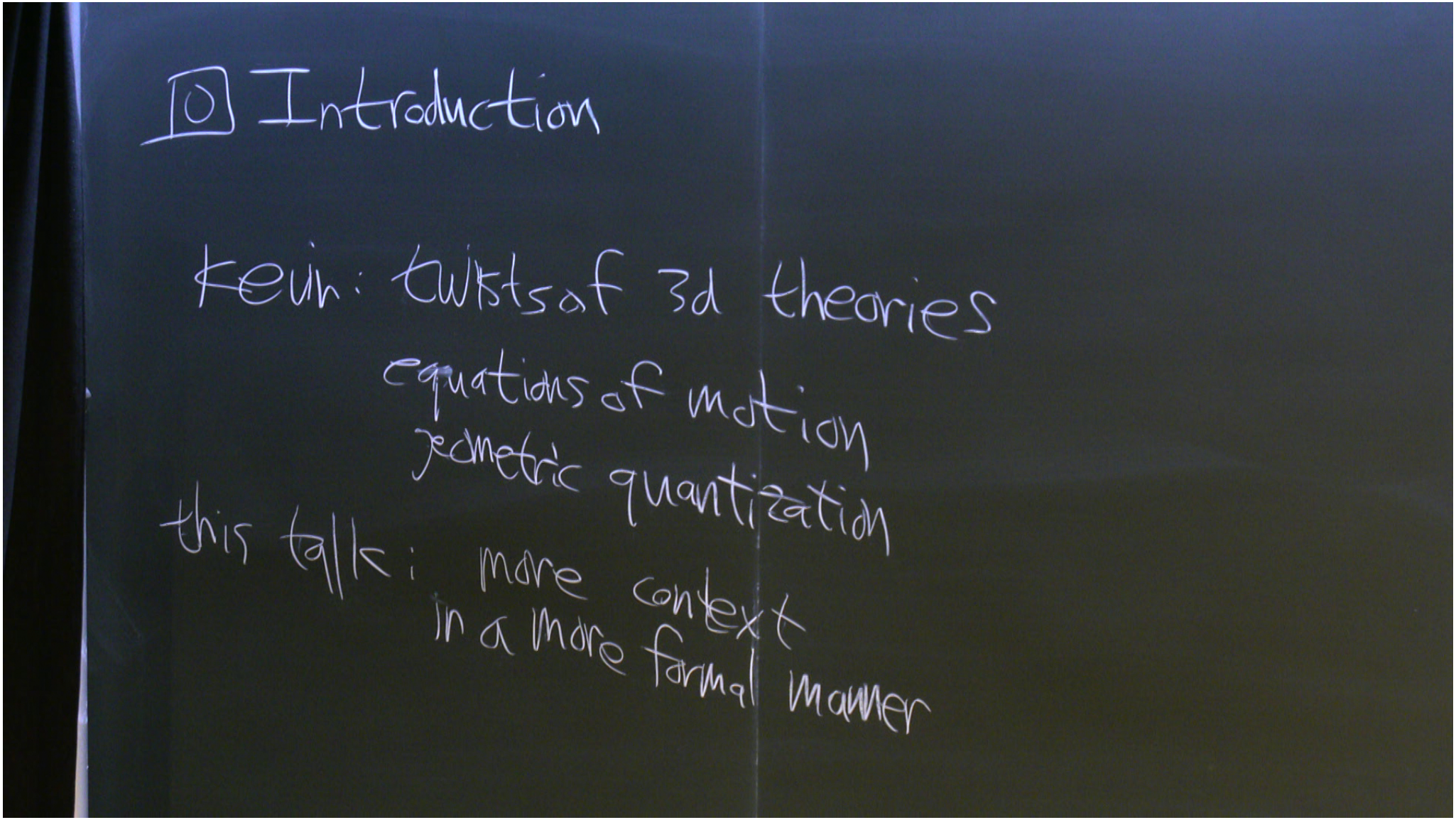
Title: Classical BV Formalism and Topological Quantum Field Theory

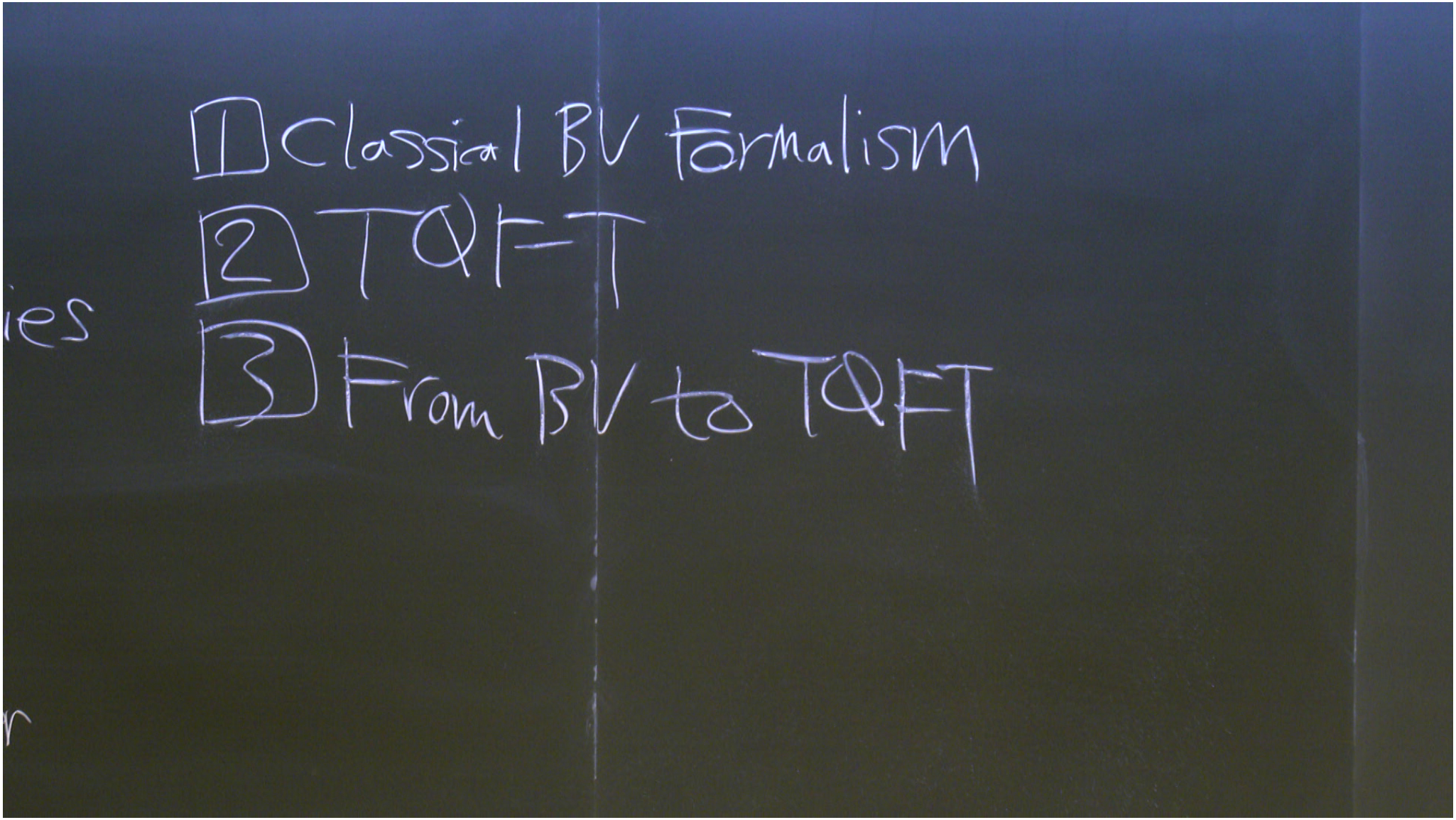
Speakers: Philsang Yoo

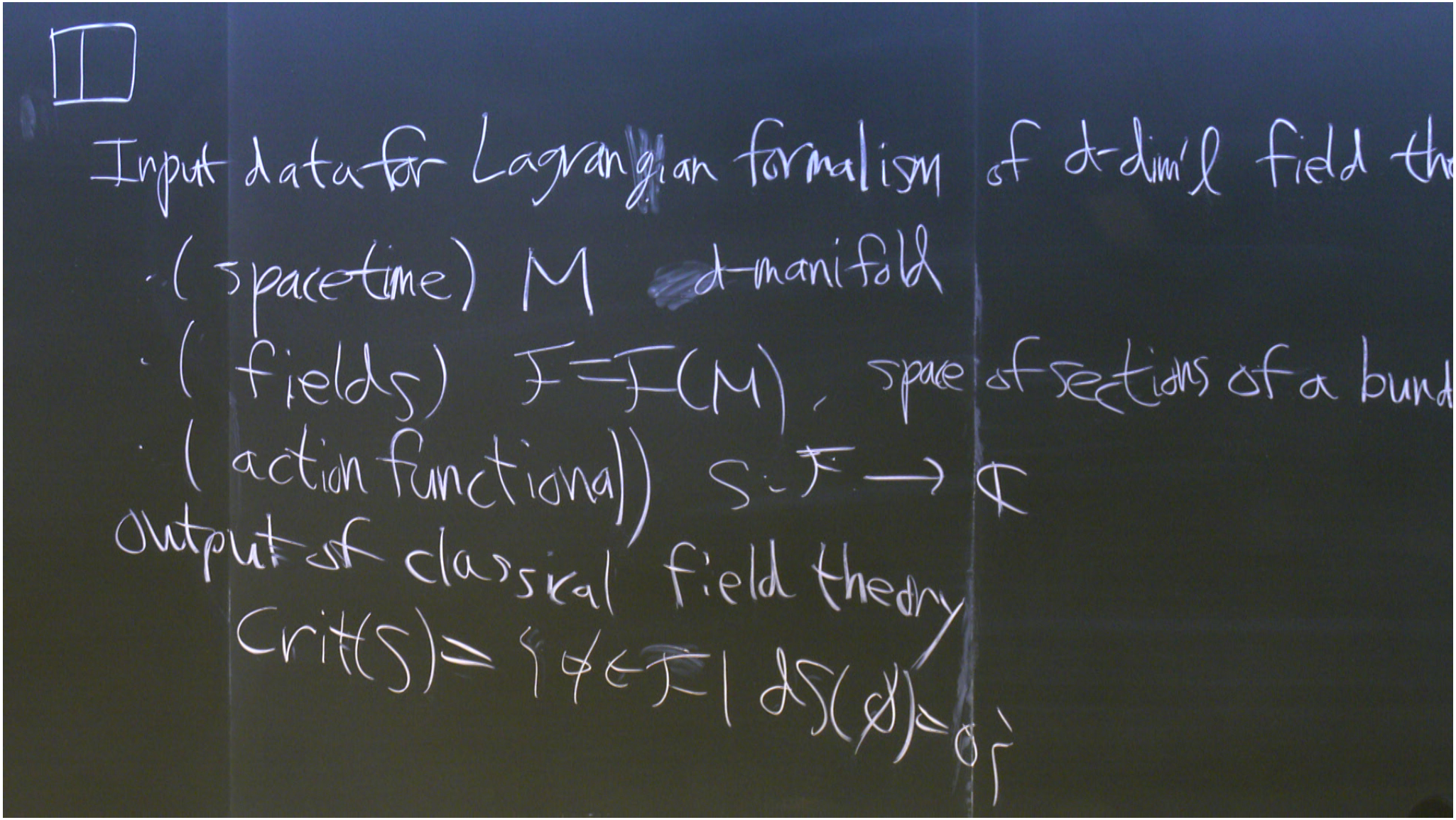
Collection: QFT for Mathematicians 2022

Date: June 20, 2022 - 3:30 PM

URL: https://pirsa.org/22060065


0 Introduction
Kevin: twists of 3d theories
equations of motion
geometric quantization
this talk: more context
in a more formal manner


1 Classical BV Formalism
2 TQFT
3 From BV to TQFT


Input data for Lagrangian formalism of d-dim'l field th
(spacetime) M d-manifold
(fields) F = F(M), space of sections of a bund
(action functional) S : F → ℝ
Output of classical field theory
Crit(S) = {φ ∈ F | dS(φ) = 0}

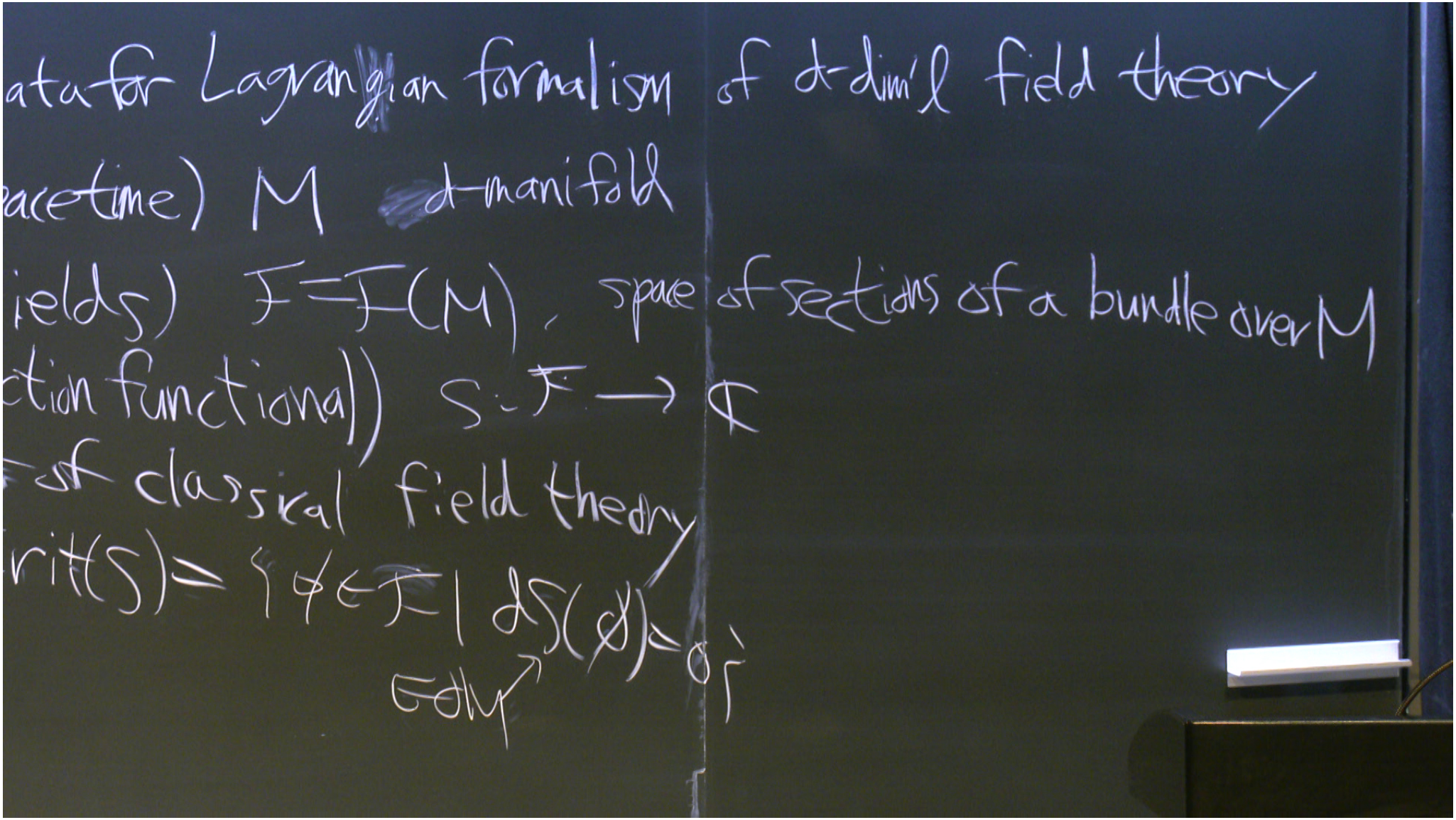
Input data for Lagrangian formalism of d-dim'l field the

