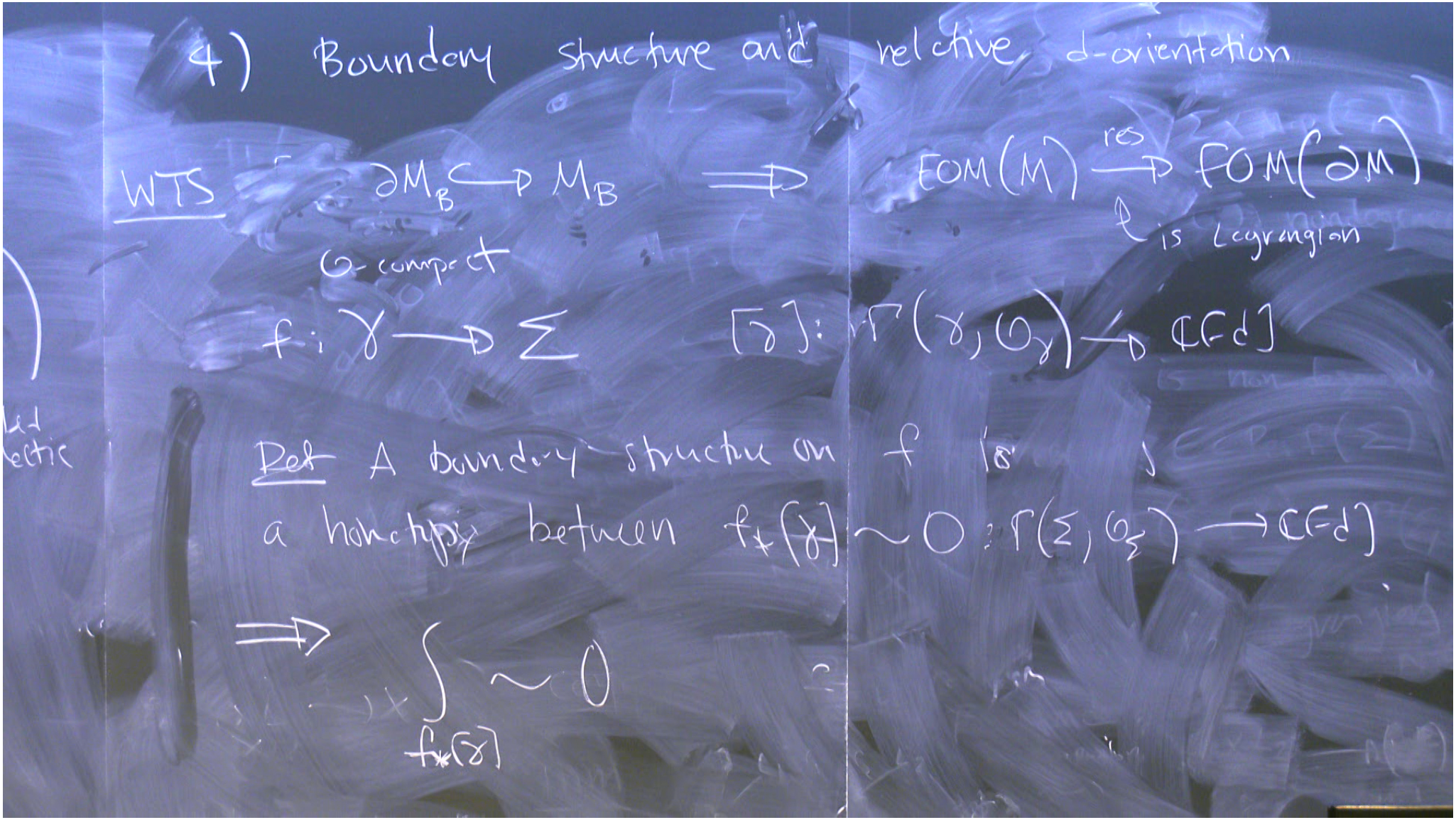

Maps (Σ(1) ×_B M²_B , T*(Σ) Y)
L_i = Maps (M²_B , N⁺_{X_i}(Σ) Y)
f_i ∘
L_1 × L_2 = Map (M²_B , T*(Σ) X_1 ×_Y X_2)
3B-model
T*(Σ) Y
= 2d B-model th
X_1 ×_Y X_2

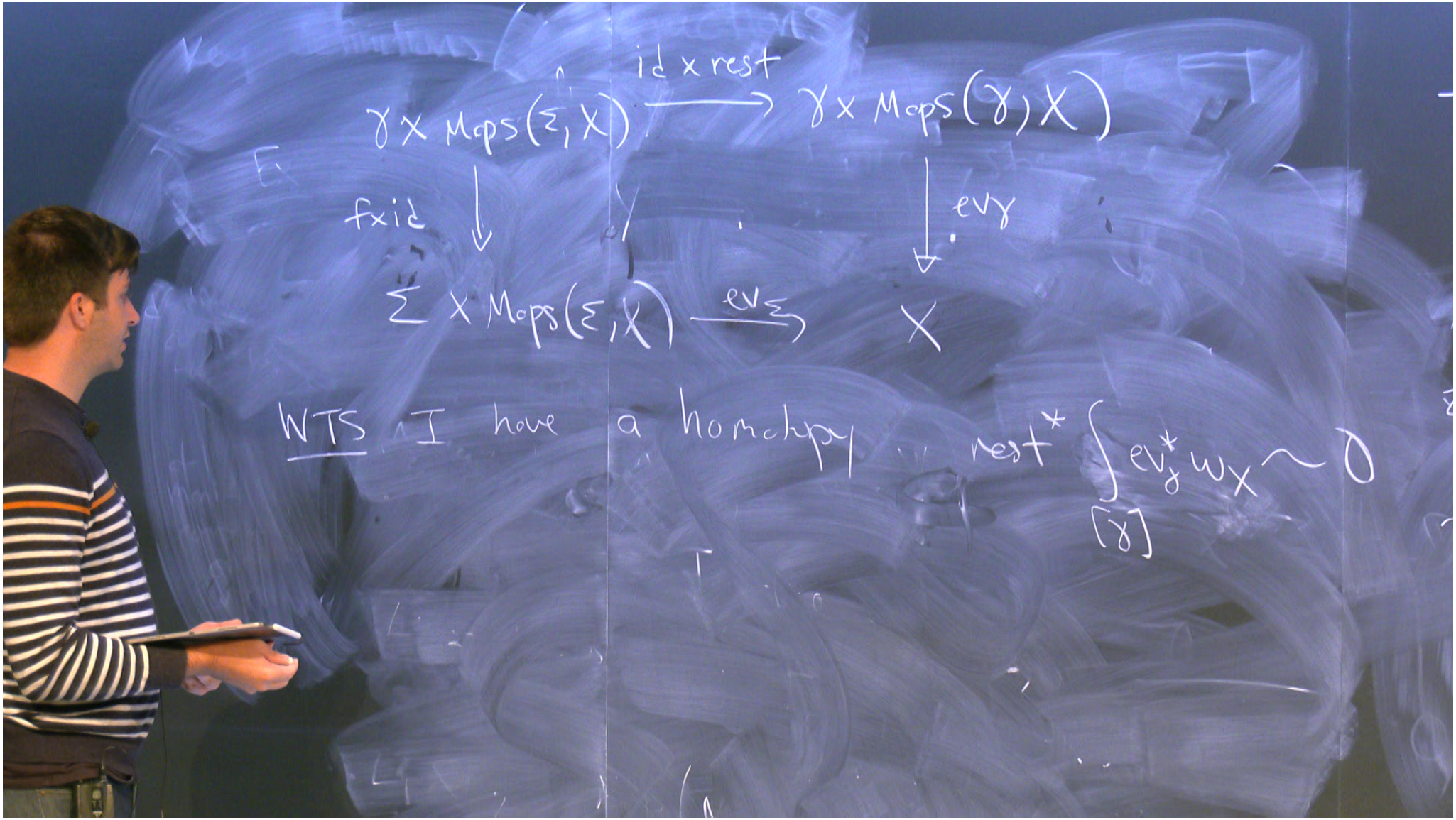

T^*Y
Γ_{dW_i}
Y
Y
$\mathrm{Mcps}\left(M^2_B, dW_1 \cap dW_2\right)$
-1 - shifted symplectic