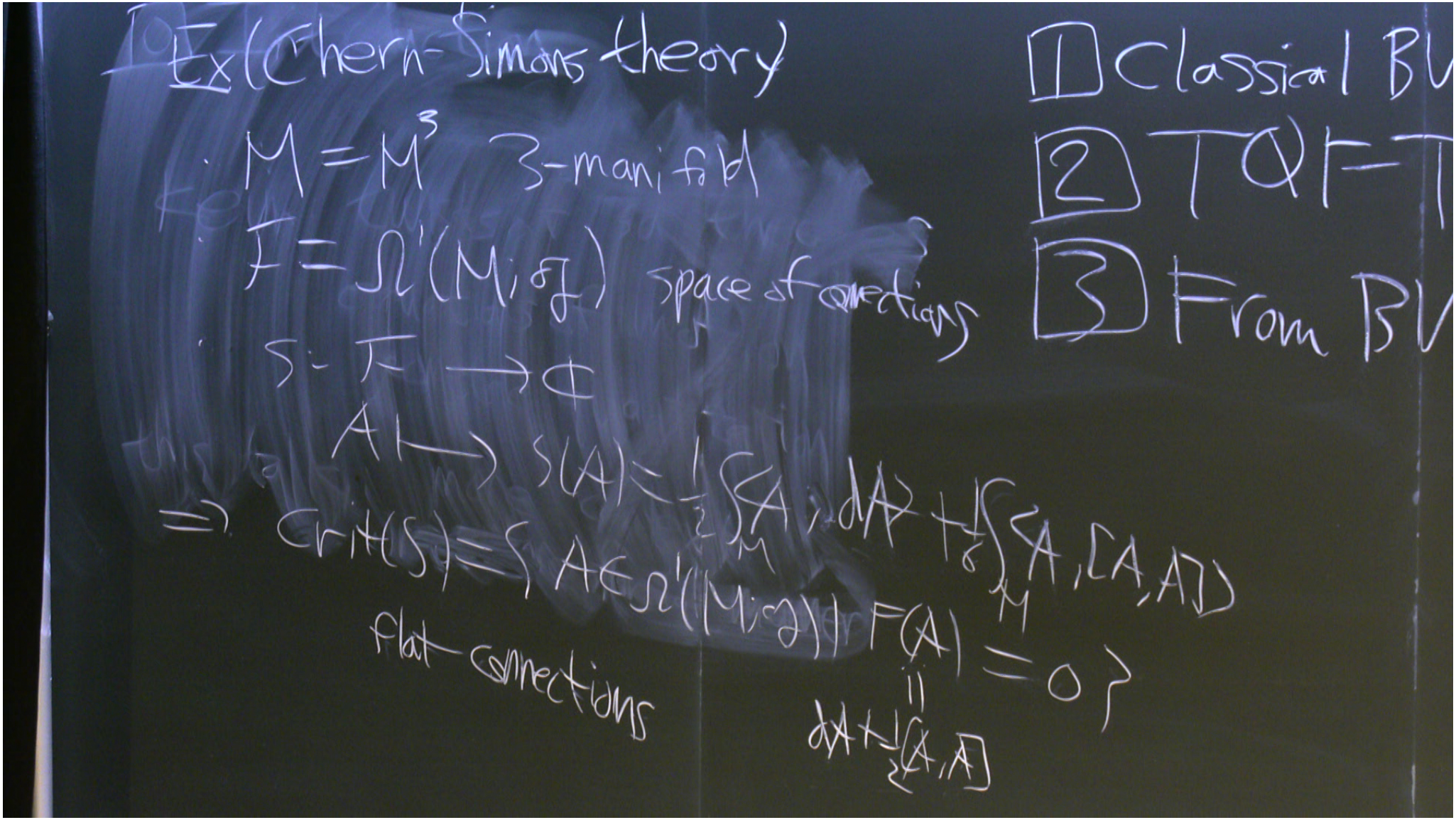
- (spacetime) M d-manifold
- (fields) $\mathcal{F} = \mathcal{F}(M)$, space of sections of a bund
- (action functional) $S : \mathcal{F} \to \mathbb{C}$

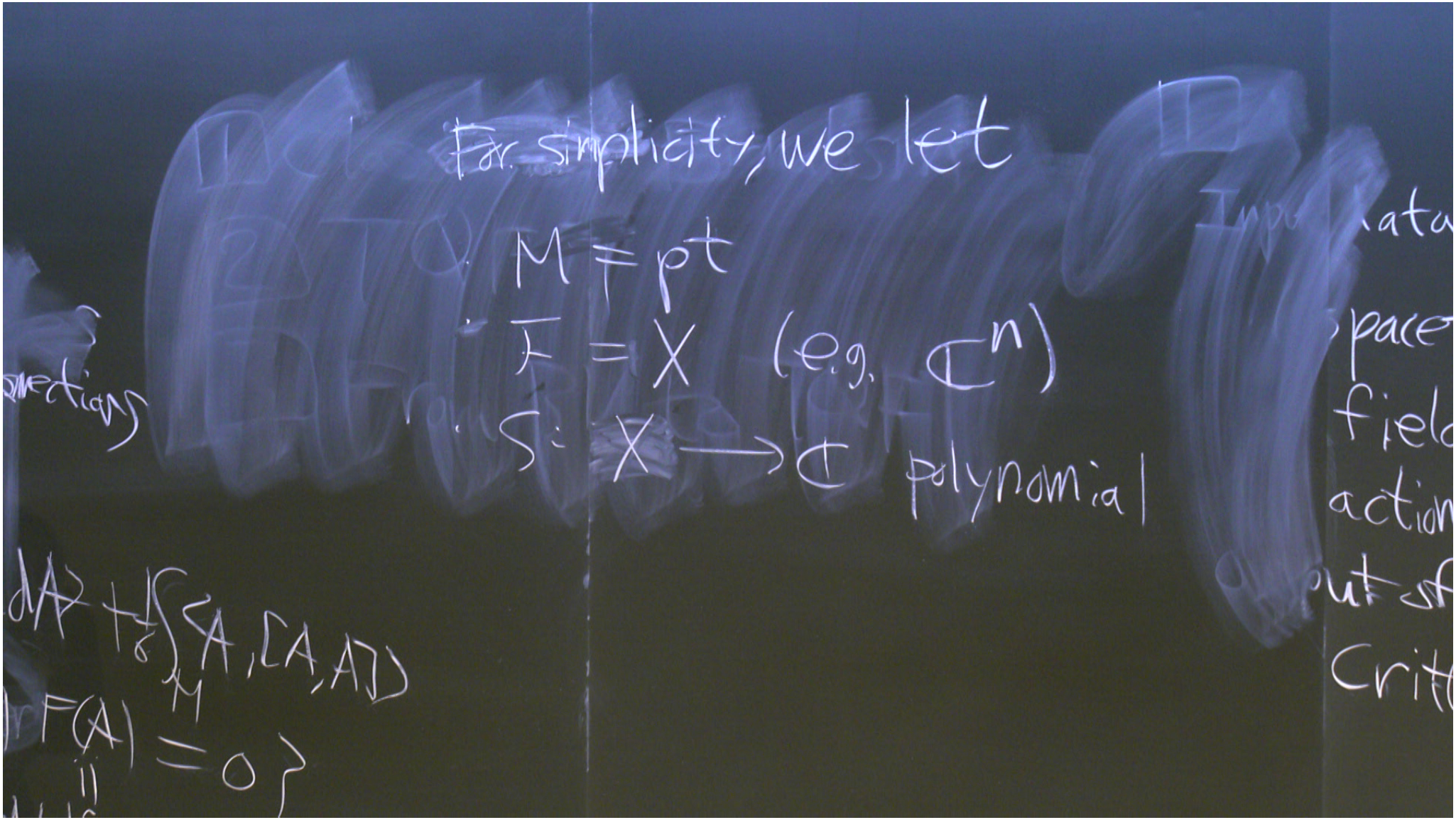
Output of classical field theory

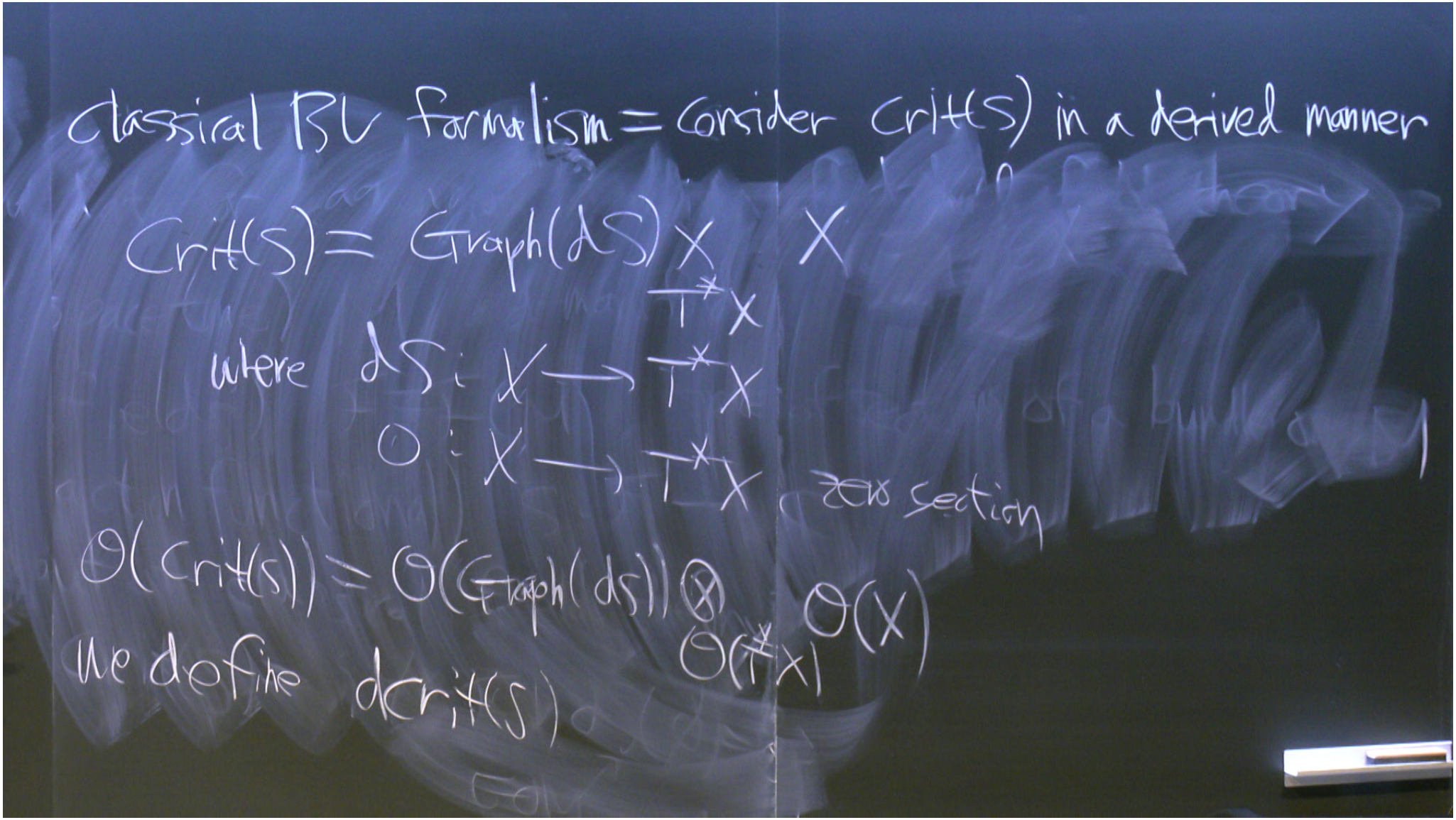
$$\mathrm{Crit}(S) = \{\phi \in \mathcal{F} \mid dS(\phi) = 0\}$$