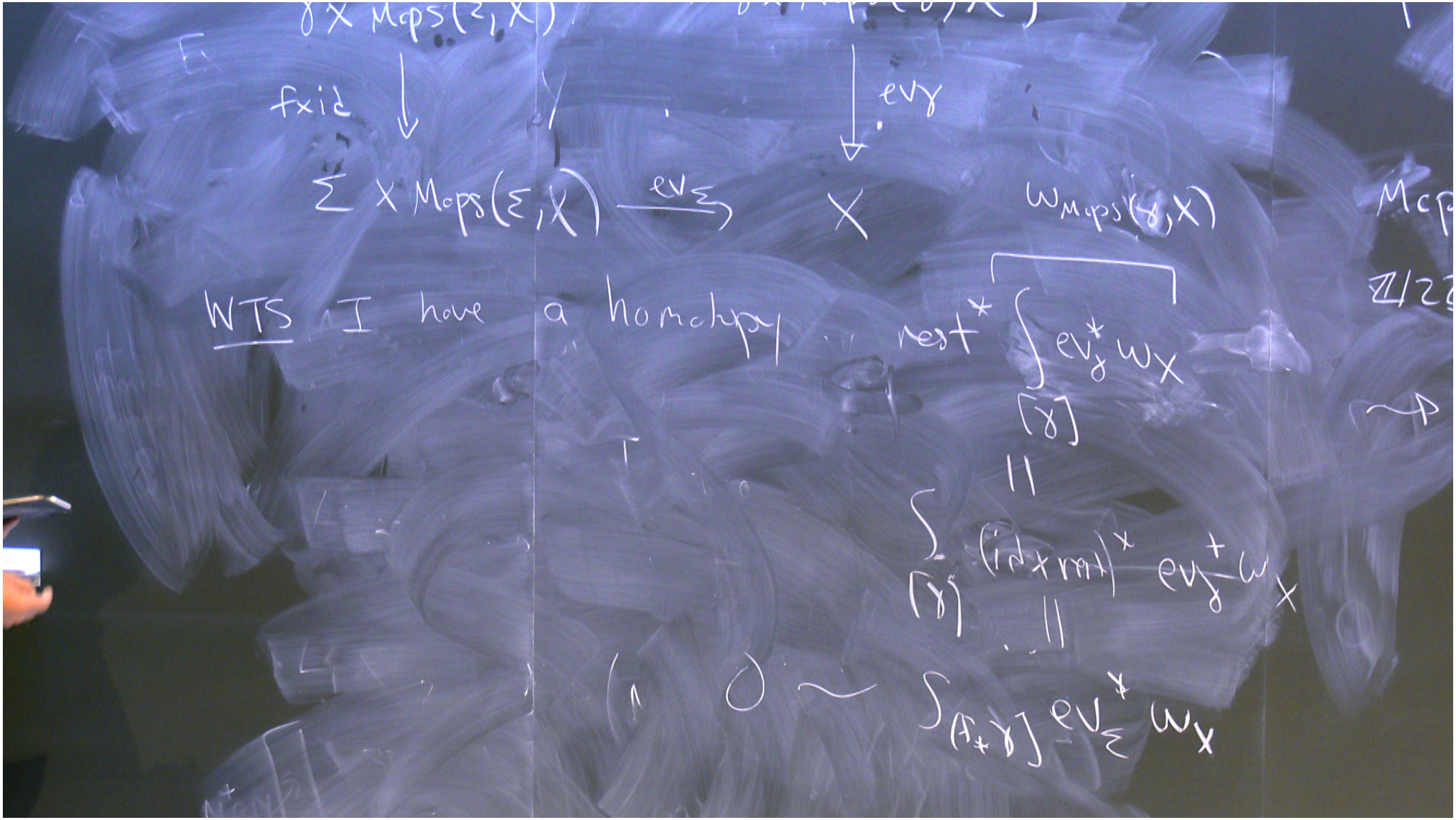

T^*Y
Y
dW_i
Y
$\mathrm{Mcps}(M_B^2, dW_1 \cap dW_2)$
$\mathbb{Z}/2\mathbb{Z}$
-1 - shifted symplectic
$\rightsquigarrow \mathrm{MF}(Y, W_1 - W_2)$

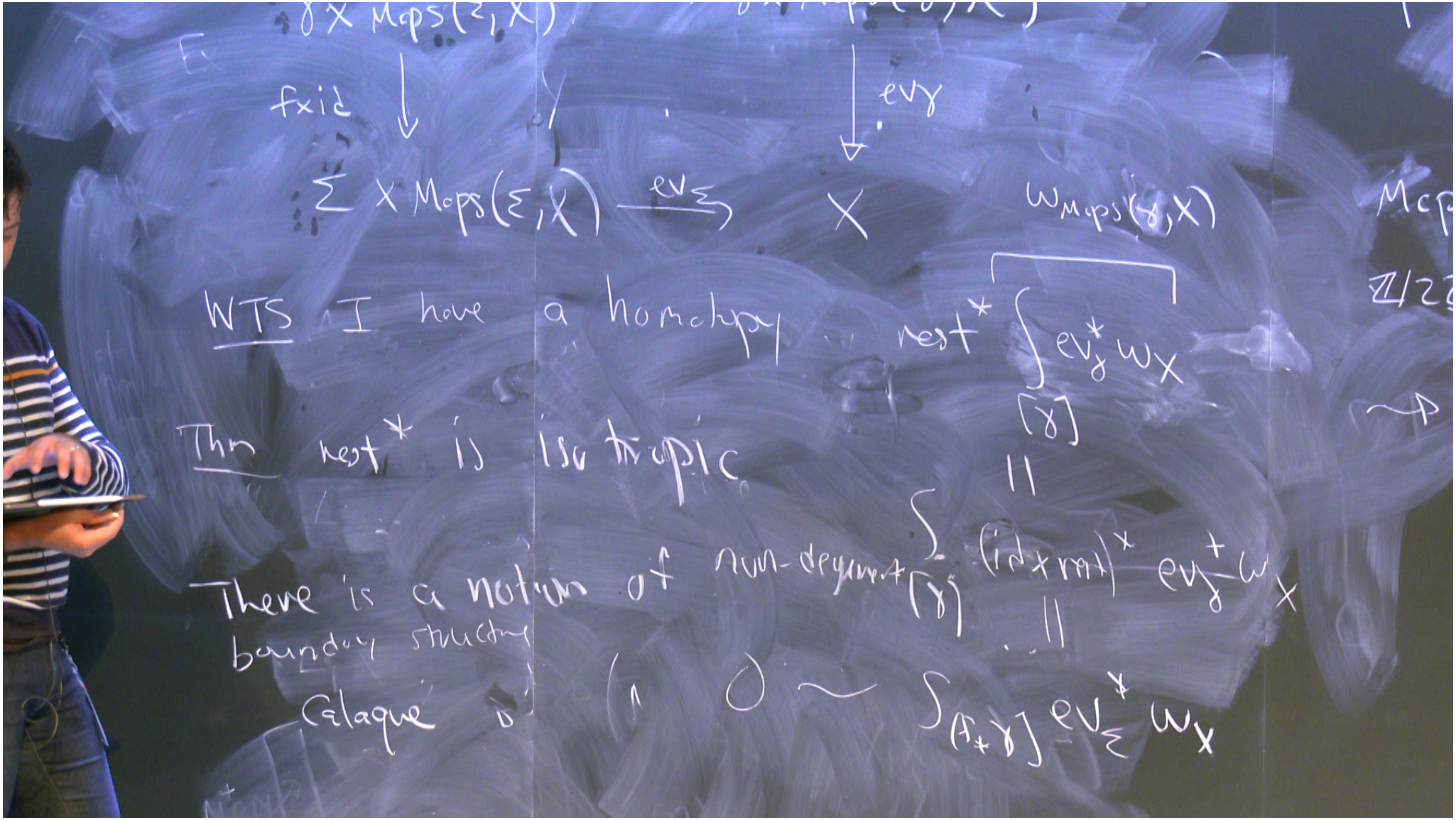

4) Boundary structure and relative d-orientation
WTS
$\partial M_B \hookrightarrow M_B \Longrightarrow$
$EOM(M) \xrightarrow{res} EOM(\partial M)$
is Lagrangian