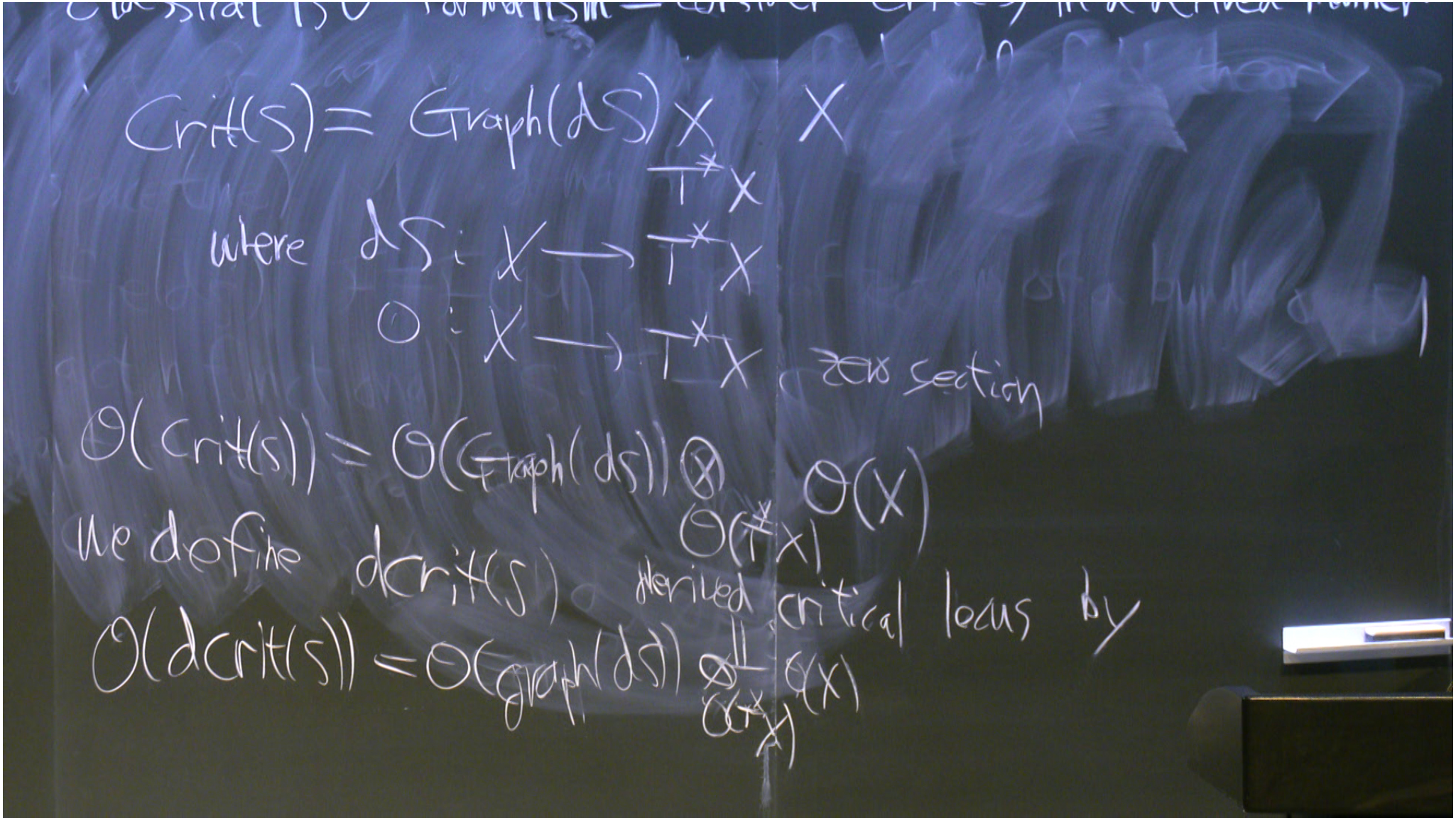
$$= \mathrm{Ker}\, d_{\mathrm{dR}}$$

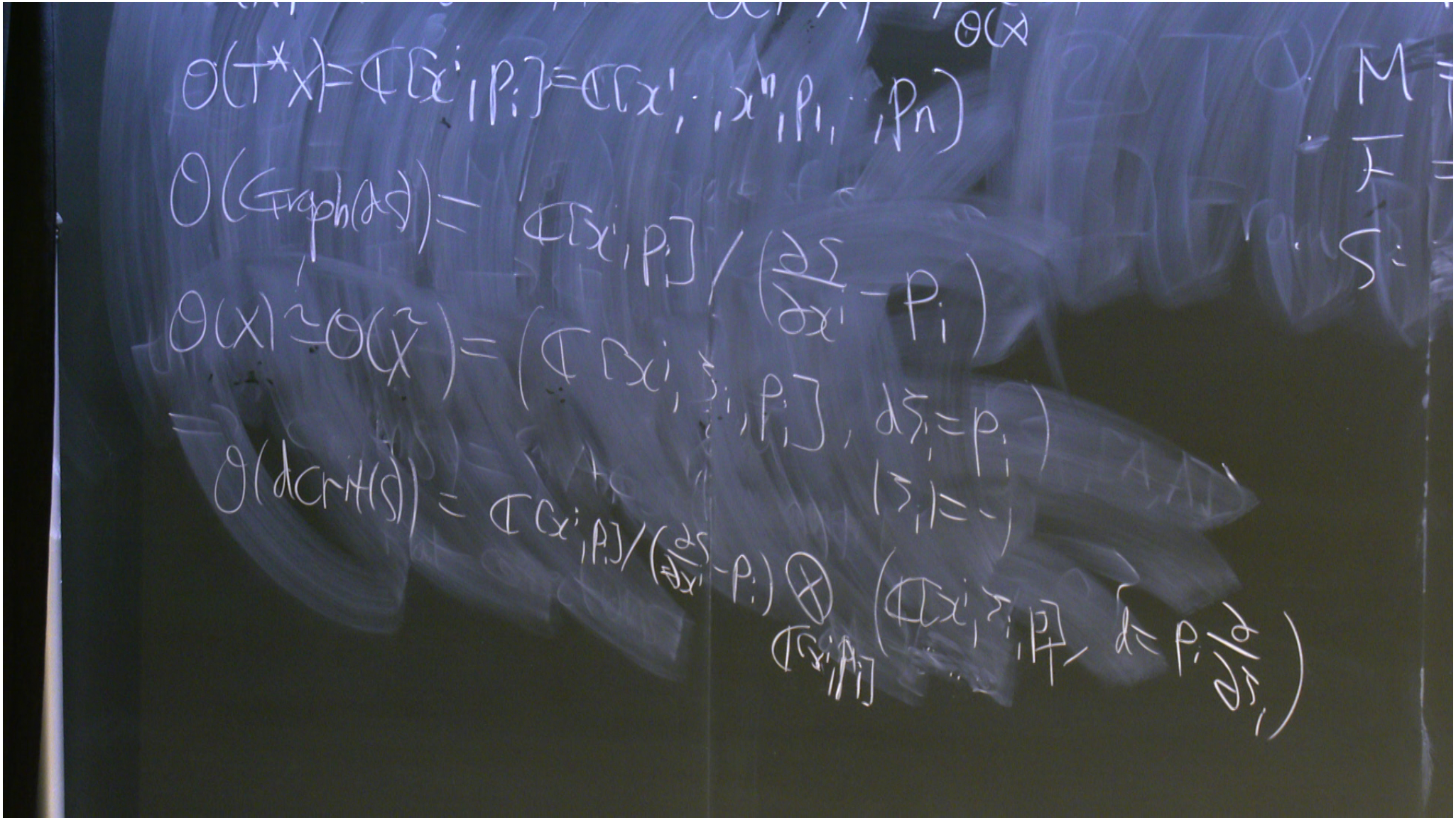

ata for Lagrangian formalism of d-dim'l field theory
acetime) M d-manifold
ields) F = F(M) space of sections of a bundle over M
ction functional) S: F → ℝ
of classical field theory
rit(S) = {φ ∈ F | dS(φ) = 0}
Eqly

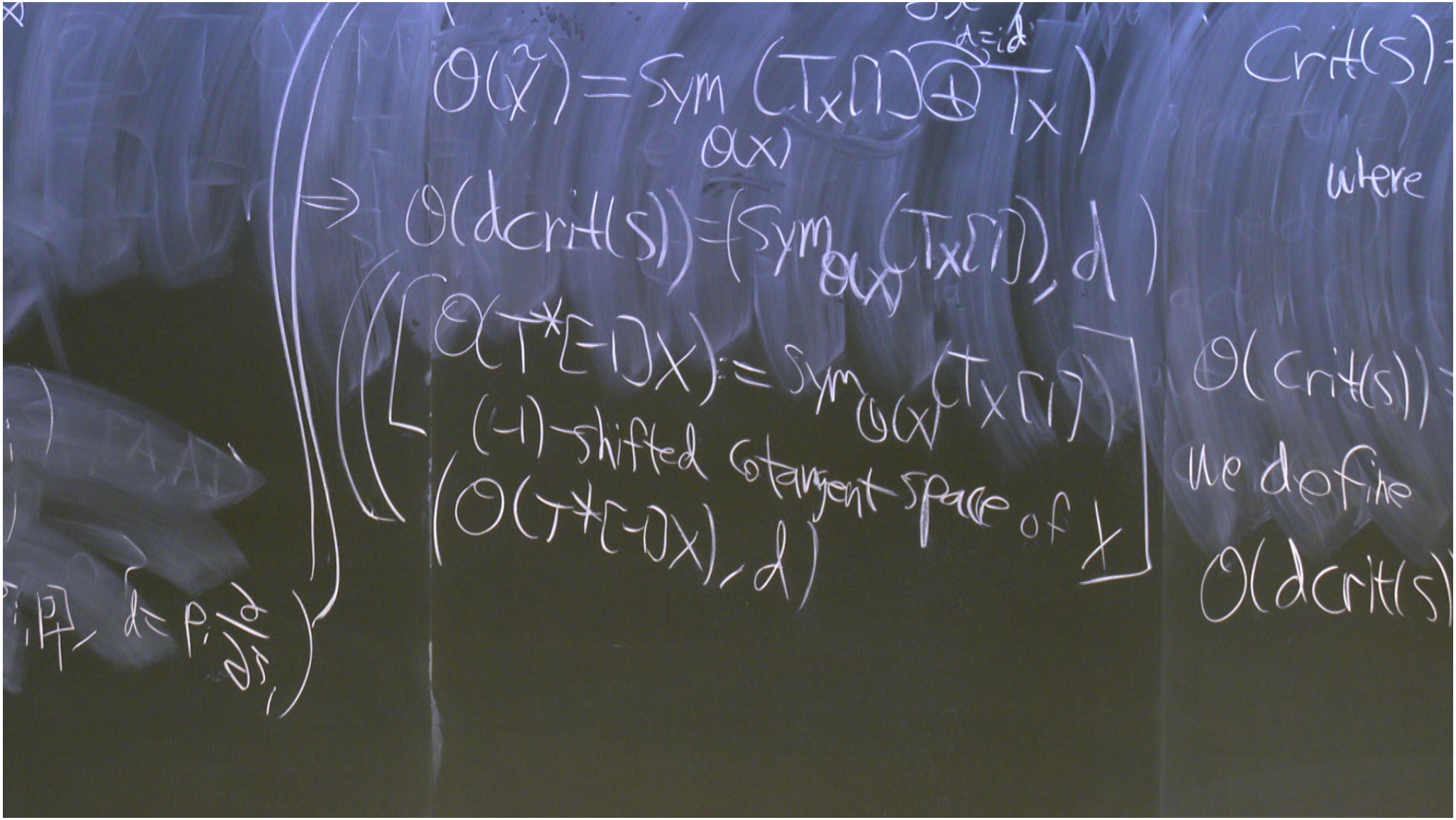

Ex (Chern-Simons theory
M = M^3 3-manifold
F = Ω^1(M; g) space of connections
S: F → C
A ⟼ S(A) = 1/2 ∫_M ⟨A, dA⟩ + 1/6 ∫_M ⟨A, [A, A]⟩
⇒ Crit(S) = { A ∈ Ω^1(M; g) | F(A) = 0 }
flat connections
F(A) = dA + 1/2 [A, A]
1 Classical BV
2 TQFT
3 From BV