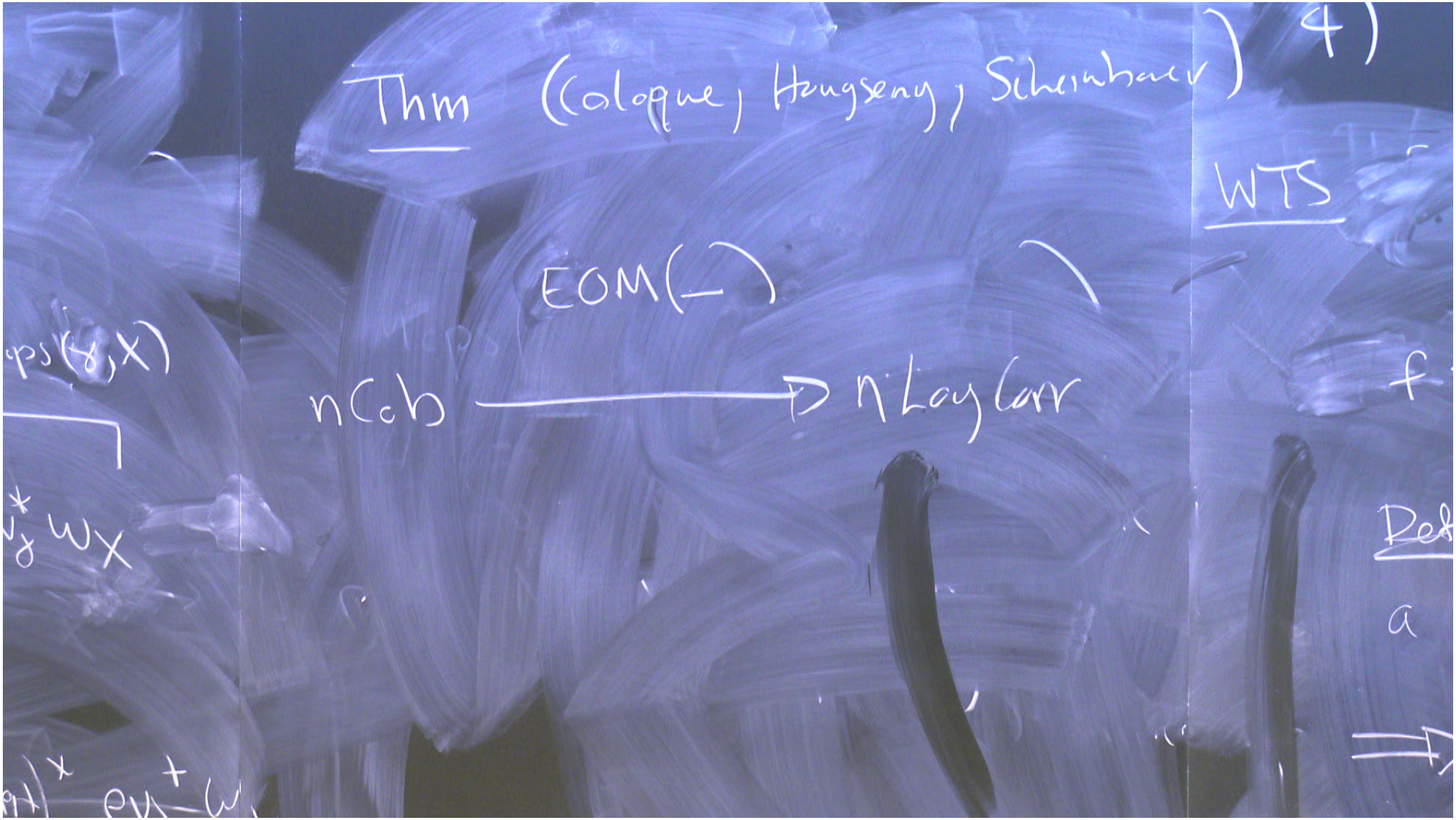

4) Boundary structure and relative d-orientation
WTS
$\partial M_B \hookrightarrow M_B \Rightarrow$
$EOM(M) \xrightarrow{res} EOM(\partial M)$
is Lagrangian
$f: \gamma \to \Sigma$
$[\gamma]: \Gamma(\gamma, \mathcal{O}_\gamma) \to \mathbb{C}[-d]$
Def A boundary structure on f is
a homotopy between $f_*[\gamma] \sim 0: \Gamma(\Sigma, \mathcal{O}_\Sigma) \to \mathbb{C}[-d]$

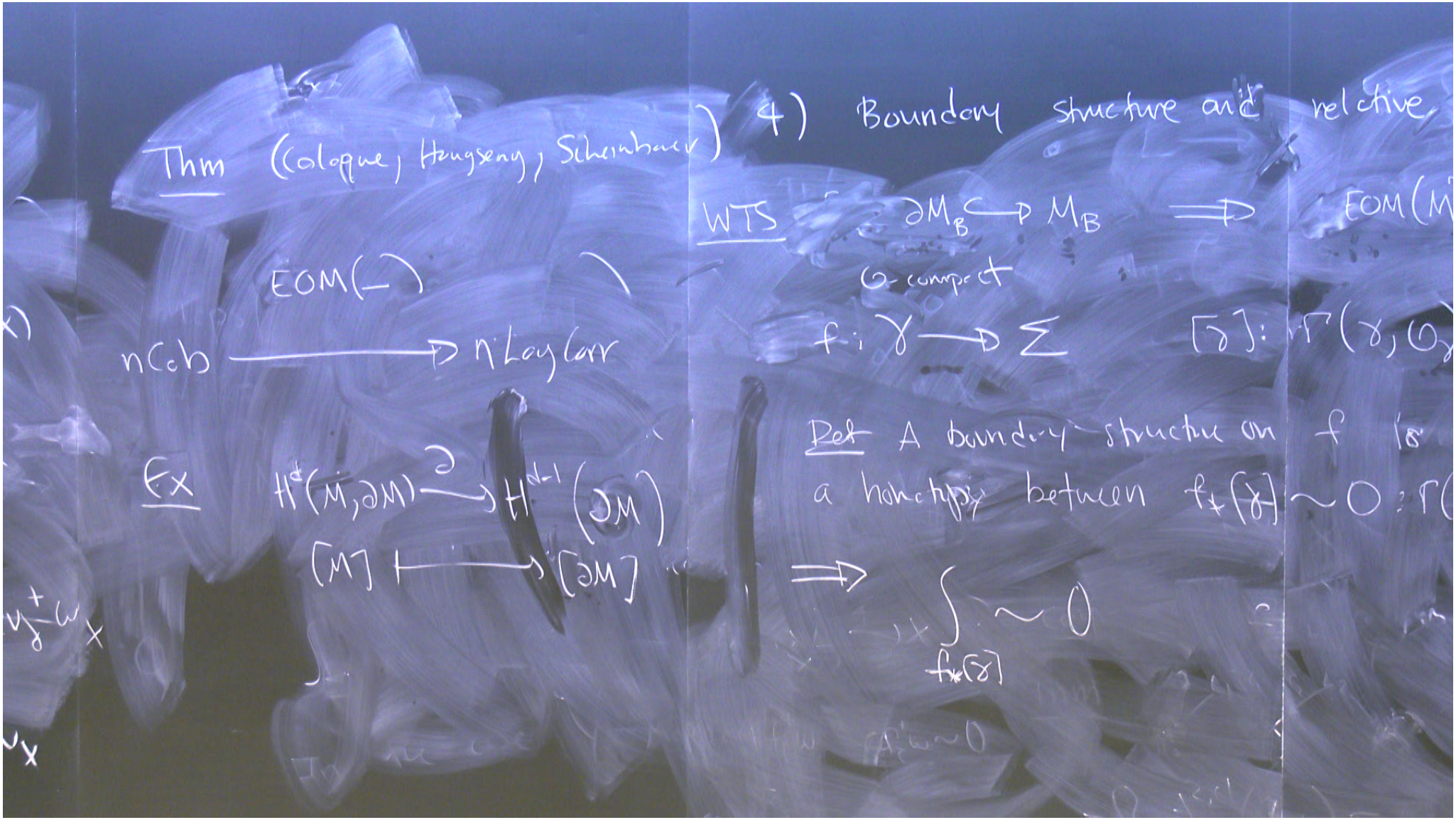

4) Boundary structure and relative d-orientation
WTS
$\partial M_B \hookrightarrow M_B \Rightarrow$
$EOM(M) \xrightarrow{res} EOM(\partial M)$
is Lagrangian
∂M_B compact
$f: \gamma \to \Sigma$
$[\gamma]: \Gamma(\gamma, \mathcal{O}_\gamma) \to \mathbb{C}[-d]$
Def A boundary structure on f is
a homotopy between $f_*[\gamma] \sim 0 : \Gamma(\Sigma, \mathcal{O}_\Sigma) \to \mathbb{C}[-d]$
$\Rightarrow \int_{f_*[\gamma]} \sim 0$