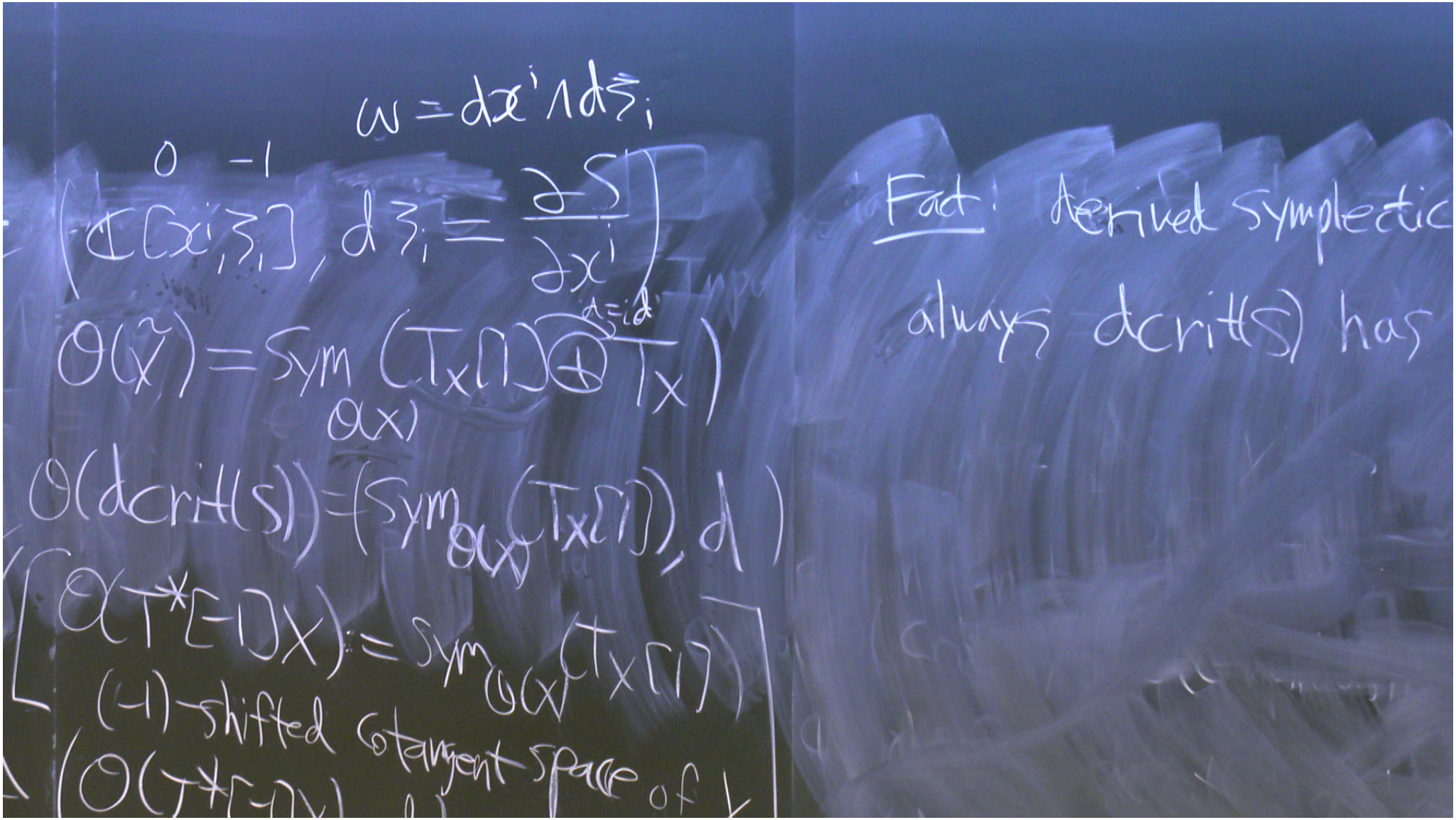

For simplicity, we let
M = pt
X = X (e.g. ℂ^n)
S : X ⟶ ℂ polynomial

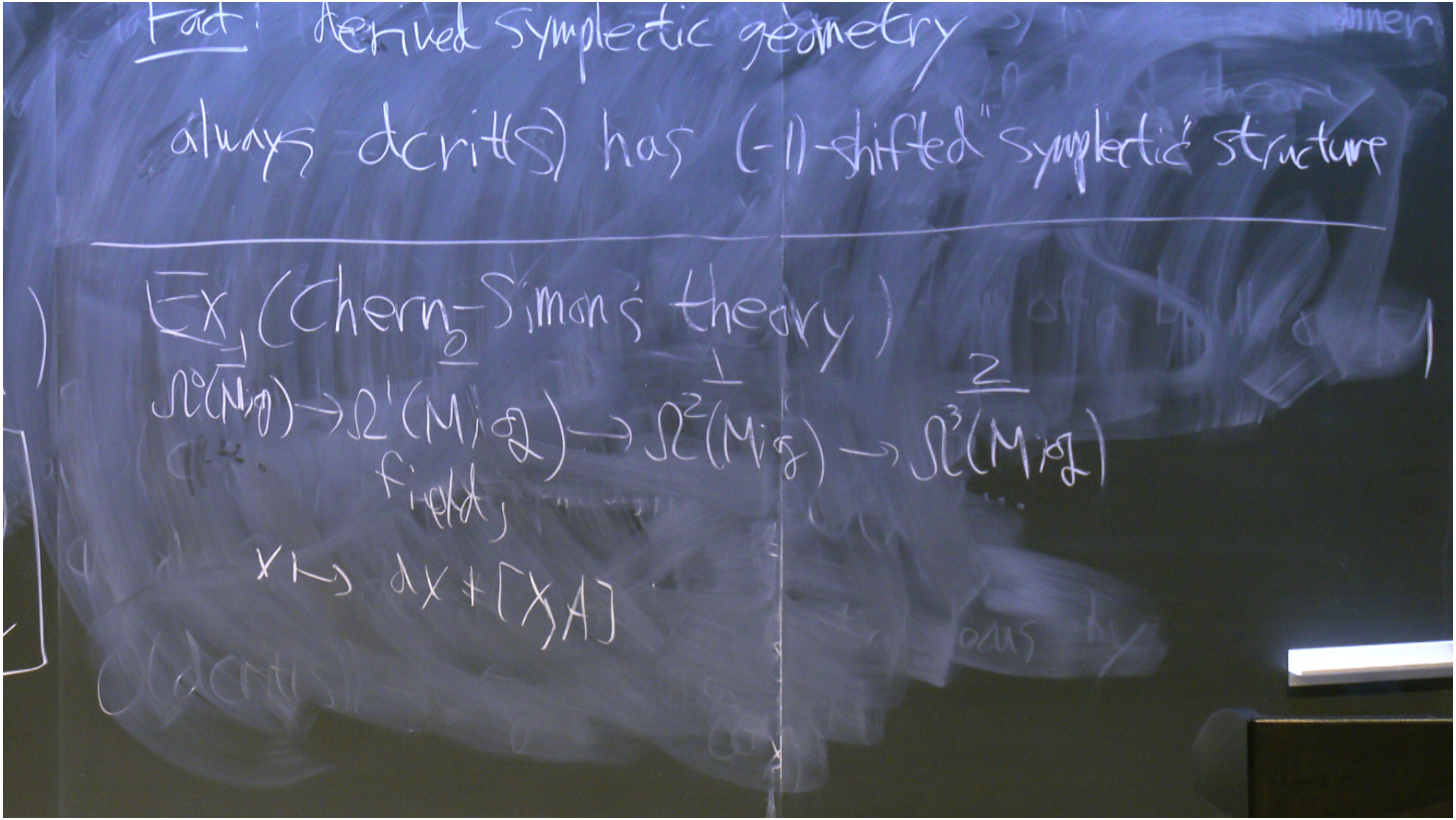

classical BV formalism = consider Crit(S) in a derived manner
Crit(S) = Graph(dS) ∩ X
T*X
where dS : X → T*X
0 : X → T*X
zero section
O(Crit(S)) = O(Graph(dS)) ⊗ O(T*X) O(X)
we define dcrit(S)

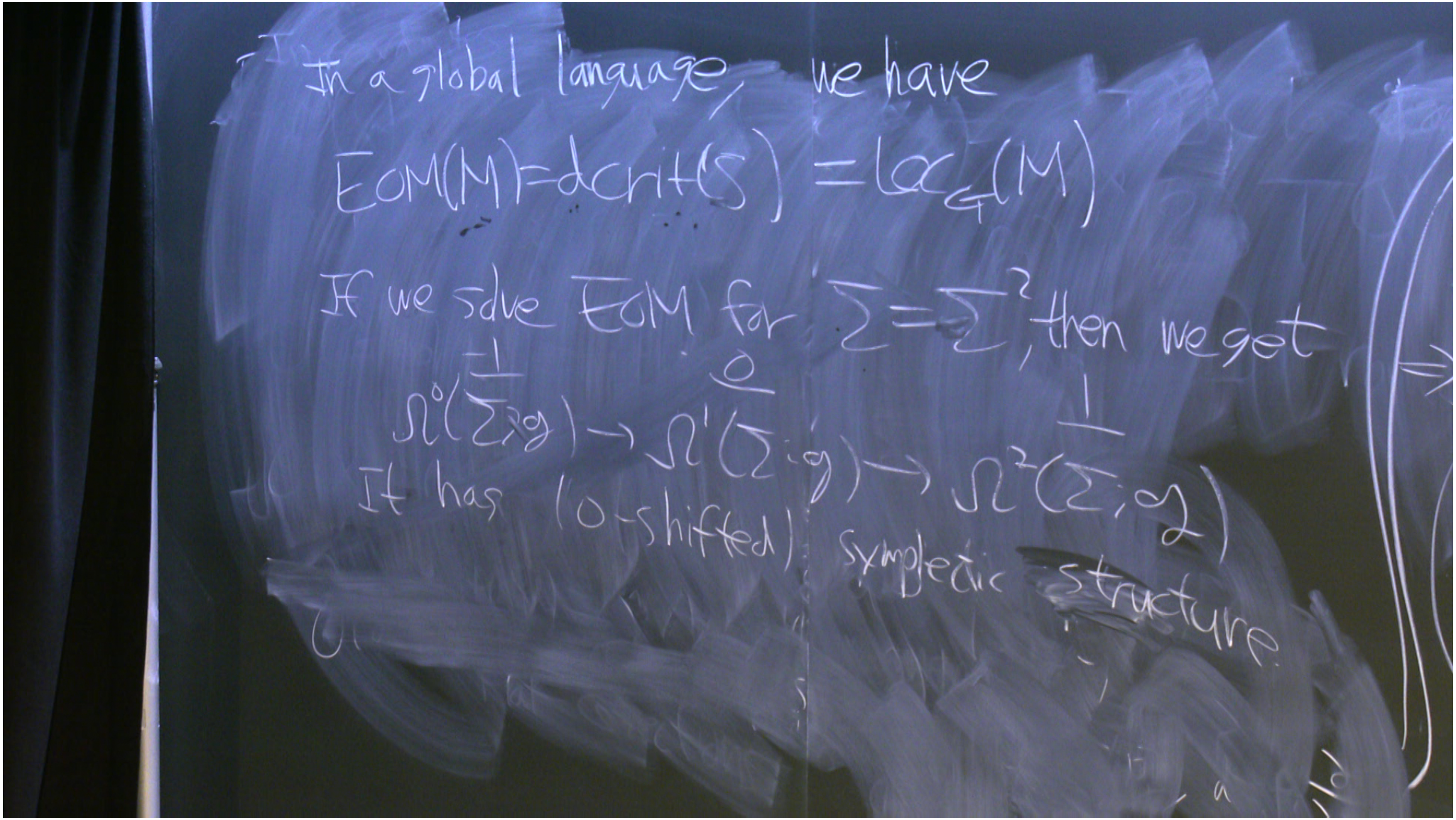

Crit(S) = Graph(dS) ∩ X
T*X
where dS : X → T*X
0 : X → T*X, zero section
O(Crit(S)) = O(Graph(dS)) ⊗ O(X)
O(T*X)
We define dCrit(S) derived critical locus by
O(dCrit(S)) = O(graph(dS)) ⊗ O(X)
O(T*X)

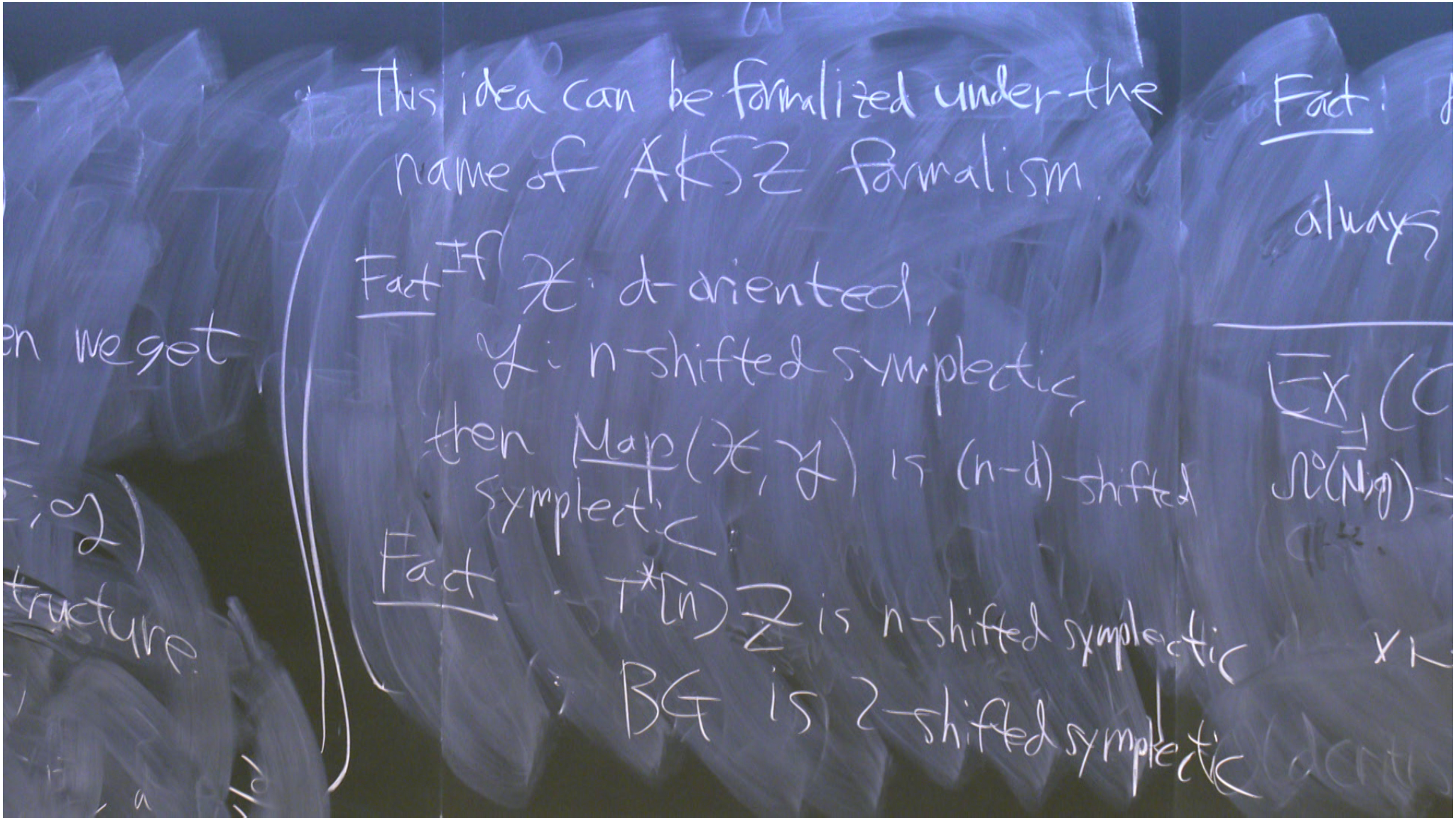

$\mathcal{O}(T^*X) = \mathbb{C}[x^i, p_i] = \mathbb{C}[x^1, \ldots, x^n, p_1, \ldots, p_n]$
$\mathcal{O}(\mathrm{Graph}(dS)) = \mathbb{C}[x^i, p_i] / \left(\frac{\partial S}{\partial x^i} - p_i\right)$
$\mathcal{O}(X) \simeq \mathcal{O}(\tilde{X}) = \left(\mathbb{C}[x^i, \xi_i, p_i],\ d\xi_i = p_i,\ \xi_i = 0\right)$
$\mathcal{O}(\mathrm{Crit}(S)) = \mathbb{C}[x^i, p_i] / \left(\frac{\partial S}{\partial x^i} - p_i\right) \otimes_{\mathbb{C}[x^i, p_i]} \left(\mathbb{C}[x^i, \xi_i, \partial_{\xi_i}],\ d - p_i \frac{\partial}{\partial \xi_i}\right)$