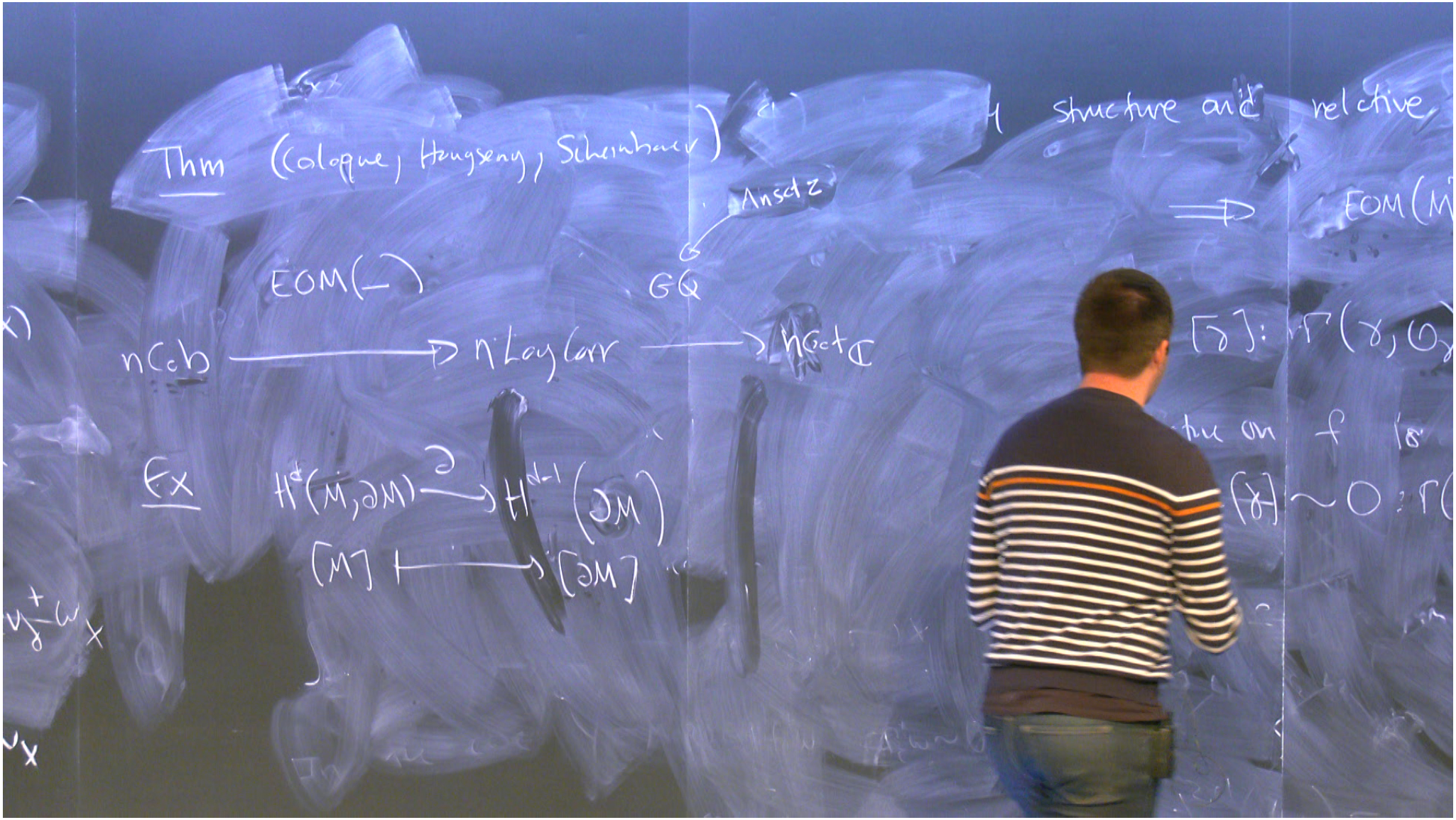

id x rest
γ x Maps(Σ, X)
γ x Maps(γ, X)
fxid
ev_γ
Σ x Maps(Σ, X)
ev_Σ
X
WTS I have a homotopy
rest* ∫_[γ] ev_γ* ω_X ~ 0

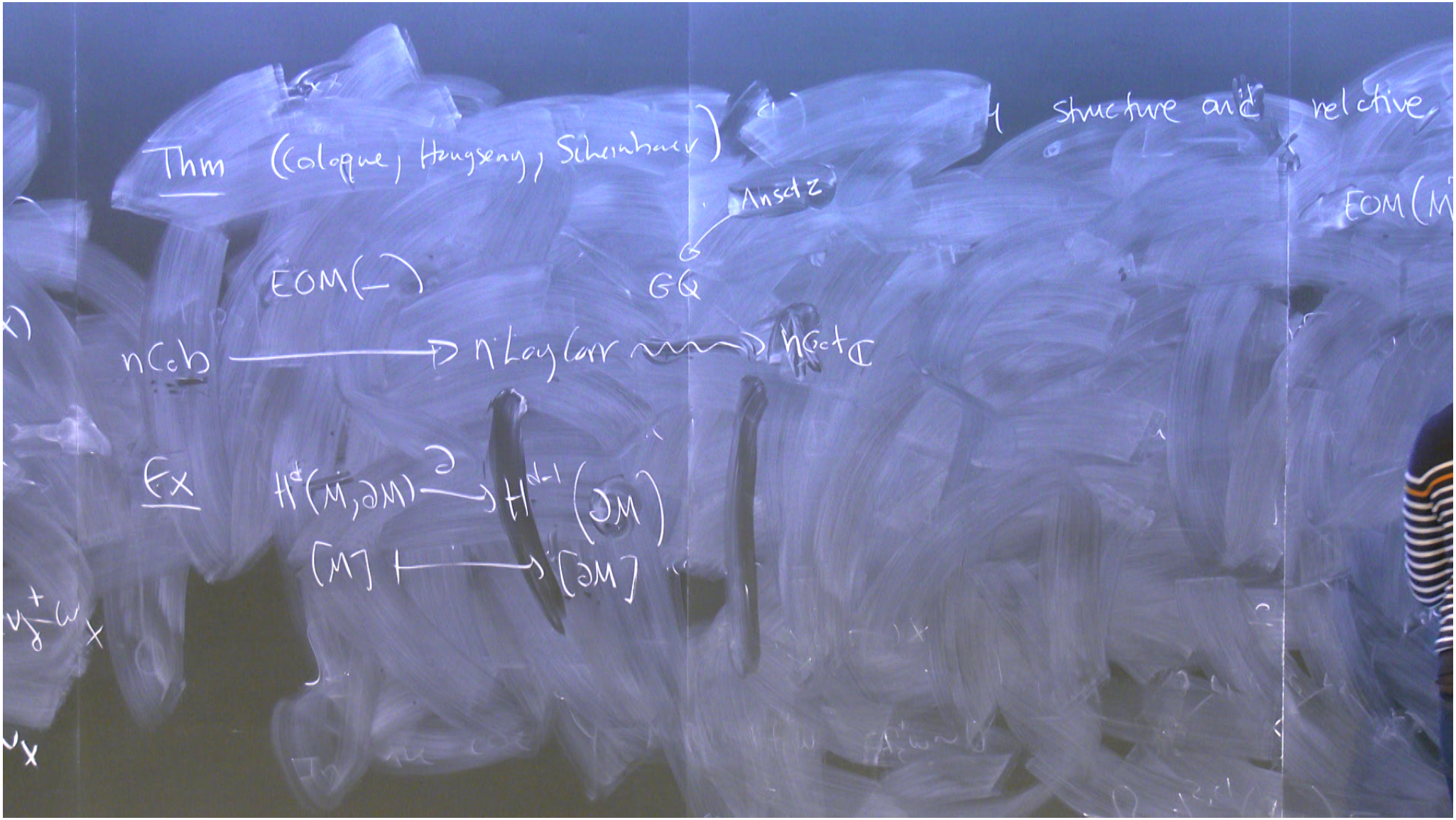

fxid
ev_γ
Σ × Maps(Σ, X)
ev_Σ
X
ω_Maps(γ,X)
WTS I have a homotopy
rest*
∫_[γ] ev_γ* ω_X
=
∫_[γ] (id × rot)^x ev_γ^+ ω_X
=
∫_[A*γ] ev_Σ* ω_X

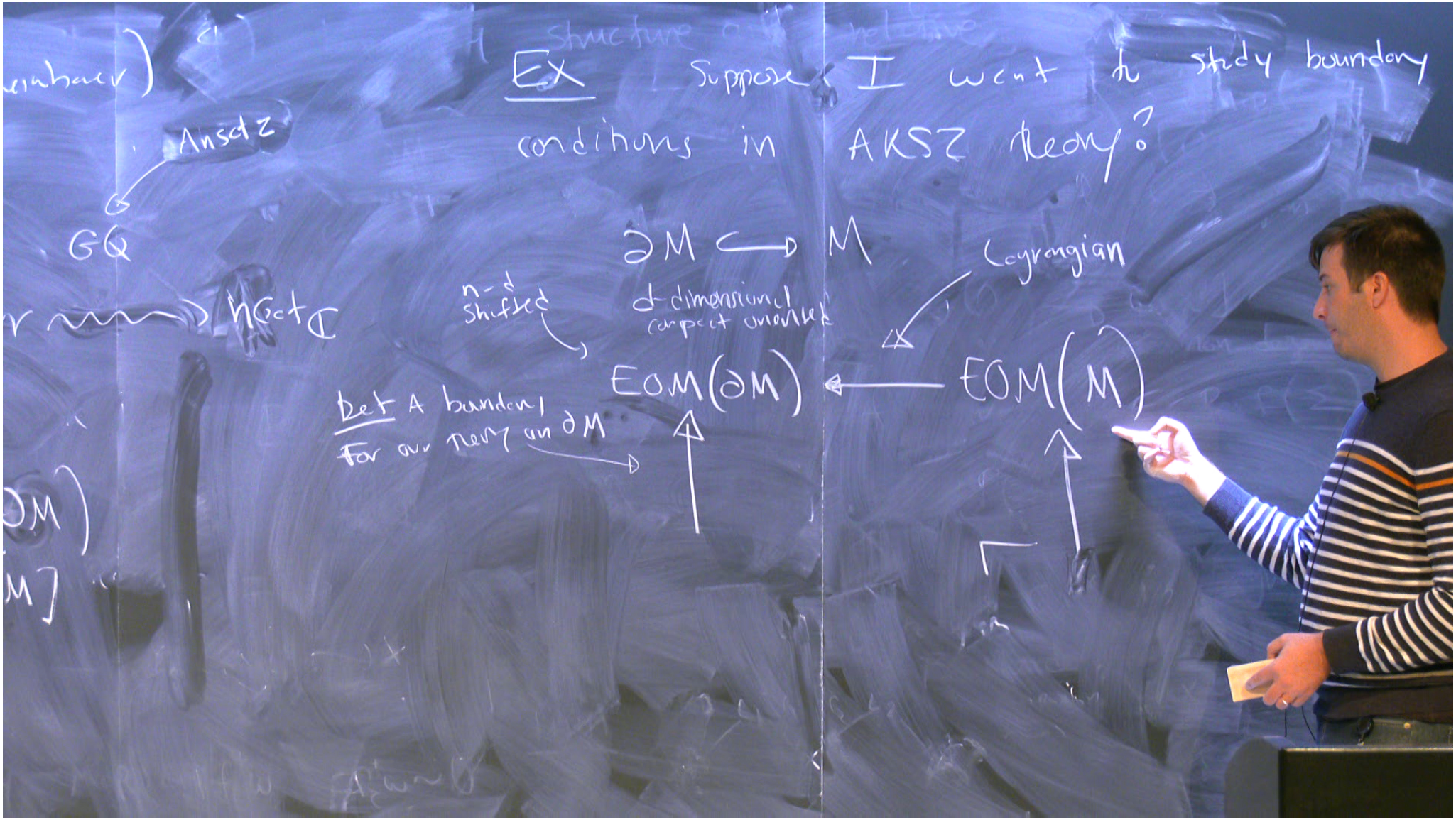

fxid
ev_γ
Σ × Maps(Σ, X)
ev_Σ
X
ω_Maps(γ,X)
WTS I have a homotopy
rest*
∫_[γ] ev_γ* ω_X
Thm rest* is isotopic
There is a notion of non-degenerate boundary structure
Calaque
∫_[Σ,γ] ev_Σ* ω_X