$\mathcal{O}(\tilde{X}) = \mathrm{Sym}_{\mathcal{O}(X)}(T_X[1] \oplus T_X)$
$\Rightarrow \mathcal{O}(\mathrm{dcrit}(S)) = (\mathrm{Sym}_{\mathcal{O}(X)}(T_X[1]), d)$
$\mathcal{O}(T^*[-1]X) := \mathrm{Sym}_{\mathcal{O}(X)}(T_X[1])$
(-1)-shifted cotangent space of X
$(\mathcal{O}(T^*[-1]X), d)$
Crit(S)
where
$\mathcal{O}(\mathrm{crit}(S))$
we define
$\mathcal{O}(\mathrm{dcrit}(S)$


$\omega = dx^i \wedge d\xi_i$
$(\deg x_i, \deg \xi_i) = (0, -1)$
$d\xi_i = \frac{\partial S}{\partial x^i}$
$\mathcal{O}(\mathcal{X}) = \mathrm{Sym}_{\mathcal{O}(X)}(T_x[1] \oplus T_x)$
$d=id$
$\mathcal{O}(\mathrm{dcrit}(S)) = (\mathrm{Sym}_{\mathcal{O}(X)}(T_x[1]), d)$
$\mathcal{O}(T^*[-1]X) := \mathrm{Sym}_{\mathcal{O}(X)}(T_x[1])$
(-1)-shifted cotangent space of
Fact: derived symplectic
always dcrit(S) has

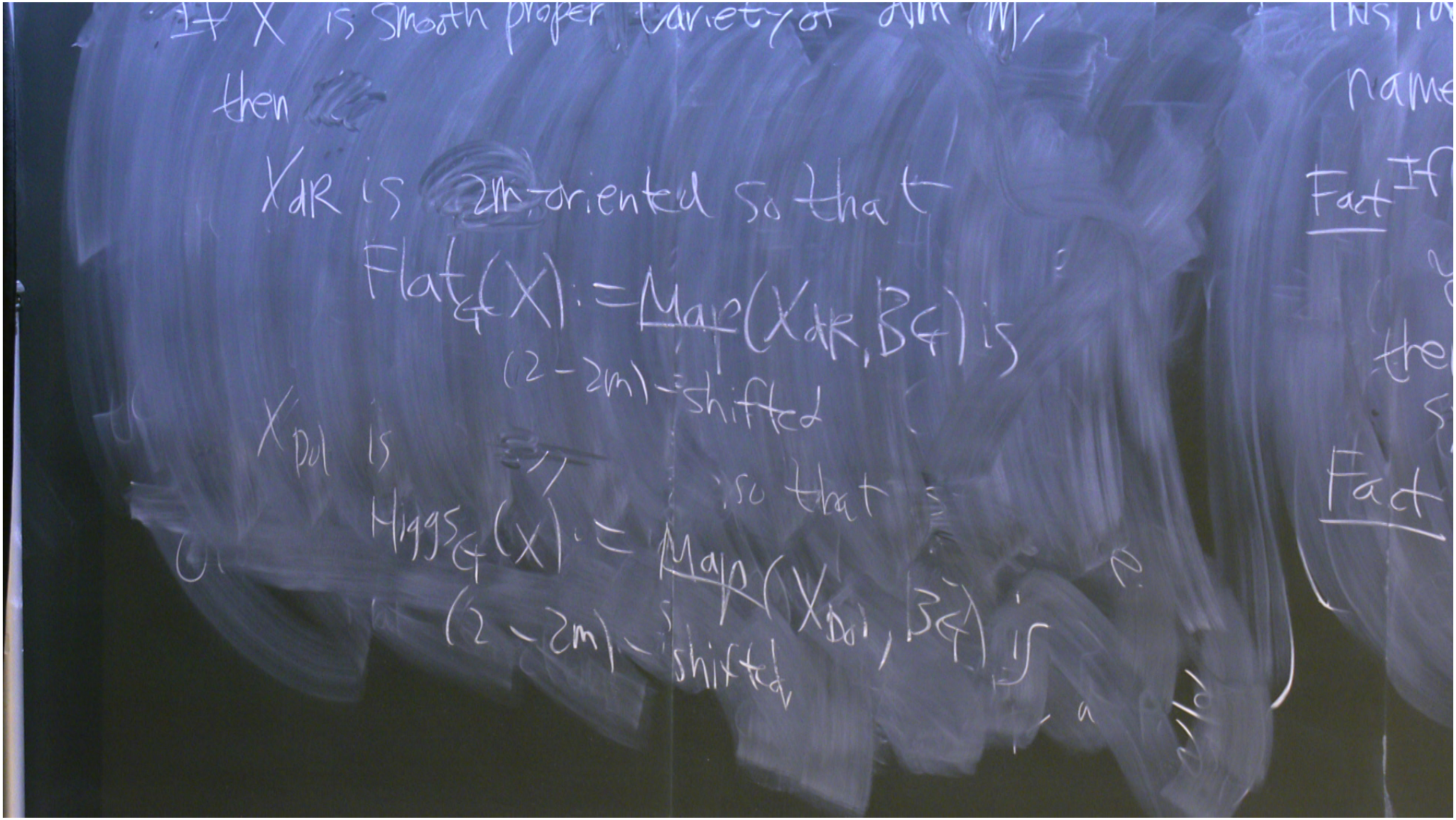

Fact: derived symplectic geometry
always dcrit(S) has (-1)-shifted "symplectic" structure
Ex (Chern-Simons theory)
$\Omega^0(M;\mathfrak{g}) \to \Omega^1(M;\mathfrak{g}) \to \Omega^2(M;\mathfrak{g}) \to \Omega^3(M;\mathfrak{g})$
1
2
field
$X \mapsto dX + [X,A]$

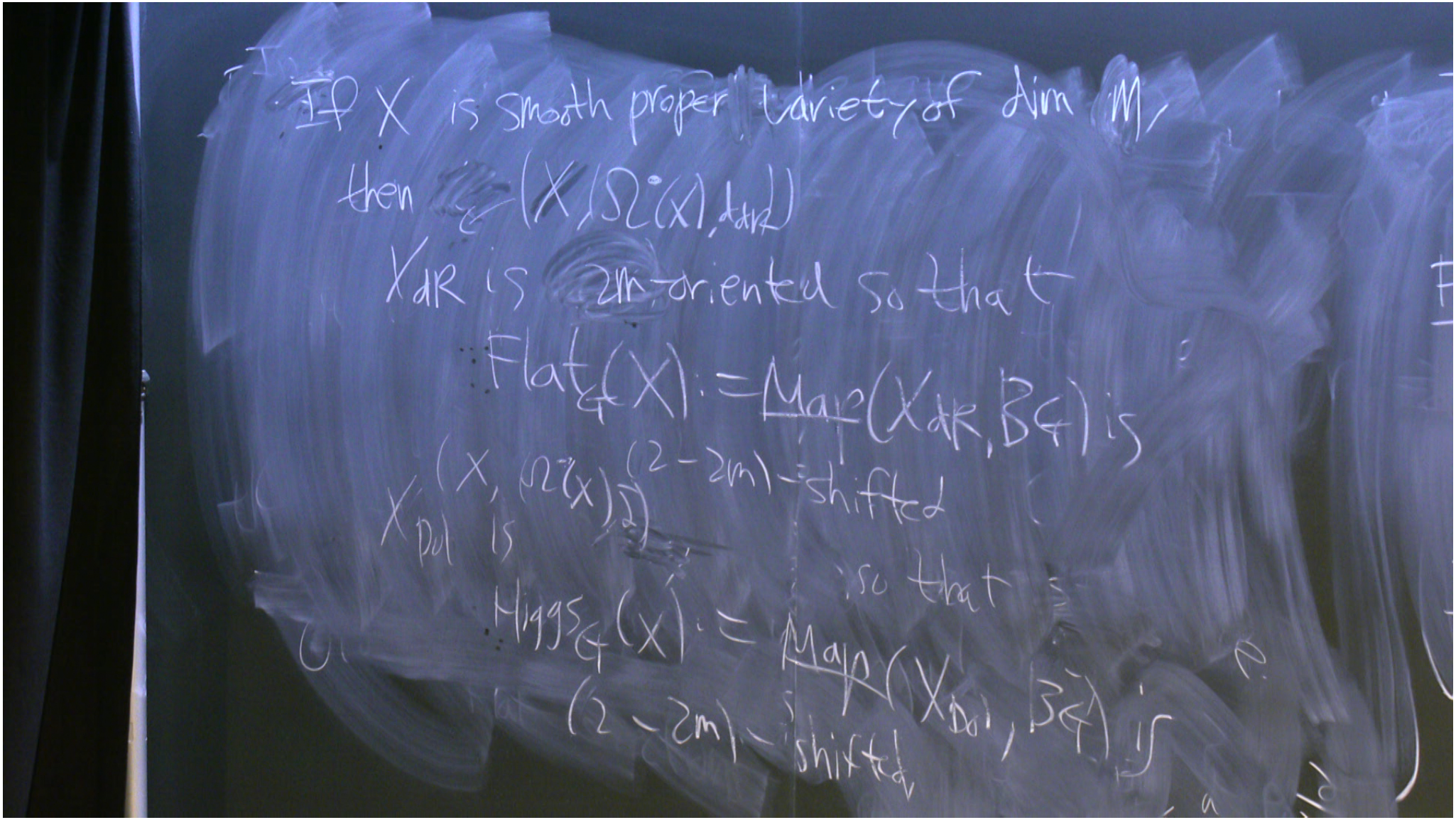

In a global language, we have
EOM(M) = dcrit(S) = loc_dR(M)
If we solve EOM for Σ = Σ², then we get
Ω⁰(Σ;g) → Ω¹(Σ;g) → Ω²(Σ;ω)
It has (0-shifted) symplectic structure.


This idea can be formalized under the name of AKSZ formalism.
Fact: If $\mathcal{X}$: d-oriented,
$\mathcal{Y}$: n-shifted symplectic,
then $\underline{\mathrm{Map}}(\mathcal{X},\mathcal{Y})$ is (n-d)-shifted symplectic.
Fact: $T^*[n]Z$ is n-shifted symplectic
BG is 2-shifted symplectic
Fact:
always
en we get
tructure

Fact. If M is compact oriented d-manifold, then M_B is d-oriented. In particular, $\mathrm{Loc}_G(M) := \mathrm{Map}(M_B, BG)$ is $(2-d)$-shifted

If X is smooth proper CY d-fold, then X is d-oriented. In particular, $\mathrm{Bun}_G(X) := \mathrm{Map}(X, BG)$ is $(2-d)$-shifted


If X is smooth proper variety of dim n,
then
X_{dR} is 2n-oriented so that
$\mathrm{Flat}_G(X) := \underline{\mathrm{Map}}(X_{dR}, BG)$ is
(2-2n)-shifted
X_{Dol} is
so that
$\mathrm{Higgs}_G(X) := \underline{\mathrm{Map}}(X_{Dol}, BG)$ is
(2-2n)-shifted
Fact


If X is smooth proper variety of dim m,
then (X, Ω°(X), d_dR)
X_dR is 2m-oriented so that
Flat_G(X) := Map(X_dR, BG) is
(X, Ω²(X))
(2-2m)-shifted
X_Dol is
so that
Higgs_G(X) := Map(X_Dol, BG) is
(2-2m)-shifted

n-shifted symplectic $\Rightarrow$ (n+1)-dim'l theory

Ex: X symplectic

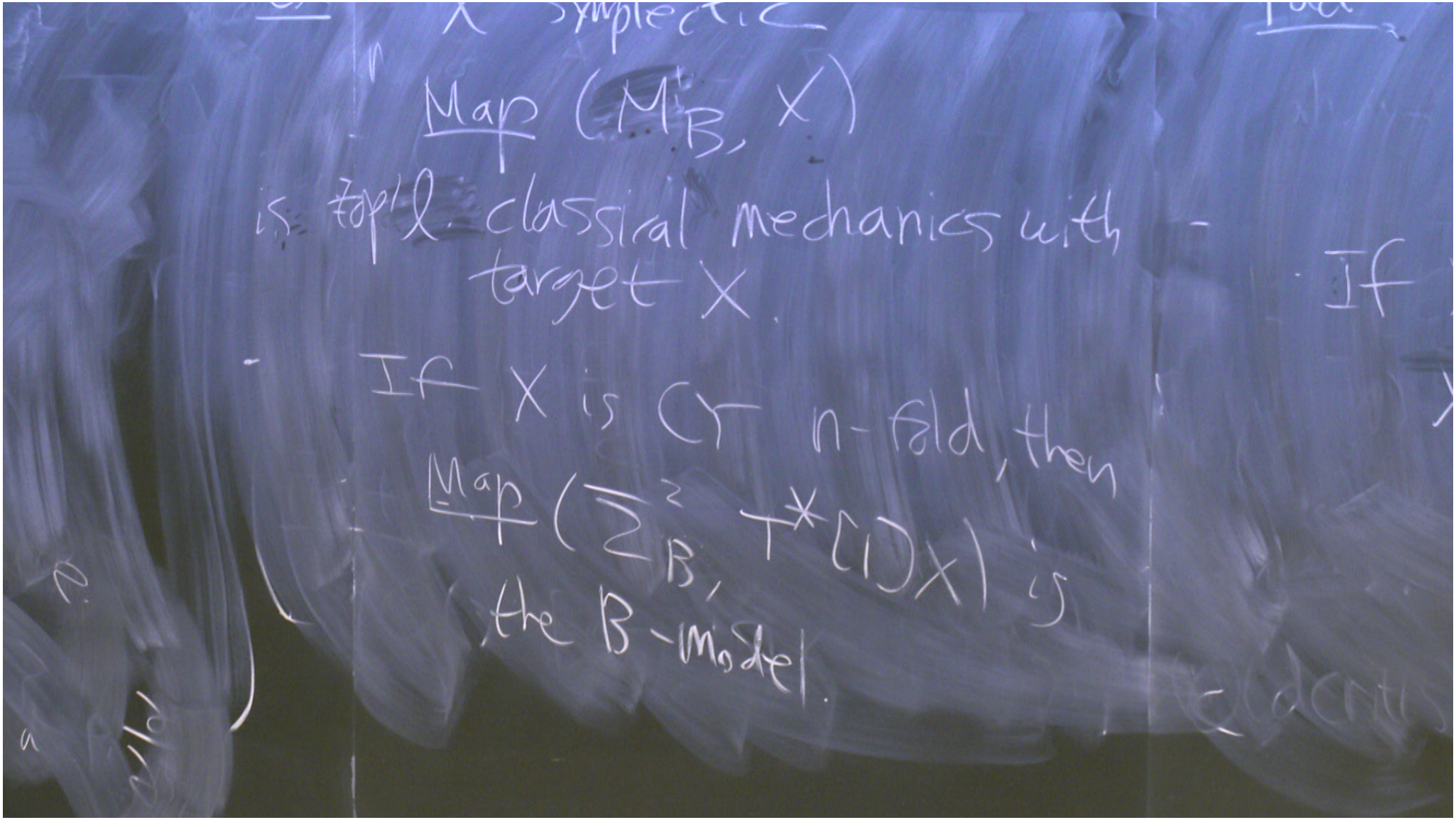
Map (M_B, X)