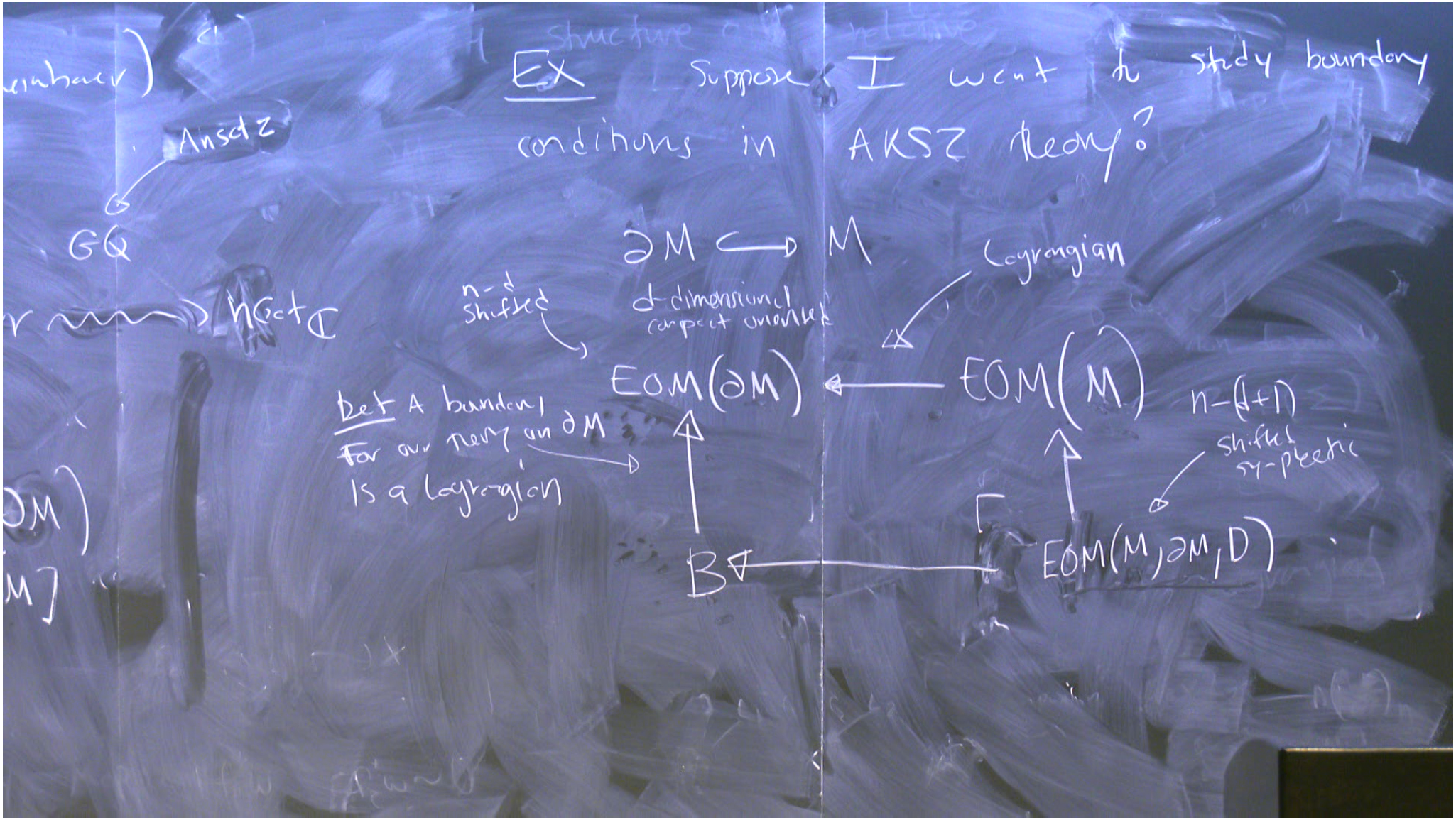

Thm (Cologne, Haugseng, Scheimbauer)
EOM(–)
nCob
n-Lay Corr
WTS


Thm (Cahiers, Haugseng, Scheimbauer)
EOM(–)
nCob ⟶ n-LaxCorr
Ex
$H^d(M,\partial M) \xrightarrow{\partial} H^{d-1}(\partial M)$
$[M] \longmapsto [\partial M]$
4) Boundary structure and relative
WTS
$\partial M_B \hookrightarrow M_B \Longrightarrow$ EOM(M
G-compact
$f : \gamma \longrightarrow \Sigma$
$[\gamma]: \Gamma(\gamma, G$
Def A boundary structure on f is
a homotopy between $f_*[\gamma] \sim 0 : \Gamma($
$\Longrightarrow \int_{f_*[\gamma]} \sim 0$


Thm (Cattaneo, Haugseng, Scheimbauer)
Ansatz
EOM(–)
GQ
nCob
nLaxCorr
hGpdC
structure and
relative
EOM(M
Ex
H^d(M, ∂M)
H^{d-1}(∂M)
[M] ⟼ [∂M]
[γ]:
[γ] ~ 0 :


Thm (Cologne, Hungseng, Scheimbauer)
Ansatz
EOM(–)
GQ
nCob
n-LaxCorr
hGot C
Ex
[M] ⟼ [∂M]


Ansatz
GQ
$\mathfrak{hGct}_{\mathbb{C}}$
Ex Suppose I want to study boundary
conditions in AKSZ theory?
$\partial M \hookrightarrow M$
n-d shifted
d-dimensional compact oriented
Lagrangian
$EOM(\partial M)$
$EOM(M)$
Def A boundary for our theory on ∂M


Ansatz
GQ
$\mathfrak{g}_{\mathrm{Get}}\mathbb{C}$
Ex Suppose I want to study boundary conditions in AKSZ theory?
$\partial M \hookrightarrow M$
d-dimensional compact oriented
n-d shifted
Lagrangian
Def A boundary for our theory on ∂M is a Lagrangian
$\mathrm{EOM}(\partial M)$
$\mathrm{EOM}(M)$
$n-(d+1)$ shifted symplectic
$B\Gamma$
$\mathrm{EOM}(M, \partial M, D)$