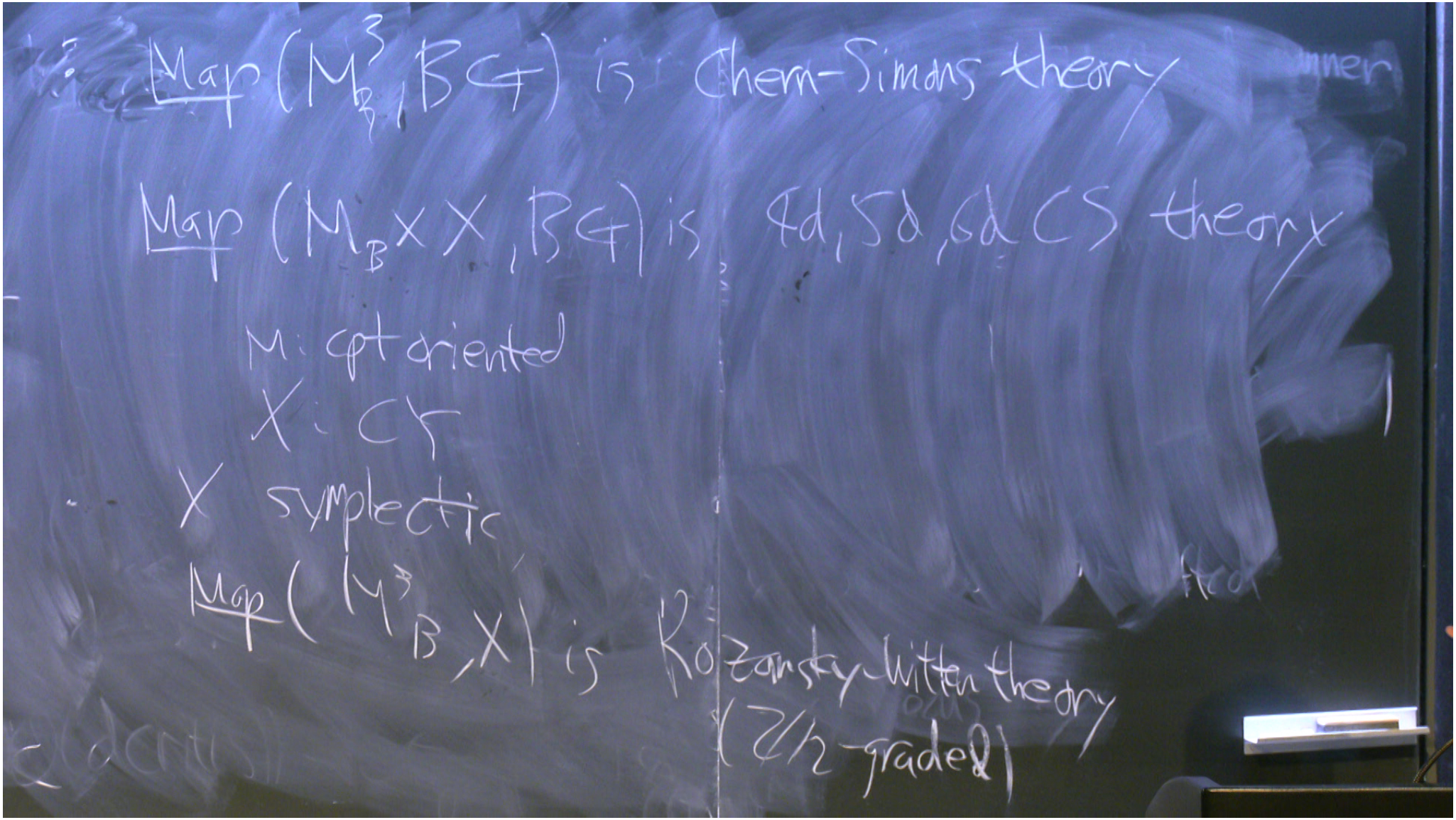
is top'l classical mechanics with target X.

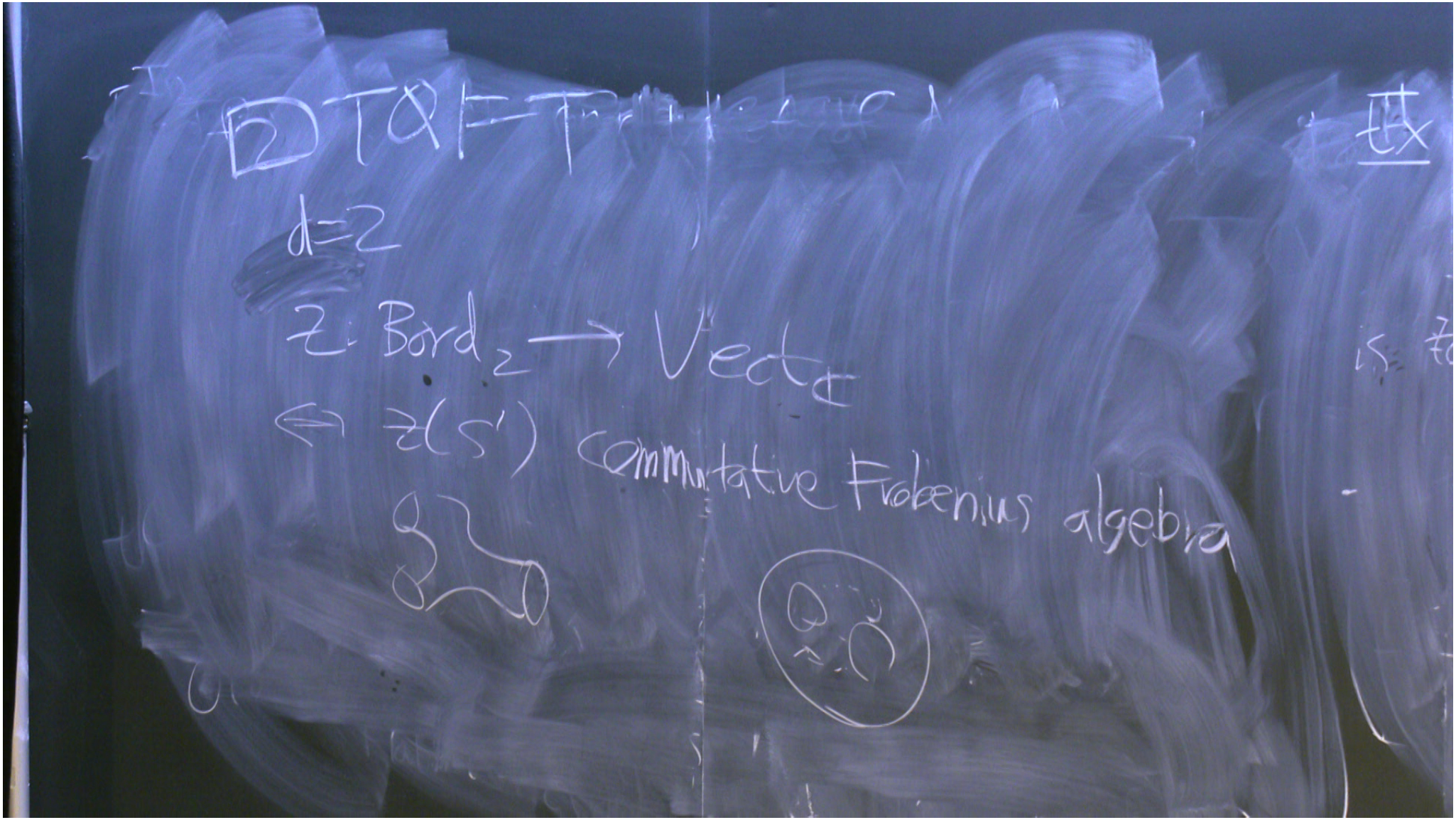
Fact:

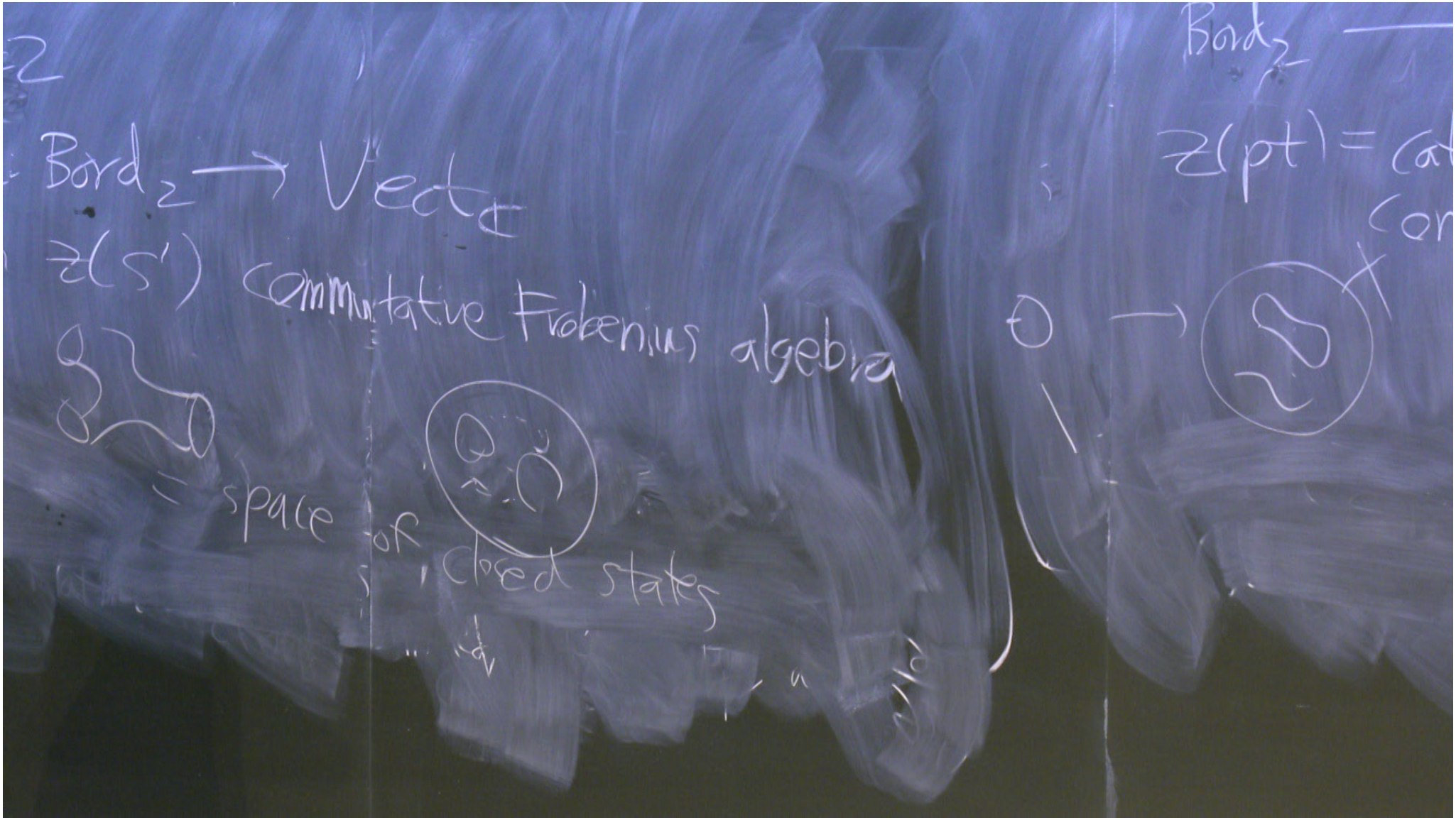
If

X


Map (M_B, X)
is top'l. classical mechanics with target X.
If X is CY n-fold, then
Map $(\Sigma^2_B, T^*[1]X)$ is the B-model.


Map (M^3_B, BG) is Chern-Simons theory
Map $(M_B \times X, BG)$ is 4d, 5d, 6d CS theory
M: spt oriented
X: CY
X symplectic
Map (M^3_B, X) is Rozansky-Witten theory
(Z/2-graded)


d=2
Z: Bord_2 → Vect_C
≅ Z(S^1) commutative Frobenius algebra


Bord$_2$ → Vect$_{\mathbb{C}}$
commutative Frobenius algebra
= space of closed states
Bord$_2$

$Z(pt)$ = category of boundary conditions



$B_1, B_2 \in Z(pt)$

$Z(-; B_1, B_2) = \mathrm{Hom}_{Z(pt)}(B_1, B_2)$

= space of open string states satisfying boundary conditions B_1, B_2 at the ends,

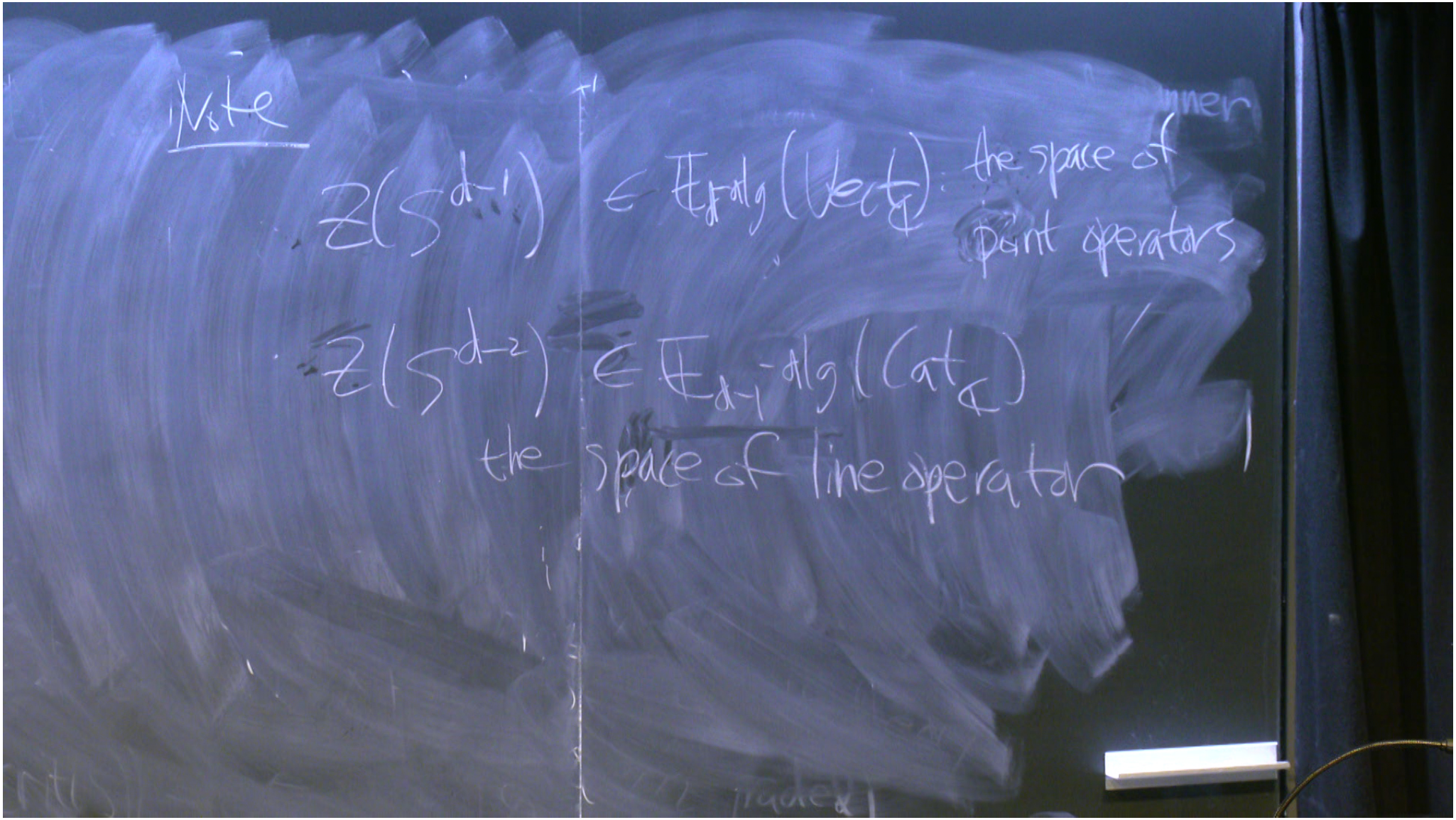
$\underline{\mathrm{Map}}(M_B \times X$

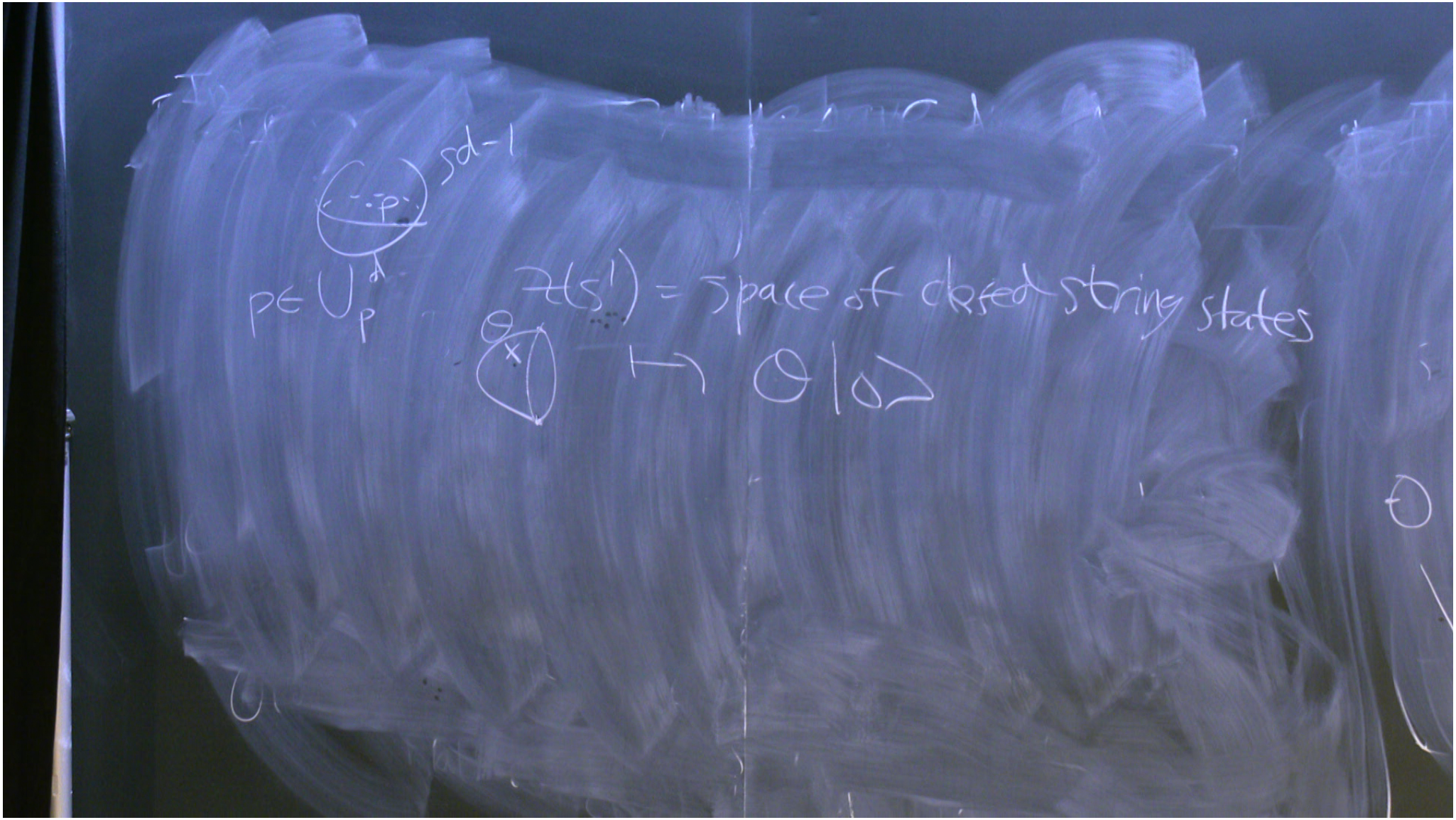
M: cpt orie

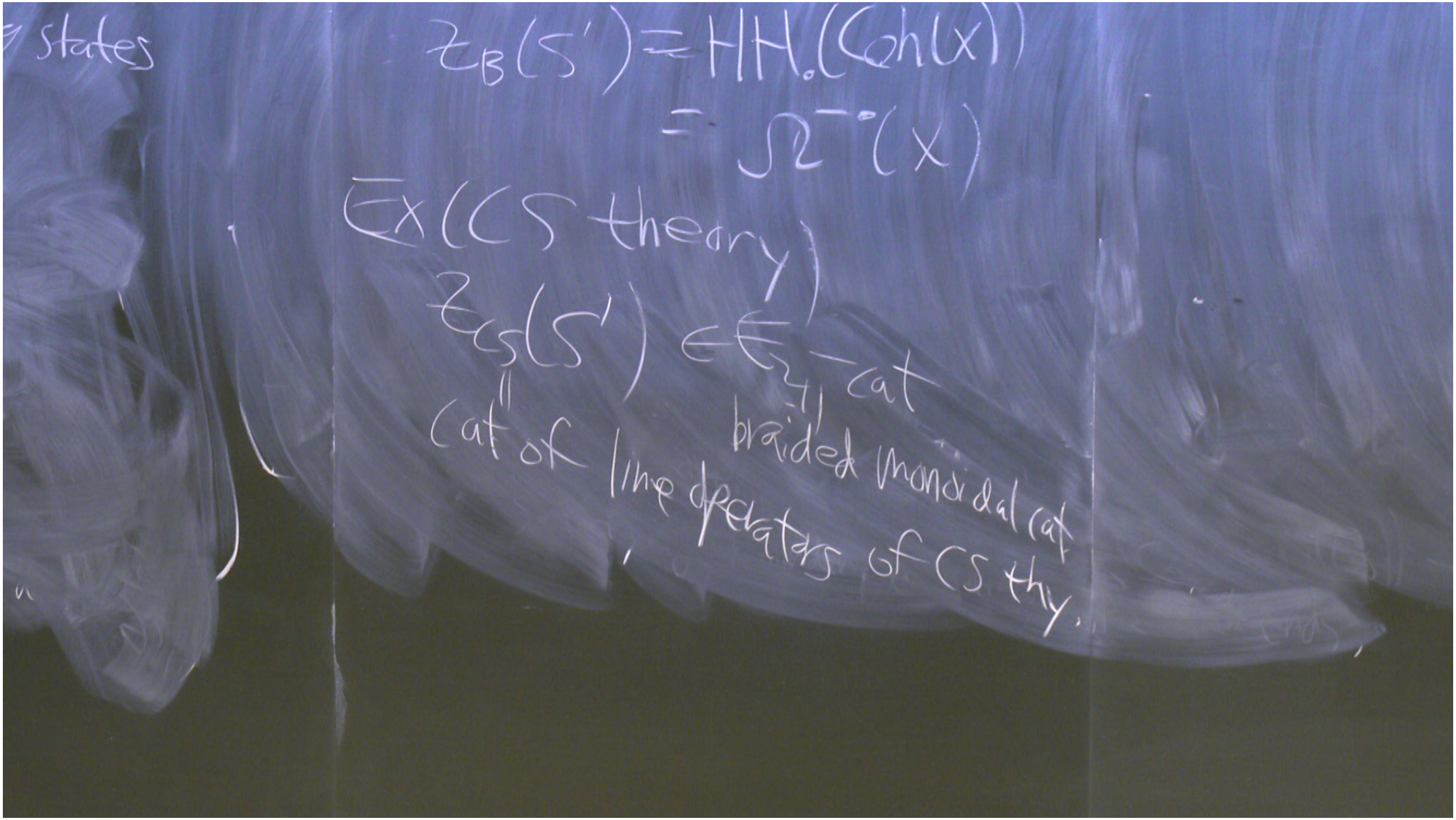
X: CY

X symplecti

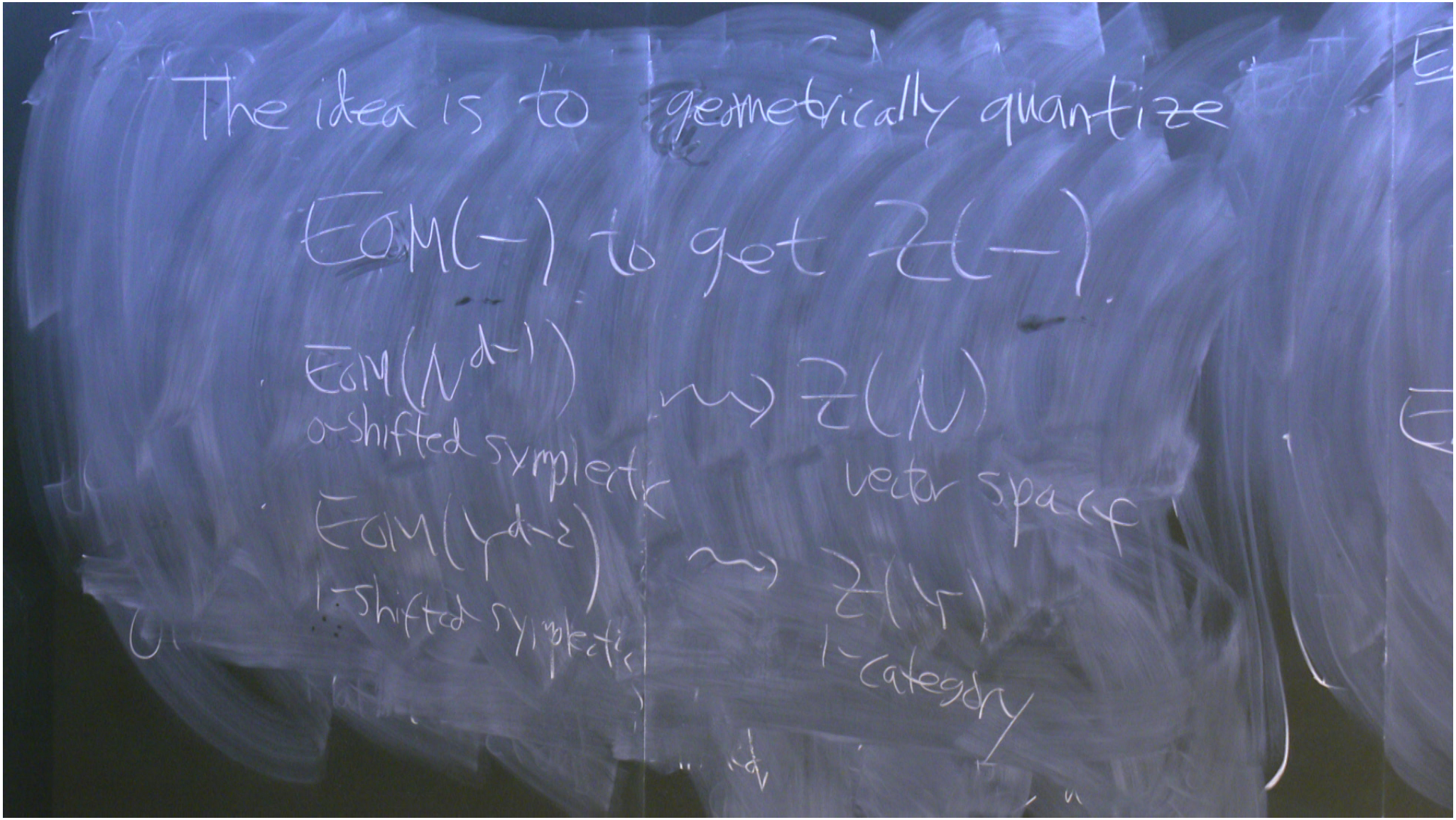
$\underline{\mathrm{Map}}(M_B,$

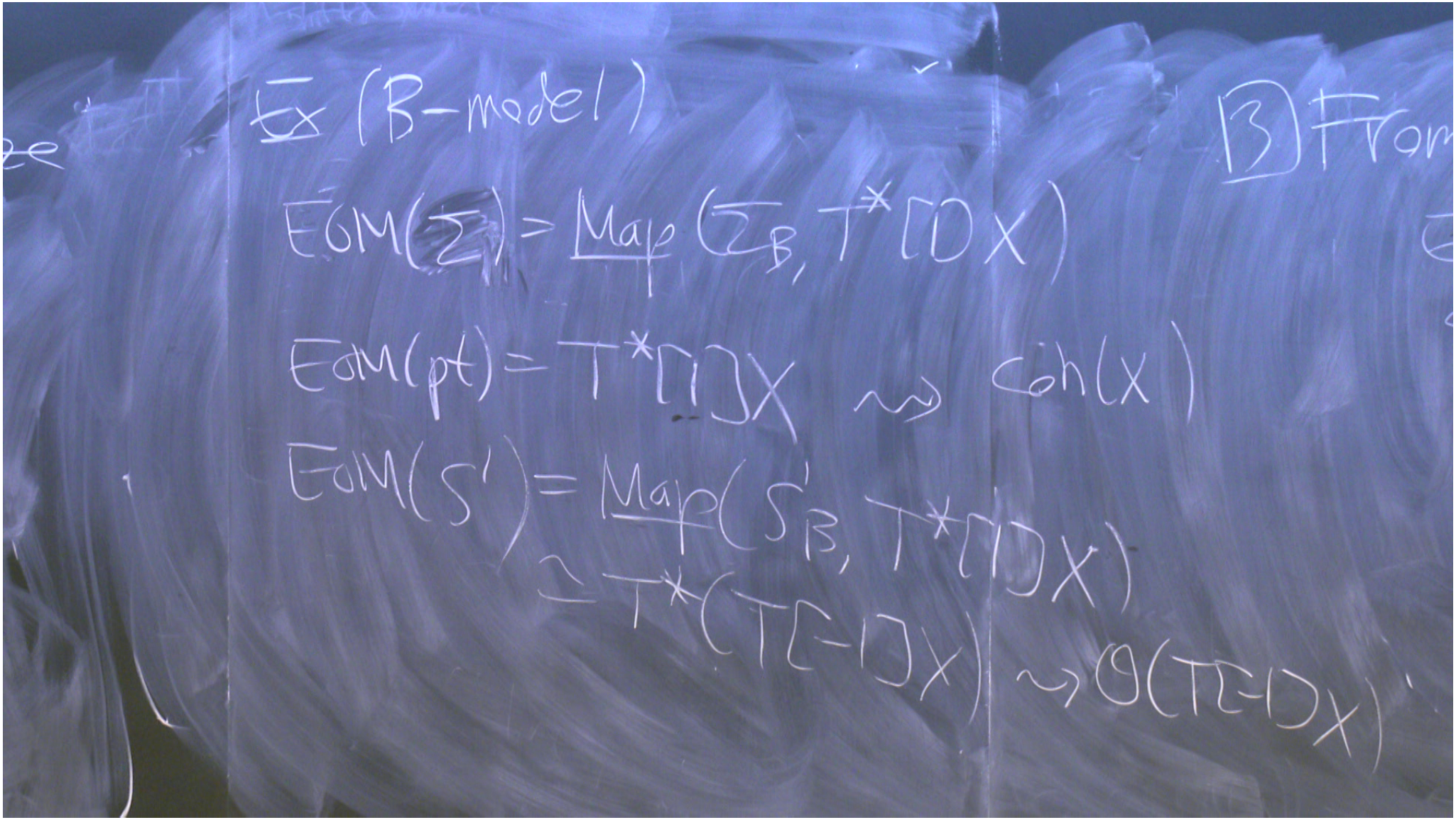

Note
$Z(S^{d-1}) \in E_{d-1}\text{-alg}(\mathrm{Vect}_{\mathbb{C}})$: the space of point operators
$Z(S^{d-2}) \in E_{d-1}\text{-alg}(\mathrm{Cat}_{\mathbb{C}})$
the space of line operator

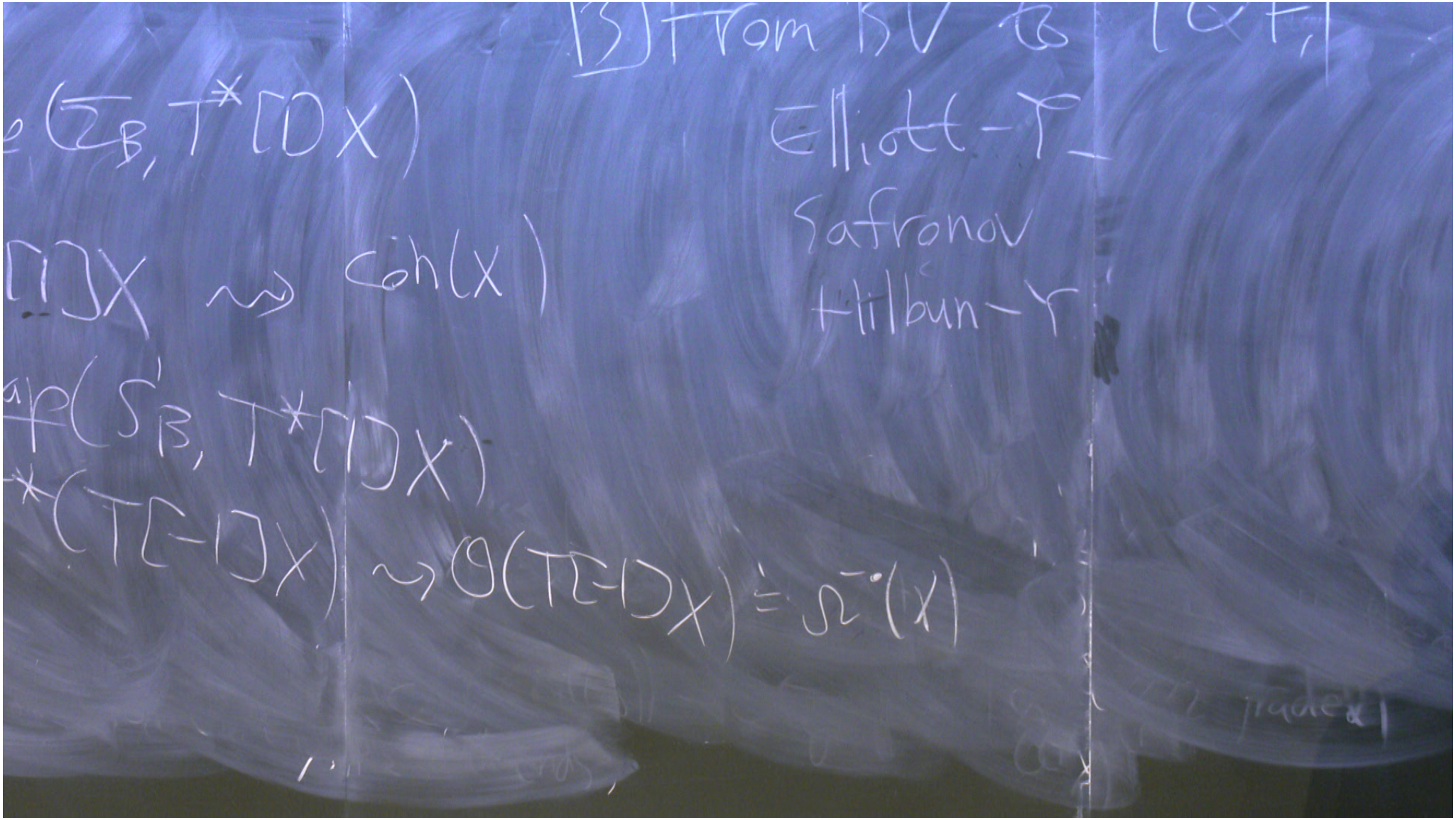

Sd-1
p
$p \in U_p$
$\mathcal{H}(S^1)$ = space of closed string states
Θ
$\Theta \mapsto \Theta|0\rangle$

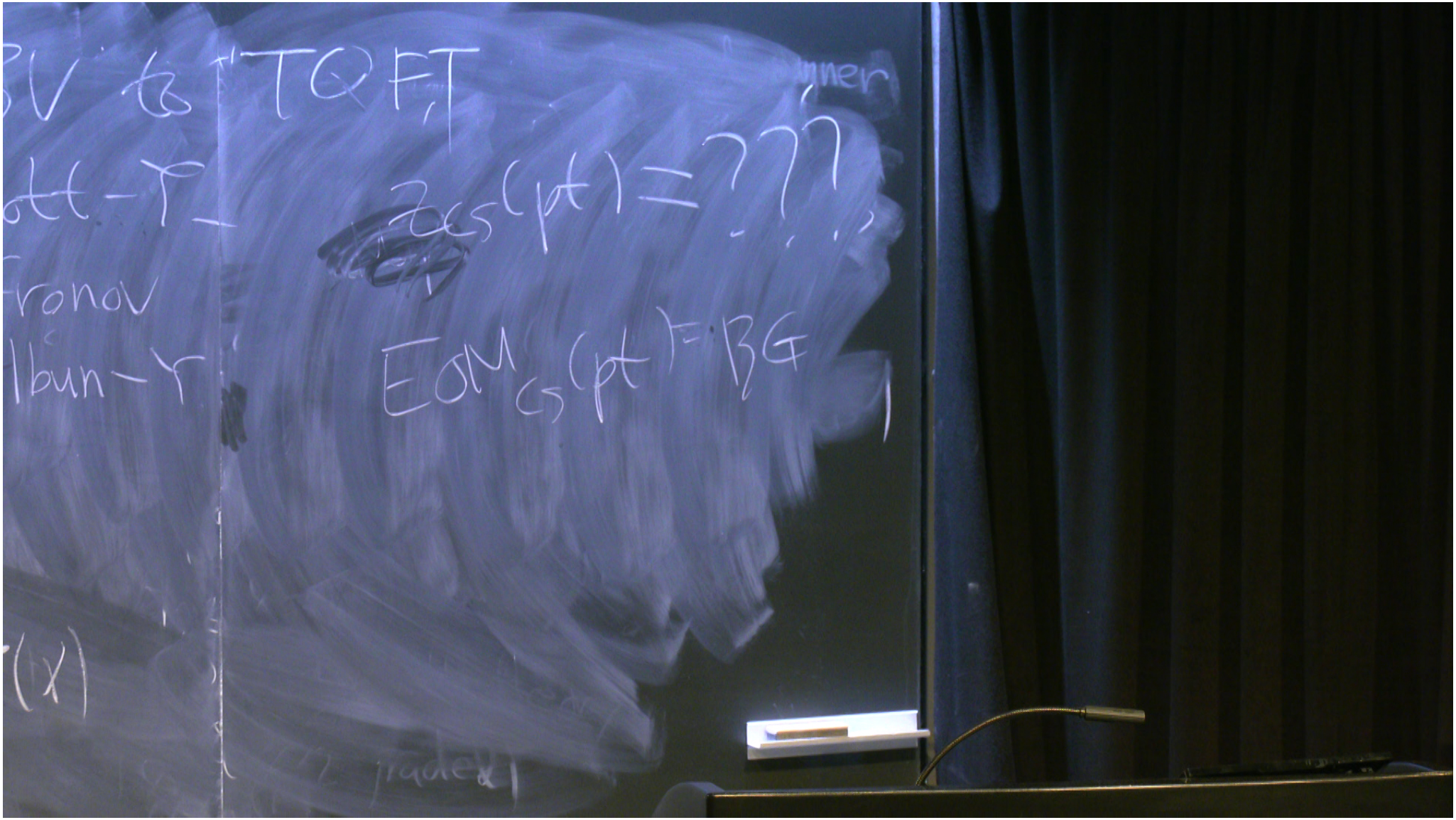

states
$Z_B(S^1) = HH_\bullet(Coh(X))$
$= \Omega^{-\infty}(X)$
$\in Ex(CS\ theory)$
$Z_{CS}(S^1) \in$
∞-cat
braided monoidal cat
Cat of line operators of CS thy.


B From BV & TQFT
Elliott-Y
Safronov
Hilbun-Y


The idea is to geometrically quantize
EoM(–) to get Z(–).
EoM(N^{d-1})
0-shifted symplectic
$\rightsquigarrow$ Z(N)
vector space
EoM(Y^{d-2})
1-shifted symplectic
$\rightsquigarrow$ Z(Y)
1-category


Ex (B-model)
EoM(Σ) = Map(Σ_B, T*[1]X)
EoM(pt) = T*[1]X ⇝ Coh(X)
EoM(S¹) = Map(S¹_B, T*[1]X)
≃ T*(TΣ[-1]X) ⇝ O(TΣ[-1]X)


From BV to
Elliott-T
Safronov
Hilbun-T
Coh(X)
Ω^{-•}(X)


TQFT
Eom_CS(pt)=BG